

Robustness of surface edge states in the presence of weak, localized disorderJosselin Garnier¹ and Basant Lal Sharma²¹*CMAP, CNRS, Ecole Polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France*²*Department of Mechanical Engineering, Indian Institute of Technology Kanpur, Kanpur, Uttar Pradesh 208016, India*

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Edge states are expected to allow for robust, nearly lossless transport along surfaces and interfaces in the presence of disorder. Here we show that this expectation can break down when localized random surface perturbation enables coupling between surface waves and bulk modes, even when it is a weak perturbation. We study the transmission of edge states in a conservative, prototypical spring-mass lattice half plane whose boundary is perturbed by finitely many random variations of surface masses or surface springs. We analyze how an incident surface wave is reflected, transmitted, and partially radiated into the bulk. Using a modal expansion on the orthonormal eigenfunctions of the unperturbed system and a multiscale analysis of the resulting random scattering problem, we apply diffusion approximation theory to characterize the statistics of reflectivity, transmissivity, and radiative loss. This approach yields explicit expressions for the moments of these quantities in the regime of weak disorder and large propagation distances. We identify structural and frequency regimes in which radiative losses into the bulk become substantial, reaching 20-80% of the incident energy flux carried by the edge state in the high-frequency range. Our results demonstrate that ballistic edge transport is not guaranteed by the mere existence of an edge mode but depends sensitively on disorder characteristics and unperturbed surface structure. These findings suggest alternative possibilities for engineering surface properties to control wave transport, with implications for phononic, photonic, and electronic systems, including graphene-based structures and nanoscale materials.

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Ballistic transport is typically associated with ordered structures, while disorder is often viewed as a prerequisite for diffusion or localization. This distinction appears across a wide range of physical systems, from crystalline solids and waveguides to photonic, seismic, and biological media. It raises a fundamental question: Under what conditions can ballistic transport persist in the presence of disorder, and when does disorder inevitably degrade it?

On the one hand, the use of ordered structures appears in studies on crystal physics [1] and in engineering [2], besides the classical engineering applications [3], tracing back to Rayleigh. On the other hand, the ubiquitous concept of diffusive transport in a lattice, due to disorder, entered the physics literature during the quest to understand the conductance in metals, via free electrons, by Drude [4]. Against this backdrop, the discovery of the absence of diffusion in certain disordered crystals [5] gave a strong impetus to research on a large number of questions related to random media. Indeed, structural perturbations, irregularities or disorder, are omnipresent in natural as well as man-made systems, motivating the questions of weak [6] or strong localization [7] in quantum mechanics, solid-state physics [8], matter waves [9], photonics [10] (with effects such as Lévy flight [11] and light harvesting [12]), oceanic systems [13], seismic systems [14], study of patterns and processes in biological systems [15,16], and several other physical systems relevant in science and technology [17,18]. Practical issues in the origin of randomness notwithstanding [19,20], it is well-known that transport,

in particular, by waves, is heavily influenced by the existence of disorder [21]. The coexistence of ballistic and diffusive transport in a static, ordered system has also been observed [22], while it is generally agreed that all systems that exhibit sub-ballistic transport are inherently disordered.

A prominent setting in which ballistic transport is expected is the propagation of edge states, as discussed in [23] and [24]. Topological insulators provide a particularly important example of such transport, exhibiting an insulating bulk and conducting boundary states protected by topology. This has been demonstrated in [25,26] and further reviewed in [27]. Similar robustness has been observed in bosonic counterparts such as topological magnon insulators; for instance, [28] showed that random defects strongly suppress bulk transport while only weakly affecting edge currents. Such findings indicate that topologically protected edge states are not unique to electronic systems but elegantly extend to photonic and spintronic platforms, as discussed in [29,30]. As an example, the magnetic and coherent transport properties of small-width zigzag graphene nanoribbons with monohydrogen edge passivation are investigated in [31] as a function of random edge-vacancy disorder and ribbon length. In the results of [32], it is also found that the edge states in the topological phase are robust with respect to spatial disorder. According to [33], neglecting absorption losses and nonlinear effects, certain edge states are also immune to backscattering from disorder, obstacles and defects. A single topologically protected edge mode is realized in [33,34] that propagated around arbitrary disorder without reflection. This robustness has led to the prevailing view that the existence of an edge state is

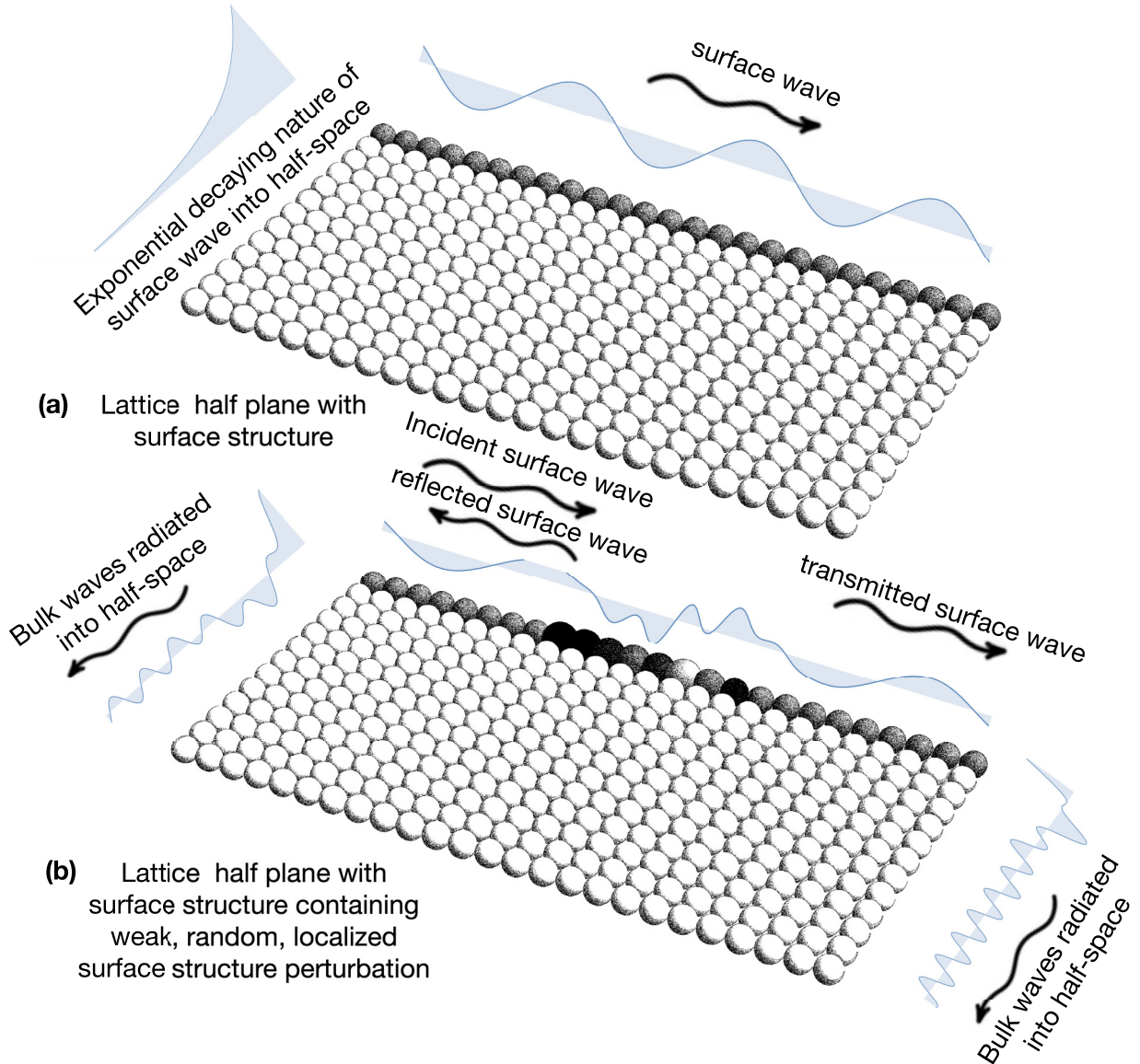


FIG. 1. A schematic presentation of (a) edge state on unperturbed lattice half plane with surface structure and (b) edge state scattering on lattice half-plane with perturbed surface structure.

sufficient to guarantee efficient surface transport. In most of these studies, however, attention has focused on backscattering along the edge, while energy leakage from surface-guided modes into bulk degrees of freedom has received comparatively little attention.

The observations mentioned in the previous paragraph raise the question of whether ballistic transport of edge states truly persists under realistic, weak surface disorder. This motivates the investigations of the present article. We investigate edge state transmission in a discrete spring-mass lattice half plane that supports surface waves (see Fig. 1). This model provides a minimal yet physically transparent setting in which surface-guided modes coexist with bulk lattice waves [35,36], allowing for the possibility of disorder-induced coupling between them. We focus on the effect of weak, localized randomness confined to a finite surface region, introduced through perturbations of surface masses or surface springs.

The methodology can be summarized as follows. We expand the wave field on the complete set of orthonormal eigenfunctions of the unperturbed lattice [37] and study the resulting random scattering problem using a multiscale approach. Applying diffusion approximation theory to the modal amplitudes, we derive explicit expressions for the statistical moments of reflectivity, transmissivity, and radiative loss. Unlike our earlier work [38], which addressed only surface mass disorder, the present analysis incorporates both surface mass and surface spring perturbations and reveals alternative mechanisms for energy leakage into the bulk. We also note that a similar multiscale approach has been applied to the acoustic or elastic wave equations in randomly scattering media to derive radiative transfer equations that account for coupling between surface waves and body waves in half-spaces or waveguide structures [39,40].

A central result of this study is that, despite the presence of an edge mode, weak surface disorder can lead to

substantial radiative losses into the bulk. In particular, we identify structural and frequency regimes in which 20-80% of the energy flux carried by a propagating surface wave is converted into bulk radiation. This finding runs counter to the common expectation that edge states necessarily support nearly lossless transport. The results have implications for electronic transport in graphene structures using tight-binding approximation and heat flux via short-wavelength surface phonons in nanoscale crystals structures under linear local interactions. Indeed, recent developments and futuristic plans in nanofabrication [41] indicate a trend toward propelling the research in materials that lies between the extremes of order and disorder that can yield innovative physical, or rather multiphysical, phenomena owing to their engineered disordered patterns [42–45]. Engineered disorder has begun to transform the fields of photonics, phononics, and the next generation of materials [42,43,46–49]. In this connection, our findings suggest a possible surface engineering tool that enables control over frequency-based radiative losses [50,51].

The article is organized as follows. In the next section, Analysis, we study the case of random surface mass and surface spring perturbations by a multiscale analysis in which the solution is expanded on the set of orthonormal eigenfunctions for the unperturbed system. A diffusion approximation theory is applied to the modal coefficients which allows us to obtain the moments of the reflectivity, transmissivity, and radiation losses in the presence of these random perturbations. In this section, in our exposition we also emphasize crucial assumptions which are stated progressively. In the Discussion section, we argue how the formulas derived in this article can be used to solve inverse problems and how the optimal surface parameters, for maximal radiation losses to the bulk, vary. This is followed by the Concluding remarks section. Supplemental Material [52] includes the proofs of the key propositions and auxiliary graphical plots.

II. ANALYSIS

A. Unperturbed lattice

The ideal spring–mass lattice model is a uniform lattice half plane [see Fig. 1(a)]. The equation of motion of the particles on the portion of lattice half plane $\mathring{\mathbb{H}} := \{(x, y) \in \mathbb{Z}^2 : y \in \mathbb{Z}^-\}$, away from its boundary $\partial\mathbb{H} := \{(x, y) \in \mathbb{Z}^2 : y = 0\}$, is

$$\Delta u_{x,y} + \omega^2 u_{x,y} = 0, \quad (x, y) \in \mathring{\mathbb{H}}, \quad (1)$$

where

$$\Delta u_{x,y} \equiv u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + u_{x,y-1} - 4u_{x,y}. \quad (2)$$

The equation of motion of the particles on the lattice half-plane boundary $\partial\mathbb{H}$ is

$$\alpha_s(u_{x+1,y} + u_{x-1,y} - 2u_{x,y}) + (u_{x,y-1} - u_{x,y}) + m_s \omega^2 u_{x,y} = 0 \quad (3)$$

for $x \in \mathbb{Z}$, $y = 0$, where $\alpha_s > 0$ denotes the elastic coefficient (spring constant) for nearest-neighbor interactions at the surface and $m_s > \alpha_s$ represents the mass of particles at $\partial\mathbb{H}$. The surface mass of particles and surface springs differ from those in the lattice half plane, which permits an existence of a surface wave band [35,36] that we briefly describe.

The solution $u : \mathbb{H} \rightarrow \mathbb{C}$, which satisfies the equation of motion (1) with the surface condition (3) on $\partial\mathbb{H}$, can be expanded as the superposition of the orthonormal modes ϕ_γ (for $\gamma \in (\omega^2 - 4, \omega^2) \cup \{\gamma_0\}$) described in the Appendix:

$$u_{x,y} = \check{u}_{\gamma_0}(x)\phi_{\gamma_0}(y) + \int_{\omega^2-4}^{\omega^2} \check{u}_\gamma(x)\phi_\gamma(y)d\gamma, \quad (x, y) \in \mathbb{H}. \quad (4)$$

Here, $\mathbb{H} = \mathring{\mathbb{H}} \cup \partial\mathbb{H}$. The modal amplitudes in (4) satisfy the uncoupled difference equations

$$\check{u}_\gamma(x+1) + \check{u}_\gamma(x-1) + (\gamma - 2)\check{u}_\gamma(x) = 0, \quad x \in \mathbb{Z}. \quad (5)$$

In (4), the component $\check{u}_{\gamma_0}(x)\phi_{\gamma_0}(y)$ has the form

$$\phi_{\gamma_0}(y) = \sqrt{\alpha_s + (e^{2\beta_0} - 1)^{-1}} e^{\beta_0 y}, \quad y \leq 0, \quad (6)$$

and

$$\check{u}_{\gamma_0}(x) = ae^{ik(\gamma_0)x} + be^{-ik(\gamma_0)x}, \quad x \in \mathbb{Z},$$

for some $a, b \in \mathbb{C}$. Here $\gamma_0 > \omega^2$ and $\beta_0 > 0$ are such that $\omega^2 = 2 + \gamma_0 - 2 \cosh \beta_0$, $m_s \omega^2 = \alpha_s \gamma_0 + 1 - e^{-\beta_0}$, and the wave number $k(\gamma_0) \in (0, \pi)$ is such that $\gamma_0 = 4 \sin^2(k(\gamma_0)/2)$. This component decays exponentially into $\mathbb{H}$ as a function of y with the attenuation coefficient β_0 and it propagates (oscillates with the wave number $k(\gamma_0)$) over arbitrarily large distances as a function of x , hence it is a propagative surface wave. The range of frequency values ω for which such a surface mode exists is $(0, \omega_{\max})$, where $\omega_{\max}$ is given by

$$\omega_{\max} = \sqrt{\left(-b_s + \sqrt{b_s^2 - 4a_s c_s}\right)/(2a_s)}, \quad (7)$$

with $a_s = m_s(m_s - 1)$, $b_s = -8\alpha_s m_s + 4\alpha_s + 4m_s + 1$, $c_s = 16\alpha_s^2 - 16\alpha_s - 4$ (note that $\omega_{\max} \in (0, 2)$). For each given $\omega \in (0, \omega_{\max})$, there exist two surface wave modes proportional to $e^{-ik(\gamma_0)x + \beta_0 y}$ and $e^{ik(\gamma_0)x + \beta_0 y}$, which can be classified as a left-going and a right-going wave based on the sign of the propagation of energy flux. From now on, we always consider frequencies $\omega \in (0, \omega_{\max})$.

In (4) the components $\check{u}_\gamma(x)\phi_\gamma(y)$ for $\gamma \in (\omega^2 - 4, \omega^2)$ are body waves, the functions $\phi_\gamma(y)$ are bounded and oscillatory as functions of y . The components obey the well-known dispersion relation described below [see also [53] and Figs. 2(b) and 2(c)].

(1) If $\gamma \in (0, \omega^2)$, then $\check{u}_\gamma(x)$ has the form

$$\check{u}_\gamma(x) = ae^{ik(\gamma)x} + be^{-ik(\gamma)x}, \quad x \in \mathbb{Z}$$

for some $a, b \in \mathbb{C}$, with

$$k(\gamma) = \arccos\left(1 - \frac{\gamma}{2}\right) \in (0, \pi), \quad \gamma \in (0, \omega^2), \quad (8)$$

which shows that it is a propagative mode [the mode $e^{ik(\gamma)x}$ is right going, $e^{-ik(\gamma)x}$ is left going].

(2) If $\gamma \in (\omega^2 - 4, 0)$, then $\check{u}_\gamma(x)$ has the form

$$\check{u}_\gamma(x) = ae^{k(\gamma)x} + be^{-k(\gamma)x}, \quad x \in \mathbb{Z}$$

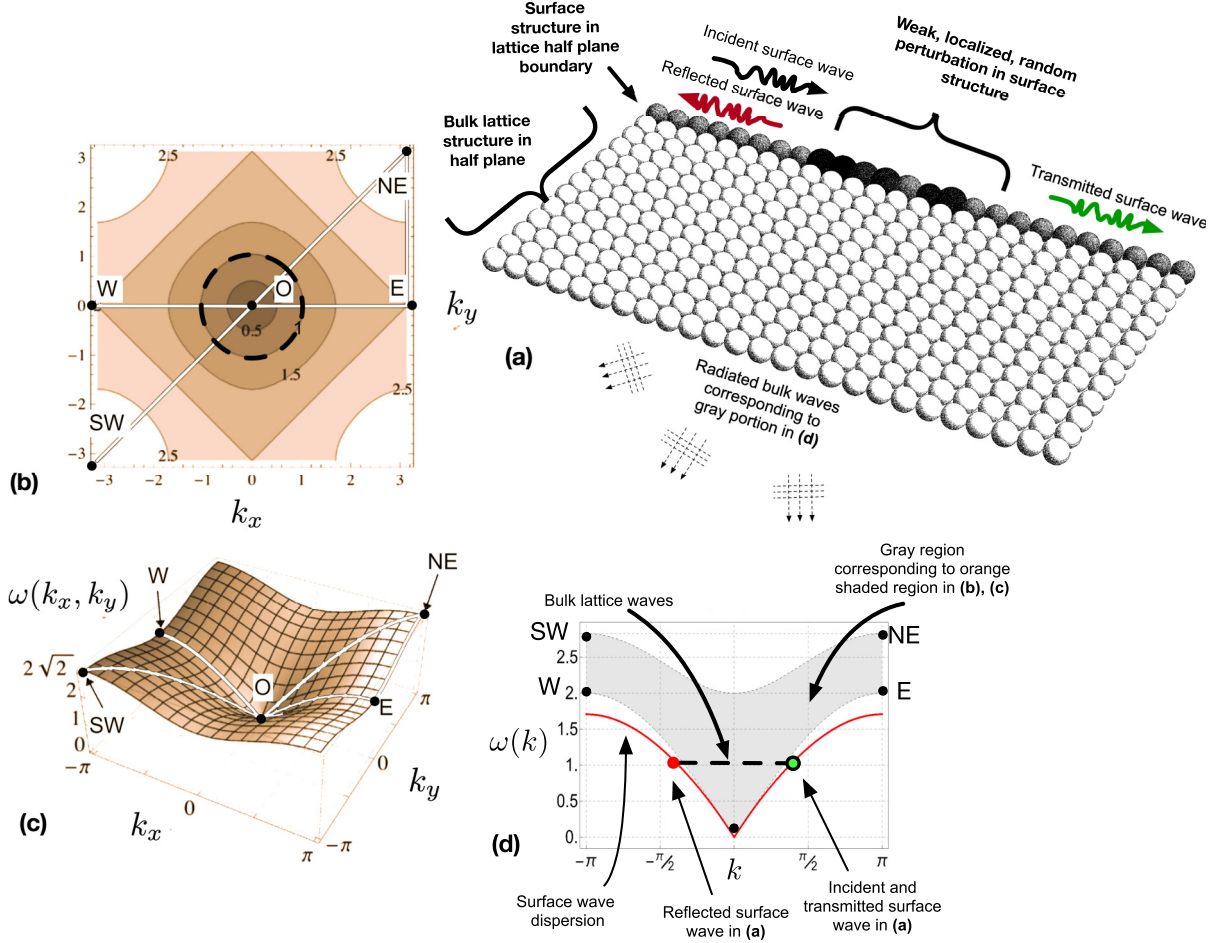


FIG. 2. (a) Schematic of lattice half plane with perturbed surface structure and scattering of a surface wave involving reflected and transmitted surface waves as well as radiated body waves. (b) Contour plot and (c) surface plot of body wave dispersion relation for square lattice. (d) Surface wave dispersion relation along with body wave pass band for square lattice with unperturbed surface structure.

for some $a, b \in \mathbb{C}$, with

$$\begin{aligned} k(\gamma) &= \operatorname{argcosh}\left(1 - \frac{\gamma}{2}\right) \\ &= \ln\left(1 - \frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma(\gamma - 4)}\right) > 0, \quad \gamma \in (\omega^2 - 4, 0), \end{aligned} \quad (9)$$

which shows that it is an evanescent mode.

The surface wave dispersion is shown in Fig. 2(d) as a red curve while the gray region depicts the body wave pass band. The black and green dot in Fig. 2(d) corresponds to a typical surface wave propagating toward the right as shown in Fig. 2(a). The maximal frequency $\omega_{\max}$ that allows for surface mode existence is plotted in Fig. 3. Refer to the Appendix for more details on the discrete spectrum γ_0 for surface waves and continuous spectrum γ for body waves.

B. Perturbed lattice

Regarding the perturbed spring-mass lattice model, we consider the two following assumptions:

Assumption 1. The lattice half plane is uniform, i.e., there is no perturbation away from its boundary. See Figs. 1(b) and

2(a). The equation of motion of the particles on the portion of lattice half-plane $\mathbb{H}$, away from its boundary $\partial\mathbb{H}$, is (1).

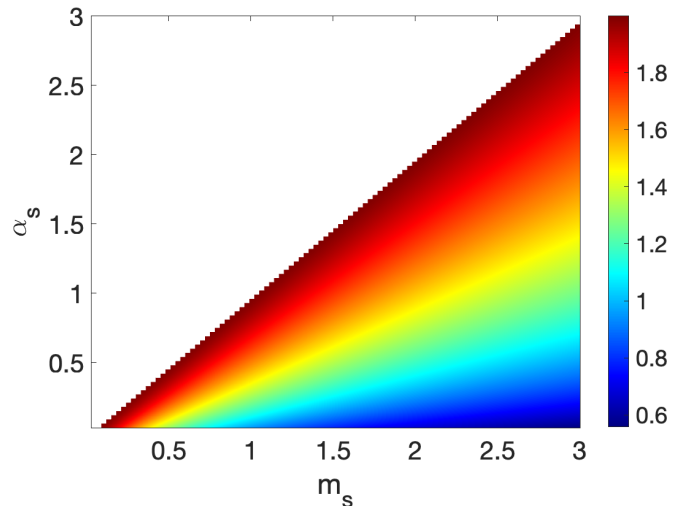


FIG. 3. The maximum allowed frequency, in the surface wave pass band, $\omega_{\max}$ as a function of m_s and α_s .

Assumption 2. The surface $\partial\mathbb{H}$ is randomly perturbed in the region $x \in [1, L] \cap \mathbb{Z}$, and the perturbations affect the surface masses and the surface springs. The total wave field u is solution of (1) for $y \neq 0$, and it satisfies

$$\alpha_s(1 + v_{x+1})(u_{x+1,y} - u_{x,y}) + \alpha_s(1 + v_x)(u_{x-1,y} - u_{x,y}) + u_{x,y-1} - u_{x,y} + m_s(1 + \mu_x)\omega^2 u_{x,y} = 0, \quad (10)$$

on $\partial\mathbb{H}$, that is, for $y = 0$, $x \in \mathbb{Z}$, where $\alpha_s > 0$ denotes the unperturbed elastic coefficient (spring constant) for nearest-neighbor interactions at the surface and $m_s > \alpha_s$ represents the unperturbed mass of particles at $\partial\mathbb{H}$. Here v_x represents the perturbation of the elastic coefficient between the particles at $(x-1, 0)$ and $(x, 0)$ and μ_x represents the perturbation of the mass of the particle at $(x, 0)$. Both kinds of perturbations are localized in the interval $x \in [1, L] \cap \mathbb{Z}$ [see Figs. 1(b) and 2(a)].

Regarding the formulation of scattering problem of a specific edge state, we consider

Assumption 3. There is a unit-amplitude right-going surface mode incoming from the left region $x \leq 0$ and satisfying the radiation condition into the right region $x \geq L+1$ [see Figs. 1(b) and 2(a)].

C. Reflection and transmission coefficients

We denote by $\hat{u}$ the propagative surface wave mode that is incident from the left side:

$$\hat{u}_{x,y} = e^{ik(\gamma_0)x} \phi_{\gamma_0}(y), \quad (x, y) \in \mathring{\mathbb{H}} \cup \partial\mathbb{H}. \quad (11)$$

We denote by $u = u - \hat{u}$ the scattered field that satisfies the radiating conditions:

(i) For $\gamma \in (0, \omega^2) \cup \{\gamma_0\}$, $\langle u_{x,\cdot}, \phi_\gamma \rangle \exp(ik(\gamma)x)$ does not depend on x for $x \leq 0$ and $\langle u_{x,\cdot}, \phi_\gamma \rangle \exp(-ik(\gamma)x)$ does not depend on x for $x \geq L+1$; and

(ii) for $\gamma \in (\omega^2 - 4, 0)$, $\langle u_{x,\cdot}, \phi_\gamma \rangle \exp(-k(\gamma)x)$ does not depend on x for $x \leq 0$ and $\langle u_{x,\cdot}, \phi_\gamma \rangle \exp(k(\gamma)x)$ does not depend on x for $x \geq L+1$, where the inner product $\langle \cdot, \cdot \rangle$ is defined by (A9).

We define by R the amplitude of the left-going surface mode in the left region $x \leq 0$ and by T the amplitude of the right-going surface mode in the right region $x \geq L+1$:

$$R = \langle u_{x,\cdot}, \phi_{\gamma_0} \rangle \exp(ik(\gamma_0)x) \text{ for } x \leq 0, \quad (12)$$

$$T = \langle u_{x,\cdot}, \phi_{\gamma_0} \rangle \exp(-ik(\gamma_0)x) \text{ for } x \geq L+1. \quad (13)$$

We refer to R , respectively, T , as the reflection, respectively, transmission, coefficient. As shown in [38] (Proposition 3.5 and Remark 3.6), the radiative loss $\mathcal{D}$ can be obtained from the reflection and transmission coefficients by the conservation of energy relation

$$\mathcal{D} = 1 - |R|^2 - |T|^2. \quad (14)$$

D. Reflectivity, transmissivity, and radiative loss

Regarding the formulation of edge state scattering problem which involves a perturbation which is weak and random, we consider

Assumption 4. The μ_x and v_x are independent random variables with mean zero and variance σ_μ^2 and σ_v^2 , respectively. Moreover, $\sigma_\mu^2 + \sigma_v^2 \ll 1$.

Our first result is the following

Proposition 1. Let us denote

$$\frac{1}{L_{\text{loc}}} = \frac{\rho(\gamma_0)^2 \mathcal{E}(\gamma_0)}{4 \sin^2(k(\gamma_0))}, \quad (15)$$

$$\Lambda = \frac{\rho(\gamma_0)}{2 \sin k(\gamma_0)} \int_0^{\omega^2} \frac{\rho(\gamma) \mathcal{E}(\gamma)}{\sin k(\gamma)} d\gamma, \quad (16)$$

with

$$\mathcal{E}(\gamma) = m_s^2 \omega^4 \sigma_\mu^2 + \alpha_s^2 \gamma^2 \sigma_v^2. \quad (17)$$

In the regime $\sigma_\mu^2 + \sigma_v^2 \ll 1$, $(\sigma_\mu^2 + \sigma_v^2)L \lesssim 1$, the moments of the reflectivity $|R|^2$, transmissivity $|T|^2$, and radiative loss $\mathcal{D}$ are functions of L/L_{loc} and ΛL_{loc} only. These functions are given in Supplemental Material S1 [52].

Proposition 1 is an extension of [38, Proposition 4.1], in which only mass perturbations were addressed. The means and standard deviations of the reflectivity $|R|^2$, transmissivity $|T|^2$ and radiative loss are plotted in Figs. 4 and 6 for different configurations. Additional figures with comparisons with full numerical simulations (based on exact solution of reflectivity $|R|^2$, transmissivity $|T|^2$ and radiative loss for each realization of spring-mass perturbation) are given in Supplemental Material S2 [52]. Note that the frequency dependencies of the effective parameters L_{loc} and Λ in Eqs. (15)–(17) are different for the two types of perturbations. As a consequence, the decomposition of the input power into reflected, transmitted, and dissipated components is different for the two types of perturbations as shown in Fig. 4.

It is possible to get closed-form expansions of the moments of the reflectivity, transmissivity, and radiative loss for small $(\sigma_\mu^2 + \sigma_v^2)L$ or for large $(\sigma_\mu^2 + \sigma_v^2)L$ as seen in [38]. We report on the second regime in the following proposition.

Proposition 2. In the regime $\sigma_\mu^2 + \sigma_v^2 \ll 1$, $(\sigma_\mu^2 + \sigma_v^2)L \gg 1$, the moments of the reflectivity $|R|^2$ and radiative loss $\mathcal{D}$ are functions of ΛL_{loc} only. The transmissivity $|T|^2$ is zero and the radiative loss is $\mathcal{D} = 1 - |R|^2$. We have, for any integer p :

$$\mathbb{E}[|R|^{2p}] = \tilde{\Lambda} \int_0^\infty \left(\frac{s}{2+s} \right)^p \exp(-\tilde{\Lambda}s) ds, \quad (18)$$

where

$$\tilde{\Lambda} = \Lambda L_{\text{loc}} = 2 \frac{\sin k(\gamma_0)}{\rho(\gamma_0) \mathcal{E}(\gamma_0)} \int_0^{\omega^2} \frac{\rho(\gamma) \mathcal{E}(\gamma)}{\sin k(\gamma)} d\gamma. \quad (19)$$

A surprising consequence of this proposition is that the amount of radiative loss in this regime does not depend on the values of the variances of the medium fluctuations (but only on the ratio σ_v^2/σ_μ^2), as we will discuss in the next section.

III. DISCUSSION

A. Application to inverse problems

The formulas derived in this article can be used to solve certain inverse problems. A classical inverse problem would consist of determining the background parameters m_s and α_s

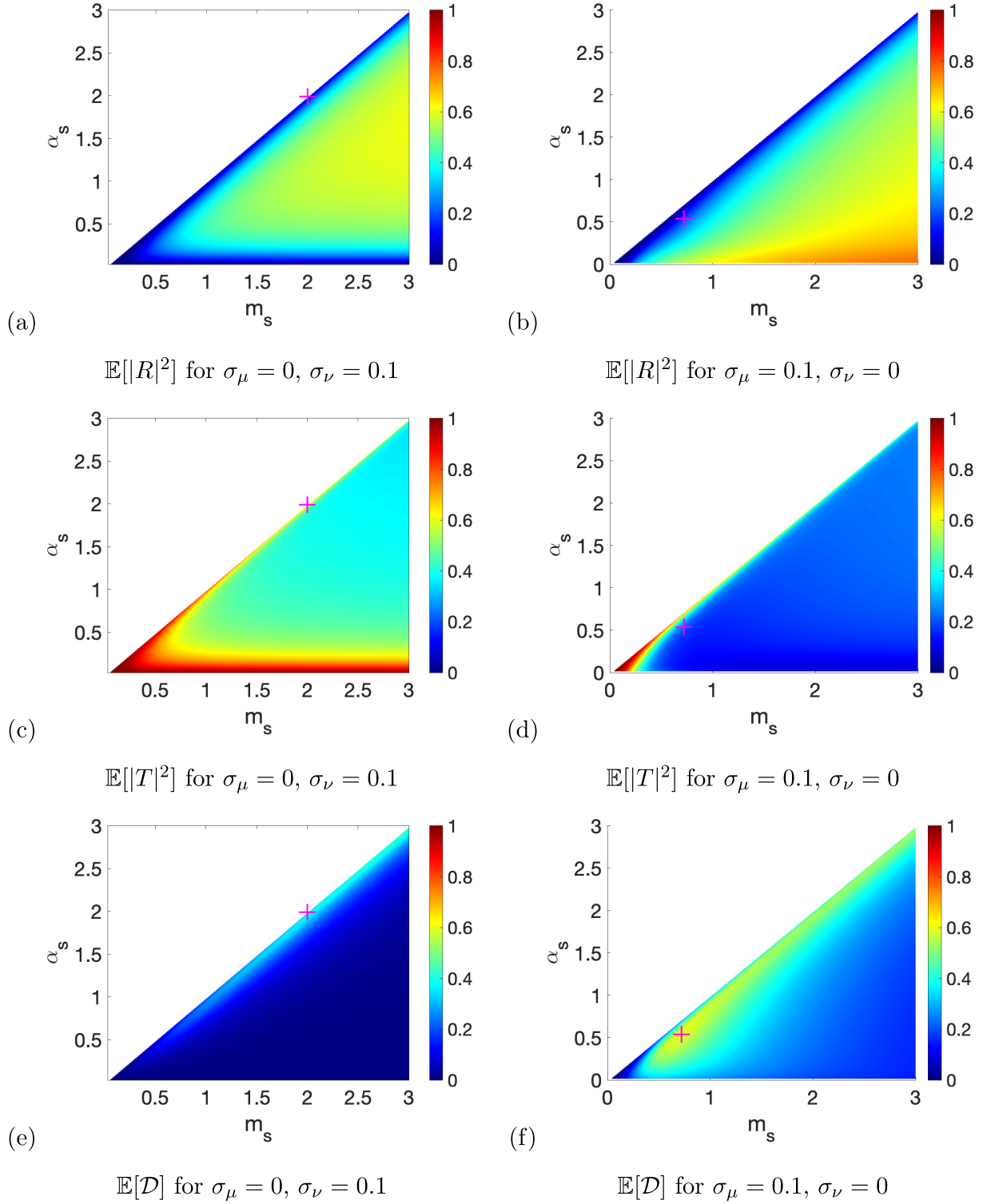


FIG. 4. Mean reflectivity, transmissivity, and radiative loss for $L = 40$ at $\omega = 0.9\omega_{\max}$. Left column ($\sigma_\mu = 0, \sigma_\nu = 0.1$): The mean radiative loss is maximal for $\alpha = m_s^-$ [the magenta cross plots the pair $(m_s, \alpha_s) = (2, 1.99)$ studied in Fig. 6(a)]. Right column ($\sigma_\mu = 0.1, \sigma_\nu = 0$): The mean radiative loss is maximal for $m_s = 0.72, \alpha = 0.54$ [magenta cross, studied in Fig. 6(b)].

from the dispersion curve (the frequency-dependent velocity) of the surface wave. This is a well-known problem that has found numerous applications in seismology, for instance [54]. Here we show that it is also possible to estimate statistical information about the fluctuations of the lattice, for instance, the variances of the surface masses and surface springs.

We consider the estimation of the properties of a perturbed region from backscattered measurements. We assume that we can measure the amplitudes of waves backscattered by a randomly perturbed half-space (i.e., the case $(\sigma_\mu^2 + \sigma_\nu^2)L \gg 1$), which gives the reflectivity $|R(\omega)|^2$ as a function of ω . From Proposition 2, the expectation of the reflectivity has the

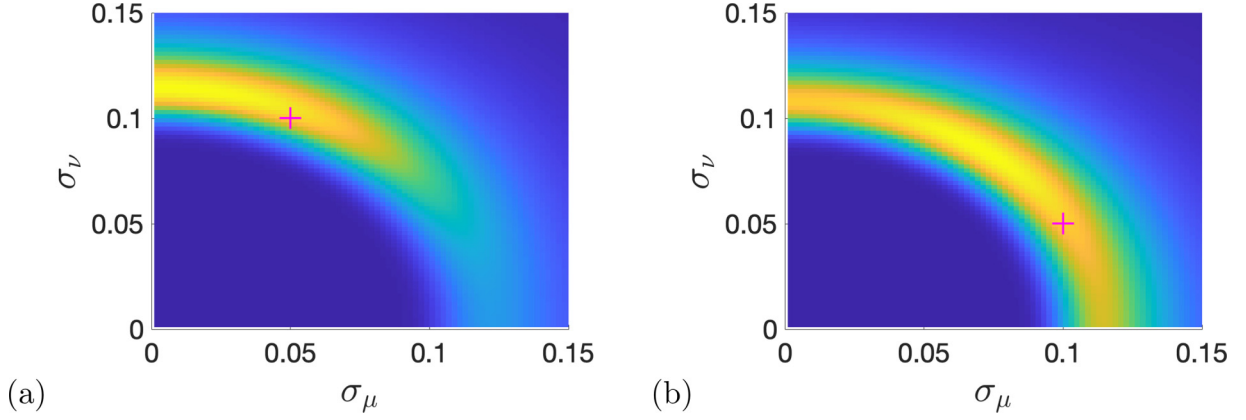


FIG. 5. Posterior densities of (σ_μ, σ_v) given measured reflectivities obtained when $\sigma_\mu = 0.05$ and $\sigma_v = 0.1$ (a) and when $\sigma_\mu = 0.1$ and $\sigma_v = 0.05$ (b). The true values of (σ_μ, σ_v) are marked by the magenta crosses. The known values of the unperturbed system parameters are $m_s = 1.96$, $\alpha_s = 1.9$, $L = 40$.

form

$$\mathbb{E}[|R(\omega)|^2] = 1 - 2\tilde{\Lambda}(\omega) \exp(2\tilde{\Lambda}(\omega)) E_1(2\tilde{\Lambda}(\omega)), \quad (20)$$

where $E_1(x) := \int_1^\infty \frac{\exp(-xt)}{t} dt$. We remark that $\tilde{\Lambda}(\omega)$ has a simple form in terms of the ratio σ_v^2/σ_μ^2 :

$$\tilde{\Lambda}(\omega) = \frac{A(\omega) + B(\omega)\sigma_v^2/\sigma_\mu^2}{1 + C(\omega)\sigma_v^2/\sigma_\mu^2},$$

where

$$\begin{aligned} A(\omega) &= 2 \frac{\sin k(\gamma_0)}{\rho(\gamma_0)} \int_0^{\omega^2} \frac{\rho(\gamma)}{\sin k(\gamma)} d\gamma, \\ B(\omega) &= 2 \frac{\alpha_s^2 \gamma_0^2}{m_s^2 \omega^4} \frac{\sin k(\gamma_0)}{\rho(\gamma_0)} \int_0^{\omega^2} \frac{\rho(\gamma)}{\sin k(\gamma)} \frac{\gamma^2}{\gamma_0^2} d\gamma, \\ C(\omega) &= \frac{\alpha_s^2 \gamma_0^2}{m_s^2 \omega^4} \end{aligned}$$

are known functions that depend only on the unperturbed characteristics of the lattice. By least-squares fitting the measured $|R(\omega)|^2$ to the theoretical formula (20), it is possible to

estimate the desired ratio, which makes it possible to determine which type of perturbations is dominant.

The previous inverse problem takes place in the framework of a randomly perturbed half-space (i.e., the case $(\sigma_\mu^2 + \sigma_v^2)L \gg 1$). In that case, it is only possible to estimate the ratio σ_μ/σ_v from the backscattered signals. For a finite randomly perturbed region with length L , the mean and variance of the reflectivity depend explicitly on σ_μ and σ_v by the formulas given in Proposition 1. By a Bayesian approach, it is then possible to infer these two parameters from the backscattered signals. In Fig. 5, we plot the posterior density of the pair (σ_μ, σ_v) given 50 measurements of the reflectivity $|R|^2$ (we consider a noninformative, uniform prior and we consider a Gaussian approximation of the likelihood by assuming that the distribution of $|R|^2$ is Gaussian with mean $\mathbb{E}[|R|^2]$ and variance $\text{Var}(|R|^2)$ as given in Proposition 1). We consider two cases: In the first case, the data were generated with $\sigma_\mu = 0.05$ and $\sigma_v = 0.1$. In the second case, the data were generated with $\sigma_\mu = 0.1$ and $\sigma_v = 0.05$. In both cases, the posterior densities are essentially supported on the same ring in the space (σ_μ, σ_v) , which means that the sum $\sigma_\mu^2 + \sigma_v^2$ can be well estimated but $\sigma_\mu^2 - \sigma_v^2$ is not so easy to estimate. We

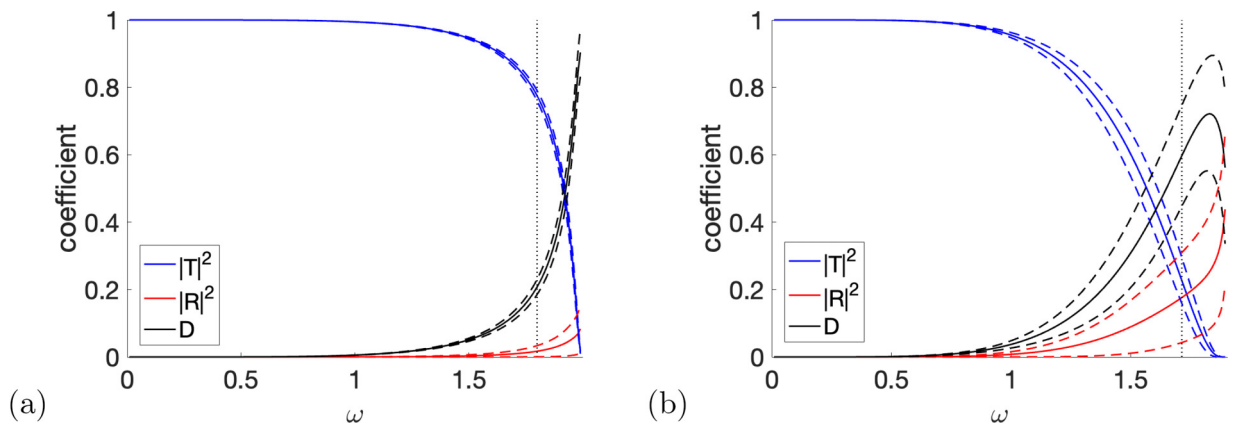


FIG. 6. Means (solid lines) and means $\pm$ standard deviations (dashed lines) of the transmissivity $|T|^2$, reflectivity $|R|^2$, and radiative loss D . In (a) $\alpha_s = 1.99$, $m_s = 2$, $\sigma_\mu = 0$, $\sigma_v = 0.1$, $L = 40$. In (b), $\alpha_s = 0.54$, $m_s = 0.72$, $\sigma_\mu = 0.1$, $\sigma_v = 0$, $L = 40$. The vertical dotted lines stand for $\omega = 0.9\omega_{\max}$.

can observe, however, that the posterior densities concentrate reasonably well around the true values in both cases.

B. Application to maximal radiative losses

When $\sigma_\mu = 0.1$, $\sigma_\nu = 0$, $L = 40$, and $\omega = 0.9\omega_{\max}$, we can see in Fig. 4(f) that the maximal mean radiative loss is reached for $\alpha_s = 0.54$ and $m_s = 0.72$. In Fig. 6(b), we plot the theoretical moments of the transmissivity $|T|^2$, reflectivity $|R|^2$, and radiative loss $\mathcal{D}$ in these conditions. Similarly, when $\sigma_\mu = 0$, $\sigma_\nu = 0.1$, $L = 40$, and $\omega = 0.9\omega_{\max}$, we can see in Fig. 4(e) that the maximal mean radiative loss is reached for $\alpha_s = m_s^-$. In Fig. 6(a), we plot the theoretical moments of the transmissivity $|T|^2$, reflectivity $|R|^2$, and radiative loss $\mathcal{D}$ when $m_s = 2$ and $\alpha_s = 1.99$. Note that in the second case, the maximal dissipative loss is one and this maximal value is reached when $\alpha_s = m_s^-$ and $\omega = \omega_{\max}$. In Supplemental Material S3 [52], we present additional plots comparing the theoretical moments of the transmissivity and reflectivity with empirical averages of full numerical simulations.

IV. CONCLUDING REMARKS

This article has examined the robustness of surface edge states in the presence of weak, localized disorder, with the goal of understanding how such imperfections affect energy transport along surfaces and interfaces. While edge states are often associated with protected, nearly lossless propagation, our analysis demonstrates that this picture is incomplete when disorder is confined to a surface layer and can couple surface modes to body waves' degrees of freedom. Using a spring-mass lattice as a prototypical model, we have shown that even weak randomness can lead to significant radiative losses, fundamentally altering the transport properties of edge states.

Our main result is the derivation of an effective description that quantitatively captures how energy carried by an edge state is redistributed between forward transmission, backscattering, and radiation into the bulk. By combining multiscale analysis with diffusion approximation theory, we obtain explicit expressions for the moments of the transmissivity, reflectivity, and radiative loss, and we identify the two dimensionless parameters that govern the transition between predominantly ballistic propagation and regimes where radiation dominates. This framework provides a clear and predictive link between microscopic disorder properties and macroscopic transport behavior.

One of the key insights of this work is that the breakdown of robust edge transport may not be a threshold phenomenon associated with strong disorder but rather a gradual process that can occur in the weak-disorder regime. In particular, we identify frequency ranges and structural configurations for which surface confinement is intrinsically fragile, despite the presence of an edge mode in the underlying periodic system. This finding challenges the common assumption that the existence of an edge state alone guarantees efficient surface transport, and highlights the need to account explicitly for disorder-induced coupling mechanisms.

Beyond its immediate implications for the specific lattice model studied here, the analysis sheds light on a more

general physical mechanism by which localized disorder enables leakage from surface-guided modes into extended body wave modes. The results suggest that similar effects should be expected in a wide class of systems supporting surface or interface waves, including phononic and photonic crystals, elastic metamaterials, and electronic systems with surface bands, such as graphenelike structures [31,48,55,56]. The tight-binding approximation in such a situation, in the presence of surface perturbations, leads to equations similar to the ones analyzed in this article. A detailed and realistic model may not permit an exact theory as developed here but the qualitative nature of the results is anticipated to be the same; see, for example [57,58] and references therein, for some recent results involving the simplified tight binding models. In these contexts, our work provides guidance for distinguishing regimes in which edge states can be exploited for reliable transport from those in which radiative losses are unavoidable.

Finally, the theoretical approach developed in this article offers a versatile framework for studying wave propagation in discrete, structured media with weak disorder. The combination of asymptotic analysis and stochastic modeling employed here can be extended to more complex geometries, higher-dimensional interfaces, and different types of randomness. We expect that these tools will be useful for future investigations of disorder-induced effects in topological and nontopological systems alike, and for the rational design of interfaces and surfaces with controlled transport properties.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: UNPERTURBED SYSTEM

The equation of motion of the particles on the portion of lattice half-plane away from its boundary $\mathring{\mathbb{H}}$ is (1). The equation of motion of the particles on the lattice half-plane boundary $\partial\mathbb{H}$ is (3) for $x \in \mathbb{Z}$, $y = 0$. Here we give the spectral decomposition of the transverse wave operator.

For any $\gamma \in \mathbb{R}$, we denote by $\psi_\gamma : \mathbb{Z}^- \cup \{0\} \rightarrow \mathbb{C}$ the unique solution of the second-order difference equation

$$(\Delta_1 + \omega^2)\psi_\gamma(y) = \gamma\psi_\gamma(y), \quad y \in \mathbb{Z}^-, \quad (\text{A1})$$

$$\psi_\gamma(y-1) - \psi_\gamma(y) + m_s\omega^2\psi_\gamma(y) = \alpha_s\gamma\psi_\gamma(y), \quad y = 0, \quad (\text{A2})$$

$$\psi_\gamma(y) = 1, \quad y = 0, \quad (\text{A3})$$

where Δ_1 is the one-dimensional discrete Laplacian

$$\Delta_1\psi(y) \equiv \psi(y+1) + \psi(y-1) - 2\psi(y), \quad y \in \mathbb{Z}, \psi : \mathbb{Z} \rightarrow \mathbb{C}. \quad (\text{A4})$$

The function ψ_γ has the following form:

If $\gamma \neq \omega^2$, then there exists a unique $\beta \neq 0$ such that $\text{Re}\beta \geq 0$, $\text{Im}\beta \geq 0$, and $e^\beta + e^{-\beta} - 2 + \omega^2 = \gamma$. Then $\psi_\gamma = A_\gamma e^{\beta y} + (1 - A_\gamma)e^{-\beta y}$, where $A_\gamma = (-\alpha_s\gamma - 1 + m_s\omega^2 + e^\beta)/(e^\beta - e^{-\beta})$ (so that the two boundary conditions Eqs. (A2)-(A3) hold).

If $\gamma = \omega^2$, then $\psi_\gamma = 1 + A_\gamma y$, where $A_\gamma = (m_s - \alpha_s)\omega^2$ [so, Eqs. (A2) and (A3) hold].

(1) If $\gamma \in (\omega^2 - 4, \omega^2)$, we have $\beta = \pm i\zeta$, $\zeta \in (-\pi/2, \pi/2)$ with $2\cos\zeta - 2 + \omega^2 = \gamma$. For $y \leq 0$, $\psi_\gamma = A_\gamma e^{i\zeta y} + \bar{A}_\gamma e^{-i\zeta y}$ (the bar stands for complex conjugate), with $A_\gamma = \frac{1}{2} + \frac{i}{2\sin\zeta}(-\cos\zeta + \alpha_s\gamma + 1 - m_s\omega^2)$, which yields

$$\psi_\gamma(y) = \frac{1}{\sin\zeta}((m_s\omega^2 - \alpha_s\gamma - 1)\sin\zeta y + \sin\zeta(y+1)). \quad (\text{A5})$$

Thus, ψ_γ is bounded on $\mathbb{Z}^-$, but it is oscillatory and is not square summable.

(2) If $\gamma < \omega^2 - 4$ or $\gamma > \omega^2$, then ψ_γ is exponentially growing on $\mathbb{Z}^-$ as $y \rightarrow -\infty$ because $A_\gamma \neq 1$ except for one value $\gamma_0 > \omega^2$ (assuming $\alpha_s < m_s$), which is such that $A_{\gamma_0} = 1$. This value γ_0 yields an eigenfunction

$$\psi_{\gamma_0}(y) = e^{\beta_0 y}, \quad (\text{A6})$$

where β_0 is the unique positive number such that

$$\omega^2 = 2 + \gamma_0 - 2\cosh\beta_0, \quad m_s\omega^2 = \alpha_s\gamma_0 + 1 - e^{-\beta_0}. \quad (\text{A7})$$

Thus, ψ_{γ_0} is square summable on $\mathbb{Z}^-$, it decays exponentially as $y \rightarrow -\infty$.

We consider the weighted l^2 space of functions $u : \mathbb{Z}^- \cup \{0\} \rightarrow \mathbb{C}$ with the norm defined by

$$\|u\|_2^2 := \alpha_s |u_0|^2 + \sum_{y \in \mathbb{Z}^-} |u_y|^2 \quad (\text{A8})$$

and the associated inner product defined by

$$\langle u, v \rangle := \alpha_s u_0 \bar{v}_0 + \sum_{y \in \mathbb{Z}^-} u_y \bar{v}_y. \quad (\text{A9})$$

The operator $\mathcal{L} : l^2 \rightarrow l^2$, defined by

$$(\mathcal{L}u)_y := \begin{cases} \Delta_1 u_y + \omega^2 u_y & \text{if } y \in \mathbb{Z}^-, \\ \frac{1}{\alpha_s}(u_{-1} - u_0 + m_s\omega^2 u_0) & \text{if } y = 0, \end{cases} \quad (\text{A10})$$

is self-adjoint in l^2 , i.e., $\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle$, for all $u, v \in l^2$ with weighted inner product defined by (A9)

$$\begin{aligned} \langle u, \mathcal{L}v \rangle &= \alpha_s u_0 (\mathcal{L}\bar{v})_0 + \sum_{y \in \mathbb{Z}^-} u_y (\mathcal{L}\bar{v})_y \\ &= u_0 (\bar{v}_{-1} - \bar{v}_0 + m_s\omega^2 \bar{v}_0) + \sum_{y \leq -1} u_y (\bar{v}_{y+1} + \bar{v}_{y-1} + (\omega^2 - 2)\bar{v}_y) \\ &= u_0 \bar{v}_{-1} - u_0 \bar{v}_0 + m_s\omega^2 u_0 \bar{v}_0 + \sum_{y \leq 0} u_{y-1} \bar{v}_y + \sum_{y \leq -2} u_{y+1} \bar{v}_y + (\omega^2 - 2) \sum_{y \leq -1} u_y \bar{v}_y \\ &= u_{-1} \bar{v}_0 - u_0 \bar{v}_0 + m_s\omega^2 u_0 \bar{v}_0 + \sum_{y \leq -1} u_{y-1} \bar{v}_y + \sum_{y \leq -1} u_{y+1} \bar{v}_y + (\omega^2 - 2) \sum_{y \leq -1} u_y \bar{v}_y \\ &= \alpha_s (\mathcal{L}u)_0 \bar{v}_0 + \sum_{y \in \mathbb{Z}^-} (\mathcal{L}u)_y \bar{v}_y = \langle \mathcal{L}u, v \rangle. \end{aligned} \quad (\text{A11})$$

Note that the choice of the weight α_s in (A9) is the one that makes the operator $\mathcal{L}$ self-adjoint for the weighted inner product, any other choice (such as that involving m_s) is not appropriate. Using standard tools of spectral decomposition, we get the following proposition (proved in [38]).

Proposition A.1. Any function $u \in l^2$ [with weighted inner product defined by (A9)] can be expanded as

$$u_y = \check{u}_{\gamma_0} \phi_{\gamma_0}(y) + \int_{\omega^2-4}^{\omega^2} \check{u}_\gamma \phi_\gamma(y) d\gamma, \quad (\text{A12})$$

with the modal amplitudes given by

$$\check{u}_{\gamma_0} = \langle u, \phi_{\gamma_0} \rangle, \quad (\text{A13a})$$

$$\check{u}_\gamma = \langle u, \phi_\gamma \rangle, \quad \gamma \in (\omega^2 - 4, \omega^2), \quad (\text{A13b})$$

and eigenmodes given by

$$\phi_{\gamma_0}(y) = \sqrt{\rho(\gamma_0)}\psi_{\gamma_0}(y), \quad (\text{A13c})$$

$$\phi_{\gamma}(y) = \sqrt{\rho(\gamma)}\psi_{\gamma}(y), \quad \gamma \in (\omega^2 - 4, \omega^2), \quad (\text{A13d})$$

where

$$\rho(\gamma_0) = \frac{1}{\alpha_s + (e^{2\beta_0} - 1)^{-1}}, \quad (\text{A13e})$$

$$\rho(\gamma) = \frac{2}{\pi} \frac{\sqrt{(\omega^2 - \gamma)(\gamma - \omega^2 + 4)}}{1 + \frac{(\gamma(2\alpha_s - 1) + \omega^2(1 - 2m_s))^2}{(\omega^2 - \gamma)(\gamma - \omega^2 + 4)}}, \quad \gamma \in (\omega^2 - 4, \omega^2). \quad (\text{A13f})$$

We also have the Parseval relation

$$\|u\|_2^2 = |\check{u}_{\gamma_0}|^2 + \int_{\omega^2 - 4}^{\omega^2} |\check{u}_{\gamma}|^2 d\gamma \quad (\text{A14})$$

for all $u \in l^2$.

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