Supplementary information on the article:
Stochastic Dynamics of Incoherent Branched Flow

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I. RAY THEORY OF BRANCHED FLOW

One of the main tools used to describe and understand the properties of branched flow has been the ray tracing method. Rays are constructed as the characteristic curves of the eikonal equation obtained by considering rapidly oscillating solution of the paraxial Eq.(1) (main text) [1]

$$\psi_z(x) = A(z,x)e^{iS(z,x)/\delta}, \tag{S1}$$

where δ is an order counting parameter that expresses the assumption that the local phase $S(z,x)$ of the wave varies rapidly in comparison with its amplitude $A(z,x)$. Expanding the wave equation in a hierarchical fashion in powers of δ^{-1} yields a system of equations for the local phase and amplitude as asymptotic series in δ. The leading-order equation is the eikonal equation:

$$-\partial_z S_0 = \alpha\left(\partial_x S_0\right)^2 + V(z,x), \tag{S2}$$

with S_0 the first term in the asymptotic series of S. The eikonal equation can be interpreted as a Hamilton-Jacobi equation for a Hamiltonian $H(z,x,k_z,k_x)$ that determines the ray trajectories. The Hamiltonian is obtained by making the substitution $(\partial_z S_0, \partial_x S_0) \to (k_z, k_x)$:

$$H = k_z + \alpha k_x^2 + V(z,x). \tag{S3}$$

The condition $H = 0$ is the local dispersion relation.

The rays are parametrized curves, $(x(s), z(s))$, that are the solutions of Hamilton's equations:

$$\dot{x} = \partial_{k_x} H, \quad \dot{z} = \partial_{k_z} H, \quad \dot{k}_x = -\partial_x H, \quad \dot{k}_z = -\partial_z H, \tag{S4}$$

where the dot represents a derivative with respect to the ray parameter s. Since the Hamiltonian is linear in k_z, we have $\dot{z} = 1$ and therefore we can choose the z coordinate to parametrize the rays. Plugging the expression for the Hamiltonian, the equations to be solved are

$$\dot{x} = 2\alpha k_x, \quad \dot{k}_x = -\partial_x V, \quad \dot{k}_z = -\partial_z V, \tag{S5}$$

together with the initial conditions $(x(z=0), k_x(z=0), k_z(z=0))$. From the definition of the momentum, we have $k_x(z=0) = \partial_x S(z=0,x)$ and k_z is then found by solving the condition $H=0$.

In the case of a coherent plane wave, $S(z=0,x)$ is constant and therefore $k_x(z=0)=0$ for all rays (all the rays are parallel to each other). In the case of an incoherent initial speckle field, $k_x \neq 0$ and the launched rays are not parallel. Figures S1-S2 report the ray dynamics and the solution of the paraxial Eq.(1) (main text) for the same realization of the random potential and for different initial conditions: coherent plane wave in Fig. S1, and incoherent speckled field in Fig. S2 (panels (a) and (b) report two different realizations of the speckle field). We can clearly see the formation of caustics which are associated with increases in wave intensities. This illustrates the fact that the rays equations can predict the positions of the maximal intensities, but the values of the maxima result from interference effects that depend on the coherence properties of the initial field. This was discussed in [2] by using a simple random phase model. The determination of the statistics of the values of the intensity maxima

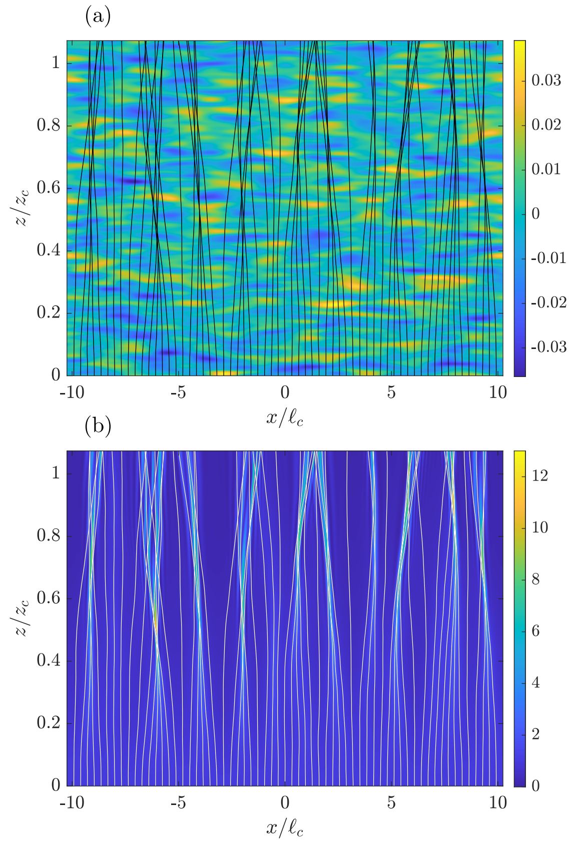

FIG. S1: Coherent initial plane wave. Example of a congruence of rays for a single realization of the random potential. Panel (a) shows a realization of the random potential $V(z,x)$ together with the rays associated with an initial coherent plane wave at $z=0$. Panel (b) shows the BF obtained by solving the paraxial Eq.(1) (main text) with the potential shown in (a), superimposed with the congruence of rays. We can see the formation of caustics which are associated with increases in wave intensity. Parameters are the same as in Fig. 1(a) (main text).

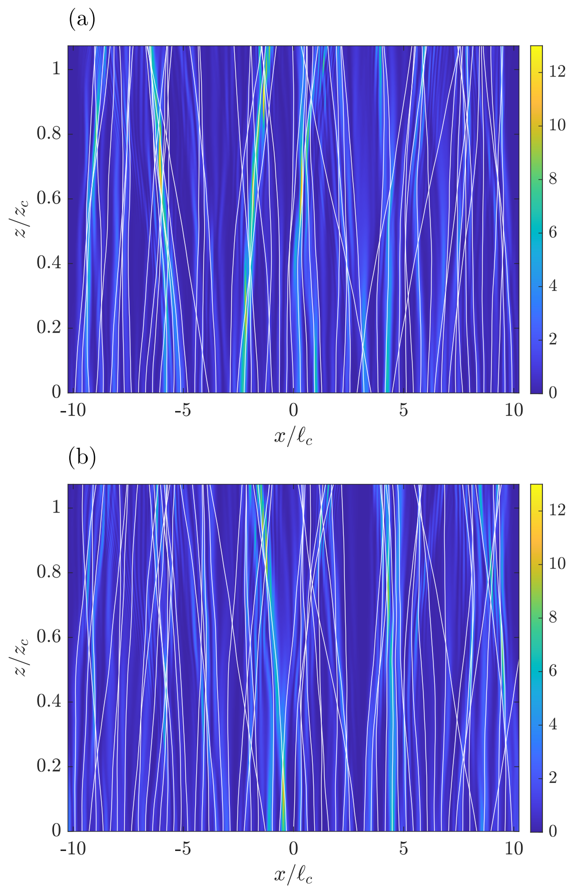

FIG. S2: Incoherent initial wave. (a) Evolution of the BF obtained by solving the paraxial Eq.(1) (main text), superimposed with the congruence of rays. (b) Same as in (a), except that a different realization of the initial incoherent speckle field is considered. The realization of the random potential $V(z,x)$ in (a) and (b) is the same as in Fig. S1. As in the case of the initial plane wave, we can see the formation of caustics despite the fact that the rays are not all launched parallel to each other due to the initial phase variations. Parameters are the same as in Fig. 3(a) (main text).

require a detailed multiscale analysis as carried out in the main text.

II. SCALED VLASOV EQUATION

We consider the Wigner transform $W_z^\varepsilon(x,k)$ defined by Eq.(12) (main text). From Eq.(10) (main text), it satisfies the scaled Vlasov-type equation

$$\begin{aligned}
&\partial_z W_z^\varepsilon + \varepsilon^{1-d-b}\partial_k\omega_k\partial_x W_z^\varepsilon \\
&+ \varepsilon^{c-b-1} i \int_{\mathbb{R}} \Big[V\big(\frac{z}{\varepsilon^b}, x+\varepsilon^d\frac{y}{2}\big) - V\big(\frac{z}{\varepsilon^b}, x-\varepsilon^d\frac{y}{2}\big)\Big] \\
&\times \Big\langle \psi^\varepsilon_{\frac{z}{\varepsilon^b}}\big(x+\varepsilon^d\frac{y}{2}\big)\overline{\psi^\varepsilon_{\frac{z}{\varepsilon^b}}}\big(x-\varepsilon^d\frac{y}{2}\big)\Big\rangle \exp(-iky)dy = 0,
\end{aligned}$$

with $\omega_k = \alpha k^2$, which gives after expansion of the last term of the left-hand side

$$\begin{aligned}
&\partial_z W_z^\varepsilon + \varepsilon^{1-d-b}\partial_k\omega_k\partial_x W_z^\varepsilon - \varepsilon^{c+d-b-1}\partial_x V\big(\frac{z}{\varepsilon^b}, x\big)\partial_k W_z^\varepsilon \\
&= O(\varepsilon^{c+3d-b-1}), \qquad \text{(S6)}
\end{aligned}$$

with the initial condition $W_{z=0}^\varepsilon(x,k) = \mathcal{W}_o(k)$. In the scaling regime $d \in (1/5, 1)$, $b = 1-d$, $c = 3(1-d)/2$, we have $1-d-b = 0$, $c+d-b-1 = -b/2$, and $c+3d-b-1 = (5d-1)/2 > 0$, so that we can neglect the remainder in (S6) and we get that W_z^ε satisfies (13) (main text) with the initial condition $W_{z=0}^\varepsilon(x,k) = \mathcal{W}_o(k)$.

Note that the scaling regime addressed here is different from the one used to derive the paraxial white-noise (or Itô-Schrödinger) model [3–6]. The paraxial white-noise model is valid when $d = 0$, $b = 1$, $c = 3/2$, that is to say, when the wavelength is much smaller than the correlation radius of the medium, which is itself of the same order as the correlation radius of the initial field.

III. PROOF OF EQUATION (15) (MAIN TEXT)

We have

$$W_z^\varepsilon(X,K) = \int_{\mathbb{R}^2} W_z^\varepsilon(x',k')\delta(x'-X)\delta(k'-K)dx'dk'.$$

We make the change of variables $(x',k') \mapsto (x,k)$ with $x' = X_z^\varepsilon(x,k)$, $k' = K_z^\varepsilon(x,k)$:

$$\begin{aligned}
W_z^\varepsilon(X,K) = &\int_{\mathbb{R}^2} W_z^\varepsilon(X_z^\varepsilon(x,k), K_z^\varepsilon(x,k))\delta(X_z^\varepsilon(x,k)-X) \\
&\times \delta(K_z^\varepsilon(x,k)-K)|\mathrm{Det}\mathbf{J}_z^\varepsilon(x,k)|dxdk,
\end{aligned}$$

where $\mathbf{J}_z^\varepsilon(x,k)$ is the Jacobian

$$\mathbf{J}_z^\varepsilon(x,k) = \begin{pmatrix} \frac{\partial X_z^\varepsilon}{\partial x}(x,k) & \frac{\partial X_z^\varepsilon}{\partial k}(x,k) \\ \frac{\partial K_z^\varepsilon}{\partial x}(x,k) & \frac{\partial K_z^\varepsilon}{\partial k}(x,k) \end{pmatrix}.$$

On the one hand we have $W_z^\varepsilon(X_z^\varepsilon(x,k), K_z^\varepsilon(x,k)) = \mathcal{W}_o(k)$ and on the other hand we can compute

$$\begin{aligned}
&\frac{d}{dz}\frac{\partial X_z^\varepsilon}{\partial x} = 2\alpha\frac{\partial K_z^\varepsilon}{\partial x}, \qquad \frac{\partial X_z^\varepsilon}{\partial x}\,|_{z=0}\,(x,k) = 1, \\
&\frac{d}{dz}\frac{\partial K_z^\varepsilon}{\partial x} = -\frac{1}{\varepsilon^{b/2}}\partial_x^2 V\big(\frac{z}{\varepsilon^b}, X_z^\varepsilon\big)\frac{\partial X_z^\varepsilon}{\partial x}, \qquad \frac{\partial K_z^\varepsilon}{\partial x}\,|_{z=0}\,(x,k) = 0, \\
&\frac{d}{dz}\frac{\partial X_z^\varepsilon}{\partial k} = 2\alpha\frac{\partial K_z^\varepsilon}{\partial k}, \qquad \frac{\partial X_z^\varepsilon}{\partial k}\,|_{z=0}\,(x,k) = 0, \\
&\frac{d}{dz}\frac{\partial K_z^\varepsilon}{\partial k} = -\frac{1}{\varepsilon^{b/2}}\partial_x^2 V\big(\frac{z}{\varepsilon^b}, X_z^\varepsilon\big)\frac{\partial X_z^\varepsilon}{\partial k}, \qquad \frac{\partial K_z^\varepsilon}{\partial k}\,|_{z=0}\,(x,k) = 1,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{d}{dz}\mathrm{Det}\mathbf{J}_z^\varepsilon = &\Big(\frac{d}{dz}\frac{\partial X_z^\varepsilon}{\partial x}\Big)\frac{\partial K_z^\varepsilon}{\partial k} + \frac{\partial X_z^\varepsilon}{\partial x}\Big(\frac{d}{dz}\frac{\partial K_z^\varepsilon}{\partial k}\Big) \\
&- \Big(\frac{d}{dz}\frac{\partial K_z^\varepsilon}{\partial x}\Big)\frac{\partial X_z^\varepsilon}{\partial k} - \frac{\partial K_z^\varepsilon}{\partial x}\Big(\frac{d}{dz}\frac{\partial X_z^\varepsilon}{\partial k}\Big) \\
= &\,2\alpha\frac{\partial K_z^\varepsilon}{\partial x}\frac{\partial K_z^\varepsilon}{\partial k} - \frac{1}{\varepsilon^{b/2}}\partial_x^2\mathcal{V}\big(\frac{z}{\varepsilon^b}, X_z^\varepsilon\big)\frac{\partial X_z^\varepsilon}{\partial x}\frac{\partial X_z^\varepsilon}{\partial k} \\
&+ \frac{1}{\varepsilon^{b/2}}\partial_x^2\mathcal{V}\big(\frac{z}{\varepsilon^b}, X_z^\varepsilon\big)\frac{\partial X_z^\varepsilon}{\partial x}\frac{\partial X_z^\varepsilon}{\partial k} - 2\alpha\frac{\partial K_z^\varepsilon}{\partial x}\frac{\partial K_z^\varepsilon}{\partial k} \\
= &\,0,
\end{aligned}$$

hence $\mathrm{Det}\mathbf{J}_z^\varepsilon = \mathrm{Det}\mathbf{J}_{z=0}^\varepsilon = \mathrm{Det}\mathbf{I} = 1$. This gives the desired result Eq.(15) (main text).

IV. DIFFUSION APPROXIMATION THEORY

This section contains the technical results that are needed to characterize the statistics of the wave field, in particular the width of the envelope, the correlation radius of the field and the scintillation index. The potential V is a smooth, stationary, random process with mean zero and integrable covariance function. Applying diffusion-approximation theory [7, Chapter 6], we can show from (14) (main text) that, for any integer n, for any $x_1, \dots, x_n \in \mathbb{R}$, for any $k_1, \dots, k_n \in \mathbb{R}$, the $\mathbb{R}^{2n}$-valued process $(X_z^\varepsilon(x_j,k_j), K_z^\varepsilon(x_j,k_j))_{j=1}^n$ converges in distribution as $\varepsilon \to 0$ to the Markov diffusion process $(X_z(x_j,k_j), K_z(x_j,k_j))_{j=1}^n$ with the infinitesimal generator

$$\mathcal{L}^{(n)} = \sum_{j=1}^n 2\alpha K_j \frac{\partial}{\partial X_j} + \frac{1}{2}\sum_{j,j'=1}^n \Gamma(X_j - X_{j'}) \frac{\partial^2}{\partial K_j \partial K_{j'}}, \tag{S7}$$

where

$$\Gamma(x) = \int_{-\infty}^{\infty} \mathbb{E}\big[\partial_x V(0,0)\partial_x V(z,x)\big] dz. \tag{S8}$$

As a particular example of smooth random medium, we can consider a potential V with Gaussian correlation function, variance σ^2 and correlation radius ℓ_c. We then have

$$\Gamma(x) = 2\sqrt{\pi}\sigma^2 \ell_c^{-1}\Big(1 - \frac{2x^2}{\ell_c^2}\Big)\exp\Big(-\frac{x^2}{\ell_c^2}\Big). \tag{S9}$$

Application $n = 1$. Let $x, k \in \mathbb{R}$. The pdf $p_z^{(1)}(X,K;x,k)$ of $(X_z(x,k), K_z(x,k))$ satisfies the Fokker-Planck equation

$$\partial_z p_z^{(1)} = (\mathcal{L}^{(1)})^* p_z^{(1)}, \tag{S10}$$

starting from $p_{z=0}^{(1)}(X,K;x,k) = \delta(X-x)\delta(K-k)$, where

$$\mathcal{L}^{(1)} = 2\alpha K \partial_X + \frac{\Gamma(0)}{2}\partial_K^2,$$

and $(\mathcal{L}^{(1)})^*$ is the adjoint of $\mathcal{L}^{(1)}$. Eq. (S10) has the form

$$\partial_z p_z^{(1)} = -2\alpha K \partial_X p_z^{(1)} + \frac{\Gamma(0)}{2}\partial_K^2 p_z^{(1)}. \tag{S11}$$

It is possible to solve this equation (by taking a Fourier transform in (X,K)) and we get the expression of the pdf of the limit process $(X_z(x,k), K_z(x,k))$:

$$\begin{aligned} p_z^{(1)}(X,K;x,k) =& \frac{1}{\sqrt{2\pi\Gamma(0)z}}\exp\Big(-\frac{(K-k)^2}{2\Gamma(0)z}\Big)\frac{1}{\sqrt{2\pi\Gamma(0)\frac{z^3}{3}}} \\ &\times \exp\Big(-\frac{3(X-x-\alpha(K+k)z)^2}{2\Gamma(0)z^3}\Big). \end{aligned} \tag{S12}$$

Application $n = 2$. Let $x_1, x_2, k_1, k_2 \in \mathbb{R}$. The pdf $p_z^{(1)}(X_1,X_2,K_1,K_2;x_1,x_2,k_1,k_2)$ of $(X_z(x_j,k_j), K_z(x_j,k_j))_{j=1}^2$ satisfies the Fokker-Planck equation

$$\partial_z p_z^{(2)} = (\mathcal{L}^{(2)})^* p_z^{(2)}, \tag{S13}$$

starting from $p_{z=0}^{(2)}(X_1,X_2,K_1,K_2;x_1,x_2,k_1,k_2) = \delta(X_1 - x_1)\delta(X_2-x_2)\delta(K_1-k_1)\delta(K_2-k_2)$, where $\mathcal{L}^{(2)}$ is the infinitesimal generator of $(X_z(x_j,k_j), K_z(x_j,k_j))_{j=1}^2$:

$$\begin{aligned} \mathcal{L}^{(2)} =& 2\alpha K_1 \frac{\partial}{\partial X_1} + 2\alpha K_2 \frac{\partial}{\partial X_2} + \frac{1}{2}\Gamma(0)\Big(\frac{\partial^2}{\partial K_1^2} + \frac{\partial^2}{\partial K_2^2}\Big) \\ &+ \Gamma(X_1 - X_2)\frac{\partial^2}{\partial K_1 \partial K_2}. \end{aligned} \tag{S14}$$

We introduce

$$R = \frac{X_1 + X_2}{2}, \quad Q = X_1 - X_2, \tag{S15}$$

$$U = \frac{K_1 + K_2}{2}, \quad V = K_1 - K_2, \tag{S16}$$

where $X_j = X_z(x_j,k_j)$, $K_j = K_z(x_j,k_j)$, $j = 1,2$. The infinitesimal generator of the process (R_z, Q_z, U_z, V_z) is

$$\mathcal{L} = 2\alpha U \partial_R + 2\alpha V \partial_Q + \frac{1}{4}\big(\Gamma(0) + \Gamma(Q)\big)\partial_U^2 + \big(\Gamma(0) - \Gamma(Q)\big)\partial_V^2. \tag{S17}$$

In particular, the process (Q_z, V_z) is Markov with generator

$$\mathcal{L} = 2\alpha V \partial_Q + \big(\Gamma(0) - \Gamma(Q)\big)\partial_V^2. \tag{S18}$$

V. EXPRESSION OF THE SCINTILLATION INDEX FOR INCOHERENT INITIAL CONDITIONS

From Eq.(15) (main text) we get the expression of the second-order moment of the Wigner transform in the limit $\varepsilon \to 0$:

$$\begin{aligned} &\lim_{\varepsilon \to 0} \mathbb{E}\big[W_z^\varepsilon(X_1,K_1) W_z^\varepsilon(X_2,K_2)\big] \\ &= \int_{\mathbb{R}^4} \mathcal{W}_o(k_1)\mathcal{W}_o(k_2) \\ &\quad \times p_z^{(2)}(X_1,K_1,X_2,K_2;x_1,k_1,x_2,k_2) dx_1 dk_1 dx_2 dk_2, \end{aligned} \tag{S19}$$

where $p_z^{(2)}$ is the solution of the Fokker-Planck equation (S13). The second-order moment of the intensity in situation `(pc)` is

$$\begin{aligned} &\mathbb{E}\Big[\Big\langle \big|\psi^\varepsilon_{\frac{z}{\varepsilon^6}}(X)\big|^2 \Big\rangle^2\Big] \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathbb{E}\big[W_z^\varepsilon(X,K_1) W_z^\varepsilon(X,K_2)\big] dK_1 dK_2. \end{aligned} \tag{S20}$$

The second-order moment of the intensity in situation `(c)` is

$$\begin{aligned} &\mathbb{E}\Big[\Big\langle \big|\psi^\varepsilon_{\frac{z}{\varepsilon^6}}(X)\big|^4 \Big\rangle\Big] \\ &= \frac{2}{(2\pi)^2}\int_{\mathbb{R}^2} \mathbb{E}\big[W_z^\varepsilon(X,K_1) W_z^\varepsilon(X,K_2)\big] dK_1 dK_2, \end{aligned} \tag{S21}$$

where we have used Isserlis' theorem [8]

$$\left\langle\psi_o^\varepsilon(x)\overline{\psi_o^\varepsilon}(y)\psi_o^\varepsilon(x')\overline{\psi_o^\varepsilon}(y')\right\rangle = \left\langle\psi_o^\varepsilon(x)\overline{\psi_o^\varepsilon}(y)\right\rangle\left\langle\psi_o^\varepsilon(x')\overline{\psi_o^\varepsilon}(y')\right\rangle + \left\langle\psi_o^\varepsilon(x)\overline{\psi_o^\varepsilon}(y')\right\rangle\left\langle\psi_o^\varepsilon(x')\overline{\psi_o^\varepsilon}(y)\right\rangle.$$

By Eq.(S19) and the change of variables (S15-S16) we then get:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big\langle\big|\psi^\varepsilon_{\frac{z}{\varepsilon^b}}(R)\big|^2\Big\rangle^2\Big]\\
&=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^6}\mathcal{W}_o(u+\frac{v}{2})\mathcal{W}_o(u-\frac{v}{2})\\
&\quad\times p_z(R,0,U,V|r,q,u,v)dUdVdrdqdudv\\
&=\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mathcal{W}_o(u+\frac{v}{2})\mathcal{W}_o(u-\frac{v}{2})du\Big]\\
&\quad\times\Big[\int_{\mathbb{R}^2}p_z(0,V|q,v)dqdV\Big]dv,
\end{aligned}\tag{S22}$$

which does not depend on R. By (S18), in the last line $p_z(Q,V|q,v)$ is the pdf solution of

$$\partial_z p_z = -2\alpha V\partial_Q p_z + \big(\Gamma(0)-\Gamma(Q)\big)\partial_V^2 p_z,\tag{S23}$$

starting from $p_{z=0}(Q,V|q,v)=\delta(Q-q)\delta(V-v)$. Eq.(S22) can be rewritten as

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big\langle\big|\psi^\varepsilon_{\frac{z}{\varepsilon^b}}(R)\big|^2\Big\rangle^2\Big] = \Pi_z(0,0)\tag{S24}$$

in terms of the function Π_z defined by

$$\Pi_z(Q,S) = \int_{\mathbb{R}^3} p_z(Q,V|q,v)\pi_o(v)e^{iSV}dqdvdV,\tag{S25}$$

with $\pi_o(v)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\mathcal{W}_o(u+\frac{v}{2})\mathcal{W}_o(u-\frac{v}{2})du$. The function Π_z is the solution of

$$\partial_z\Pi_z = 2i\alpha\partial_Q\partial_S\Pi_z - \big(\Gamma(0)-\Gamma(Q)\big)S^2\Pi_z,\tag{S26}$$

starting from $\Pi_{z=0}(Q,S)=\int\pi_o(v)e^{iSv}dv = |\mathcal{C}_o(S)|^2$. This gives Eqs.(16-17) (main text).

VI. PROOF OF THE SMALL z-EXPANSION

Let $\tilde\Pi_{\tilde z}$ be the solution of (17) (main text). We consider the functions

$$\tilde M_{j,\tilde z}(\tilde x) = (-i)^j\partial_{\tilde y}^j\tilde\Pi_{\tilde z}(\tilde x,\tilde y)\mid_{\tilde y=0}.$$

They satisfy the equations

$$\begin{aligned}
\partial_{\tilde z}\tilde M_{0,\tilde z} &= -\partial_{\tilde x}\tilde M_{1,\tilde z},\\
\partial_{\tilde z}\tilde M_{1,\tilde z} &= -\partial_{\tilde x}\tilde M_{2,\tilde z},\\
\partial_{\tilde z}\tilde M_{2,\tilde z} &= -\partial_{\tilde x}\tilde M_{3,\tilde z} + \big(\tilde\Gamma(0)-\tilde\Gamma(\tilde x)\big)\tilde M_{0,\tilde z},
\end{aligned}$$

starting from $\tilde M_{j,\tilde z=0}(\tilde x) = \tilde M_{j,o} := (-iX_o)^j\tilde\pi_o^{(j)}(0)$. For small $\tilde z$ and using the fact that $\tilde M_{j,o}$ does not depend on $\tilde x$, we get successively:

$$\begin{aligned}
\tilde M_{2,\tilde z}(\tilde x) &= \tilde M_{2,o} + \big(\tilde\Gamma(0)-\tilde\Gamma(\tilde x)\big)\tilde M_{0,o}\tilde z + o(\tilde z),\\
\tilde M_{1,\tilde z}(\tilde x) &= \tilde M_{1,o} + \frac{1}{2}\partial_{\tilde x}\tilde\Gamma(\tilde x)\tilde M_{0,o}\tilde z^2 + o(\tilde z^2),\\
\tilde M_{0,\tilde z}(\tilde x) &= \tilde M_{0,o} - \frac{1}{6}\partial_{\tilde x}^2\tilde\Gamma(\tilde x)\tilde M_{0,o}\tilde z^3 + o(\tilde z^3).
\end{aligned}$$

By Eq.(16) (main text) this gives the desired result for the small z-expansions of $S_z^{(\mathrm{c})}$ and $S_z^{(\mathrm{pc})}$ since $\tilde M_{0,o}=1$. More specifically, we get $S_z^{(\mathrm{pc})} = \frac{\tilde\gamma_4}{6}\frac{z^3}{z_c^3}+o\big(\frac{z^3}{z_c^3}\big)$, $S_z^{(\mathrm{c})} = 1+\frac{\tilde\gamma_4}{3}\frac{z^3}{z_c^3}+o\big(\frac{z^3}{z_c^3}\big)$, with $\tilde\gamma_4 = -\partial_{\tilde x}^2\tilde\Gamma(0) = \partial_{\tilde x}^4\tilde\gamma(0)$, where $\gamma(x)=\int_{\mathbb{R}}\mathbb{E}[V(0,0)V(z,x)]dz = \sigma^2\ell_c\tilde\gamma\big(x/\ell_c\big)$. For a medium with Gaussian correlation function, $\tilde\gamma(\tilde x)=\sqrt{\pi}\exp(-\tilde x^2)$ and $\tilde\gamma_4 = 12\sqrt{\pi}$.

VII. EXTENSION TO THE THREE-DIMENSIONAL CASE

The results described in this paper can be readily extended to the three-dimensional paraxial wave equation:

$$i\partial_z\psi_z = -\alpha(\partial_{x_1}^2+\partial_{x_2}^2)\psi_z + V(z,\boldsymbol{x})\psi,\tag{S27}$$

for $z>0$, $\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}^2$. As an illustration, let us assume that that the initial field is Gaussian with Wigner transform independent of $\boldsymbol{x}$:

$$\int_{\mathbb{R}^2}\Big\langle\psi_o(\boldsymbol{x}+\frac{\boldsymbol{y}}{2})\overline{\psi_o}(\boldsymbol{x}-\frac{\boldsymbol{y}}{2})\Big\rangle e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}d\boldsymbol{y} = \tilde{\mathcal{W}}_o(\rho_o\boldsymbol{k}),\tag{S28}$$

and that the medium fluctuations have Gaussian covariance function:

$$\mathbb{E}\big[V(0,\boldsymbol{0})V(z,\boldsymbol{x})\big] = \sigma^2\exp\Big(-\frac{|\boldsymbol{x}|^2+z^2}{\ell_c^2}\Big).\tag{S29}$$

Under such circumstances, in the situation (pc) the scintillation index defined by (4) (main text) has the form

$$S_z^{(\mathrm{pc})} = \tilde\Pi_{z/z_c}(\boldsymbol{0},\boldsymbol{0})-1,\tag{S30}$$

while in the situation (c) the scintillation index defined by (3) (main text) has the form

$$S_z^{(\mathrm{c})} = 2\tilde\Pi_{z/z_c}(\boldsymbol{0},\boldsymbol{0})-1.\tag{S31}$$

The scintillation index in situations (c) and (pc) depends on the function $\tilde\Pi_{\tilde z}$ that is the solution of:

$$\partial_{\tilde z}\tilde\Pi_{\tilde z} = i\nabla_{\tilde{\boldsymbol{x}}}\cdot\nabla_{\tilde{\boldsymbol{y}}}\tilde\Pi_{\tilde z} - \frac{1}{2}\sum_{j,l=1}^{2}\big(\tilde\Gamma_{jl}(\boldsymbol{0})-\tilde\Gamma_{jl}(\tilde{\boldsymbol{x}})\big)\tilde y_j\tilde y_l\tilde\Pi_{\tilde z},\tag{S32}$$

starting from $\tilde\Pi_{\tilde z=0}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}) = |\tilde{\mathcal{C}}_o(\tilde{\boldsymbol{y}}/X_o)|^2/\tilde{\mathcal{C}}_o(\boldsymbol{0})^2$, where $\tilde{\mathcal{C}}_o$ is the inverse Fourier transform of $\tilde{\mathcal{W}}_o$ and

$$\tilde{\boldsymbol{\Gamma}}(\tilde{\boldsymbol{x}}) = 2\sqrt{\pi}\Big(\mathbf{I}-2\begin{pmatrix}\tilde x_1^2 & \tilde x_1\tilde x_2\\ \tilde x_1\tilde x_2 & \tilde x_2^2\end{pmatrix}\Big)\exp\big(-|\tilde{\boldsymbol{x}}|^2\big).\tag{S33}$$

VIII. NORMALIZATION AND SIMULATIONS

We performed numerical simulations of the paraxial wave equation Eq.(1) (main text) by normalizing the spatial variables with respect to the wavelength λ:

$$i\partial_{z'}\psi_{z'}(x') = -\alpha'\partial_{x'}^2\psi_{z'} + V'(z',x')\psi_{z'},\tag{S34}$$

where $x' = x/\lambda$, $z' = z/\lambda$, $V' = \lambda V = \pi(n_o^2 - n^2(z', x'))/n_o$, and $\alpha' = \alpha/\lambda = 1/(4\pi n_o) \simeq 0.053$ with a reference refractive index of $n_o = 1.5$. Accordingly, the normalized initial correlation length is $\rho_o' = \rho_o/\lambda$, and the normalized variance of the random potential is $\sigma'^2 = \mathbb{E}[V'^2] = \lambda^2\sigma^2$. Note that the relevant parameters are invariant with respect to the normalization, $X_c = X_c' = \sigma'^{2/3}\ell_c'/\alpha'^{1/3}$, $X_o = X_o' = \sigma'^{2/3}\rho_o'/\alpha'^{1/3}$, and $z_c/\lambda = z_c' = 1/(2\sigma'^{2/3}\alpha'^{2/3})$.

The normalized paraxial Eq.(S34) is solved using a pseudo-spectral split-step method, with a frequency cutoff of the spectral grid $k_c' = 2\pi$ (i.e., $k_c = 2\pi/\lambda = k_o$ refers to the light wavenumber in dimensional units), so that the spatial discretization is $dx' = 1/2$ (i.e., $dx = \lambda/2$ in dimensional units). In all simulations, the size of the spatial window, $T_{x'}$, is chosen to be much larger than ℓ_c'. Typically, we take $T_{x'}/\ell_c' \simeq 40$. Each realization of the random processes $V'(x', z')$ and $\psi_{z'=0}(x')$ are defined in the spectral domain using Gaussian correlation functions characterized by ℓ_c' and ρ_o' respectively. The results presented in the main part of the text are the results of the numerical simulations, averaged over 1000 realizations in cases: 1) of an initial plane wave and 2) of a coherent speckled field, corresponding to situation `(c)`. In the case of a partially coherent speckled initial field, situation `(pc)`, we perform 300 realizations of the potential $V'(x', z')$. For each of those realizations, we perform an average over 400 realizations of the initial field $\psi_{z'=0}(x')$. The different realizations are performed in parallel using HPC resources from DNUM CCUB (Centre de Calcul de l'Université de Bourgogne).

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