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EFFECTIVE DYNAMICS IN LATTICES WITH RANDOM MASS PERTURBATIONS*

JOSSELIN GARNIER† AND BASANT LAL SHARMA‡

Abstract. We consider a one-dimensional mono-atomic lattice with random perturbations of masses spread over a finite number of particles. Assuming Newtonian dynamics and linear nearest-neighbour interactions and allowing for a provision of pinning due to substrate interaction, we discuss lattice dynamics in two frameworks: a transient dynamics problem and a time-harmonic transmission problem. By a stochastic, multiscale analysis we provide expressions for the transient displacement field, that propagates through the random perturbations, in the first case and for the time-harmonic transmission coefficients in the second case. These theoretical predictions are supported by illustrations of their agreements with numerical simulations.

Keywords. Random perturbations; lattice dynamics; discrete Schrödinger operators; multiscale analysis.

AMS subject classifications. 34F05; 60H10; 34C15; 37K60.

1. Introduction

In this paper we consider a one-dimensional chain of particles. Each particle interacts through a nearest-neighbor potential. The difference equation that governs the dynamics of the one-dimensional lattice is deduced from Newton's law. In the *time-dependent* framework, the problem for the displacement field has the form

$$(1+\mathfrak{m}_{\mathtt{x}})\ddot{u}_{\mathtt{x}}(t)=u_{\mathtt{x}+1}(t)+u_{\mathtt{x}-1}(t)-(2+\varpi_{\mathrm{s}})u_{\mathtt{x}}(t),\quad \mathtt{x}\in\mathbb{Z},t\in\mathbb{R}, \tag{1.1}$$

where the dot stands for the time derivative, the masses of the particles are $1+\mathfrak{m}_{\mathtt{x}}$, the parameter $\varpi_{\mathrm{s}}\geq 0$ represents the pinning due to substrate interaction, and the initial condition is

$$u_{\mathtt{x}}(0)=\mathtt{u}^{(0)}_{\mathtt{x}},\quad \dot{u}_{\mathtt{x}}(0)=\mathtt{v}^{(0)}_{\mathtt{x}},\quad \mathtt{x}\in\mathbb{Z}, \tag{1.2}$$

with a specified $\mathtt{u}^{(0)}$ and $\mathtt{v}^{(0)}$ in $\ell_2(\mathbb{Z})$. In this framework, we are particularly interested in solving for $u:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ with the initial condition

$$u_{\mathtt{x}_0}(0)=1, u_{\mathtt{x}}(0)=0,\ \ \mathtt{x}\in\mathbb{Z}\setminus\{\mathtt{x}_0\},\qquad \dot{u}_{\mathtt{x}}(0)=0,\ \ \mathtt{x}\in\mathbb{Z}, \tag{1.3}$$

that corresponds to an initial displacement localized at $\mathtt{x}_0\in\mathbb{Z}$.

In connection with the history of this problem, we mention that Equation (1.1) with $\mathfrak{m}_{\mathtt{x}}\equiv 0$ was studied in [29], where several remarkable analyses and features were proposed (such as a closed-form expression of the solution when $\varpi_{\mathrm{s}}=0$); see also [9,20]. Equation (1.1) describes the vibration of an infinite mono-atomic chain with nearest-neighbour interactions and belongs to a class of problems that appear in the study of the dynamics of crystal lattices [6].

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†Centre de Mathématiques Appliquées, Ecole polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France (josselin.garnier@polytechnique.edu).

‡Department of Mechanical Engineering, Indian Institute of Technology Kanpur, Kanpur, 208016 UP, India (bls@iitk.ac.in).

Figures 1.1 and 1.2 present trajectories of particles in the lattice obtained by solving (1.1), (1.3) using a standard numerical method assuming $\varpi_s = 0$ and $\varpi_s \neq 0$, respectively. In the unperturbed case $\mathfrak{m}_x \equiv 0$ one can observe the propagation of a displacement front whose velocity is not the one of the long wavelength limit (continuum limit) of (1.1) (except when $\varpi_s = 0$), but can be obtained by an asymptotic analysis as we describe in Section 2. In the perturbed case $\mathfrak{m}_x$ are independent and identically distributed with mean zero and variance σ^2 in the section $x \in [1, L] \cap \mathbb{Z}$, and $\mathfrak{m}_x = 0$ outside the section $[1, L] \cap \mathbb{Z}$. One can observe that the mass perturbations induce perturbations in the dynamics that we describe in Section 3.

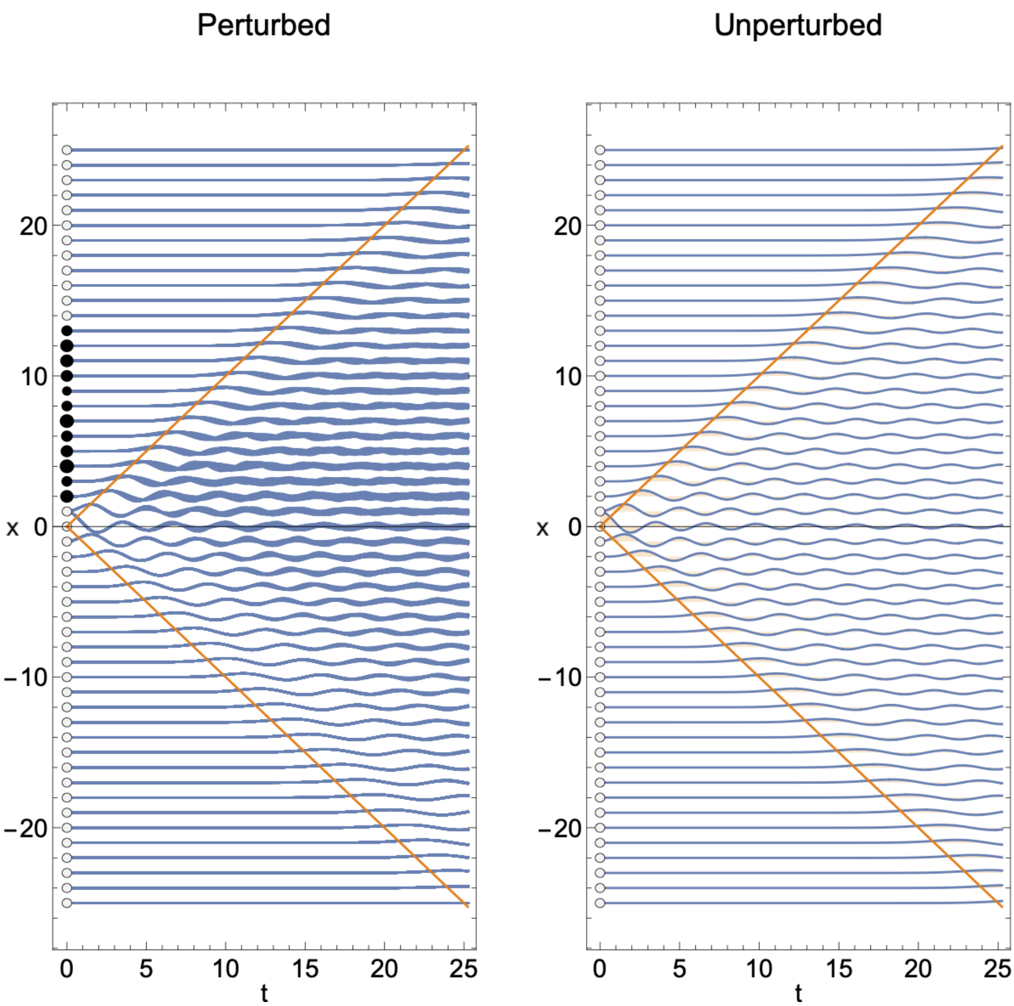


FIG. 1.1. *Time domain problem with (left) and without (right) mass perturbations. Here* $L = 12, \sigma = 0.15$ *and* $\varpi_s = 0$ *(the number of curves on the left is 51 corresponding to a set of realizations of random mass perturbations with the same statistics). Perturbed masses are indicated as solid black disks on the schematic, adjacent to* x*-axis, whereas empty dots refer to the regular lattice. Mass perturbations are located from* $x = 2$ *to* $x = 13$*. Initial condition (1.3) is supported outside the mass defect with* $x_0 = 0$*. The orange lines indicate the boundaries of the cone with unit speed of propagation.*

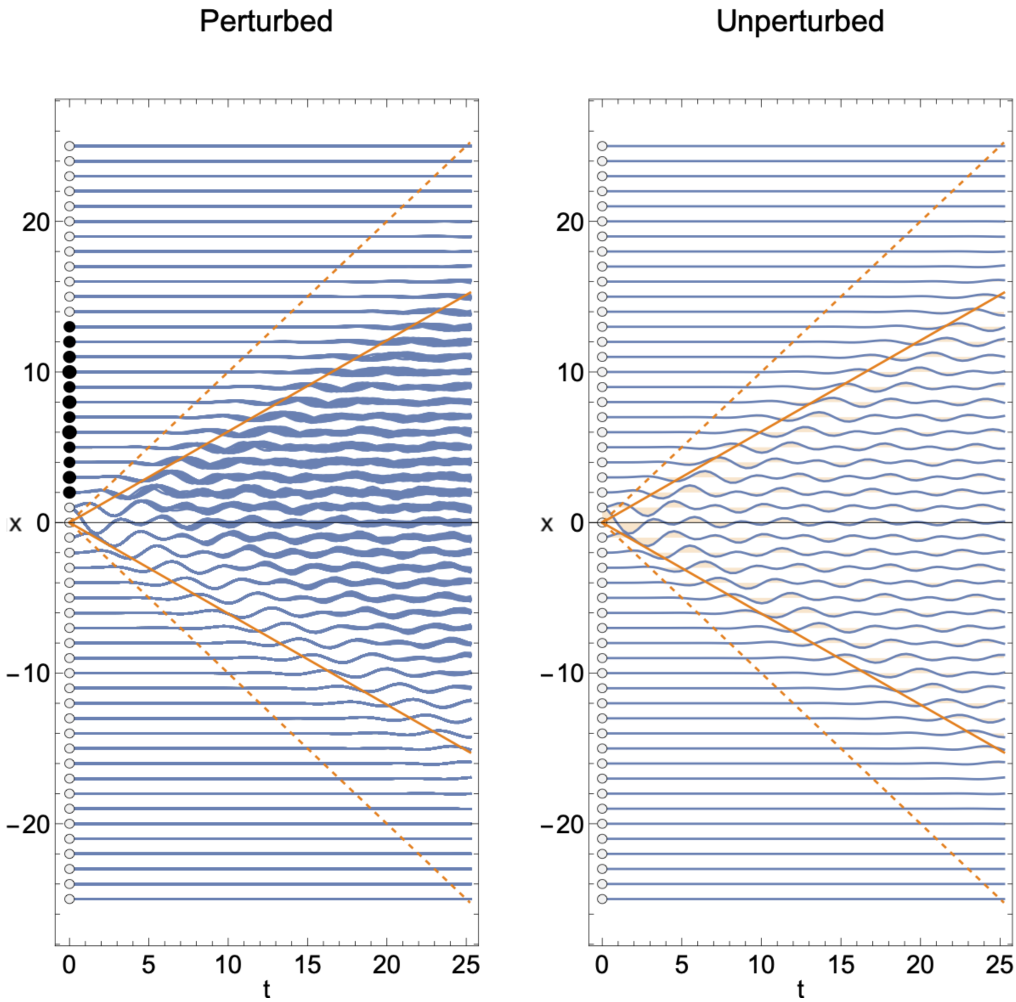


FIG. 1.2. *Time domain problem with (left) and without (right) mass perturbations. Here* $\varpi_s = 1.1$ *and the other parameters and details are the same ones as in Figure 1.1. The dashed orange lines indicate the boundaries of the cone with unit speed of propagation, the solid orange lines indicate the boundaries of the cone with speed of propagation* $1/\alpha_s$ *(see Equation (2.10)).*

In the *time-harmonic* framework with frequency ω the solution of (1.1) has the form

$$u_{\mathtt{x}}(t) = e^{-i\omega t}\hat{u}_{\mathtt{x}}, \quad \mathtt{x} \in \mathbb{Z}, t \in \mathbb{R}, \tag{1.4}$$

where $\hat{u}: \mathbb{Z} \to \mathbb{C}$ is a solution of

$$-\omega^2(1+\mathfrak{m}_{\mathtt{x}})\hat{u}_{\mathtt{x}} = (\hat{u}_{\mathtt{x}+1} + \hat{u}_{\mathtt{x}-1} - 2\hat{u}_{\mathtt{x}}) - \varpi_s \hat{u}_{\mathtt{x}}, \quad \mathtt{x} \in \mathbb{Z}. \tag{1.5}$$

As $\mathtt{x} \to +\infty$ (in fact, as soon as $\mathtt{x} > L$) the solution of (1.5) takes the form $\hat{u}_{\mathtt{x}} = Te^{ik\mathtt{x}}$, where k and T are functions of ω, and T depends, additionally, on the random mass perturbations $\mathfrak{m}_{\mathtt{x}}$ in the section $\mathtt{x} \in [1, L] \cap \mathbb{Z}$, see Section 4. In this framework, Figure 1.3 presents the transmittance $|T|^2$ as a function of ω obtained by solving (1.5) using

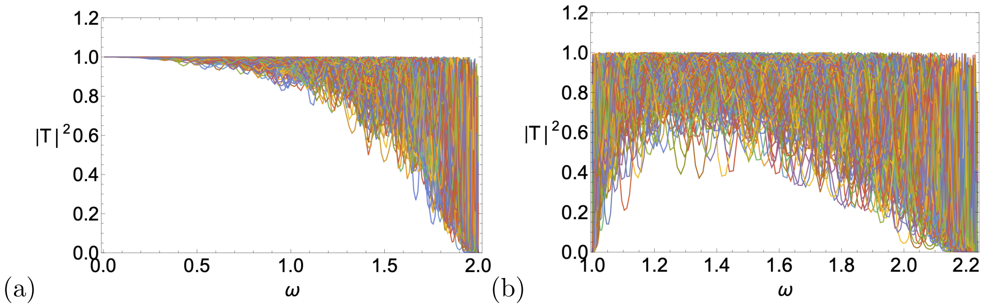


FIG. 1.3. *Numerical evaluations of* $|T|^2$ *using the exact expression* (A.10) *described in Appendix A.1 with (a)* $\varpi_s = 0$ *(b)* $\varpi_s = 1$*,* $\sigma = 0.05, L = 40$ *and* 200 *different realizations of the mass perturbations* $\mathtt{m}_\mathtt{x}$*,* $\mathtt{x} \in [1, L]$ *(in the absence of mass perturbation on* $[1, L] \cap \mathbb{Z}$*, we have* $T \equiv 1$*).*

numerical methods for different realizations of the random mass perturbations. One can observe that the transmittance has strong fluctuations and we describe its statistics in Section 4.

In connection with the history of this problem, we mention that, besides its application to lattice vibrations, the Equation (1.5) also belongs to a class of discrete scattering problems in the context of the discrete Schrödinger equation (within tight-binding model of the electrons in crystals) [3, 30]. It has also played a crucial role in the discovery of significant phenomena such as the famous localization result of [1]. In the domain of electrical engineering, network synthesis and filter design [18], LC circuits based lattice structures also involve similar difference operators as in (1.1) and (1.5), while such operators also appear in the lumped circuit models for electromagnetic metamaterials [11, 17]. The Equation (1.5) naturally emerges in case of time-harmonic lattice waves in one dimension [4–6].

In the simple physical model that we study in this article, the mass perturbations are random, but time-independent and spread over a finite number of particles. Regarding the notion of such random perturbations, as assumed in both frameworks analyzed in this article, it is known that impurities and vacancies play an important role in the thermal conductivity of materials specially due to local mass distortions around point defects; see [19, 31] as examples of some recent studies from a physics related viewpoint involving mass disorder in crystalline materials. Indeed, over the last century, till this date, the analysis of one-dimensional lattice models, while also accounting for disorder and randomness, has been a part of several physics-oriented and mathematical research works [12, 16]. In a sense, we initiate in this paper the development of a stochastic framework for capturing effective dynamical behaviours of discrete random media by using a prototype of one-dimensional lattice model; recently, we extended some of these results to lattice half-spaces [15]. The questions posed in the present article concerning (1.1) and (1.5) can be also potentially generalized to quasi-one-dimensional problems of waveguides, such as the Anderson localization of states due to mass disorder [24] or the problem of electronic transport through several interfaces and junctions [25–28]. On the other hand, the question of inverse problem is also related to such forward analysis of discrete media and attaining information about statistics of the perturbation can be useful. A recent result on inverse scattering on lattices, in particular in one dimension, is given in [21, 22]; see references therein for the general problem of inverse scattering in discrete framework.

The main results that we report in this paper are the following ones. An initial condition of the form (1.3) generates a displacement field that has the form of a dispersive wave whose front propagates with a velocity that depends on $\varpi_{\rm s}$. In the presence of random mass perturbations, the wave front is not perturbed (when $\varpi_{\rm s}=0$) but the tail of the wave field experiences damping (see Theorems 3.1-3.4). This is in contrast with the results known for the continuous scalar wave equation in random media in which the wave front experiences two different phenomena: a deterministic attenuation and spreading and a random time shift [23]. Moreover, the statistics of the transmittance can be fully characterized and exhibits a very strong frequency-dependence (see Theorem 4.1).

The paper is organized as follows. In Section 2 the unperturbed problem is addressed and asymptotic solutions are presented, which describe the behaviors of the waves in homogeneous media. The time-dependent, resp. time-harmonic, problem with random mass perturbations is studied in Section 3, resp. Section 4. Proofs are given in the following sections.

2. Preliminary results

In this section we consider the unperturbed case $\mathfrak{m}_{\tt x}=0$ for all ${\tt x}\in\mathbb{Z}$. The following lemmas are proved in Section 5.

LEMMA 2.1 (Solution without perturbation for $\varpi_{\rm s}=0$). *The solution of (1.1) with the initial conditions (1.3) in the absence of mass perturbations $\mathfrak{m}_{\tt x}\equiv 0$ and with $\varpi_{\rm s}=0$ is of the form*

$$u_{\tt x}(t)=\frac{1}{2\pi}\int_0^{+\infty}\hat{u}_{\tt x}(\omega)e^{-i\omega t}d\omega+c.c.,\qquad t\geq 0,\quad {\tt x}\in\mathbb{Z}, \tag{2.1}$$

where the Fourier components are given by

$$\hat{u}_{\tt x}(\omega)=\hat{c}(\omega)\cos\big(k(\omega)({\tt x}-{\tt x}_0)\big), \tag{2.2}$$

with

$$k(\omega)=2\arcsin\Big(\frac{\omega}{2}\Big),\qquad \hat{c}(\omega)=\frac{2}{\sqrt{4-\omega^2}}\mathbf{1}_{(0,2)}(|\omega|). \tag{2.3}$$

We can also write, since $\hat{u}_{\tt x}(-\omega)=\overline{\hat{u}_{\tt x}(\omega)}$, that

$$u_{\tt x}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}_{\tt x}(\omega)e^{-i\omega t}d\omega,\qquad t\geq 0,\quad {\tt x}\in\mathbb{Z}. \tag{2.4}$$

REMARK 2.1. We can make the change of variable $\omega\mapsto 2\sin s$ in (2.4) and we obtain

$$u_{\tt x}(t)=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\big(e^{2is({\tt x}-{\tt x}_0)}+e^{-2is({\tt x}-{\tt x}_0)}\big)e^{-2i\sin(s)t}ds,\qquad t\geq 0,\quad {\tt x}\in\mathbb{Z}, \tag{2.5}$$

which gives

$$u_{\tt x}(t)=J_{2({\tt x}-{\tt x}_0)}(2t),\qquad t\geq 0,\quad {\tt x}\in\mathbb{Z}, \tag{2.6}$$

where J is the Bessel function of the first kind. This expression is valid only for $\varpi_{\rm s}=0$ and there is no equivalent expression for $\varpi_{\rm s}>0$.

We can study the asymptotic behavior of $u_{\mathtt{x}_0+\mathtt{x}}(t)$ for large t and $\mathtt{x}$. The objective is to describe theoretically the behavior of the wave field that can be observed in the numerical experiments reported in Figures 1.1 and 1.2 (right): the wave field is essentially supported in the cone delimited by the space-time lines $\mathtt{x}=\mathtt{x}_0\pm t$ when $\varpi_{\rm s}=0$ (Figure 1.1 right) and $\mathtt{x}=\mathtt{x}_0\pm t/\alpha_s$ when $\varpi_{\rm s}\neq 0$, with α_s defined by (2.10) (Figure 1.2 right).

Let us consider $\mathtt{x}\gg 1$ and $t=\alpha\mathtt{x}$, $\alpha\in(1,+\infty)$. A stationary phase argument gives the following result that determines the form of the wave field inside the cone delimited by the space-time lines $\mathtt{x}=\mathtt{x}_0\pm t$ and that is characterized by an oscillatory behavior and an algebraic decay as $1/\sqrt{\mathtt{x}}$.

LEMMA 2.2 (Asymptotic behavior $\varpi_{\rm s}=0$). *For any $\alpha\in(1,+\infty)$ we have*

$$u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x})=\frac{1}{\sqrt{\pi\mathtt{x}}\sqrt[4]{\alpha^2-1}}\cos\Big(\frac{\pi}{4}+2\big(\arccos(1/\alpha)-\sqrt{\alpha^2-1}\big)\mathtt{x}\Big)+o\big(\frac{1}{\sqrt{\mathtt{x}}}\big),\qquad(2.7)$$

as $\mathtt{x}\to+\infty$.

For $\alpha\in[0,1)$ there is no stationary point which implies that $u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x})$ is much smaller than $1/\sqrt{\mathtt{x}}$. In other words, there is no signal outside the cone.

When $\alpha\searrow 1$, i.e. at times close to $\mathtt{x}$, formula (2.7) blows up, which indicates that the amplitude of the wave close to the front (the boundary of the cone) is larger than $1/\sqrt{\mathtt{x}}$. A refined study (see Section 5) gives the behavior of the front field, which has a duration of the order of $\sqrt[3]{\mathtt{x}}$ and an amplitude of the order of $1/\sqrt[3]{\mathtt{x}}$.

LEMMA 2.3 (Asymptotic front behavior $\varpi_{\rm s}=0$). *For any $\beta\in\mathbb{R}$ we have*

$$u_{\mathtt{x}_0+\mathtt{x}}(\mathtt{x}+\sqrt[3]{\mathtt{x}}\beta)=\frac{1}{\sqrt[3]{\mathtt{x}}}Ai(-2\beta)+o\big(\frac{1}{\sqrt[3]{\mathtt{x}}}\big),\qquad(2.8)$$

as $\mathtt{x}\to+\infty$, where Ai is the Airy function.

REMARK 2.2. Recall that $Ai(x)\sim e^{-(2/3)x^{3/2}}/(2\sqrt{\pi}\sqrt[4]{x})$ for $x\gg 1$ which shows that the field $u_{\mathtt{x}_0+\mathtt{x}}(t)$ is vanishing before the arrival time $\mathtt{x}$ and the field is essentially contained in the cone with unit speed of propagation as seen in Figure 1.1 right. Moreover $Ai(-x)\sim\cos[(\pi/4)-(2/3)x^{3/2}]/(\sqrt{\pi}\sqrt[4]{x})$ for $x\gg 1$, which shows that the two expansions (2.7) and (2.8) match.

If $\varpi_{\rm s}>0$, then the same procedure gives the expression (2.1-2.2) for the solution $u_{\mathtt{x}}(t)$ with modified expressions for $k(\omega)$ and $\hat{c}(\omega)$.

LEMMA 2.4 (Solution without perturbation for $\varpi_{\rm s}>0$). *If $\varpi_{\rm s}>0$, then the solution $u_{\mathtt{x}}(t)$ has the form (2.1-2.2) with*

$$k(\omega)=2\arcsin\Big(\frac{\sqrt{\omega^2-\varpi_{\rm s}}}{2}\Big),\quad \hat{c}(\omega)=\frac{2|\omega|}{\sqrt{\omega^2-\varpi_{\rm s}}\sqrt{4+\varpi_{\rm s}-\omega^2}}\mathbf{1}_{(\sqrt{\varpi_{\rm s}},\sqrt{\varpi_{\rm s}+4})}(|\omega|).\qquad(2.9)$$

We can study the asymptotic behavior of $u_{\mathtt{x}_0+\mathtt{x}}(t)$ for large $t>0$ and $\mathtt{x}\in\mathbb{Z}$. Let us consider $\mathtt{x}\gg 1$ and $t=\alpha\mathtt{x}$, $\alpha\in(\alpha_s,+\infty)$, with

$$\alpha_s=\frac{\sqrt{2}}{\sqrt{2+\varpi_{\rm s}-\omega_s^2}},\qquad \omega_s=\sqrt[4]{4\varpi_{\rm s}+\varpi_{\rm s}{}^2}.\qquad(2.10)$$

Note that α_s is larger than one. A stationary phase argument gives the following result.

LEMMA 2.5 (Asymptotic behavior $\varpi_s>0$). *For any $\alpha\in(\alpha_s,+\infty)$ we have*

$$
\begin{aligned}
u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x}) = & \frac{\sqrt{2}\omega_\alpha^{+3/2}}{\sqrt{\pi\alpha(\omega_\alpha^{+4}-\omega_s^4)\mathtt{x}}}\cos\Big(\frac{\pi}{4}+[k(\omega_\alpha^+)-\omega_\alpha^+\alpha]\mathtt{x}\Big)\\
&+\frac{\sqrt{2}\omega_\alpha^{-3/2}}{\sqrt{\pi\alpha(\omega_s^4-\omega_\alpha^{-4})\mathtt{x}}}\cos\Big(-\frac{\pi}{4}+[k(\omega_\alpha^-)-\omega_\alpha^-\alpha]\mathtt{x}\Big)+o\big(\frac{1}{\sqrt{\mathtt{x}}}\big),
\end{aligned}
\tag{2.11}
$$

as $\mathtt{x}\to+\infty$, where

$$
\omega_\alpha^{\pm2}=\frac{2}{\alpha^2}\Big[(1+\frac{\varpi_s}{2})\alpha^2-1\pm\sqrt{\alpha^4-(2+\varpi_s)\alpha^2+1}\Big]. \tag{2.12}
$$

If $\alpha\in[0,\alpha_s)$ then there is no stationary point so we can conclude that $u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x})$ is much smaller than $1/\sqrt{\mathtt{x}}$. When $\alpha\searrow\alpha_s$, i.e. at times close to $\alpha_s\mathtt{x}$, formula (2.11) blows up which indicates that the amplitude of the wave close to the front (the boundary of the cone) is larger than $1/\sqrt{\mathtt{x}}$. A refined study gives the behavior of the front field, which has a duration of the order of $\sqrt[3]{\mathtt{x}}$ and an amplitude of the order of $1/\sqrt[3]{\mathtt{x}}$.

LEMMA 2.6 (asymptotic front behavior $\varpi_s>0$). *For any $\beta\in\mathbb{R}$ we have*

$$
u_{\mathtt{x}_0+\mathtt{x}}(\alpha_s\mathtt{x}+\beta\sqrt[3]{\mathtt{x}})=\tfrac{\sqrt[3]{2}}{\sqrt[3]{\mathtt{x}}}Ai\Big(-\sqrt[3]{2}\tfrac{\beta}{\alpha_s}\Big)\cos\Big([k(\omega_s)-\omega_s\alpha_s]\mathtt{x}-\omega_s\beta\sqrt[3]{\mathtt{x}}\Big)+o\big(\tfrac{1}{\sqrt[3]{\mathtt{x}}}\big), \tag{2.13}
$$

as $\mathtt{x}\to+\infty$.

Since $Ai(x)$ decays very fast for $x\gg1$, this confirms that the field is vanishing before the arrival time $\alpha_s\mathtt{x}$. The field is essentially contained in the cone with speed of propagation $1/\alpha_s$ as seen in Figure 1.2 right. Note that the function $\varpi_s\in[0,+\infty)\mapsto 1/\alpha_s\in(0,1]$ is decreasing, which means that pinning reduces the speed of propagation of the wave field.

REMARK 2.3. It is possible to get the front velocity $1/\alpha_s$ by a group velocity analysis. Indeed from (2.9) we get the dispersion relation $\omega(k)=\sqrt{\varpi_s+4\sin^2(k/2)}$. The group velocity is $v_g(k)=\partial_k\omega(k)$. The maximal group velocity is obtained at $k_s=2\arcsin(\sqrt{\omega_s^2-\varpi_s}/2)$ (which corresponds to the frequency $\omega(k_s)=\omega_s$) and its value is precisely $1/\alpha_s$. This means that the wave front is made of modes whose wavenumbers k are close to k_s, which are the ones that propagate the fastest. We finally remark that in the long wavelength limit (continuum limit) of (1.1), the dispersion relation is $\omega(k)=\sqrt{\varpi_s+k^2}$ and the maximal group velocity is 1 whatever the value of ϖ_s. This shows that the dynamics of the discrete system is different from its long wavelength limit even for large propagation distances and times.

3. Effective dynamics for the time-dependent problem

In this section we assume that, for $\mathtt{x}\in[1,L]\cap\mathbb{Z}$, the variables $\mathfrak{m}_\mathtt{x}$ are independent and identically distributed with mean zero and variance σ^2:

$$
\mathbb{E}[\mathfrak{m}_\mathtt{x}^2]=\sigma^2. \tag{3.1}
$$

The forthcoming results are obtained by a multiscale analysis that is valid when $\sigma\ll1$ and L is of the order of σ^{-2} (so that $\sigma^2L=O(1)$) and they are proved in Section 7. We consider the initial condition (1.3) with $\mathtt{x}_0 = 0$. We define

$$\gamma(\omega) = \frac{\sigma^2 \omega^4}{4 \sin^2 k(\omega)}, \tag{3.2}$$

where $k(\omega)$ is given by (2.9).

THEOREM 3.1 (Mean field with random mass perturbations and $\varpi_{\mathrm{s}} = 0$). *If $\sigma \ll 1$, $\mathtt{x} > L \gg 1, \mathtt{x} \in \mathbb{Z}$, then the mean field of (1.1), (1.3) has the form:*

$$\mathbb{E}\big[u_{\mathtt{x}}(\alpha \mathtt{x})\big] = \frac{1}{\sqrt{\pi \mathtt{x}}\sqrt[4]{\alpha^2 - 1}} \cos\Big(\frac{\pi}{4} + 2\big(\arccos(1/\alpha) - \sqrt{\alpha^2 - 1}\big)\mathtt{x}\Big) e^{-\gamma(\omega_\alpha) L} + o\big(\frac{1}{\sqrt{\mathtt{x}}}\big), \tag{3.3}$$

for any $\alpha \in (1, +\infty)$, where

$$\omega_\alpha = \frac{2\sqrt{\alpha^2 - 1}}{\alpha}, \qquad \gamma(\omega_\alpha) = \sigma^2(\alpha^2 - 1). \tag{3.4}$$

THEOREM 3.2 (Mean front with random mass perturbations and $\varpi_{\mathrm{s}} = 0$). *If $\sigma \ll 1$, $\mathtt{x} > L \gg 1, \mathtt{x} \in \mathbb{Z}$, then we have*

$$\mathbb{E}\big[u_{\mathtt{x}}(\mathtt{x} + \sqrt[3]{\mathtt{x}}\beta)\big] = \frac{1}{\sqrt[3]{\mathtt{x}}} Ai(-2\beta) + o\big(\frac{1}{\sqrt[3]{\mathtt{x}}}\big), \tag{3.5}$$

for any $\beta \in \mathbb{R}$.

The expression (3.5) shows that the mean field for times t close to the front $\mathtt{x}$ is not affected to leading order by the random mass perturbations, while the expression (3.3) shows that the mean field following the front for times t larger than $\mathtt{x}$ is affected. This is in contrast with the results known for the scalar wave equation in which the wave front experiences two different phenomena: a deterministic attenuation and spreading and a random time shift. The attenuation and spreading is described by a deterministic kernel determined by the statistics of the random medium. The random time shift has Gaussian statistics with mean zero and variance that depends on the statistics of the random medium. The stabilization of the wave front in one-dimensional random media (or in three-dimensional randomly layered media) was first noted by O'Doherty and Anstey in a geophysical context [23]. A time-domain integral equation approach was given in [7, 8]. A frequency-domain approach was presented in [10, 14].

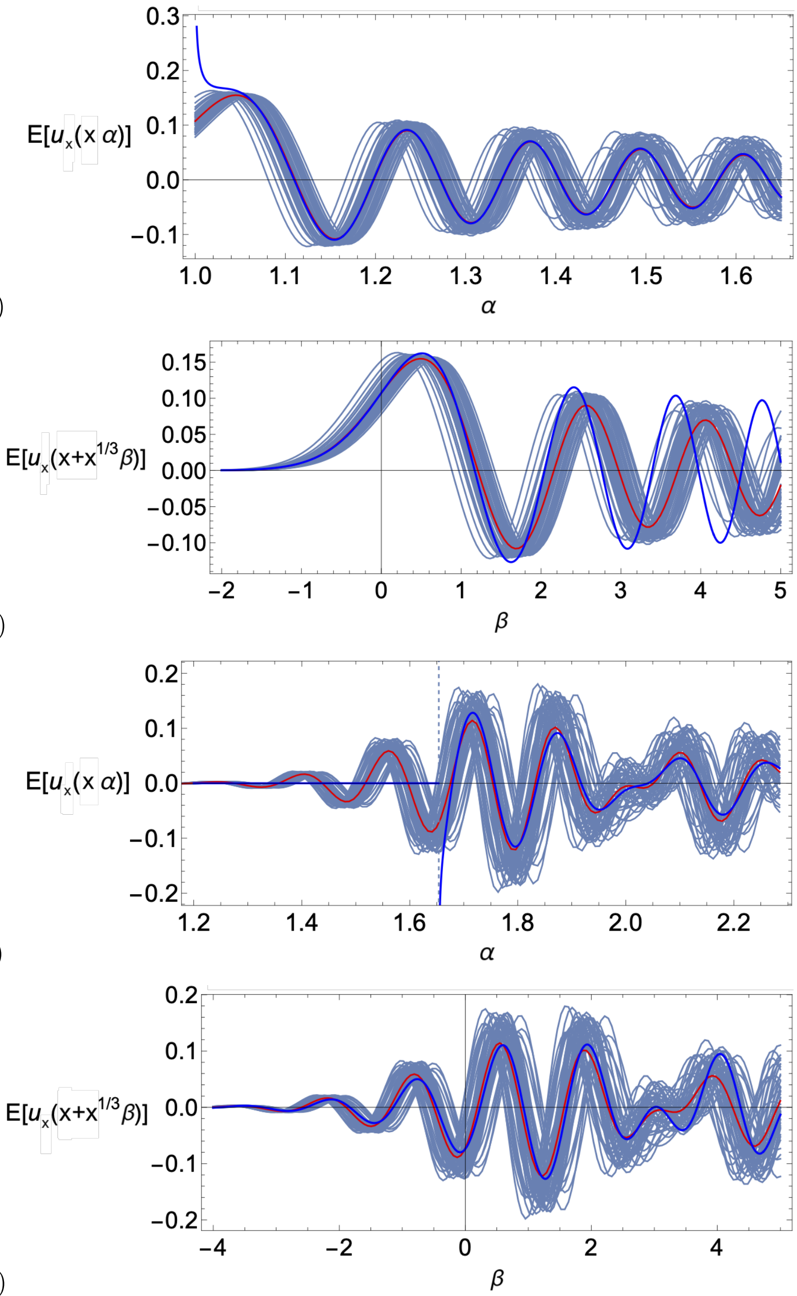
REMARK 3.1 (Transmitted field with random mass perturbations and $\varpi_{\mathrm{s}} = 0$). In the region $\mathtt{x} > L, \mathtt{x} \in \mathbb{Z}$, the solution has the form

$$u_{\mathtt{x}}(\alpha \mathtt{x}) = \frac{1}{\sqrt{\pi \mathtt{x}}\sqrt[4]{\alpha^2 - 1}} \cos\Big(\frac{\pi}{4} + 2\big(\arccos(1/\alpha) - \sqrt{\alpha^2 - 1}\big)\mathtt{x} + \sqrt{\gamma(\omega_\alpha)} W_L\Big) \times e^{-\frac{\gamma(\omega_\alpha)}{2} L} + o\big(\frac{1}{\sqrt{\mathtt{x}}}\big), \tag{3.6}$$

where $W_L \sim \mathcal{N}(0, L)$. At times close to $\mathtt{x} > 0, \mathtt{x} \in \mathbb{Z}$, we have

$$u_{\mathtt{x}}(\mathtt{x} + \sqrt[3]{\mathtt{x}}\beta) = \frac{1}{\sqrt[3]{\mathtt{x}}} Ai(-2\beta) + o\big(\frac{1}{\sqrt[3]{\mathtt{x}}}\big), \quad \beta \in \mathbb{R}. \tag{3.7}$$

These expressions show that the field for times t close to the front $\mathtt{x}$ is not affected to leading order by the random perturbations, but the field following the front for times t larger than $\mathtt{x}$ is affected.



(a)

(b)

(c)

(d)

FIG. 3.1. *Time domain problem with random mass perturbations with $\sigma = 0.15$, comparing ensemble of numerical solutions (in grayish blue shade with empirical mean curve in red) with asymptotic formulas (in blue). (a) Mean field (3.3), $\varpi_s = 0$, $L = 16$. (b) Mean front (3.5), $\varpi_s = 0$, $L = 16$. (c) Mean field (3.8), $\varpi_s = 1.1$, $L = 8$. (d) Mean front (3.10), $\varpi_s = 1.1$, $L = 8$.*

We now address the case when $\varpi_s > 0$. Let α_s be defined by (2.10).

THEOREM 3.3 (Mean field with random mass perturbations and $\varpi_s > 0$). *If $\sigma \ll 1$, $\mathtt{x} > L \gg 1, \mathtt{x} \in \mathbb{Z}$, the mean field has the form:*

$$\mathbb{E}\big[u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x})\big] = \frac{\sqrt{2}\omega_\alpha^{+\,3/2}}{\sqrt{\pi\alpha(\omega_\alpha^{+\,4} - \omega_s^4)\mathtt{x}}} \cos\Big(\frac{\pi}{4} + [k(\omega_\alpha^+) - \omega_\alpha^+\alpha]\mathtt{x}\Big) e^{-\gamma(\omega_\alpha^+)L}$$
$$+ \frac{\sqrt{2}\omega_\alpha^{-\,3/2}}{\sqrt{\pi\alpha(\omega_s^4 - \omega_\alpha^{-\,4})\mathtt{x}}} \cos\Big(-\frac{\pi}{4} + [k(\omega_\alpha^-) - \omega_\alpha^-\alpha]\mathtt{x}\Big) e^{-\gamma(\omega_\alpha^-)L} + o\Big(\frac{1}{\sqrt{\mathtt{x}}}\Big), \tag{3.8}$$

for any $\alpha \in (\alpha_s, +\infty)$, where $k(\omega)$ is given by (2.9), ω_s is defined by (2.10), $\omega_\alpha^\pm$ is defined by (2.12) and

$$\gamma(\omega_\alpha^\pm) = \frac{\sigma^2\alpha^2\omega_\alpha^{\pm\,2}}{4}. \tag{3.9}$$

THEOREM 3.4 (Mean front with random mass perturbations and $\varpi_s > 0$). *If $\sigma \ll 1$, $\mathtt{x} > L \gg 1, \mathtt{x} \in \mathbb{Z}$, then we have*

$$\mathbb{E}\big[u_{\mathtt{x}}(\alpha_s\mathtt{x} + \sqrt[3]{\mathtt{x}}\beta)\big]$$
$$= \frac{\sqrt[3]{2}}{\sqrt[3]{\mathtt{x}}} Ai\Big(-\sqrt[3]{2}\frac{\beta}{\alpha_s}\Big) \cos\Big([k(\omega_s) - \omega_s\alpha_s]\mathtt{x} - \omega_s\beta\sqrt[3]{\mathtt{x}}\Big) \times e^{-\gamma(\omega_s)L} + o\Big(\frac{1}{\sqrt[3]{\mathtt{x}}}\Big), \tag{3.10}$$

for any $\beta \in \mathbb{R}$, where $k(\omega)$ is defined by (2.9), ω_s is defined by (2.10), and

$$\gamma(\omega_s) = \frac{\sigma^2\alpha_s^2\omega_s^2}{4} = \frac{\sigma^2\sqrt{\varpi_s}\sqrt{4+\varpi_s}}{(\sqrt{4+\varpi_s} - \sqrt{\varpi_s})^2}. \tag{3.11}$$

As $\varpi_s \to 0^+$, it is noted that the expression (3.10) reduces to the earlier result (3.5) valid for $\varpi_s = 0$ and the attenuation drops to zero. Thus the case $\varpi_s > 0$ is characterized by an attenuation of the front field for times close to $\alpha_s\mathtt{x}$, which is different from the behaviour in the case $\varpi_s = 0$.

In Figure 3.1 we compare the empirical averages of numerical simulations with the theoretical predictions for the mean field and the mean front (i.e. (3.3), (3.5) when $\varpi_s = 0$ and (3.8), (3.10) when $\varpi_s > 0$). We obtain excellent agreement which demonstrates the accuracy of the asymptotic approach.

4. Effective dynamics for the time-harmonic problem

Here we assume that the variables $\mathfrak{m}_{\mathtt{x}}$ in the section $\mathtt{x} \in [1, L] \cap \mathbb{Z}$ are identically distributed with mean zero and variance σ^2:

$$\mathbb{E}[\mathfrak{m}_{\mathtt{x}}^2] = \sigma^2, \tag{4.1}$$

with $\sigma \ll 1$, and L is of the order of σ^{-2}.

We consider a time-harmonic wave (1.4) with $\omega \in (\sqrt{\varpi_s}, \sqrt{\varpi_s + 4})$ (propagative regime). When $\mathfrak{m}_{\mathtt{x}} = 0$ for $\mathtt{x} \le 0, \mathtt{x} \in \mathbb{Z}$, and for $\mathtt{x} > L, \mathtt{x} \in \mathbb{Z}$, and a unit-amplitude right-going input wave is incoming from the left, the solution has the form

$$\hat{u}_{\mathtt{x}} = e^{ik\mathtt{x}} + Re^{-ik\mathtt{x}} \qquad \text{for } \mathtt{x} \le 0, \mathtt{x} \in \mathbb{Z}, \tag{4.2}$$

$$\hat{u}_{\mathtt{x}} = T e^{ik\mathtt{x}} \qquad \text{for } \mathtt{x} > L, \mathtt{x} \in \mathbb{Z}, \tag{4.3}$$

where $k(\omega)$ is the solution of the dispersion relation $-\omega^2 = 2\cos k - 2 - \varpi_{\mathrm{s}}$, which has the form (2.9). The time-harmonic field $\hat{u}_{\mathtt{x}}$ satisfies (1.5) for $\mathtt{x} \in [1,L] \cap \mathbb{Z}$. The complex coefficient R, resp. T, is the reflection, resp. transmission, coefficient of the perturbed section $[1,L] \cap \mathbb{Z}$.

4.1. Independent perturbations. In this subsection we assume that the variables $\mathfrak{m}_{\mathtt{x}}$ are independent and identically distributed.

THEOREM 4.1. *For a given $\omega \in (\sqrt{\varpi_{\mathrm{s}}}, \sqrt{\varpi_{\mathrm{s}}+4})$, the square modulus of the transmission coefficient $\tau = |T|^2$ behaves as a Markov diffusion process as a function of L with the infinitesimal generator*

$$\mathcal{L} = \gamma \big[\tau^2 (1-\tau) \partial_\tau^2 - \tau^2 \partial_\tau \big], \tag{4.4}$$

starting from $\tau_0 = 1$, where $\gamma(\omega)$ is defined by (3.2) and $k(\omega)$ is defined by (2.9).

This theorem is proved in Section 6.1 by a multiscale analysis in the scaling regime $\sigma \ll 1$ and $L = O(\sigma^{-2})$. The form of the infinitesimal generator is similar to the one obtained for the square modulus of the transmission coefficient of the one-dimensional wave equation in random medium [14, Theorem 7.3], except for the frequency dependence which is different here and which follows from the particular dispersion relation of the discrete lattice.

Using the results of [14, Section 7.1.5] we obtain that, for any $n \geq 1$:

$$\mathbb{E}\big[|T|^{2n}\big] = e^{-\frac{\gamma L}{4}} \int_0^\infty e^{-\gamma L s^2} \frac{2\pi s \sinh(\pi s)}{\cosh^2(\pi s)} \phi_n(s) ds, \tag{4.5}$$

with

$$\phi_1(s) = 1, \qquad \phi_n(s) = \prod_{j=1}^{n-1} \frac{s^2 + (j-\frac{1}{2})^2}{j^2}, \quad n \geq 2. \tag{4.6}$$

As a result of (4.5), we can obtain the mean transmission $\mathbb{E}\big[|T|^2\big]$ and its variance $\mathrm{Var}\big(|T|^2\big) = \mathbb{E}\big[|T|^4\big] - \mathbb{E}\big[|T|^2\big]^2$.

In Figure 4.1, we compare the empirical averages of numerical simulations with the theoretical predictions for the expectation $\mathbb{E}[|T|^2]$ and the standard deviation $\mathrm{Std}(|T|^2) = \mathrm{Var}(|T|^2)^{1/2}$. The numerical simulations are based on the exact solution (A.10) and the theoretical predictions are based on (4.5). We obtain excellent agreement which confirms the accuracy of the asymptotic approach. We can observe that the behavior of the transmittance close to the left endpoint of the propagative band $(\sqrt{\varpi_{\mathrm{s}}}, \sqrt{\varpi_{\mathrm{s}}+4})$ is very different in the cases $\varpi_{\mathrm{s}} = 0$ and $\varpi_{\mathrm{s}} \neq 0$. The transmittance goes to zero as $\omega \to \sqrt{\varpi_{\mathrm{s}}}$ when $\varpi_{\mathrm{s}} > 0$ and it goes to one when $\varpi_{\mathrm{s}} = 0$.

4.2. Correlated perturbations. The previous results can be extended to the case where the variables $\mathfrak{m}_{\mathtt{x}}$ are identically distributed with mean zero, variance σ^2, and integrable covariance function:

$$\mathbb{E}[\mathfrak{m}_{\mathtt{x}} \mathfrak{m}_{\mathtt{x}'}] = \sigma^2 \Gamma(\mathtt{x} - \mathtt{x}'), \quad \mathtt{x}, \mathtt{x}' \in \mathbb{Z}, \tag{4.7}$$

with $\sigma \ll 1$, and L is of the order of σ^{-2}. The function Γ is assumed to be integrable $\sum_{j \in \mathbb{Z}} |\Gamma(j)| < +\infty$. We can then apply the diffusion approximation theory set forth

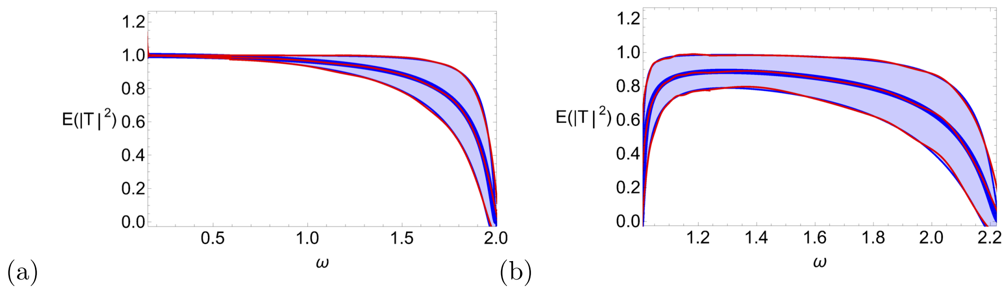


FIG. 4.1. *Theoretical mean transmittance* $\mathbb{E}[|T|^2]$ *(thick blue line) with theoretical spread* $\pm\mathrm{Std}(|T|^2)$ *(thin blue lines) given by* (4.5) *vs frequency* ω. *(a)* $\varpi_s = 0.02$ *(b)* $\varpi_s = 1.01$. *The thick red lines plot the empirical averages of numerical simulations using the exact expression* (A.10) *described in Appendix A.1, the thin red lines plot the numerical spread. Ensemble details are the same ones as in Figure 1.3. In the absence of mass perturbation on* $[1,L]\cap\mathbb{Z}$, *we have* $|T|^2 \equiv 1$.

in [14, Chapter 6] (other approaches based on Duhamel series expansions exist but will not be used here [2]). We find that $\tau = |T|^2$ behaves as a diffusion process with the infinitesimal generator (4.4) where

$$\gamma(\omega) = \frac{\omega^4 \sigma^2}{4\sin^2 k(\omega)} \check{\Gamma}(2k(\omega)), \qquad \check{\Gamma}(k) = \Gamma(0) + 2\sum_{j=1}^{\infty} \cos(kj)\Gamma(j). \tag{4.8}$$

The moments of $|T|^2$ are still given by (4.5) with the new expression (4.8) of γ. Note that $\check{\Gamma}(2k)$ is non-negative by Wiener-Khintchine theorem but it may be non-monotonous as a function of k. In other words the spatial correlation of the variables $\mathfrak{m}_{\mathtt{x}}$ has a non-trivial impact onto the parameter $\gamma(\omega)$. As a consequence, the observation of $\gamma(\omega)$ for $\omega \in (\sqrt{\varpi_s}, \sqrt{\varpi_s + 4})$ makes it possible to retrieve $\check{\Gamma}(k)$ for $k \in (0, 2\pi)$ which characterizes the correlation function $\Gamma(j)$. This opens the way towards an original method to estimate the statistics of random mass perturbations by measuring the frequency-dependent transmission coefficient of a section of a discrete lattice.

4.3. Scattering in non-matched medium. The previous results can be revisited in the case where

$$\mathfrak{m}_{\mathtt{x}} = \Delta_0 \text{ for } \mathtt{x} \le 0, \tag{4.9}$$

$$\mathfrak{m}_{\mathtt{x}} = \Delta_1 \text{ for } \mathtt{x} > L, \tag{4.10}$$

and $\mathtt{x} \in \mathbb{Z}$. This means that the masses in the two unperturbed half-spaces may be different from the average mass in the perturbed section $[1,L]\cap\mathbb{Z}$. We may then anticipate that the boundaries of the perturbed section can generate reflections themselves.

We introduce the wavenumbers k_0 and k_1 solutions of the dispersion relations

$$-(1+\Delta_j)\omega^2 = 2\cos \mathrm{k}_j - 2 - \varpi_s, \quad j = 0, 1. \tag{4.11}$$

Here we assume the regime is propagative, i.e. the frequency ω is such that $(2 + \varpi_s - \omega^2(1+\Delta_j))/(2) \in (-1,1)$ for $j = 0,1$ so that there is a unique solution $\mathrm{k}_j \in (0,\pi)$ to (4.11). If a right-going input wave is incoming from the left, the time-harmonic field in the two unperturbed half-spaces has the form

$$\hat{u}_{\mathtt{x}} = e^{i\mathrm{k}_0 \mathtt{x}} + Re^{-i\mathrm{k}_0 \mathtt{x}} \qquad \text{for } \mathtt{x} \le 0, \tag{4.12}$$

$$\hat{u}_{\mathtt{x}} = T e^{i\mathrm{k}_1 \mathtt{x}} \qquad \text{for } \mathtt{x} > L, \tag{4.13}$$

and $\hat{u}$ satisfies (1.5) for $0 < \mathtt{x} \leq L$ with $\mathtt{x} \in \mathbb{Z}$.

We assume that the random variables $\mathfrak{m}_{\mathtt{x}}$ are identically distributed with mean zero, variance σ^2, and integrable covariance function (4.7). We can give explicit formulas in two special cases (see Section 6.2).

(1) If $\Delta_1 = 0$, then we get

$$\begin{aligned}\mathbb{E}[|T|^2] &= \frac{2\sin^2 \mathrm{k}_0}{1-\cos(k+\mathrm{k}_0)} \sum_{m=0}^{\infty} \Big(\frac{1-\cos(k-\mathrm{k}_0)}{1-\cos(k+\mathrm{k}_0)}\Big)^m \Big(\mathbb{E}[|\tilde{R}|^{2m}] - \mathbb{E}[|\tilde{R}|^{2m+2}]\Big) \\ &= \frac{2\sin^2 \mathrm{k}_0}{1-\cos(k+\mathrm{k}_0)} \sum_{n=0}^{\infty} (-1)^n \mathbb{E}[|\tilde{T}|^{2n+2}] \Big[\sum_{m=n}^{\infty} \binom{m}{n} \Big(\frac{1-\cos(k-\mathrm{k}_0)}{1-\cos(k+\mathrm{k}_0)}\Big)^m\Big],\end{aligned} \tag{4.14}$$

where $\mathbb{E}[|\tilde{T}|^{2n}]$ is given by (4.5). When there is no random mass perturbation, we have simply $|T|^2 = \frac{2\sin^2 \mathrm{k}_0}{1-\cos(k+\mathrm{k}_0)}$ (see also (A.23)). We have similarly

$$\begin{aligned}\mathbb{E}[|T|^4] = &\Big(\frac{2\sin^2 \mathrm{k}_0}{1-\cos(k+\mathrm{k}_0)}\Big)^2 \sum_{n=0}^{\infty} (-1)^n \mathbb{E}[|\tilde{T}|^{2n+4}] \\ &\times \Big[\sum_{m=n}^{\infty} \binom{m}{n} (1+m)^2 \Big(\frac{1-\cos(k-\mathrm{k}_0)}{1-\cos(k+\mathrm{k}_0)}\Big)^m\Big],\end{aligned} \tag{4.15}$$

which makes it possible to get $\mathrm{Var}(|T|^2) = \mathbb{E}[|T|^4] - \mathbb{E}[|T|^2]^2$.

(2) If $\Delta_0 = 0$, then we have

$$\mathbb{E}[|T|^2] = \frac{\sin k}{\sin \mathrm{k}_1} e^{-\frac{\gamma L}{4}} \int_0^\infty e^{-\gamma L s^2} \frac{2\pi s \sinh(\pi s)}{\cosh^2(\pi s)} P_{-1/2+is}\Big(\frac{1-\cos k \cos \mathrm{k}_1}{\sin k \sin \mathrm{k}_1}\Big) ds. \tag{4.16}$$

When there is no random mass perturbation, we have simply $|T|^2 = \frac{2\sin^2 k}{1-\cos(k+\mathrm{k}_1)}$ (see also (A.23)). More generally, for any $n \geq 1$,

$$\begin{aligned}\mathbb{E}[|T|^{2n}] = \Big(\frac{\sin k}{\sin \mathrm{k}_1}\Big)^n e^{-\frac{\gamma L}{4}} &\int_0^\infty e^{-\gamma L s^2} \frac{2\pi s \sinh(\pi s)}{\cosh^2(\pi s)} \phi_n(s) \\ &\times P_{-1/2+is}\Big(\frac{1-\cos k \cos \mathrm{k}_1}{\sin k \sin \mathrm{k}_1}\Big) ds,\end{aligned} \tag{4.17}$$

where $\phi_n(s)$ is defined by (4.6) and

$$P_{-1/2+is}(\eta) = \frac{\sqrt{2}}{\pi} \cosh(\pi s) \int_0^\infty \frac{\cos(st)}{\sqrt{\cosh(t+\eta)}} dt. \tag{4.18}$$

REMARK 4.1. It is possible to address the general case where both Δ_0 and Δ_1 are not zero, but the expressions become complicated.

REMARK 4.2. It is possible to address the case where ω is outside the pass band on the right half-space, i.e. $(2+\varpi_{\mathrm{s}} - \omega^2(1+\Delta_1))/(2) \notin (-1,1)$. The results are given in Subsection 6.2. As can be expected, the reflection coefficients have modulus one since the wave cannot propagate in the right half-space. Note that this happens for all frequencies when $\Delta_1 \to +\infty$.

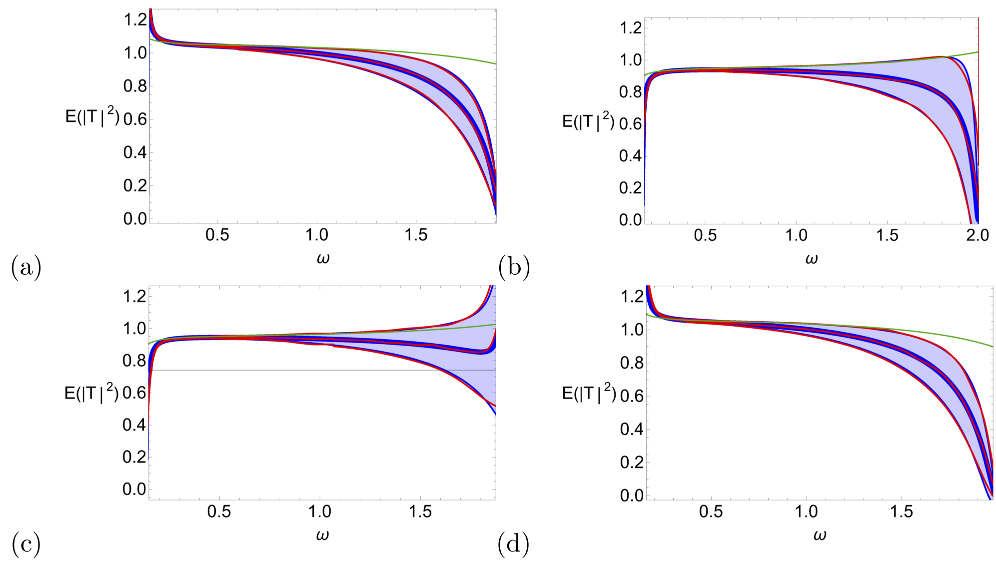


FIG. 4.2. *Mean transmittance* $\mathbb{E}[|T|^2]$ *(thick lines) with spread* $\pm\mathrm{Std}(|T|^2)$ *(thin lines) vs frequency* ω*. (a)* $\Delta_0=0.1, \Delta_1=0$*, (b)* $\Delta_0=-0.1, \Delta_1=0$*, (c)* $\Delta_0=0, \Delta_1=0.1$*, (d)* $\Delta_0=0, \Delta_1=-0.1$*. In all cases* $L=40, \sigma=0.05, \varpi_s=0.02$*. Blue: theoretical formulas* (4.14)-(4.15) *for* $\Delta_0\neq 0$, $\Delta_1=0$ *and* (4.16)-(4.17) *for* $\Delta_0=0$, $\Delta_1\neq 0$*, obtained by multiscale analysis. Red: empirical averages based on the exact expression* (A.15) *for (a), (b) and* (A.21) *for (c), (d); the ensemble size is* 151*. Green: exact formula* (A.23) *for* $|T|^2$ *for the case* (6.21) *without mass perturbation on* $[1,L]\cap\mathbb{Z}$*; similarities between the green lines in (a), (c) (and in (b), (d)) is due to the symmetry in the expression of* $|T|^2$ *with respect to index* 0 *and* 1 *in the absence of mass perturbation.*

In Figure 4.2 we compare the empirical averages of numerical simulations with the theoretical predictions for the expectation $\mathbb{E}[|T|^2]$ and the standard deviation $\mathrm{Std}(|T|^2)$ for different mismatched media. The numerical simulations are based on the exact solution (A.15) for $\Delta_0\neq 0$, $\Delta_1=0$ and (A.21) for $\Delta_0=0$, $\Delta_1\neq 0$. The theoretical predictions are based on (4.14) and (4.15) for $\Delta_0\neq 0$, $\Delta_1=0$ and on (4.17) for $\Delta_0=0$, $\Delta_1\neq 0$. We obtain excellent agreement which confirms the accuracy of the asymptotic approach. We can observe that the medium mismatch has a dramatic impact on the transmittance, in particular close to the endpoints of the propagative band $(\sqrt{\varpi_s}, \sqrt{\varpi_s+4})$.

5. Proofs of Preliminary results

Proof. **(Proof of Lemma 2.1).** Since $k(\omega)$ satisfies $-\omega^2=2(\cos(k(\omega))-1)$, we find that $\hat{u}_{\mathtt{x}}(\omega)$ satisfies

$$\hat{u}_{\mathtt{x}+1}(\omega)+\hat{u}_{\mathtt{x}-1}(\omega)-2\hat{u}_{\mathtt{x}}(\omega)=2(\cos(k(\omega))-1)\hat{u}_{\mathtt{x}}(\omega)=-\omega^2\hat{u}_{\mathtt{x}}(\omega) \tag{5.1}$$

for any $\mathtt{x}\in\mathbb{Z}$, hence $u_{\mathtt{x}}(t)$ satisfies (1.1). It remains to show that $u_{\mathtt{x}}(t)$ satisfies the appropriate initial conditions. On the one hand (using the change of variable $\omega\mapsto k(\omega)$) we have

$$\begin{aligned} u_{\mathtt{x}}(0) &= \frac{1}{2\pi}\int_0^2 \hat{u}_{\mathtt{x}}(\omega)d\omega + c.c. \\ &= \frac{2}{\pi}\int_0^2 \frac{\cos(k(\omega)(\mathtt{x}-\mathtt{x}_0))}{\sqrt{4-\omega^2}}d\omega = \frac{1}{\pi}\int_0^\pi \cos(k(\mathtt{x}-\mathtt{x}_0))dk = \delta_{\mathtt{x},\mathtt{x}_0}. \end{aligned} \tag{5.2}$$

On the other hand we have

$$\dot{u}_{\mathtt{x}}(0) = -\frac{i}{2\pi}\int_0^2 \omega\,\hat{u}_{\mathtt{x}}(\omega)d\omega + c.c. = 0, \tag{5.3}$$

which completes the proof. ∎

Proof. **(Proof of Lemma 2.2).** For $\mathtt{x} \gg 1, \mathtt{x} \in \mathbb{Z}$, the expression

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x}) = & \frac{1}{2\pi}\int_0^2 \frac{1}{\sqrt{4-\omega^2}} e^{i(k(\omega)\mathtt{x}-\omega\alpha\mathtt{x})}d\omega \\ & + \frac{1}{2\pi}\int_0^2 \frac{1}{\sqrt{4-\omega^2}} e^{i(-k(\omega)\mathtt{x}-\omega\alpha\mathtt{x})}d\omega + c.c. \end{aligned} \tag{5.4}$$

involves the value of an integral (in ω) with a fast phase $[\pm k(\omega) - \omega\alpha]\mathtt{x}$ and it can be evaluated by the stationary phase method. We compute

$$k'(\omega) = \frac{2}{\sqrt{4-\omega^2}}, \qquad k''(\omega) = \frac{2\omega}{(4-\omega^2)^{3/2}}. \tag{5.5}$$

The phase $[k(\omega)-\omega\alpha]\mathtt{x}$ has a unique stationary point at $\omega_\alpha \in (0,2)$ such that $k'(\omega_\alpha) = \alpha$, i.e.

$$\omega_\alpha = \frac{2\sqrt{\alpha^2-1}}{\alpha}, \tag{5.6}$$

for which we have $k''(\omega_\alpha) = \alpha^2\sqrt{\alpha^2-1}/2$. We then obtain (after the change of variable $\omega = \omega_\alpha + \mathtt{x}^{-1/2}s$)

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x}) &= \frac{1}{2\pi\sqrt{4-\omega_\alpha^2}\sqrt{\mathtt{x}}} e^{i[k(\omega_\alpha)-\omega_\alpha\alpha]\mathtt{x}} \int_{-\infty}^{\infty} e^{i\frac{k''(\omega_\alpha)}{2}s^2} ds + c.c. + o\Big(\frac{1}{\sqrt{\mathtt{x}}}\Big) \\ &= \frac{1}{2\pi\sqrt{4-\omega_\alpha^2}\sqrt{\mathtt{x}}} e^{i[k(\omega_\alpha)-\omega_\alpha\alpha]\mathtt{x}} \frac{\sqrt{2\pi}e^{i\pi/4}}{\sqrt{k''(\omega_\alpha)}} + c.c. + o\Big(\frac{1}{\sqrt{\mathtt{x}}}\Big), \end{aligned} \tag{5.7}$$

which gives (2.7). ∎

Proof. **(Proof of Lemma 2.3).** We now look for an asymptotic expansion of field $u_{\mathtt{x}_0+\mathtt{x}}(t)$ around time $\mathtt{x}$ with $\mathtt{x} > 0, \mathtt{x} \in \mathbb{Z}$:

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\mathtt{x}+\sqrt[3]{\mathtt{x}}\beta) = & \frac{1}{2\pi}\int_0^2 \frac{1}{\sqrt{4-\omega^2}} e^{i(k(\omega)\mathtt{x}-\omega\mathtt{x}-\omega\beta\sqrt[3]{\mathtt{x}})}d\omega \\ & + \frac{1}{2\pi}\int_0^2 \frac{1}{\sqrt{4-\omega^2}} e^{i(-k(\omega)\mathtt{x}-\omega\mathtt{x}-\omega\beta\sqrt[3]{\mathtt{x}})}d\omega + c.c.. \end{aligned} \tag{5.8}$$

For $\mathtt{x} \gg 1$, we find that the phase $[k(\omega)\mathtt{x} - \omega\mathtt{x} - \omega\beta\sqrt[3]{\mathtt{x}}]$ has a unique stationary point at $\omega_0 = 0$ such that $k'(\omega_0) = 1$ and it is localized at the border of the interval $(0,2)$. We then obtain (using $k'''(0) = 1/4$ and the change of variable $\omega = \mathtt{x}^{-1/3}s$)

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\mathtt{x}+\sqrt[3]{\mathtt{x}}\beta) &= \frac{1}{4\pi\sqrt[3]{\mathtt{x}}}\int_0^\infty e^{i\frac{k'''(0)}{6}s^3 - is\beta} ds + c.c. + o\Big(\frac{1}{\sqrt[3]{\mathtt{x}}}\Big) \\ &= \frac{1}{\pi\sqrt[3]{\mathtt{x}}}\int_0^\infty \cos\Big(\frac{s^3}{3} - 2\beta s\Big) ds + o\Big(\frac{1}{\sqrt[3]{\mathtt{x}}}\Big), \end{aligned} \tag{5.9}$$

which completes the proof since $Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xs + \frac{s^3}{3}\right)ds$. ∎

Proof. **(Proof of Lemma 2.5).** For $\mathtt{x} \gg 1$, the expression

$$u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x}) = \frac{1}{4\pi}\int_0^\infty \hat{c}(\omega)e^{i(k(\omega)\mathtt{x}-\omega\alpha\mathtt{x})}d\omega + \frac{1}{4\pi}\int_0^\infty \hat{c}(\omega)e^{i(-k(\omega)\mathtt{x}-\omega\alpha\mathtt{x})}d\omega + c.c. \quad (5.10)$$

is the value of an integral (in ω) with a fast phase $[\pm k(\omega) - \omega\alpha]\mathtt{x}$ and it can be evaluated by the stationary phase method. We compute

$$k'(\omega) = \frac{2\omega}{\sqrt{\omega^2 - \varpi_{\mathrm{s}}}\sqrt{4+\varpi_{\mathrm{s}}-\omega^2}}, \qquad k''(\omega) = \frac{2(\omega^4 - \omega_s^4)}{(\omega^2-\varpi_{\mathrm{s}})^{3/2}(4+\varpi_{\mathrm{s}}-\omega^2)^{3/2}}. \quad (5.11)$$

If $\alpha < \alpha_s$ then the phase $[k(\omega)-\omega\alpha]\mathtt{x}$ does not have any stationary point. For any $\alpha \in (\alpha_s, +\infty)$ the phase $[k(\omega)-\omega\alpha]\mathtt{x}$ has two stationary points at $\omega_\alpha^\pm \in (\sqrt{\varpi_{\mathrm{s}}}, \sqrt{4+\varpi_{\mathrm{s}}})$ such that $k'(\omega_\alpha^\pm) = \alpha$. We have in fact $\omega_\alpha^+ \in (\omega_s, \sqrt{4+\varpi_{\mathrm{s}}})$ and $\omega_\alpha^- \in (\sqrt{\varpi_{\mathrm{s}}}, \omega_s)$, and

$$k''(\omega_\alpha^\pm) = \frac{\alpha^3({\omega_\alpha^\pm}^4 - \omega_s^4)}{4{\omega_{\pm,\alpha}}^3}, \quad (5.12)$$

which is positive for ω_α^+ and negative for ω_α^-. We then obtain

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\alpha\mathtt{x}) &= \frac{1}{4\pi\sqrt{\mathtt{x}}}\hat{c}(\omega_\alpha^+)e^{i[k(\omega_\alpha^+)-\omega_\alpha^+\alpha]\mathtt{x}}\int_{-\infty}^\infty e^{i\frac{k''(\omega_\alpha^+)}{2}s^2}ds \\ &\quad + \frac{1}{4\pi\sqrt{\mathtt{x}}}\hat{c}(\omega_\alpha^-)e^{i[k(\omega_\alpha^-)-\omega_\alpha^-\alpha]\mathtt{x}}\int_{-\infty}^\infty e^{i\frac{k''(\omega_\alpha^-)}{2}s^2}ds + c.c. + o(\frac{1}{\sqrt{\mathtt{x}}}) \\ &= \frac{\hat{c}(\omega_\alpha^+)}{\sqrt{2\pi k''(\omega_\alpha^+)\mathtt{x}}}\cos\left(\frac{\pi}{4} + [k(\omega_\alpha^+) - \omega_\alpha^+\alpha]\mathtt{x}\right) \\ &\quad + \frac{\hat{c}(\omega_\alpha^-)}{2\sqrt{\pi|k''(\omega_\alpha^-)|\mathtt{x}}}\cos\left(-\frac{\pi}{4} + [k(\omega_\alpha^-) - \omega_\alpha^-\alpha]\mathtt{x}\right) + o(\frac{1}{\sqrt{\mathtt{x}}}), \end{aligned} \quad (5.13)$$

which gives (2.11) using $\hat{c}(\omega_\alpha^\pm) = \alpha$. ∎

Proof. **(Proof of Lemma 2.6).** We now look for an asymptotic expansion of field $u_{\mathtt{x}_0+\mathtt{x}}(t)$ around time $\alpha_s\mathtt{x}$ with $\mathtt{x} > 0, \mathtt{x} \in \mathbb{Z}$:

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\alpha_s\mathtt{x} + \beta\sqrt[3]{\mathtt{x}}) &= \frac{1}{4\pi}\int_0^\infty \hat{c}(\omega)e^{i(k(\omega)\mathtt{x}-\omega\alpha_s\mathtt{x}-\omega\beta\sqrt[3]{\mathtt{x}})}d\omega \\ &\quad + \frac{1}{4\pi}\int_0^\infty \hat{c}(\omega)e^{i(-k(\omega)\mathtt{x}-\omega\alpha_s\mathtt{x}-\omega\beta\sqrt[3]{\mathtt{x}})}d\omega + c.c. \end{aligned} \quad (5.14)$$

For $\mathtt{x} \gg 1$, we find that the phase $[k(\omega)\mathtt{x} - \omega\alpha_s\mathtt{x} - \omega\beta\sqrt[3]{\mathtt{x}}]$ has a unique stationary point at ω_s such that $k'(\omega_s) = \alpha_s$. It satisfies $k''(\omega_s) = 0$ and ω_s is localized in the interior of the interval $(\sqrt{\varpi_{\mathrm{s}}}, \sqrt{4+\varpi_{\mathrm{s}}})$. We then obtain by the change of variable $\omega = \omega_s + \mathtt{x}^{-1/3}s$ (and using $k'''(\omega_s) = \alpha_s^3$, $\hat{c}(\omega_s) = \alpha_s$)

$$\begin{aligned} u_{\mathtt{x}_0+\mathtt{x}}(\alpha_s\mathtt{x}+\beta\sqrt[3]{\mathtt{x}}) &= \frac{\hat{c}(\omega_s)}{4\pi\sqrt[3]{\mathtt{x}}}e^{i[k(\omega_s)-\omega_s\alpha_s]\mathtt{x}-i\omega_s\beta\sqrt[3]{\mathtt{x}}}\int_{-\infty}^\infty e^{i\frac{k'''(\omega_s)}{6}s^3 - is\alpha_s}ds + c.c. + o(\frac{1}{\sqrt[3]{\mathtt{x}}}) \\ &= \frac{\sqrt[3]{2}}{2\pi\sqrt[3]{\mathtt{x}}}\int_0^\infty \cos\left(\frac{s^3}{3} - \sqrt[3]{2}\frac{\beta}{\alpha_s}s\right)ds\, e^{i[k(\omega_s)-\omega_s\alpha_s]\mathtt{x}-i\omega_s\beta\sqrt[3]{\mathtt{x}}} + c.c. + o(\frac{1}{\sqrt[3]{\mathtt{x}}}), \end{aligned} \quad (5.15)$$

which completes the proof of (2.13). ∎

6. Proof of Theorem 4.1

6.1. Scattering in matched medium $\Delta_0 = \Delta_1 = 0$. When $\mathfrak{m}_{\mathtt{x}} \equiv 0$, the solution is of the form

$$\hat{u}_{\mathtt{x}} = \alpha e^{ik\mathtt{x}} + \beta e^{-ik\mathtt{x}}, \quad \mathtt{x} \in \mathbb{Z}, \tag{6.1}$$

with k solution of the dispersion relation

$$-\omega^2 = 2\cos k - 2 - \varpi_{\mathrm{s}}. \tag{6.2}$$

Here we assume the regime is propagative, i.e. the frequency ω is such that $(2+\varpi_{\mathrm{s}} - \omega^2)/(2) \in (-1,1)$ so that there is a unique solution $k \in (0,\pi)$ to (6.2).

When $\mathfrak{m}_{\mathtt{x}} = 0$ for $\mathtt{x} \leq 0$ and for $\mathtt{x} > L$ and a right-going input wave is incoming from the left, the solution has the form

$$\hat{u}_{\mathtt{x}} = e^{ik\mathtt{x}} + Re^{-ik\mathtt{x}} \quad \text{for } \mathtt{x} \leq 0, \tag{6.3}$$

$$\hat{u}_{\mathtt{x}} = Te^{ik\mathtt{x}} \quad \text{for } \mathtt{x} > L, \tag{6.4}$$

and $\hat{u}$ satisfies (1.5) for $0 < \mathtt{x} \leq L$ with $\mathtt{x} \in \mathbb{Z}$.

We introduce

$$\alpha_{\mathtt{x}} = \frac{e^{-ik\mathtt{x}}}{2i\sin k}\big(\hat{u}_{\mathtt{x}+1} - e^{-ik}\hat{u}_{\mathtt{x}}\big), \tag{6.5}$$

$$\beta_{\mathtt{x}} = -\frac{e^{ik\mathtt{x}}}{2i\sin k}\big(\hat{u}_{\mathtt{x}+1} - e^{ik}\hat{u}_{\mathtt{x}}\big). \tag{6.6}$$

We then have $(\alpha_{\mathtt{x}}, \beta_{\mathtt{x}}) = (1, R)$ for $\mathtt{x} \leq 0$, $(\alpha_{\mathtt{x}}, \beta_{\mathtt{x}}) = (T, 0)$ for $\mathtt{x} > L$, and

$$\hat{u}_{\mathtt{x}} = \alpha_{\mathtt{x}} e^{ik\mathtt{x}} + \beta_{\mathtt{x}} e^{-ik\mathtt{x}}, \tag{6.7}$$

for any $\mathtt{x} \in \mathbb{Z}$.

After some algebra, using the fact that

$$\hat{u}_{\mathtt{x}+1} - \hat{u}_{\mathtt{x}} = \alpha_{\mathtt{x}} e^{ik\mathtt{x}}(e^{ik} - 1) + \beta_{\mathtt{x}} e^{-ik\mathtt{x}}(e^{-ik} - 1), \tag{6.8}$$

we find that $(\alpha_{\mathtt{x}}, \beta_{\mathtt{x}})$ satisfies the system:

$$\alpha_{\mathtt{x}-1} = \alpha_{\mathtt{x}} - \frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\big(\alpha_{\mathtt{x}} + \beta_{\mathtt{x}} e^{-2ik\mathtt{x}}\big), \tag{6.9}$$

$$\beta_{\mathtt{x}-1} = \beta_{\mathtt{x}} + \frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\big(\alpha_{\mathtt{x}} e^{2ik\mathtt{x}} + \beta_{\mathtt{x}}\big), \tag{6.10}$$

for $0 < \mathtt{x} \leq L$, with the boundary conditions

$$\alpha_L = T, \quad \beta_L = 0, \quad \alpha_0 = 1, \quad \beta_0 = R. \tag{6.11}$$

Let $(\tilde{\alpha}_{\mathtt{x}}, \tilde{\beta}_{\mathtt{x}})$ be the solution of the same system

$$\tilde{\alpha}_{\mathtt{x}-1} = \tilde{\alpha}_{\mathtt{x}} - \frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\big(\tilde{\alpha}_{\mathtt{x}} + \tilde{\beta}_{\mathtt{x}} e^{-2ik\mathtt{x}}\big), \tag{6.12}$$

$$\tilde{\beta}_{\mathtt{x}-1} = \tilde{\beta}_{\mathtt{x}} + \frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\big(\tilde{\alpha}_{\mathtt{x}} e^{2ik\mathtt{x}} + \tilde{\beta}_{\mathtt{x}}\big), \tag{6.13}$$

for $0<\mathtt{x}\le L$, $\mathtt{x}\in\mathbb{Z}$, but with the terminal conditions:

$$\tilde{\alpha}_L=1,\quad \tilde{\beta}_L=0. \tag{6.14}$$

Then, by linearity, we have

$$T=\frac{1}{\tilde{\alpha}_0},\quad R=\frac{\tilde{\beta}_0}{\tilde{\alpha}_0}. \tag{6.15}$$

Remark 6.1. *We can check that*

$$|\tilde{\alpha}_{\mathtt{x}-1}|^2-|\tilde{\beta}_{\mathtt{x}-1}|^2=|\tilde{\alpha}_{\mathtt{x}}|^2-|\tilde{\beta}_{\mathtt{x}}|^2, \tag{6.16}$$

which shows that $|\tilde{\alpha}_{\mathtt{x}}|^2-|\tilde{\beta}_{\mathtt{x}}|^2=1$ *for all* $\mathtt{x}\in\mathbb{Z}$*, and therefore we get the energy conservation relation*

$$|R|^2+|T|^2=1. \tag{6.17}$$

Remark 6.2. If the variables $\mathfrak{m}_{\mathtt{x}}$ are independent and identically distributed with mean zero, then $(\tilde{\alpha}_{\mathtt{x}},\tilde{\beta}_{\mathtt{x}})$ is a martingale in the sense that: if we denote $\mathcal{F}_{\mathtt{x}}=\sigma(\mathfrak{m}_{\mathtt{x}'},\mathtt{x}<\mathtt{x}'\le L)$, then $(\tilde{\alpha}_{\mathtt{x}},\tilde{\beta}_{\mathtt{x}})$ is $\mathcal{F}_{\mathtt{x}}$-adapted and $\mathbb{E}\big[(\tilde{\alpha}_{\mathtt{x}-1},\tilde{\beta}_{\mathtt{x}-1})|\mathcal{F}_{\mathtt{x}}\big]=(\tilde{\alpha}_{\mathtt{x}},\tilde{\beta}_{\mathtt{x}})$.

We apply the diffusion approximation theory (see Appendix B). We find that $\tau_L:=|T|^2$ behaves as a diffusion process as stated in Theorem 4.1. This means that the probability density function of τ_L satisfies the Fokker Planck equation

$$\partial_L p_L(\tau)=\mathcal{L}^* p_L=\gamma\big[\partial_\tau^2\big(\tau^2(1-\tau)p_L\big)+\partial_\tau\big(\tau^2 p_L\big)\big], \tag{6.18}$$

starting from $p_{L=0}(\tau)=\delta(\tau-1)$, where δ denotes the Dirac delta distribution. In particular we have for any $n\ge1$:

$$\mathbb{E}\big[|T|^{2n}\big]=\mathbb{E}\big[\tau_L^n\big]=\int\tau^n p_L(\tau)d\tau, \tag{6.19}$$

which yields (4.5).

6.2. Scattering in non-matched medium. In this section we assume that

$$\mathfrak{m}_{\mathtt{x}}=\Delta_0 \text{ for } \mathtt{x}\le 0, \tag{6.20}$$

$$\mathfrak{m}_{\mathtt{x}}=\Delta_1 \text{ for } \mathtt{x}> L, \tag{6.21}$$

$\mathtt{x}\in\mathbb{Z}$, and we introduce the wavenumbers k_0 and k_1 solutions of the dispersion relations

$$-(1+\Delta_j)\omega^2=2\cos\mathrm{k}_j-2-\varpi_{\mathrm{s}},\quad j=0,1. \tag{6.22}$$

Here we assume the regime is propagative, i.e. the frequency ω is such that $(2+\varpi_{\mathrm{s}}-\omega^2(1+\Delta_j))/(2)\in(-1,1)$ for $j=0,1$ so that there is a unique solution $\mathrm{k}_j\in(0,\pi)$ to (4.11).

The wavenumber k is still defined by (6.2). We introduce

$$\alpha_{\mathtt{x}}=\frac{e^{-ik\mathtt{x}}}{2i\sin k}\big(\hat{u}_{\mathtt{x}+1}-e^{-ik}\hat{u}_{\mathtt{x}}\big), \tag{6.23}$$

$$\beta_{\mathtt{x}}=-\frac{e^{ik\mathtt{x}}}{2i\sin k}\big(\hat{u}_{\mathtt{x}+1}-e^{ik}\hat{u}_{\mathtt{x}}\big), \tag{6.24}$$

$\mathtt{x}\in\mathbb{Z}$. We then have

$$\alpha_L = e^{i(\mathrm{k}_1-k)L}\frac{e^{i\mathrm{k}_1}-e^{-ik}}{2i\sin k}T, \tag{6.25}$$

$$\beta_L = -e^{i(\mathrm{k}_1+k)L}\frac{e^{i\mathrm{k}_1}-e^{ik}}{2i\sin k}T, \tag{6.26}$$

and

$$\hat{u}_{\mathtt{x}} = \alpha_{\mathtt{x}} e^{ik\mathtt{x}} + \beta_{\mathtt{x}} e^{-ik\mathtt{x}}, \tag{6.27}$$

for any $\mathtt{x}\in\mathbb{Z}$. The variables $(\alpha_{\mathtt{x}},\beta_{\mathtt{x}})$ satisfy the system:

$$\alpha_{\mathtt{x}-1} = \alpha_{\mathtt{x}} - \frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\left(\alpha_{\mathtt{x}} + \beta_{\mathtt{x}} e^{-2ik\mathtt{x}}\right), \tag{6.28}$$

$$\beta_{\mathtt{x}-1} = \beta_{\mathtt{x}} + \frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\left(\alpha_{\mathtt{x}} e^{2ik\mathtt{x}} + \beta_{\mathtt{x}}\right), \tag{6.29}$$

for $\mathtt{x}\le L$, with the terminal conditions (6.25-6.26) at $\mathtt{x}=L$.

Moreover, we can express the reflection coefficient R and the coefficient α_0 in terms of β_0/α_0:

$$R = -\frac{B + A\frac{\beta_0}{\alpha_0}}{\overline{A} + \overline{B}\frac{\beta_0}{\alpha_0}}, \tag{6.30}$$

$$\alpha_0 = -\frac{2i\sin\mathrm{k}_0}{\overline{A} + \overline{B}\frac{\beta_0}{\alpha_0}}, \tag{6.31}$$

with

$$A = e^{i\mathrm{k}_0} - e^{-ik}, \qquad B = e^{i\mathrm{k}_0} - e^{ik}. \tag{6.32}$$

Proof. We have $\hat{u}_0 = 1 + R = \alpha_0 + \beta_0$, therefore

$$\alpha_0 + \beta_0 = 1 + R. \tag{6.33}$$

We have $\hat{u}_{-1} = e^{-i\mathrm{k}_0} + Re^{i\mathrm{k}_0} = \alpha_{-1}e^{-ik} + \beta_{-1}e^{ik}$. We can express (α_{-1},β_{-1}) in terms of (α_0,β_0) by (6.28-6.29) evaluated at $\mathtt{x}=0$ (remember that $\mathfrak{m}_0=\Delta_0$), so that we get

$$\alpha_0(e^{-ik} - \omega^2\Delta_0) + \beta_0(e^{ik} - \omega^2\Delta_0) = e^{-i\mathrm{k}_0} + Re^{i\mathrm{k}_0}. \tag{6.34}$$

By combining the two equations we can get the two relations

$$\alpha_0(e^{-ik} - e^{i\mathrm{k}_0} - \omega^2\Delta_0) + \beta_0(e^{ik} - e^{i\mathrm{k}_0} - \omega^2\Delta_0)\beta_0 = -2i\sin\mathrm{k}_0, \tag{6.35}$$

$$\alpha_0(-e^{-i\mathrm{k}_0} + e^{-ik} - \omega^2\Delta_0) + \beta_0(-e^{-i\mathrm{k}_0} + e^{ik} - \omega^2\Delta_0)\beta_0 = 2iR\sin\mathrm{k}_0, \tag{6.36}$$

which give:

$$R = \frac{(e^{-i\mathrm{k}_0} - e^{-ik} + \omega^2\Delta_0) + (e^{-i\mathrm{k}_0} - e^{ik} + \omega^2\Delta_0)\frac{\beta_0}{\alpha_0}}{(e^{-ik} - e^{i\mathrm{k}_0} - \omega^2\Delta_0) + (e^{ik} - e^{i\mathrm{k}_0} - \omega^2\Delta_0)\frac{\beta_0}{\alpha_0}}, \tag{6.37}$$

$$\alpha_0 = \frac{-2i\sin\mathrm{k}_0}{(e^{-ik} - e^{i\mathrm{k}_0} - \omega^2\Delta_0) + (e^{ik} - e^{i\mathrm{k}_0} - \omega^2\Delta_0)\frac{\beta_0}{\alpha_0}}. \tag{6.38}$$

We then get the desired result by remarking that $\frac{\Delta_0\omega^2}{2}=\cos k-\cos \mathrm{k}_0$, which gives $e^{-ik}-e^{i\mathrm{k}_0}-\omega^2\Delta_0=-e^{ik}+e^{-i\mathrm{k}_0}=\overline{A}$ and $e^{ik}-e^{i\mathrm{k}_0}-\omega^2\Delta_0=e^{-i\mathrm{k}_0}-e^{-ik}=\overline{B}$. □

Let $(\tilde{\alpha}_{\mathtt{x}},\tilde{\beta}_{\mathtt{x}})$ be the solution of the same system (6.28-6.29)

$$\tilde{\alpha}_{\mathtt{x}-1}=\tilde{\alpha}_{\mathtt{x}}-\frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\left(\tilde{\alpha}_{\mathtt{x}}+\tilde{\beta}_{\mathtt{x}}e^{-2ik\mathtt{x}}\right), \tag{6.39}$$

$$\tilde{\beta}_{\mathtt{x}-1}=\tilde{\beta}_{\mathtt{x}}+\frac{i\omega^2}{2\sin k}\mathfrak{m}_{\mathtt{x}}\left(\tilde{\alpha}_{\mathtt{x}}e^{2ik\mathtt{x}}+\tilde{\beta}_{\mathtt{x}}\right), \tag{6.40}$$

for $0<\mathtt{x}\le L,\mathtt{x}\in\mathbb{Z}$, but with the terminal conditions at $\mathtt{x}=L$:

$$\tilde{\alpha}_L=e^{i(\mathrm{k}_1-k)L}\frac{e^{i\mathrm{k}_1}-e^{-ik}}{2i\sin k},\quad \tilde{\beta}_L=-e^{i(\mathrm{k}_1+k)L}\frac{e^{i\mathrm{k}_1}-e^{ik}}{2i\sin k}. \tag{6.41}$$

Then, by linearity, we have $\tilde{\beta}_0/\tilde{\alpha}_0=\beta_0/\alpha_0$ so we get from (6.30):

$$R=-\frac{A}{\overline{A}}\frac{\frac{B}{A}+\tilde{R}}{1+\frac{\overline{B}}{\overline{A}}\tilde{R}},\ \text{with}\ \tilde{R}=\frac{\tilde{\beta}_0}{\tilde{\alpha}_0}. \tag{6.42}$$

By linearity, we have $\alpha_0=\tilde{\alpha}_0T$ so that we get from (6.31):

$$T=-\frac{2i\sin \mathrm{k}_0}{\overline{A}}\frac{1}{1+\frac{\overline{B}}{\overline{A}}\tilde{R}}\tilde{T},\ \text{with}\ \tilde{T}=\frac{1}{\tilde{\alpha}_0}. \tag{6.43}$$

We can check that

$$|\tilde{\alpha}_{\mathtt{x}-1}|^2-|\tilde{\beta}_{\mathtt{x}-1}|^2=|\tilde{\alpha}_{\mathtt{x}}|^2-|\tilde{\beta}_{\mathtt{x}}|^2, \tag{6.44}$$

which shows that $|\tilde{\alpha}_{\mathtt{x}}|^2-|\tilde{\beta}_{\mathtt{x}}|^2=|\tilde{\alpha}_L|^2-|\tilde{\beta}_L|^2=\frac{\sin \mathrm{k}_1}{\sin k}$ for all $\mathtt{x}$, and therefore we get the energy conservation relation

$$|R|^2+\frac{\sin \mathrm{k}_1}{\sin k}|T|^2=1. \tag{6.45}$$

In case $\Delta_0\neq\Delta_1,\Delta_0\Delta_1=0$, we have

$$|T|^2=\frac{\sin k}{\sin \mathrm{k}_1}|\tilde{T}|^2, \tag{6.46}$$

where $|\tilde{T}|^2$ behaves as the diffusion process τ_L with the infinitesimal generator (4.4) starting from

$$\tau_0=\frac{2\sin k\sin \mathrm{k}_1}{1-\cos(k+\mathrm{k}_1)}. \tag{6.47}$$

We get the following representation of the probability density function of $|\tilde{T}|^2$:

$$p_L(\tau)=\frac{2}{\tau^2}e^{-\frac{\gamma L}{4}}\int_0^\infty s\tanh(\pi s)P_{-1/2+is}\Big(\frac{2}{\tau}-1\Big)P_{-1/2+is}\Big(\frac{2}{\tau_0}-1\Big)e^{-s^2\gamma L}ds, \tag{6.48}$$

where $P_{-1/2+is}(\eta)$, $\eta\ge1$, $s\ge0$ is the Legendre function of the first kind, which is the solution of

$$\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta}P_{-1/2+is}(\eta)=-\left(s^2+\frac{1}{4}\right)P_{-1/2+is}(\eta), \tag{6.49}$$

starting from $P_{-1/2+is}(1)=1$. It has the integral representation (4.18). In particular, we have (4.16) and (4.17).

REMARK 6.3. It is possible to address the case where ω is outside the common pass band on the right half-space, i.e. $(2+\varpi_s-\omega^2(1+\Delta_1))/(2)\notin(-1,1)$. If $-1<\Delta_1<0$ is such that $(2+\varpi_s-\omega^2(1+\Delta_j))/(2)>1$, then the wave has the form $\hat{u}_x=Te^{-k_1x}$ for $x>L$ instead of (4.13), where k_1 is given by

$$\cosh(k_1)=\frac{2+\varpi_s-\omega^2(1+\Delta_1)}{2}. \tag{6.50}$$

We can then proceed as above and find that, instead of (6.25-6.26), we have

$$\begin{aligned}\alpha_L&=e^{(-k_1-ik)L}\frac{e^{-k_1}-e^{-ik}}{2i\sin k}T,\\ \beta_L&=-e^{(-k_1+ik)L}\frac{e^{-k_1}-e^{ik}}{2i\sin k}T.\end{aligned} \tag{6.51}$$

This implies that $|\tilde{\alpha}_x|^2-|\tilde{\beta}_x|^2=|\tilde{\alpha}_L|-|\tilde{\beta}_L|^2=0$ for all $0\le x\le L, x\in\mathbb{Z}$, and therefore $|\tilde{R}|=1$ and $|R|=1$. The wave is totally reflected, the random section only changes the phase of the reflected component compared to the case without random perturbation:

$$\hat{u}_x=e^{ik_0x}+Re^{-ik_0x}\quad\text{for } x\le 0, x\in\mathbb{Z}. \tag{6.52}$$

If $\Delta_1>0$ is such that $Q:=(2+\varpi_s-\omega^2(1+\Delta_1))/(2)<-1$, then the wave has the form $\hat{u}_x=T\rho^x$ for $x>L$ instead of (4.13), where ρ is given by $\rho=Q+\sqrt{Q^2-1}$ (it is the unique solution in $(-1,1)$ of the dispersion relation $-2Q+\rho+\rho^{-1}=0$). We obtain the same conclusion: the reflection coefficient R has modulus one. Note that this happens for all frequencies ω when $\Delta_1\to+\infty$ for instance.

7. Proofs of Theorems 3.1, 3.2, 3.3, 3.4

Here we consider that $x_0=0$ and that in the section $[1,L]\cap\mathbb{Z}$ the variables $\mathfrak{m}_x$ are independent and identically distributed with mean zero and variance σ^2. The transmitted wave for $x>L$ has the form

$$u_x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}_x(\omega)e^{-i\omega t}d\omega,\qquad t\ge 0,\quad x\in\mathbb{Z}, \tag{7.1}$$

where the Fourier components are given by

$$\hat{u}_x(\omega)=\hat{c}(\omega)T(\omega)\cos\big(k(\omega)(x-x_0)\big), \tag{7.2}$$

with $k(\omega)$ and $\hat{c}(\omega)$ given by (2.3) when $\varpi_s=0$ and by (2.9) for $\varpi_s\ge 0$. The statistics of the transmission coefficient $T(\omega)$ at a fixed frequency has been studied in the previous section. We have

$$\mathbb{E}\big[T(\omega)\big]=e^{-\gamma(\omega)L}, \tag{7.3}$$

$$\mathbb{E}\big[|T(\omega)|^2\big]=e^{-\frac{1}{4}\gamma(\omega)L}\int_0^\infty e^{-\gamma(\omega)Ls^2}\frac{2\pi s\sinh(\pi s)}{\cosh^2(\pi s)}ds, \tag{7.4}$$

where $\gamma(\omega)$ is given by (4.8). In order to characterize the time-dependent field, we need to characterize the statistics of the vector $(T(\omega_j))_{j=1}^n$ for any set of distinct frequencies

$(\omega_j)_{j=1}^n$, more exactly, we need to characterize the moments

$$\mathbb{E}\Big[\prod_{j=1}^n T(\omega_j)\Big] \tag{7.5}$$

because they in turn characterize all moments of the field

$$\mathbb{E}\big[u_{\mathtt{x}}(t)^n\big] = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathbb{E}\Big[\prod_{j=1}^n T(\omega_j)\Big]\prod_{j=1}^n \hat{c}(\omega_j)\cos\big(k(\omega_j)(\mathtt{x}-\mathtt{x}_0)\big) \\ e^{-i\sum_{j=1}^n \omega_j t}d\omega_1\cdots d\omega_n. \tag{7.6}$$

Proceeding as in [14, Chapter 8], we can show that, for any set of distinct frequencies $(\omega_j)_{j=1}^n$, $(T(\omega_j))_{j=1}^n$ has the martingale representation

$$T(\omega_j) = M(\omega_j)\tilde{T}(\omega_j), \tag{7.7}$$

where

$$\tilde{T}(\omega) = \exp\big(i\sqrt{\gamma(\omega)}W_L - \gamma(\omega)L/2\big), \tag{7.8}$$

$W_L \sim \mathcal{N}(0,L)$, and $M(\omega_j)$ are independent complex martingales (and independent of W_L) with mean one. As a consequence,

$$\mathbb{E}\Big[\prod_{j=1}^n T(\omega_j)\Big] = \mathbb{E}\Big[\prod_{j=1}^n \tilde{T}(\omega_j)\Big], \tag{7.9}$$

and $u_{\mathtt{x}}(t)$ can be written as (more exactly, it has the same moments as)

$$u_{\mathtt{x}}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{T}(\omega)\hat{c}(\omega)\cos\big(k(\omega)(\mathtt{x}-\mathtt{x}_0)\big)e^{-i\omega t}d\omega. \tag{7.10}$$

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Appendix A. Exact solutions.

A.1. Matched medium. Using the Green’s function for the one-dimensional model, the scattered field $\hat{u}^{\mathrm{s}}_{\mathtt{x}} = \hat{u}_{\mathtt{x}} - e^{ik\mathtt{x}}$ is given by

$$\hat{u}^{\mathrm{s}}_{\mathtt{x}} = -\omega^2\mathtt{C}(k)\sum_{q=1}^{L}\mathfrak{m}_q(\hat{u}^{\mathrm{s}}_q + z^q)z^{|\mathtt{x}-q|}, \quad \mathtt{x}\in\mathbb{Z}, \tag{A.1}$$

with

$$\mathtt{C}(k) := \frac{1}{2i\sin k}, \quad z := e^{ik}\in\mathbb{C}. \tag{A.2}$$

The reduced set of equations for $(\hat{u}_1^{\rm s}, \hat{u}_2^{\rm s}, \ldots, \hat{u}_L^{\rm s})^T$ is therefore

$$\hat{u}_p^{\rm s} = -\omega^2 \mathtt{C}(k) \sum_{q=1}^{L} \mathfrak{m}_q z^{|p-q|} (\hat{u}_q^{\rm s} + z^q), \quad p = [1, L] \cap \mathbb{Z}. \tag{A.3}$$

Using the definitions

$$\hat{\boldsymbol{u}}^{\rm s} := (\hat{u}_1^{\rm s}, \hat{u}_2^{\rm s}, \ldots, \hat{u}_L^{\rm s})^T \in \mathbb{C}^L, \tag{A.4}$$

$$\boldsymbol{z}_L(\mathfrak{z}) := (1, \mathfrak{z}, \mathfrak{z}^2, \ldots, \mathfrak{z}^{L-1})^T \in \mathbb{C}^L, \quad \mathfrak{z} \in \mathbb{C}, \tag{A.5a}$$

$$\boldsymbol{D}(\mathfrak{m}) := -\omega^2 \,\mathrm{diag}(\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_L) \in \mathbb{C}^{L \times L}, \tag{A.5b}$$

$$\boldsymbol{I} := \mathrm{diag}(1, \ldots, 1) \in \mathbb{C}^{L \times L} \quad \text{(identity matrix)}, \tag{A.5c}$$

$$\boldsymbol{T}(z) := \mathtt{C}(k) \mathrm{Toeplitz}(\boldsymbol{z}_L(z)^T) \in \mathbb{C}^{L \times L}, \tag{A.6}$$

the form of the scattered field can therefore be obtained formally by solving (A.3):

$$\hat{\boldsymbol{u}}^{\rm s} = (\boldsymbol{I} - \boldsymbol{T}(z) \boldsymbol{D}(\mathfrak{m}))^{-1} \boldsymbol{T}(z) \boldsymbol{D}(\mathfrak{m}) z \boldsymbol{z}_L(z), \tag{A.7}$$

where z is defined in $(\text{A.2})_2$ and $\mathtt{C}$ in $(\text{A.2})_1$. For $\mathtt{x} > L, \mathtt{x} \in \mathbb{Z}$, according to (A.1),

$$\hat{u}_{\mathtt{x}}^{\rm s} = -\omega^2 \mathtt{C}(k) \sum_{q=1}^{L} \mathfrak{m}_q z^{\mathtt{x}-q} (\hat{u}_q^{\rm s} + z^q), \tag{A.8}$$

so that, using the definitions (A.2),(A.5a), (A.5) and the expression (A.7), the transmission coefficient T in

$$\hat{u}_{\mathtt{x}}^{\rm s} + z^{\mathtt{x}} = T z^{\mathtt{x}}, \tag{A.9}$$

is

$$T = 1 + \mathtt{C}(k) \boldsymbol{z}_L(z^{-1}) \cdot \boldsymbol{D}(\mathfrak{m}) (\boldsymbol{I} - \boldsymbol{T}(z) \boldsymbol{D}(\mathfrak{m}))^{-1} \boldsymbol{z}_L(z) \in \mathbb{C}. \tag{A.10}$$

In the absence of mass perturbation on $[1, L] \cap \mathbb{Z}$, it is clear that $\boldsymbol{D}(\mathfrak{m}) = \boldsymbol{0}$ implies $T = 1$, as expected.

A.2. Non-matched medium. Recall (6.21) and (6.22) regarding the definitions of Δ_0, Δ_1 and $\mathrm{k}_0, \mathrm{k}_1$, respectively.

A.2.1. Case 1: $\Delta_0 \neq 0, \Delta_1 = 0$. Let

$$\hat{\boldsymbol{u}}^* := (1 + \mathtt{C}_2(\mathrm{k}_0, k)) (\boldsymbol{I} - \boldsymbol{T} \boldsymbol{D}(\mathfrak{m}))^{-1} z \boldsymbol{z}_L(z) \in \mathbb{C}^L, \tag{A.11}$$

using the definitions (A.5) and $(\text{A.2})_2$, and

$$\boldsymbol{T} = [T_{pq}]_{p,q=1}^{L} := [G_{p,q}]_{p,q=1}^{L} \in \mathbb{C}^{L \times L}, \tag{A.12}$$

with

$$G_{p,q} := \begin{cases} e^{-i\mathrm{k}_0(p-1)} e^{i\mathrm{k}_0} \mathtt{C}_3(\mathrm{k}_0, k) e^{ik(q-1)}, & p < 1 \\ \left(e^{ik(p-1)} \mathtt{C}_1(\mathrm{k}_0, k) e^{2ik} + e^{-ik(p-1)} \mathtt{C}(k)\right) e^{ik(q-1)}, & 1 \le p \le q \\ e^{ik(p-1)} \left(e^{ik(q-1)} \mathtt{C}_1(\mathrm{k}_0, k) e^{2ik} + e^{-ik(q-1)} \mathtt{C}(k)\right), & p > q, \end{cases} \tag{A.13}$$

$p,q\in\mathbb{Z}$, where, in addition to (A.2)$_1$, we employ the definitions

$$\mathtt{C}_1(\mathrm{k}_0,k):=\mathtt{C}(k)\mathtt{C}_2(k,\mathrm{k}_0),\quad \mathtt{C}_2(\mathrm{k}_0,k):=-\frac{1-e^{i(k-\mathrm{k}_0)}}{1-e^{i(k+\mathrm{k}_0)}},\quad \mathtt{C}_3(\mathrm{k}_0,k):=-\frac{e^{i(k-\mathrm{k}_0)}}{1-e^{i(\mathrm{k}_0+k)}}. \tag{A.14}$$

For $\mathtt{x}>L,\mathtt{x}\in\mathbb{Z}$, as $\mathtt{x}\to+\infty$, using (A.13)$_3$, the transmission coefficient in (A.9) is found to be

$$T=1+\mathtt{C}_2(\mathrm{k}_0,k)+T_1+T_2\in\mathbb{C}, \tag{A.15}$$

with

$$T_1:=\mathtt{C}_1(\mathrm{k}_0,k)z\boldsymbol{z}_L(z)\cdot\boldsymbol{D}(\mathfrak{m})\hat{\boldsymbol{u}}^*,\quad T_2:=\mathtt{C}(k)z^{-1}\boldsymbol{z}_L(z^{-1})\cdot\boldsymbol{D}(\mathfrak{m})\hat{\boldsymbol{u}}^*, \tag{A.16}$$

where we also use the definitions (A.2), (A.5), (A.11), (A.14).

Remark A.1. As $\Delta_0\to0$, we get $\mathtt{C}_1(\mathrm{k}_0,k)=\mathtt{C}(k)\mathtt{C}_2(k,\mathrm{k}_0)\to0$, $\mathtt{C}_3(\mathrm{k}_0,k)\to -1/\left(1-e^{2ik}\right),\mathtt{C}_2(\mathrm{k}_0,k)\to0$. Hence, $G_{p,q}\to\mathtt{C}(k)e^{ik|p-q|}$, and (A.10) follows from (A.15) after simplifying (A.11).

A.2.2. Case 2: $\Delta_0=0,\Delta_1\neq0$. Let

$$\hat{\boldsymbol{u}}^*:=(\boldsymbol{I}-\boldsymbol{T}\boldsymbol{D}(\mathfrak{m}))^{-1}(z\boldsymbol{z}_L(z)+\mathtt{C}_2(k,\mathrm{k}_1)z^{2L-1}\boldsymbol{z}_L(z^{-1}))\in\mathbb{C}^L, \tag{A.17}$$

using the definitions (A.5) and (A.2), (A.14), and

$$\boldsymbol{T}=[T_{pq}]_{p,q=1}^L:=[G_{p,q}]_{p,q=1}^L\in\mathbb{C}^{L\times L}, \tag{A.18}$$

with

$$G_{p,q}:=\begin{cases} e^{-ik(q-L-1)}e^{ik}\mathtt{C}_3(k,\mathrm{k}_1)e^{i\mathrm{k}_1(p-L-1)}, & p>L,\\ e^{-ik(q-L-1)}\left(e^{-ik(p-L-1)}\mathtt{C}_1(\mathrm{k}_1,k)+e^{ik(p-L-1)}\mathtt{C}(k)\right), & q\le p\le L,\\ e^{-ik(p-L-1)}\left(e^{-ik(q-L-1)}\mathtt{C}_1(\mathrm{k}_1,k)+e^{ik(q-L-1)}\mathtt{C}(k)\right), & p<q,\end{cases} \tag{A.19}$$

$p,q\in\mathbb{Z}$.

For $\mathtt{x}>L,\mathtt{x}\in\mathbb{Z}$, as $\mathtt{x}\to+\infty$, using (A.19)$_1$, the transmission coefficient in

$$\hat{u}^{\mathrm{s}}_{\mathtt{x}}+z^{\mathtt{x}}=Te^{i\mathrm{k}_1\mathtt{x}}, \tag{A.20}$$

is obtained as

$$T=z^Le^{-i\mathrm{k}_1L}(1+\mathtt{C}_2(k,\mathrm{k}_1)+T_3)\in\mathbb{C}, \tag{A.21}$$

with

$$T_3:=e^{-i\mathrm{k}_1}\mathtt{C}_3(k,\mathrm{k}_1)z\boldsymbol{z}_L(z^{-1})\cdot\boldsymbol{D}(\mathfrak{m})\hat{\boldsymbol{u}}^*, \tag{A.22}$$

where the definitions (A.2), (A.5), (A.17), (A.14) are employed.

Remark A.2. As $\Delta_1\to0$, we get $\mathtt{C}_1(\mathrm{k}_1,k)=\mathtt{C}(k)\mathtt{C}_2(k,\mathrm{k}_1)\to0$, $\mathtt{C}_3(k,\mathrm{k}_1)\to -1/\left(1-e^{2ik}\right),\mathtt{C}_2(k,\mathrm{k}_1)\to0$. Hence, $G_{p,q}\to\mathtt{C}(k)e^{ik|p-q|}$, and (A.10) follows from (A.21) after simplifying (A.17).

REMARK A.3. Due to the definitions (A.16) and (A.22), in the absence of mass perturbation on $[1,L]\cap\mathbb{Z}$, it is clear that $\boldsymbol{D}(\mathfrak{m})=\mathbf{0}$ so that (A.15) and (A.21) lead to $|T|=|1+\mathtt{C}_2(\mathrm{k}_0,\mathrm{k}_1)|$. Upon simplifying $|T|^2$ further it is easy find that

$$|T|^2=\frac{1-\cos(2\mathrm{k}_0)}{1-\cos(\mathrm{k}_0+\mathrm{k}_1)}. \tag{A.23}$$

Indeed, $|T|=1$ in the matched case without mass perturbation as $\mathrm{k}_0=\mathrm{k}_1$.

Appendix B. Diffusion-approximation. Here we give a few elements to the proof of the convergence of the process $|T|^2$ to the diffusion Markov process with the generator given by Equation (4.4) when L is of the order of σ^{-2} and $\sigma\to 0$. The proof consists in showing that $(\tilde{\alpha}_{\mathtt{x}},\tilde{\beta}_{\mathtt{x}})$ converges to a diffusion Markov process, that $\tilde{\beta}_{\mathtt{x}}/\tilde{\alpha}_{\mathtt{x}}$ is itself a diffusion Markov process, that $|\tilde{\beta}_{\mathtt{x}}/\tilde{\alpha}_{\mathtt{x}}|^2$ is itself a diffusion Markov process, and then compute the moments of $|T|^2=1-|\tilde{\beta}_0/\tilde{\alpha}_0|^2$.

The first diffusion-approximation theorem is the following one:

PROPOSITION B.1. *Let X_j be a $\mathbb{R}^d$-valued random sequence solution of*

$$X_{j+1}=X_j+\epsilon Y_j F(X_j), \tag{B.1}$$

where Y_j are independent and identically distributed with mean zero and variance σ^2 and $F:\mathbb{R}^d\to\mathbb{R}^d$ is smooth. Let $X^\epsilon(z)=X_{[z\epsilon^{-2}]}$, where $[\cdot]$ stands for the integer part. As $\epsilon\to 0$ the process X^ϵ converges (in the space of the cadlag functions) to the diffusion process $\bar{X}$ with the infinitesimal generator

$$\mathcal{L}=\frac{\sigma^2}{2}\sum_{i,i'=1}^{d}F_i(x)F_{i'}(x)\partial^2_{x_i x_{i'}}, \tag{B.2}$$

or, equivalently, solution of the stochastic differential equation

$$d\bar{X}=\sigma F(\bar{X})dW(z), \tag{B.3}$$

where W is a Brownian motion and the stochastic integral is Itô.

We can use the diffusion-approximation Theorem 6.1 in [14, Chapter 6] to prove this result. We need to pay attention to the fact that we here deal with a discrete system, not a continuous one. We need to introduce the correct approximation of X^ϵ which is:

$$\frac{d\tilde{X}^\epsilon(z)}{dz}=\frac{1}{\varepsilon}Y_{[\epsilon^{-2}z]}F(\tilde{X}^\epsilon)-\frac{1}{2}\sum_{i=1}^{d}\partial_{x_i}F(\tilde{X}^\epsilon)F_i(\tilde{X}^\epsilon)Y^2_{[\epsilon^{-2}z]}. \tag{B.4}$$

Then we can show that $\tilde{X}^\epsilon(z)-X^\epsilon(z)$ converges to zero and by [14, Theorem 6.1] that $\tilde{X}^\epsilon(z)$ converges to a diffusion process with the infinitesimal generator

$$\begin{aligned}\mathcal{L}&=\frac{\sigma^2}{2}\sum_{i,i'=1}^{d}F_i(x)\partial_{x_i}\big(F_{i'}(x)\partial_{x_{i'}}\big)-\frac{\sigma^2}{2}\sum_{i,i'=1}^{d}\partial_{x_i}F_{i'}(x)F_i(x)\partial_{x_{i'}}\\&=\frac{\sigma^2}{2}\sum_{i,i'=1}^{d}F_i(x)F_{i'}(x)\partial^2_{x_i x_{i'}}.\end{aligned} \tag{B.5}$$

In fact, the result can also be obtained from [13, Chapter 7, Corollary 4.2] which addresses directly discrete systems. In the framework of [13], $\epsilon = 1/\sqrt{n}$ and the transition function is

$$\mu_n(x,\Gamma) = \mathbb{P}\big(X_{j+1} \in \Gamma | X_j = x\big) = \mathbb{P}\big(x + \sqrt{n}^{-1} Y F(x) \in \Gamma\big), \tag{B.6}$$

which is such that

$$n \int (\tilde{x} - x)\mu_n(x, d\tilde{x}) = 0, \tag{B.7}$$

$$n \int (\tilde{x} - x)(\tilde{x} - x)^T \mu_n(x, d\tilde{x}) = \sigma^2 F(x) F(x)^t. \tag{B.8}$$

We need in fact a version of the diffusion-approximation theorem with a periodic component. The diffusion-approximation theorem that we need is the following one:

PROPOSITION B.2. *Let X_j be a $\mathbb{R}^d$-valued random sequence solution of*

$$X_{j+1} = X_j + \epsilon Y_j F(X_j, j), \tag{B.9}$$

where Y_j are independent and identically distributed with mean zero and variance σ^2 and F is smooth with respect to its first entry and periodic with respect to its second entry. Let $X^\epsilon(z) = X_{[z\epsilon^{-2}]}$, where $[\cdot]$ stands for the integer part. As $\epsilon \to 0$ the process X^ϵ converges to the diffusion process $\bar{X}$ with the infinitesimal generator

$$\mathcal{L} = \frac{\sigma^2}{2} \sum_{i,i'=1}^{d} \langle F_i(x,\cdot) F_{i'}(x,\cdot)\rangle \partial^2_{x_i x_{i'}}, \tag{B.10}$$

where $\langle\cdot\rangle$ is an average over the periodic component. Equivalently, $\bar{X}$ is solution of the stochastic differential equation

$$d\bar{X} = \sum_{j=1}^{d'} \sigma \bar{F}^{(j)}(\bar{X}) dW^{(j)}(z), \tag{B.11}$$

where the $W^{(j)}$ are independent Brownian motions, the stochastic integrals are Itô and we have identified $d' \geq 1$ functions $F^{(j)} : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\sum_{j=1}^{d'} \bar{F}^{(j)}(x) \bar{F}^{(j)}(x)^T = \big\langle F(x,\cdot) F(x,\cdot)^T \big\rangle. \tag{B.12}$$

We can use the diffusion-approximation Theorem 6.5 in [14, Chapter 6] to prove this result. When $L = [z_0/\sigma^2]$ for some $z_0 > 0$, the application of this theorem (proceeding as in [14, Chapter 7]) gives that $X^\sigma(z) = (\tilde{\alpha}_{[z_0/\sigma^2 - z/\sigma^2]}, \tilde{\beta}_{[z_0/\sigma^2 - z/\sigma^2]})^T$ converges as $\sigma \to 0$ to a diffusion Markov process $\bar{X}(z)$, $z \in [0, z_0]$, solution of the stochastic differential equation

$$d\bar{X} = \frac{i\omega^2\sigma}{2\sin k} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{X} dW_0(z) - \frac{\omega^2 \sigma}{2\sqrt{2}\sin k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{X} dW_1(z) - \frac{i\omega^2\sigma}{2\sqrt{2}\sin k} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{X} d\tilde{W}_1(z), \tag{B.13}$$

where W_0, W_1 and $\tilde{W}_1$ are independent Brownian motions.

Denoting $\tilde{R}^\sigma(z)=\tilde{\beta}_{[z_0/\sigma^2-z/\sigma^2]}/\tilde{\alpha}_{[z_0/\sigma^2-z/\sigma^2]}$, we deduce that $\tilde{R}^\sigma(z)$ converges as $\sigma\to 0$ to a diffusion Markov process $\tilde{R}(z)$, $z\in[0,z_0]$, solution of the stochastic differential equation

$$\begin{aligned} d\tilde{R}=&-\frac{i\omega^2\sigma}{\sin k}\tilde{R}dW_0(z)+\frac{\omega^2\sigma}{2\sqrt{2}\sin k}\big(\tilde{R}^2-1\big)dW_1(z)\\ &+\frac{i\omega^2\sigma}{2\sqrt{2}\sin k}\big(\tilde{R}^2+1\big)d\tilde{W}_1(z)-\frac{3\omega^4\sigma^2}{4\sin^2 k}\tilde{R}dz. \end{aligned} \tag{B.14}$$

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