Inverse Problems and Imaging

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First lecture: friday, january 16, 2026, 9:00-12:00 (ENS, room 1Z25).
Material on the course website.
Validation: project (notebook jupyter + oral presentation).

Sensor array imaging

• Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, radar, etc) has two steps:
- data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver array.
- data processing: the recorded signals are processed to identify the quantities of interest (reflector locations, etc).

• Example:
Ultrasound echography

$\longrightarrow$

• Standard imaging techniques require:
- good receivers,
- suitable conditions for wave propagation (ideally, the “target” is embedded in a homogeneous medium),
- controlled and known sources.

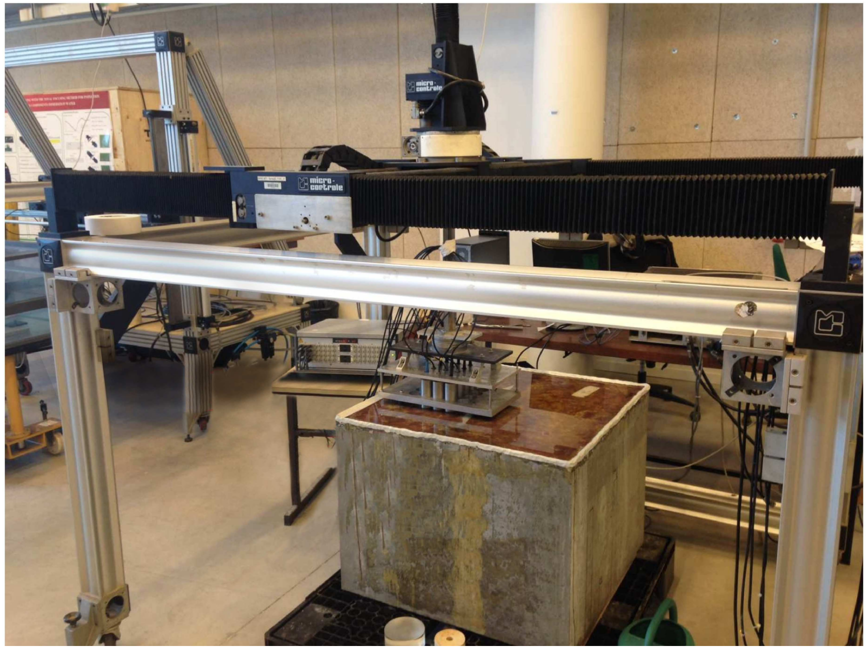
Sensor array imaging

• Goal: Propose and study imaging techniques that are robust with respect to:
- measurement noise,
- the complexity of the medium (heterogeneous medium),
- the control and the knowledge of the sources.

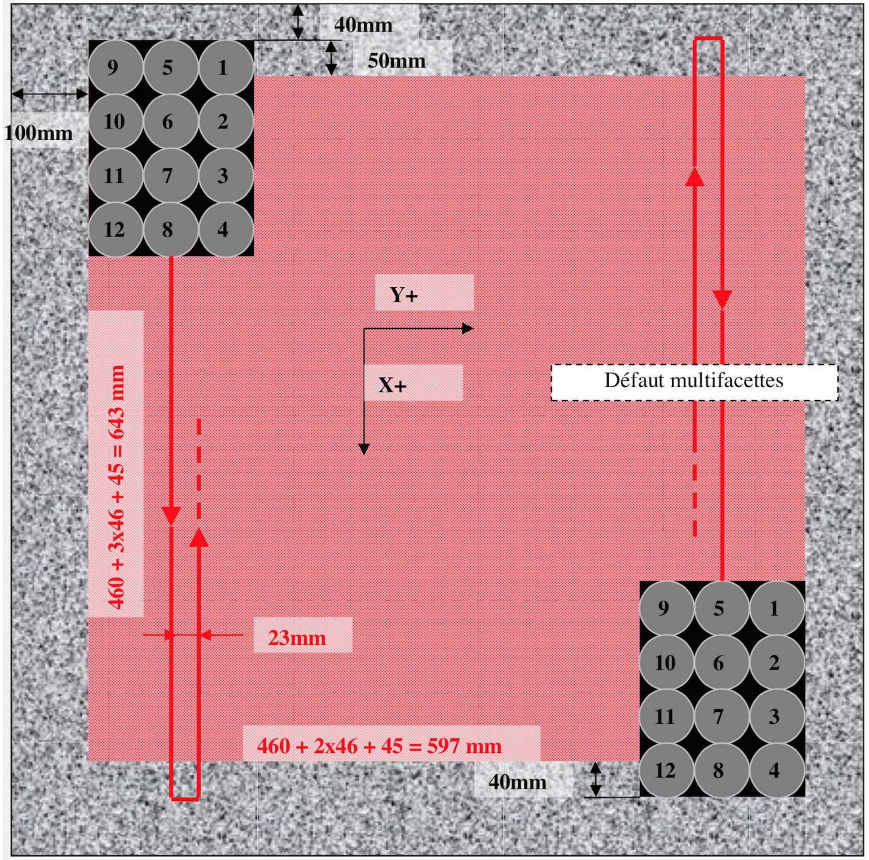
• More generally: resolution of ill-posed inverse problems.

↪ Introduce probabilistic and statistical techniques:
- Least squares optimization,
- Bayesian analysis,
- Random matrix theory,
- Spectral theory for stationary processes,
- Gaussian processes.

Application 1: Ultrasound echography in concrete

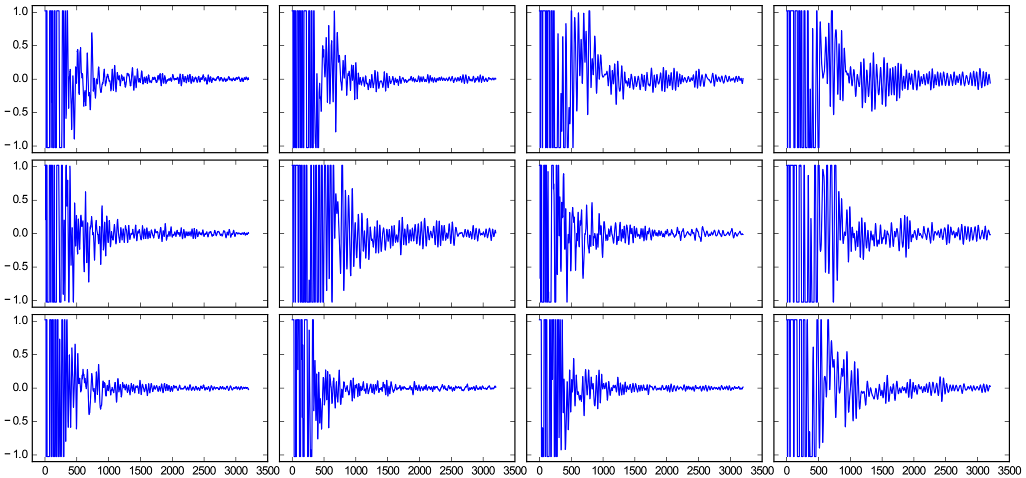


Experimental configuration



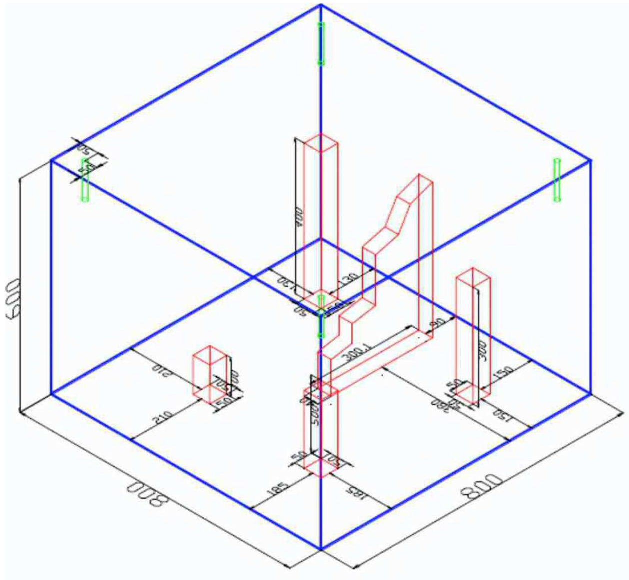
Top view of the acquisition geometry

Application 1: Ultrasound echography in concrete



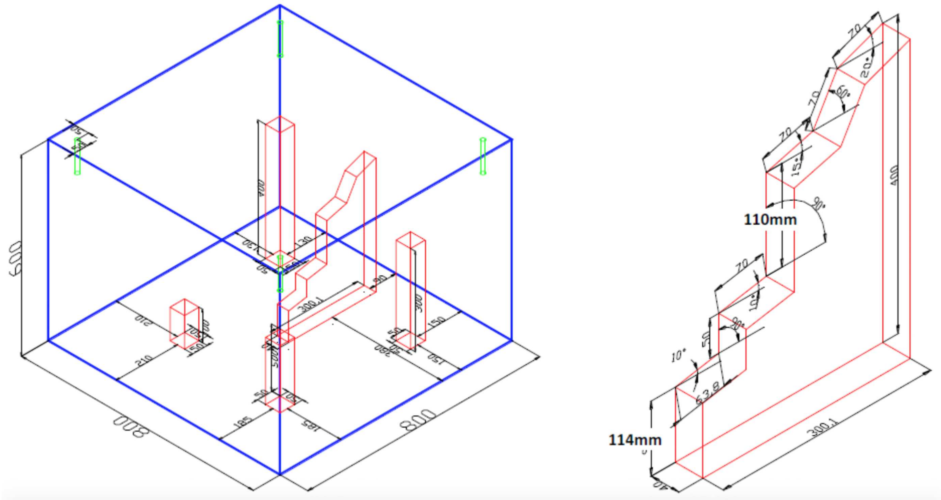
Data

$\xrightarrow{?}$

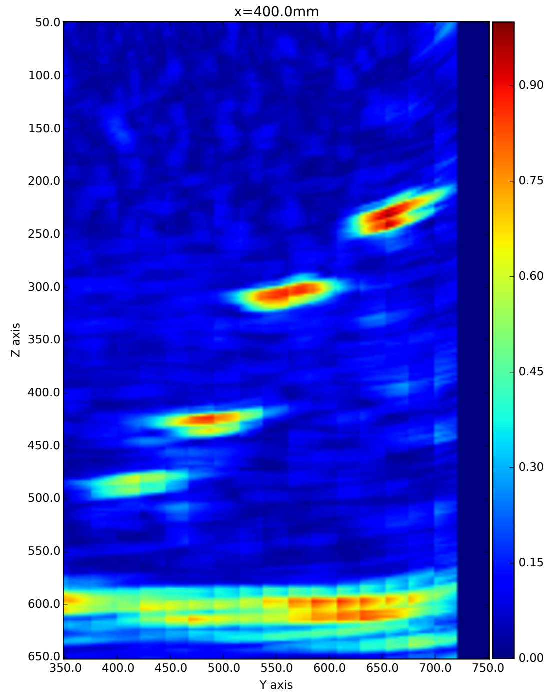


Real configuration

Application 1: Ultrasound echography in concrete

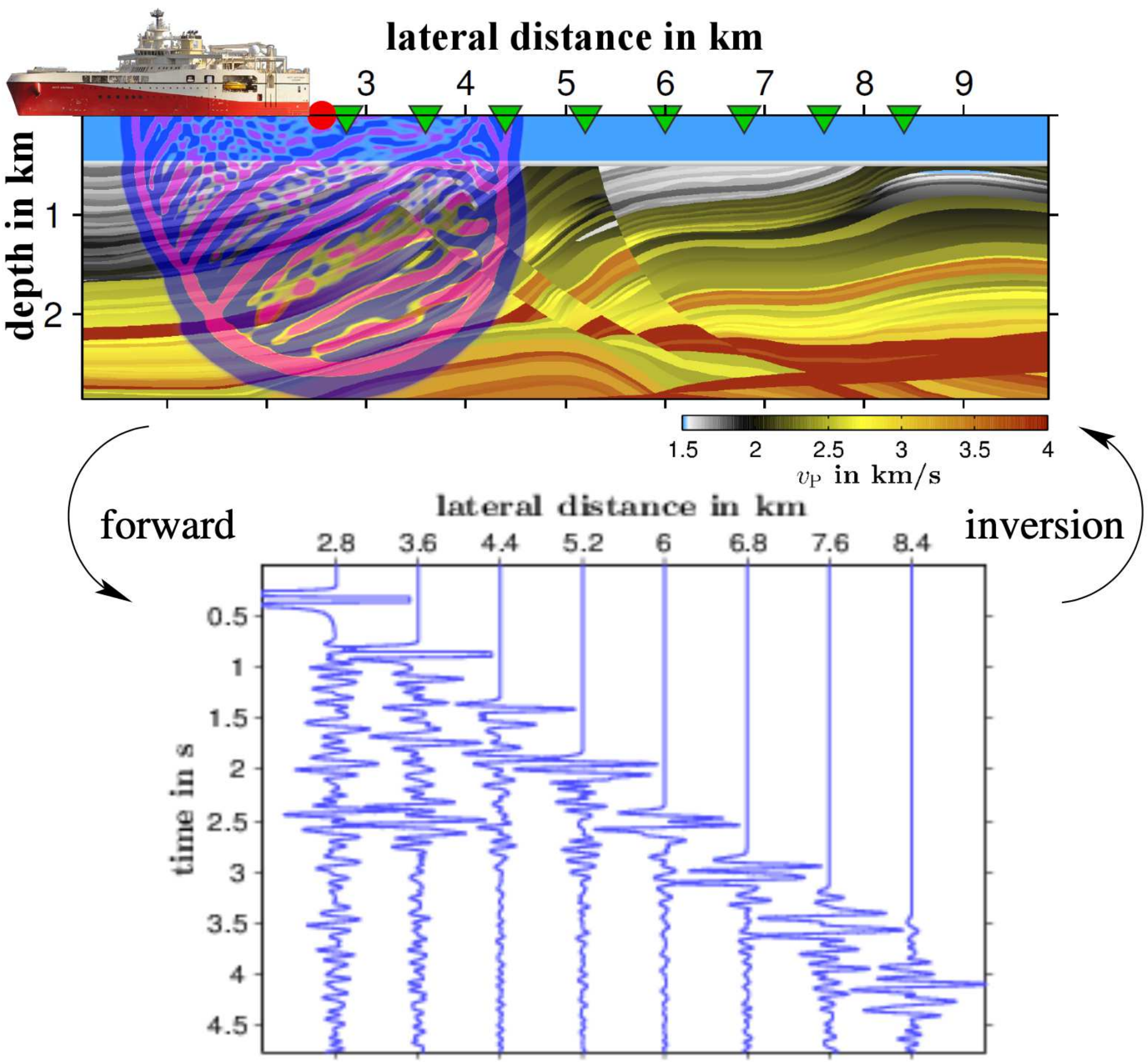


Real configuration



2D Image (along the complex defect plane)

Application 2: Reflection seismology



An inverse problem: Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x\in\Omega}$ of the medium, compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t,x) = f(t)\delta(x - x_s), \quad t \in \mathbb{R}, x \in \Omega,$$

starting from $p^{(s)}(t,x) = 0$, $t \ll 0$.
At the locations of the receivers:

$$d_{r,s}(t) = p^{(s)}(t, x_r), \quad r, s = 1, .., N$$

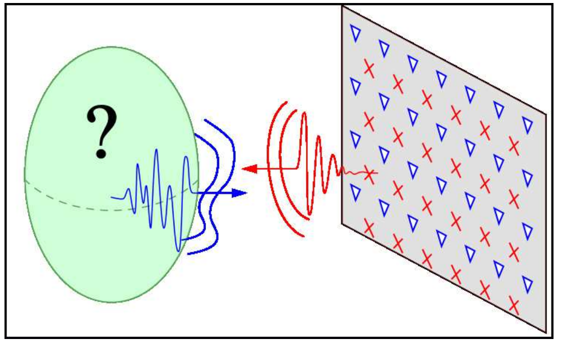
$\hookrightarrow$ forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t\in[t_{\min},t_{\max}]}$, is the array response matrix.

- *Inverse problem:*

Given the time-resolved measurements $\mathbf{d}$, determine the velocity map c.



Full Waveform Inversion (FWI)

- FWI fits the model prediction $\mathcal{D}[\boldsymbol{c}]$ with the measured data $\mathbf{d}_{meas}$ (least-square minimization):

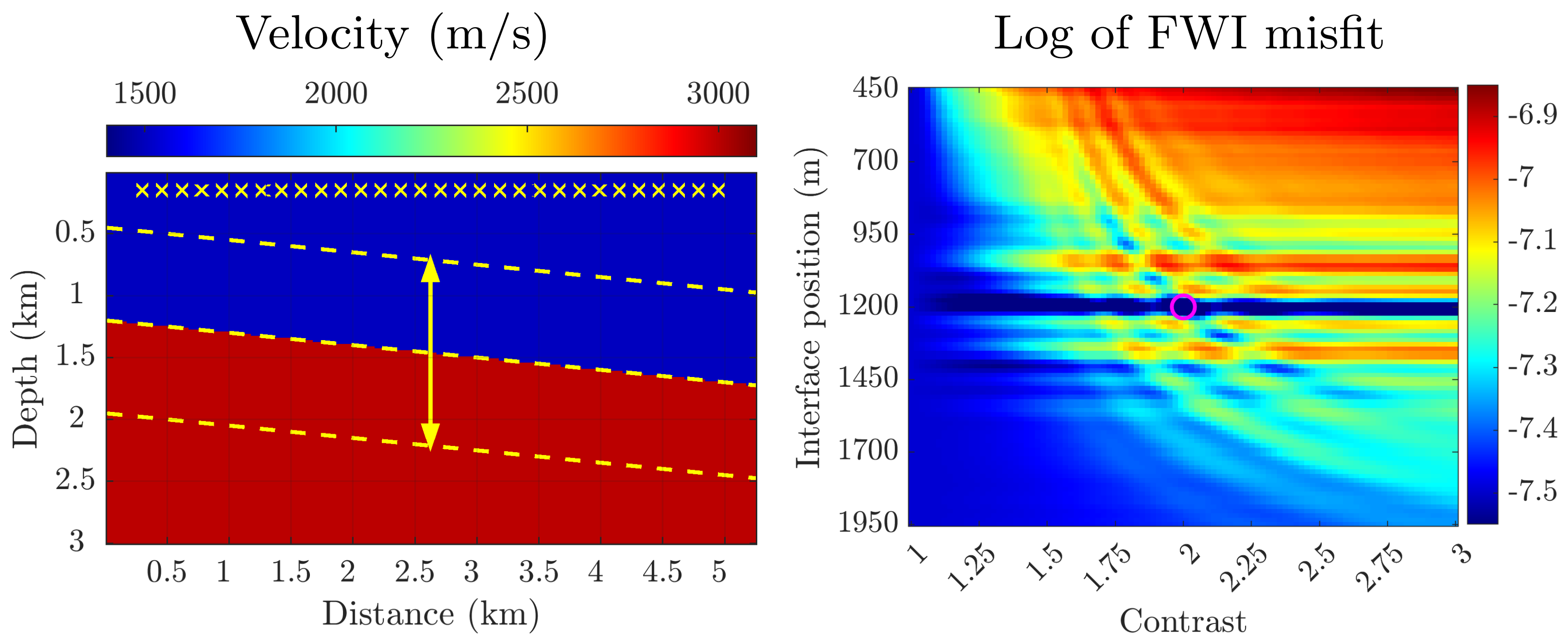
$$\hat{\boldsymbol{c}} = \underset{\boldsymbol{c}}{\text{argmin}} \ \mathcal{O}_{FWI}[\boldsymbol{c}],$$

$$\mathcal{O}_{FWI}[\boldsymbol{c}] = \|\mathcal{D}[\boldsymbol{c}] - \mathbf{d}_{meas}\|^2 = \sum_{r,s=1}^{N} \int_{t_{\min}}^{t_{\max}} |\mathcal{D}[\boldsymbol{c}](t)_{r,s} - d_{meas}(t)_{r,s}|^2 dt,$$

with $\boldsymbol{c} = (c(x))_{x\in\Omega}$.

- Mathematical formulation: a PDE constrained minimization problem.
- Resolution by iterative methods (Newton, Gauss-Newton, steepest-descent, ...).
- Problem: The objective function $\mathcal{O}_{FWI}[\boldsymbol{c}]$ is not convex in $\boldsymbol{c}$.
$\hookrightarrow$ optimization needs hard to get good initial guess.

Topography of the FWI objective function



- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$.
- $N = 30$ sensors and $N_{\mathrm{t}} = 39$ time samples at interval $\tau = 0.0435$s.
- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2$$

A short history of Full Waveform Inversion (FWI)

- Regularization [Virieux and Operto 2009]:

$$\hat{\boldsymbol{c}} = \text{argmin}_{\boldsymbol{c}} \left\{ \| \mathcal{D}[\boldsymbol{c}] - \mathbf{d}_{meas} \|^2 + \lambda \text{Reg}[\boldsymbol{c}] \right\},$$

  with $\text{Reg}[\boldsymbol{c}] = \|\boldsymbol{c}\|_{L^2}^2, \|\boldsymbol{c}\|_{L^1}, \|\boldsymbol{c}\|_{\text{TV}}, \ldots$ (Bayesian interpretation).

- Progressive time continuation (layer stripping): Proceed hierarchically from the shallow part to the deep part [Sheng et al. 2006].

- Progressive frequency continuation: Successive inversion of subdata sets of increasing high-frequency content [Bunks et al. 1995].

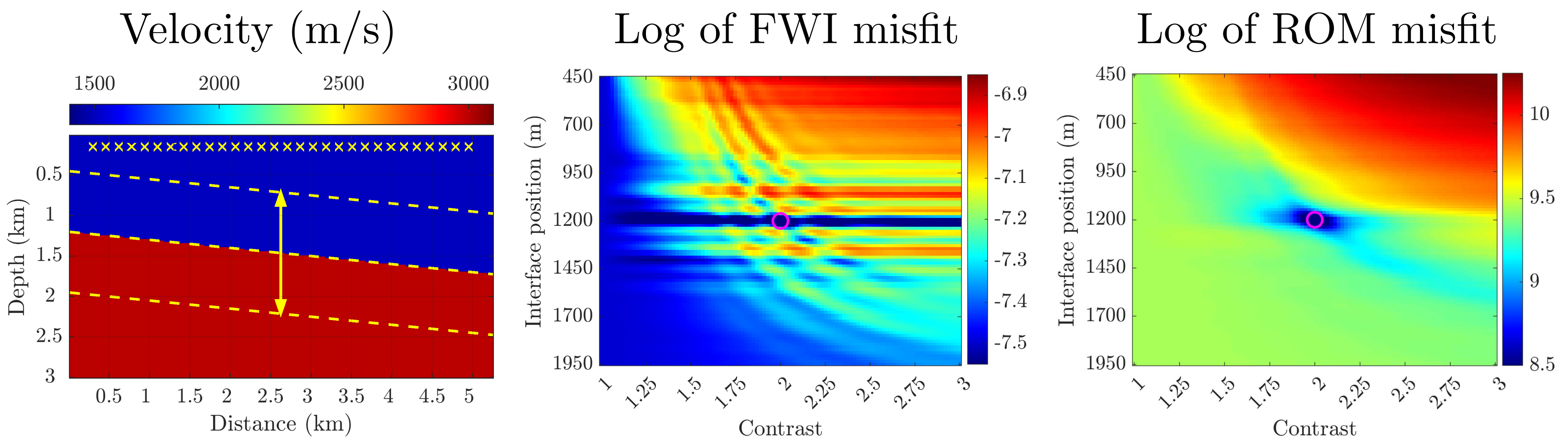
- Optimal transport: Wasserstein distance instead of least-squares [Engquist et al. 2016].

- Extension (or relaxation) strategies: model-space extension [Symes 2008], source-space extension [Huang et al. 2018], receiver-space extension [Benziane et al. 2025].

- Use of reduced-order models [Borcea et al., 2024].

$\hookrightarrow$ What about robustness/stability with respect to measurement noise, complexity of the medium, control of the sources ?

Topographies of the FWI and ROM objective functions



- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).

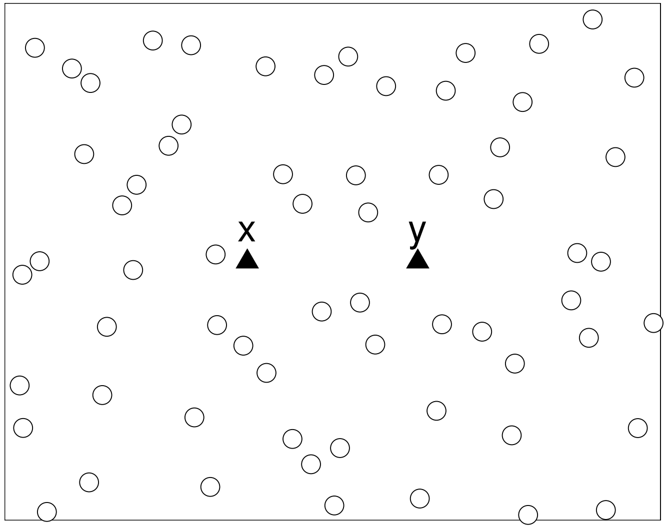
- FWI objective function:

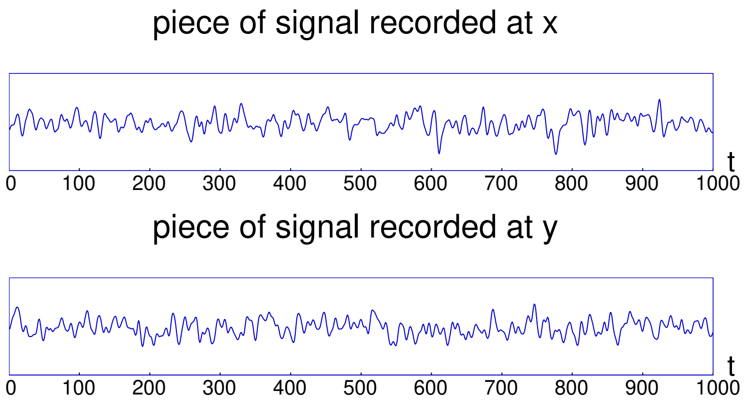
$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}^{meas}\|_2^2$$

- ROM objective function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$

Theory: Cross correlation of signals transmitted by noise sources





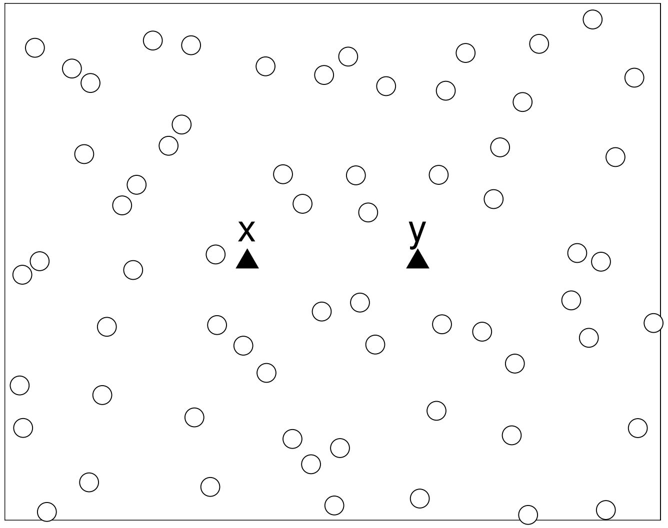
Numerical simulation of wave propagation

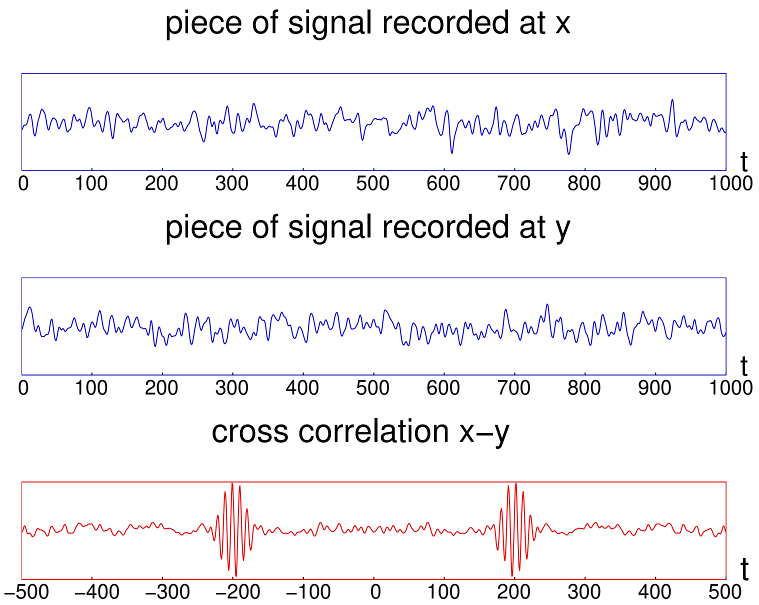
with many noise sources (○) and two receivers at $\boldsymbol{x}$ and $\boldsymbol{y}$ (▲)

How to extract information from the recorded signals $u_{\boldsymbol{x}}(t)$ and $u_{\boldsymbol{y}}(t)$?

These signals are just noise !

Theory: Cross correlation of signals transmitted by noise sources





Numerical simulation of wave propagation

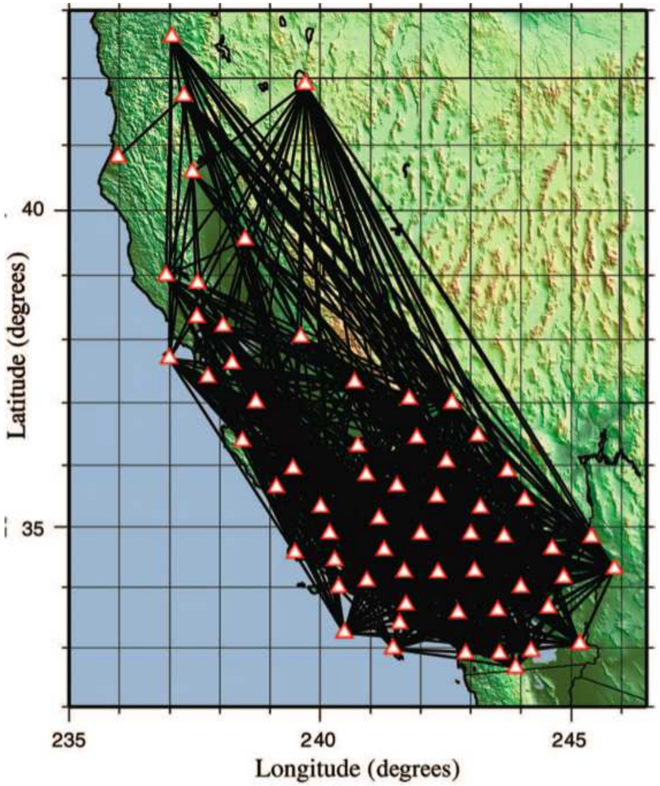
with many noise sources ($\circ$) and two receivers at $\boldsymbol{x}$ and $\boldsymbol{y}$ ($\blacktriangle$)

$\hookrightarrow$ Compute the cross correlation of the recorded signals

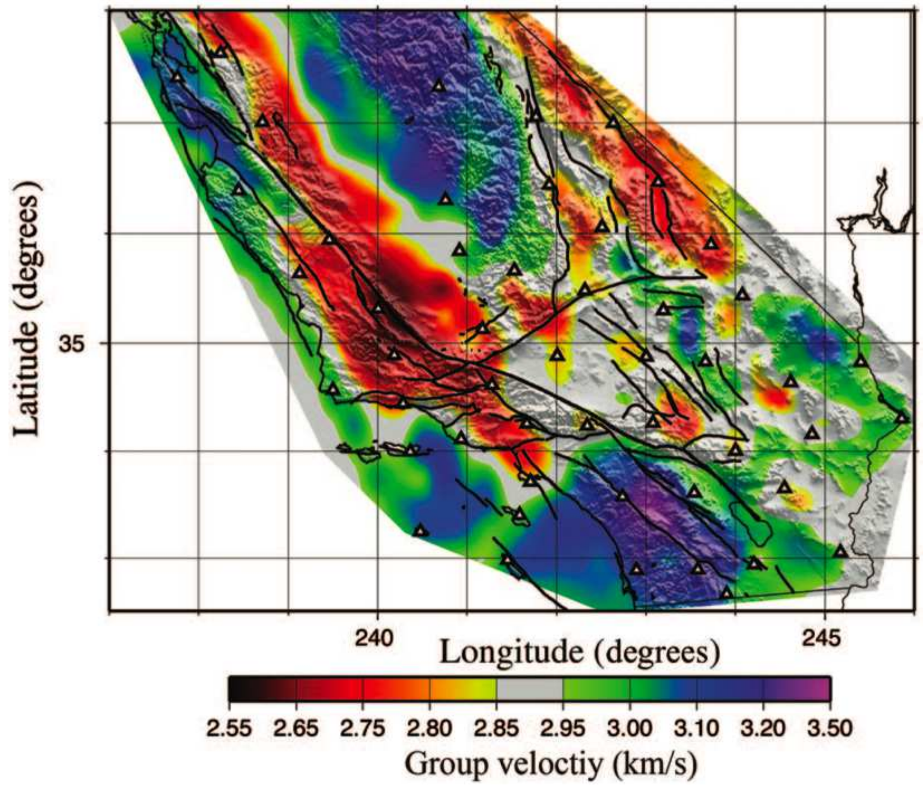
$$C^T_{\boldsymbol{x},\boldsymbol{y}}(t) = \frac{1}{T}\int_0^T u_{\boldsymbol{x}}(s)u_{\boldsymbol{y}}(s+t)ds$$

and extract the travel time between the receivers at $\boldsymbol{x}$ and $\boldsymbol{y}$.

Application 3: Seismic interferometry

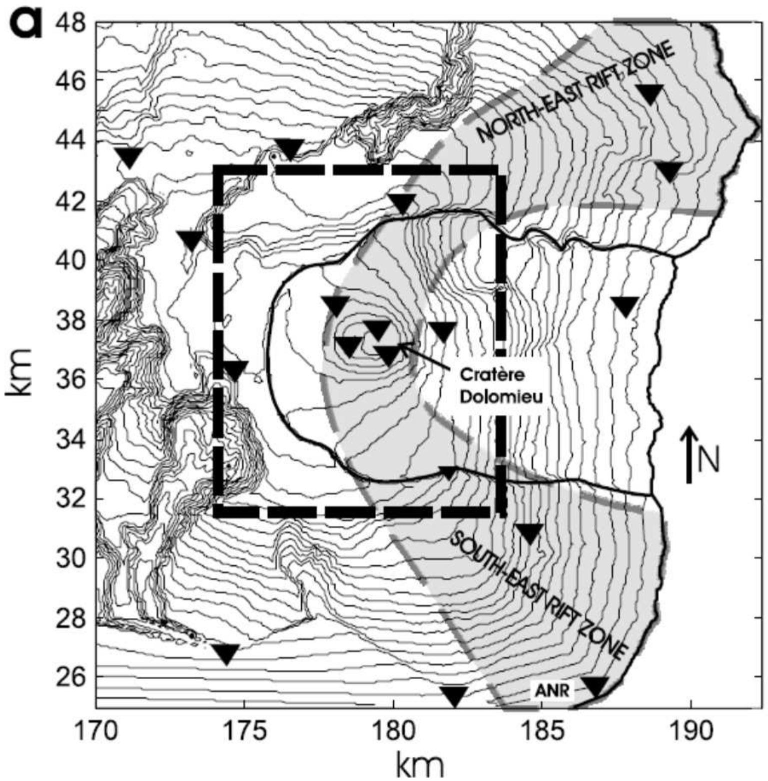


Travel time estimation

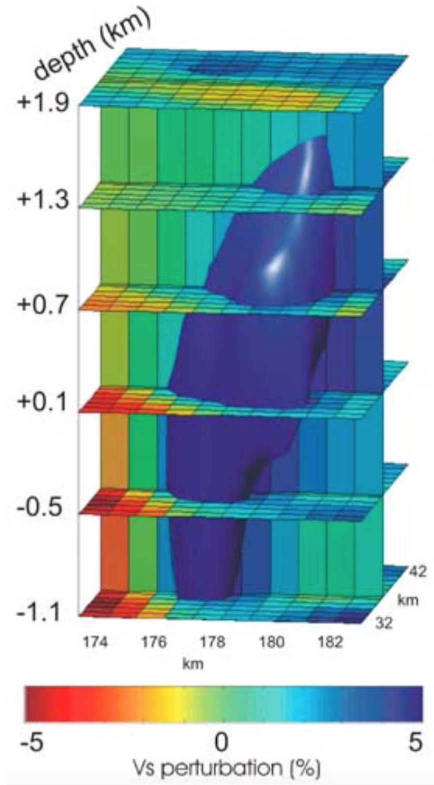


Background velocity estimation

Application 3: Seismic interferometry



Piton de la Fournaise



Eruption warning system