Inverse Problems and Imaging

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First lecture: friday, january 17, 2025, 9:00-12:00 (ENS, room 1Z25).
Material on the course website.
Validation: project (notebook jupyter + oral presentation).

Sensor array imaging

• Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, radar, etc) has two steps:
- data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver array.
- data processing: the recorded signals are processed to identify the quantities of interest (reflector locations, etc).

• Example:
Ultrasound echography

$\longrightarrow$

• Standard imaging techniques require:
- good receivers,
- suitable conditions for wave propagation (ideally, the "target" is embedded in a homogeneous medium),
- controlled and known sources.

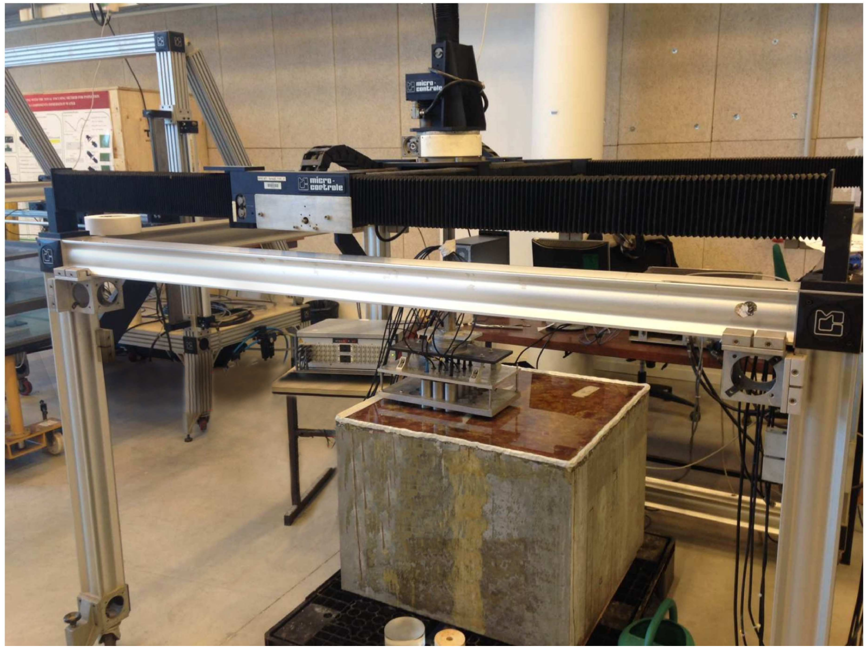
Sensor array imaging

• Goal: Propose and study imaging techniques that are robust with respect to:
- measurement noise,
- the complexity of the medium (heterogeneous medium),
- the control and the knowledge of the sources.

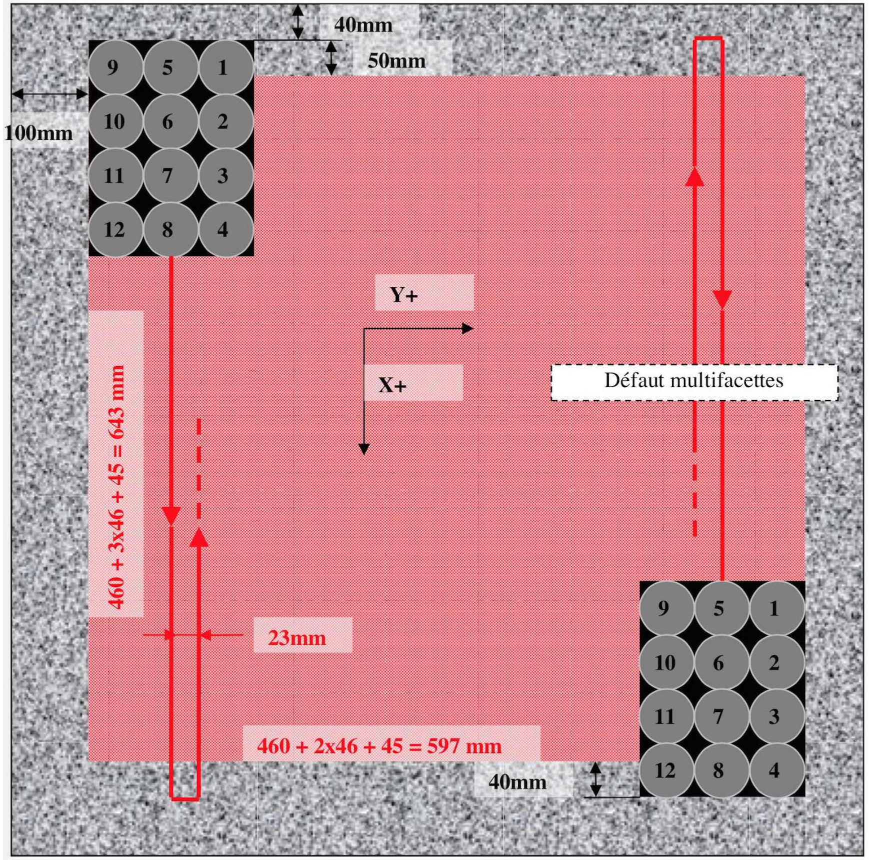
• More generally: resolution of ill-posed inverse problems.

$\hookrightarrow$ Introduce probabilistic and statistical techniques:
- Bayesian analysis,
- Random matrix theory,
- Spectral theory for stationary processes,
- Gaussian processes.

Application 1: Ultrasound echography in concrete

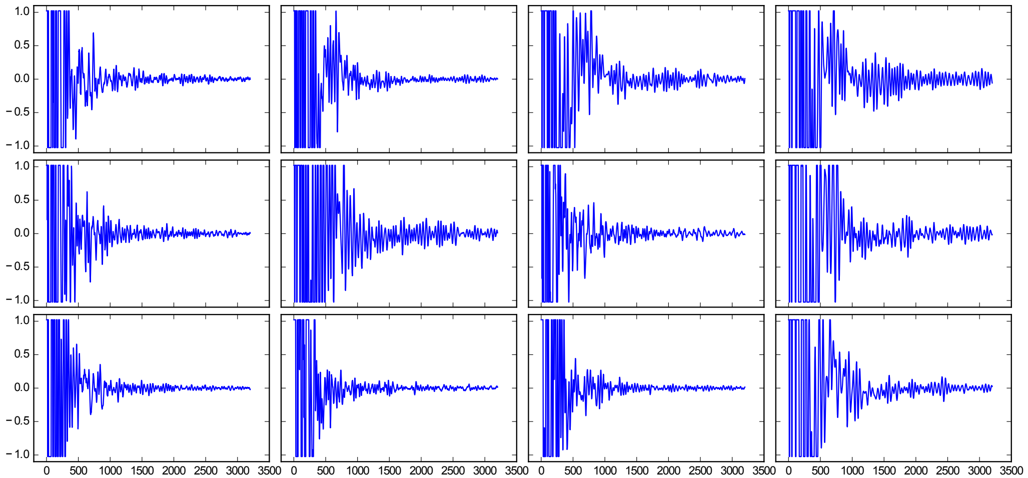


Experimental configuration



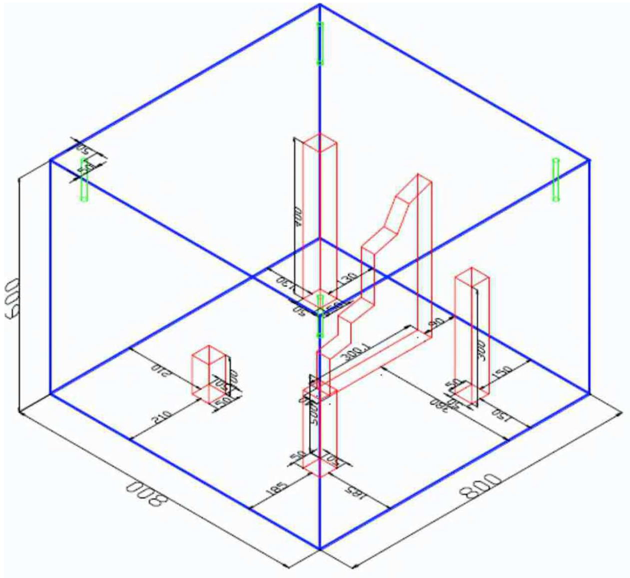
Top view of the acquisition geometry

Application 1: Ultrasound echography in concrete



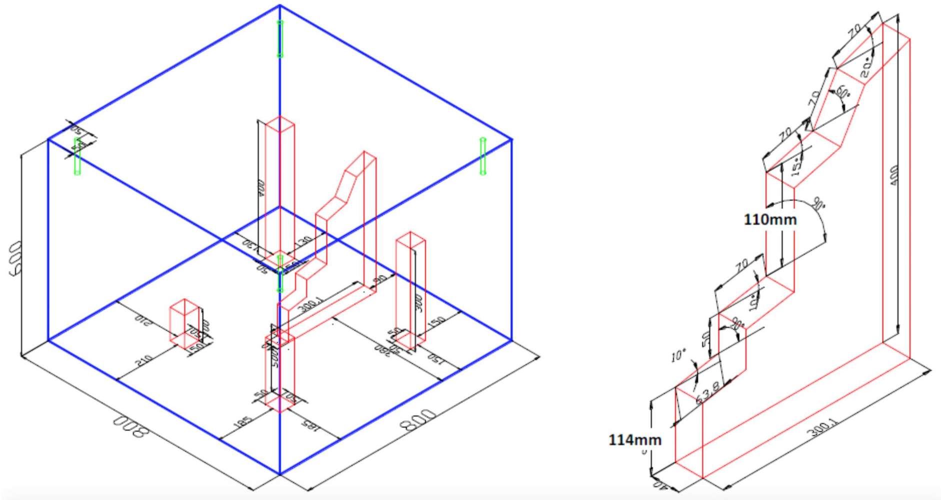
Data

$\xrightarrow{?}$

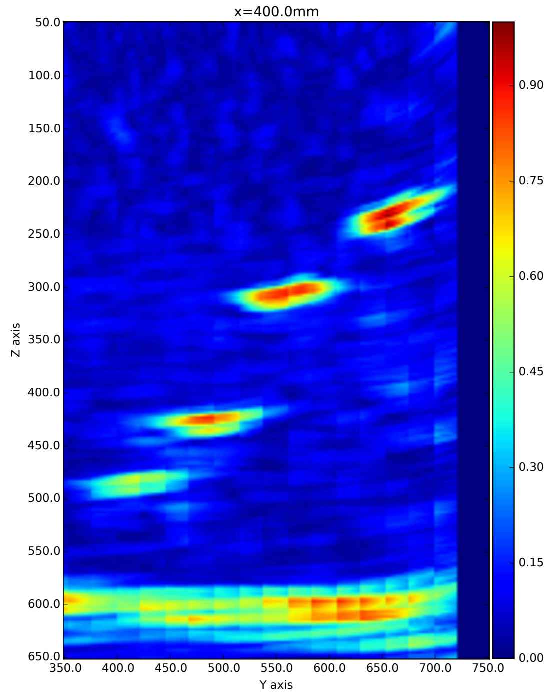


Real configuration

Application 1: Ultrasound echography in concrete

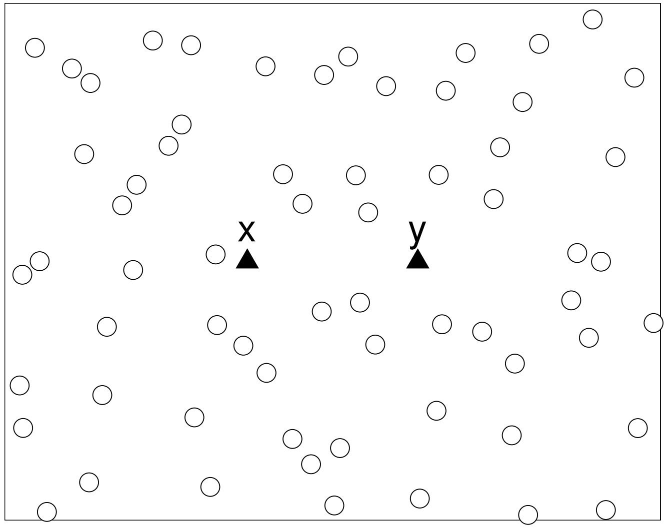


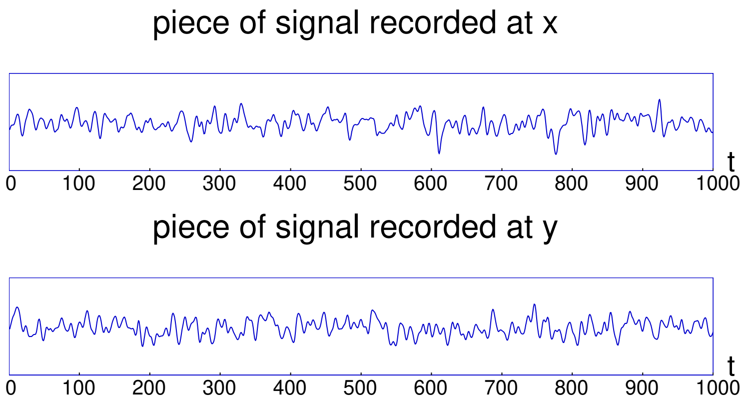
Real configuration



2D Image (along the complex defect plane)

Theory: Cross correlation of signals transmitted by noise sources





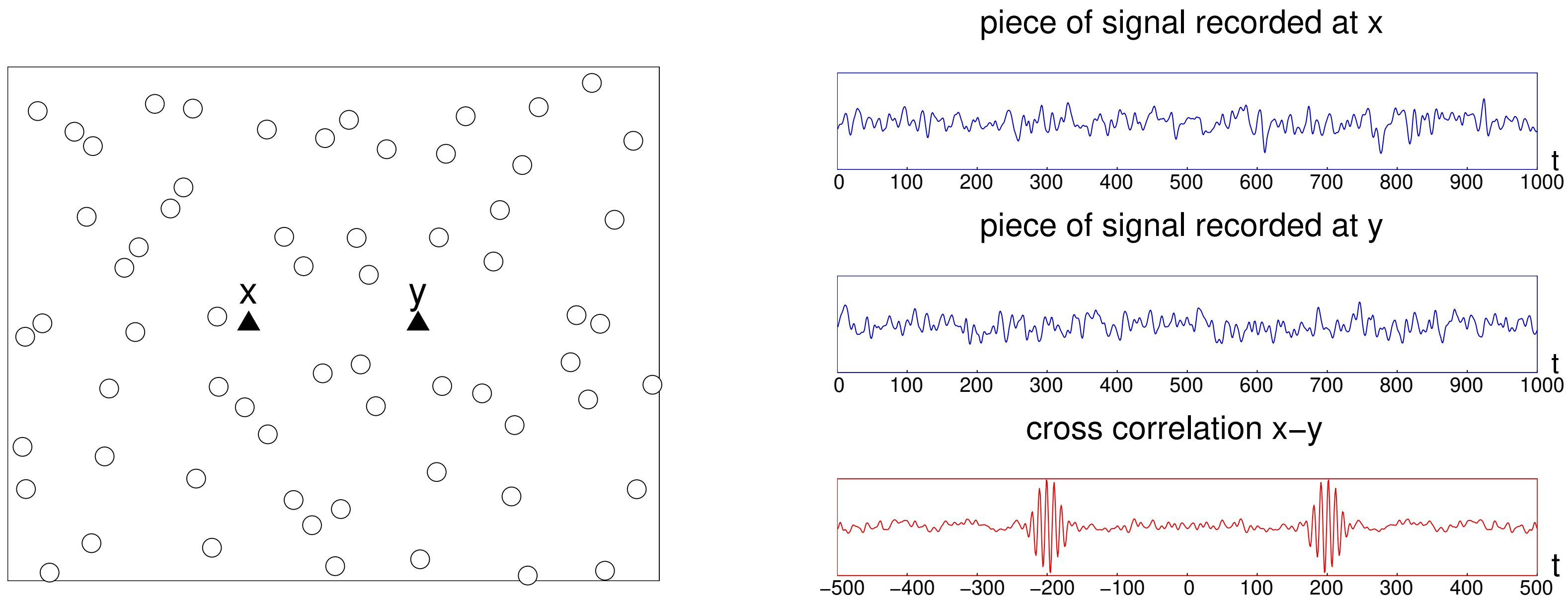
Numerical simulation of wave propagation

with many noise sources (○) and two receivers at $\boldsymbol{x}$ and $\boldsymbol{y}$ (▲)

How to extract information from the recorded signals $u_{\boldsymbol{x}}(t)$ and $u_{\boldsymbol{y}}(t)$?

These signals are just noise !

Theory: Cross correlation of signals transmitted by noise sources



Numerical simulation of wave propagation

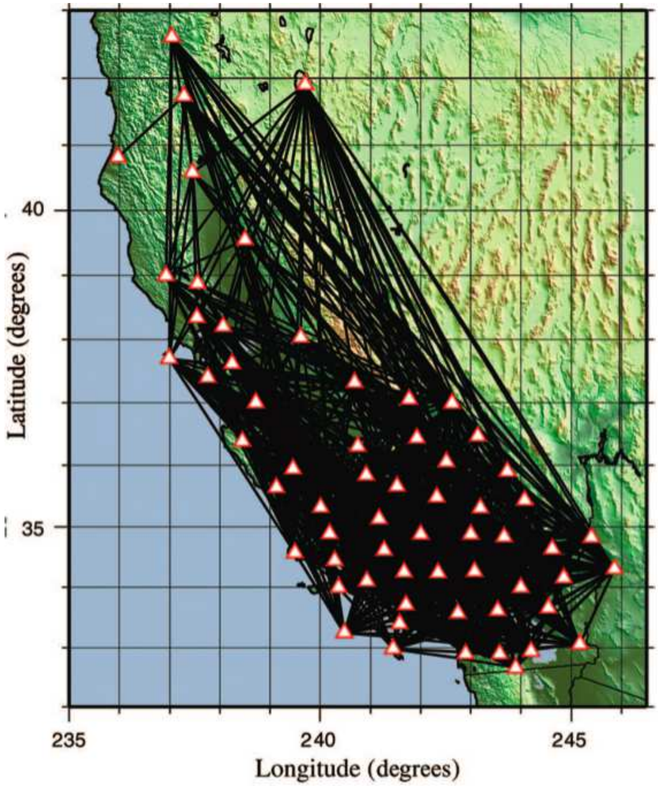
with many noise sources ($\circ$) and two receivers at $\boldsymbol{x}$ and $\boldsymbol{y}$ ($\blacktriangle$)

$\hookrightarrow$ Compute the cross correlation of the recorded signals

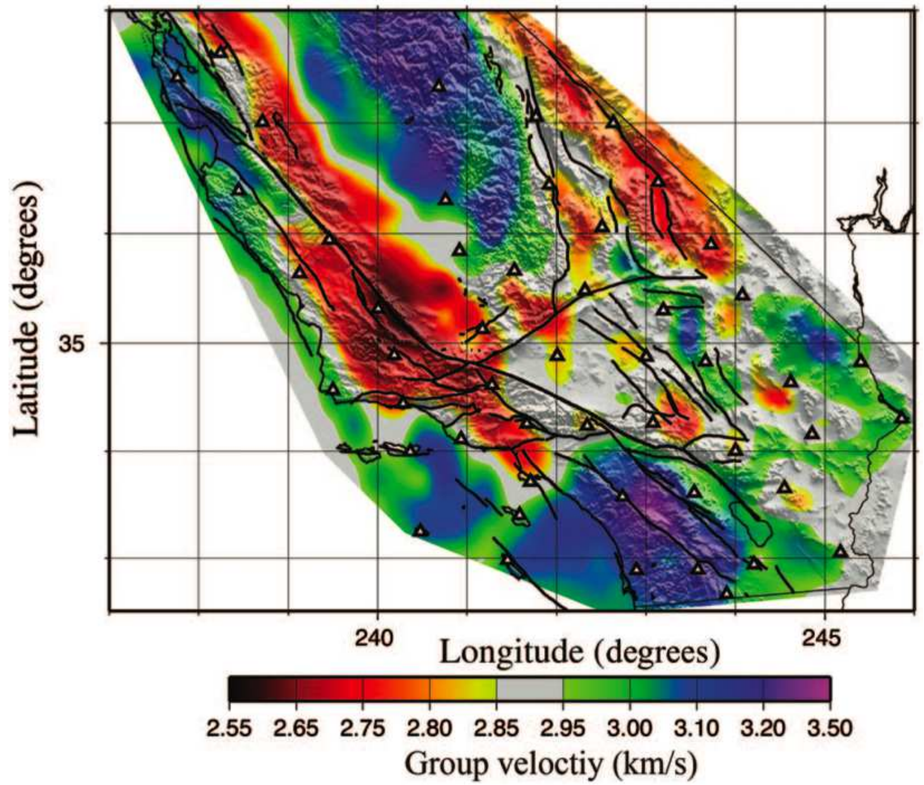
$$C^T_{\boldsymbol{x},\boldsymbol{y}}(t) = \frac{1}{T}\int_0^T u_{\boldsymbol{x}}(s)u_{\boldsymbol{y}}(s+t)ds$$

and extract the travel time between the receivers at $\boldsymbol{x}$ and $\boldsymbol{y}$.

Application 2: Seismic interferometry

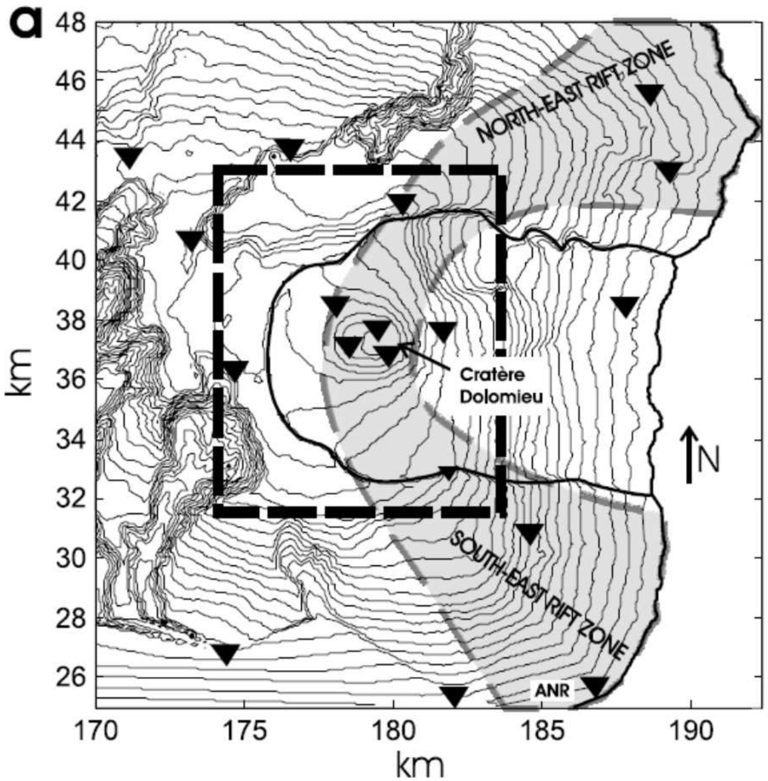


Travel time estimation

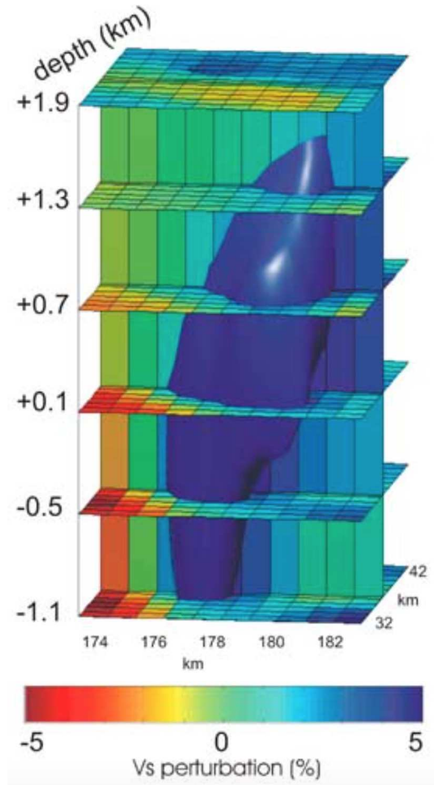


Background velocity estimation

Application 2: Seismic interferometry



Piton de la Fournaise



Eruption warning system

An inverse problem: Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x\in\Omega}$ of the medium, compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t,x) = f(t)\delta(x-x_s), \quad t\in\mathbb{R},\, x\in\Omega,$$

starting from $p^{(s)}(t,x)=0$, $t \ll 0$.
At the locations of the receivers:

$$d_{r,s}(t) = p^{(s)}(t,x_r), \quad r,s=1,..,N$$

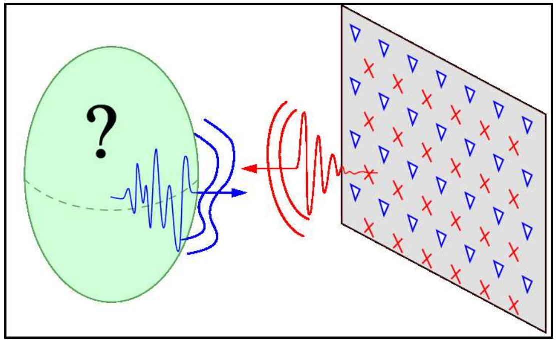
$\hookrightarrow$ forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t\in[t_{\min},t_{\max}]}$, is the array response matrix.

- *Inverse problem:*

Given the time-resolved measurements $\mathbf{d}$, determine the velocity map c.



Full Waveform Inversion (FWI)

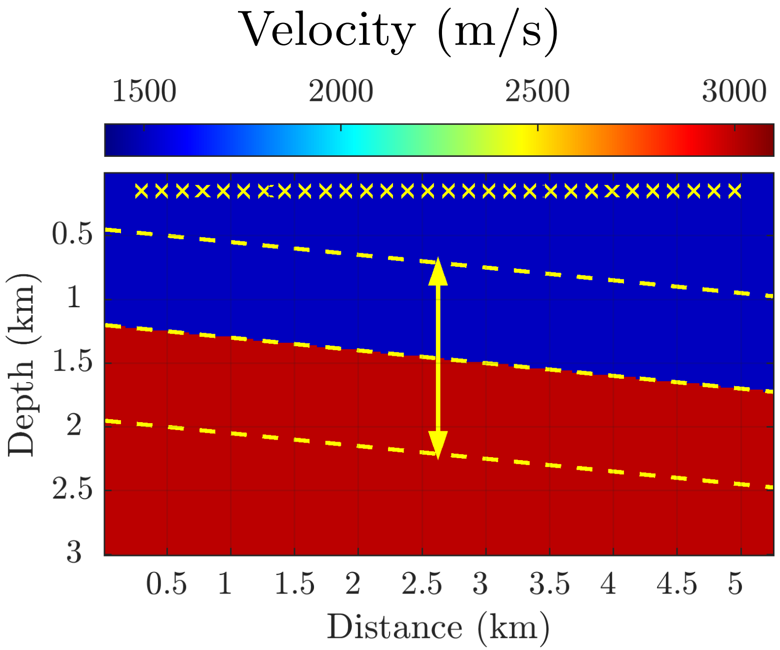
- FWI fits data with its model prediction in L^2-norm (least-square minimization):

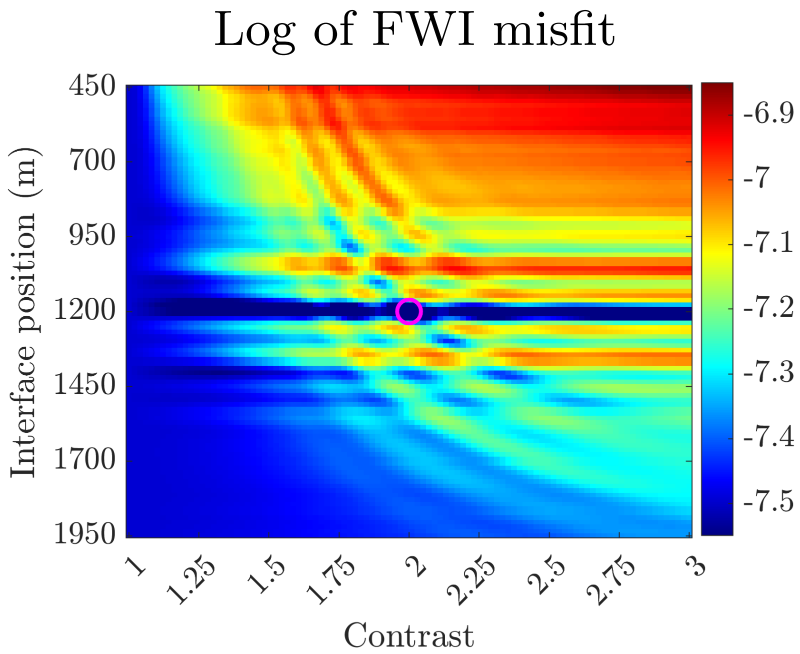
$$\hat{c} = \underset{c}{\operatorname{argmin}} \ \mathcal{O}_{FWI}[c],$$

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2 = \sum_{r,s=1}^{N} \int_{t_{\min}}^{t_{\max}} |d_{meas}(t)_{r,s} - \mathcal{D}[c](t)_{r,s}|^2 dt$$

- The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
$\hookrightarrow$ optimization needs hard to get good initial guess.

Topography of the FWI objective function





- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$.
- $N = 30$ sensors and $N_{\rm t} = 39$ time samples at interval $\tau = 0.0435$s.
- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2$$

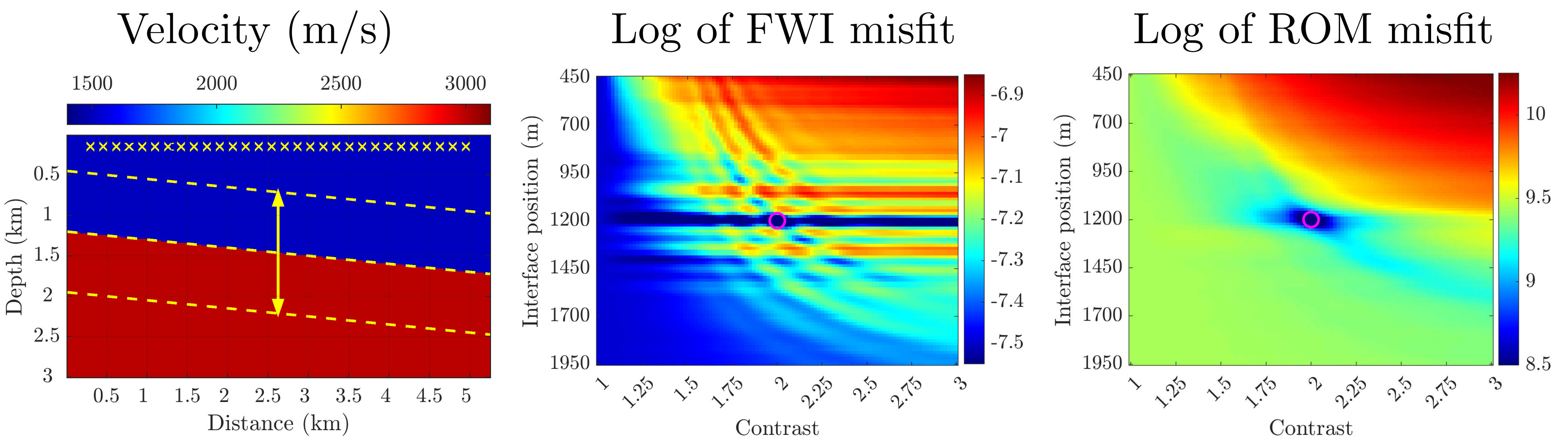
Main objective

- Our main objective: find a convex formulation of FWI.
- Proposed approach: Find a nonlinear mapping $\mathcal{R}$: data $\mathbf{d} \mapsto$ reduced order model (ROM) $\mathbf{A}^{rom}$ such that minimization of ROM misfit is better for velocity estimation.
- We can think of the data to ROM mapping $\mathcal{R}$ as a preconditioner of the forward mapping $\mathcal{D}$

$$c \overset{\mathcal{D}}{\mapsto} \mathbf{d} \overset{\mathcal{R}}{\mapsto} \mathbf{A}^{rom}$$

  because the composition $\mathcal{R} \circ \mathcal{D}$, which gives $\mathbf{A}^{rom} = \mathcal{R}\big(\mathcal{D}[c]\big)$, is easier to "invert".

Topographies of the FWI and ROM objective functions



- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).
- FWI objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}^{meas}\|_2^2$$

- ROM objective function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$