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Enhanced wave transmission in random media with mirror symmetry

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We present an analysis of enhanced wave transmission through random media with mirror symmetry about a reflecting barrier. The mathematical model is the acoustic wave equation, and we consider two setups, where the wave propagation is along a preferred direction: in a randomly layered medium and in a randomly perturbed waveguide. We use the asymptotic stochastic theory of wave propagation in random media to characterize the statistical moments of the frequency-dependent random transmission and reflection coefficients, which are scalar-valued in layered media and matrix-valued in waveguides. With these moments, we can quantify explicitly the enhancement of the net mean transmitted intensity, induced by wave interference near the barrier.

1. Introduction

Multiple scattering of waves travelling through disordered media is a serious impediment for applications like imaging and free space communications. This has motivated the pursuit of strategies for wave transmission enhancement and mitigation of scattering effects.

At propagation distances (depths) that do not exceed a few scattering mean free paths, the wave field retains some coherence. Mitigation strategies seek to enhance this coherence by: filtering the incoherent wave components, like in optical coherence tomography [1] and in imaging in waveguides with rough boundary [2]; correcting wavefront distortion in adaptive optics [3]; or using coherent interferometry [4,5].

Beyond a few scattering mean free paths, the wave field is incoherent and it is typically described by the radiative transfer theory [6–8] or the diffusion theory [9]. These theories neglect wave interference effects that cause phenomena like coherent backscattering enhancement, a.k.a. weak localization [10,11] and Anderson localization [12]. Such interference effects can be exploited for enhancing transmission through a strongly scattering medium. In [13], it was shown, using random matrix theory, that in a disordered three-dimensional metallic body, some of the eigenvalues of the transmission matrix are close to one. The eigenvectors for such eigenvalues are known as open channels and their existence has been demonstrated in optics experiments in [14]. If the transmission matrix can be measured, the open channels can be determined and used for delivering waves deep inside disordered media [14–16] and improved focusing in space [17] and in space–time [18]. However, these methods require accurate measurements of the complex wavefield and spatial light modulators for wavefront shaping.

Recent developments show that interesting wave interference phenomena can also be induced by mirror symmetry in chaotic cavities and in waveguides filled with disordered media. Large conductance enhancement through a reflecting barrier has been demonstrated for arbitrary incident fields in [19] for symmetric quantum dots and in [20,21] for symmetric chaotic cavities. Experimental demonstration of broadband wave transmission enhancement through diffusive, symmetric slabs with a barrier in the middle is given in [22,23] without using transmission or reflection matrix measurements and without wavefront shaping. Symmetric media are also encountered when studying waves propagating in a random half-space with Dirichlet boundary condition. The method of images replaces this half-space problem with a full-space problem with symmetric sources and media [24].

Our goal in this paper is to study mathematically wave transmission enhancement in disordered systems with mirror symmetry about a barrier. The analysis can be carried out for any type of linear waves. For simplicity, we consider acoustic waves that propagate along a preferred direction in either a layered medium or a waveguide filled with a disordered medium. We prove that in both cases, the mirror symmetry has a beneficial effect on the transmission. This complements the experimental results in [22,23], carried out in diffusive slabs. Our quantitative analysis shows that the transmission enhancement comes from constructive interference between symmetric scattering processes. The enhancement is more striking when the obstruction at the barrier is strong.

We model the disordered medium by random fluctuations of the coefficients of the wave equation. These fluctuations are mirror symmetric with respect to a reflecting barrier. The interaction of the waves with the barrier and the random medium is described by frequency-dependent reflection and transmission coefficients, which are scalar-valued in the layered case and matrix-valued in waveguides. We use the stochastic asymptotic theory of wave propagation [25,26] to write the statistical moments of these coefficients and thus quantify explicitly the net mean transmission intensity for various opacities of the barrier. We find that the mean transmitted intensity in the presence of symmetric random media is greater than that through the barrier alone. We also show that if the media on both sides of the barrier are statistically independent, then the mean transmitted intensity is smaller than that through the barrier alone.

We organize the analysis and results in two main sections: we begin in §2 with the case of waves propagating at normal incidence through a randomly layered medium. This lets us introduce the main ideas in a simple, one-dimensional setting, so we can analyse the enhanced transmission in great detail. Then, we study in §3 transmission through random waveguides. The statistical moments of the reflection and transmission matrices in random waveguides are known, but their computation is much more complicated than in the one-dimensional case [26]. Thus, for waveguides, we consider a regime of weak scattering in the random medium, where we can get an explicit approximation of the net transmitted intensity. The involved calculations needed to derive the results in §2 and 3 are given in appendixes. We end with a summary in §4.

2. Enhanced transmission in randomly layered media

We give here the analysis of wave transmission in one-dimensional (layered) random media with mirror symmetry. We begin in §2a with the setup and the wave decomposition in forward and backward going modes. The analysis of the reflection and transmission of these modes at the reflecting barrier is in §2b and in the random medium is in §2c. We gather the results in §2d to quantify the transmission enhancement.

(a) Setup

One-dimensional wave propagation along the z-axis is described by the first-order system

$$\left[\begin{pmatrix}\rho(z) & 0\\ 0 & K^{-1}(z)\end{pmatrix}\partial_t + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\partial_z\right]\begin{pmatrix}u(t,z)\\ p(t,z)\end{pmatrix} = \mathbf{0}, \quad t\in\mathbb{R},\ z\in\mathbb{R}, \tag{2.1}$$

where p is the acoustic pressure and u is the velocity. The medium is modelled by the variable density ρ and bulk modulus K, which determine the local wave speed c and impedance ζ,

$$c(z) = \sqrt{\frac{K(z)}{\rho(z)}} \quad \text{and} \quad \zeta(z) = \sqrt{K(z)\rho(z)}. \tag{2.2}$$

The medium contains a thin barrier at $z\in(-d/2, d/2)$, sandwiched between two randomly perturbed, symmetric regions at $d/2 \le |z| \le L$. Assuming that the z-axis is horizontal, we call the region $z < -d/2$ the left side of the barrier and the region $z > d/2$ the right side of the barrier. The medium is modelled by

$$\rho(z) = \begin{cases}\rho_0 & \text{if } |z| \ge d/2,\\ \rho_1 & \text{if } |z| < d/2,\end{cases} \quad \text{and} \quad \frac{1}{K(z)} = \begin{cases}\dfrac{1}{K_0} & \text{if } |z| > L,\\[2mm] \dfrac{1}{K_0}[1+\mu(|z|)] & \text{if } |z| \in \left[\dfrac{d}{2}, L\right],\\[2mm] \dfrac{1}{K_1} & \text{if } |z| < d/2,\end{cases} \tag{2.3}$$

where ρ_j and K_j are positive constants, for $j = 0, 1$ and μ is a mean zero, mixing random process, satisfying the uniform bound $|\mu| < 1$, so that the bulk modulus is a positive function [25, ch. 6]. Note that only the bulk modulus has random fluctuations in our model. This simplifies the presentation and unifies it with that in the next section, because (2.1) reduces to the standard second-order wave equation for the pressure at $|z| > d/2$ and at $|z| < d/2$. Random fluctuations of the density can be included, and the results are qualitatively the same [25, ch. 17].

The interaction of the waves with the medium depends on frequency, so we Fourier transform with respect to time,

$$\widehat{p}(\omega, z) = \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} p(t,z) \quad \text{and} \quad \widehat{u}(\omega, z) = \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} u(t,z), \tag{2.4}$$

and then decompose the wave field into right (forward) going and left (backward) going modes [25, ch. 7]. The decomposition at $|z| \ge d/2$ is

$$\widehat{a}(\omega, z) = [\zeta_0^{-1/2}\widehat{p}(\omega,z) + \zeta_0^{1/2}\widehat{u}(\omega,z)]\mathrm{e}^{-\mathrm{i}\omega(z/c_o)} \tag{2.5}$$

and

$$\widehat{b}(\omega, z) = [-\zeta_0^{-1/2}\widehat{p}(\omega,z) + \zeta_0^{1/2}\widehat{u}(\omega,z)]\mathrm{e}^{\mathrm{i}\omega(z/c_o)}, \tag{2.6}$$

where $c_0 = \sqrt{K_0/\rho_0}$ and $\zeta_0 = \sqrt{K_0\rho_0}$. The decomposition at $|z| < d/2$ is similar, except that c_0 and ζ_0 are replaced by $c_1 = \sqrt{K_1/\rho_1}$ and $\zeta_1 = \sqrt{K_1\rho_1}$. Note that equations (2.4)–(2.6) give

$$p(t,z) = \frac{\zeta_0^{1/2}}{4\pi}\int_{\mathbb{R}} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t}[\widehat{a}(\omega,z)\,\mathrm{e}^{\mathrm{i}\omega(z/c_o)} - \widehat{b}(\omega,z)\,\mathrm{e}^{-\mathrm{i}\omega(z/c_o)}] \tag{2.7}$$

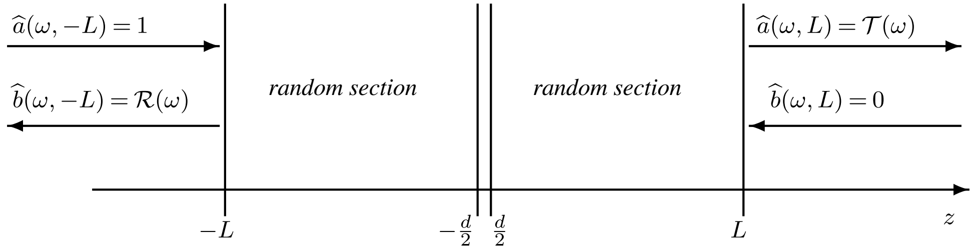

Figure 1. Schematic of transmission and reflection in the random medium with mirror symmetry about the thin barrier located at $z \in (-d/2, d/2)$.

and

$$u(t,z) = \frac{\zeta_0^{-1/2}}{4\pi}\int_{\mathbb{R}} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t}[\widehat{a}(\omega,z)\,\mathrm{e}^{\mathrm{i}\omega(z/c_o)} + \widehat{b}(\omega,z)\,\mathrm{e}^{-\mathrm{i}\omega(z/c_o)}]. \tag{2.8}$$

This is a decomposition in monochromatic waves propagating along the z-axis in the right direction, with amplitude $\widehat{a}$, and the left direction, with amplitude $\widehat{b}$.

The wave excitation specifies $\widehat{a}(\omega,-L)$, and corresponds to a wave impinging on the heterogeneous medium at $z=-L$. The goal is to quantify the wave emerging at $z=L$, with amplitude $\widehat{a}(\omega,L)$ (figure 1). Since the medium is homogeneous at $z>L$, the wave is outgoing there, i.e. $\widehat{b}(\omega,z)=\widehat{b}(\omega,L)=0$ for $z \geq L$.

(b) Model of the barrier

The mapping of the wave modes on the left of the barrier, at $z=-d/2$, to the modes on the right of the barrier, at $z=d/2$, is given by the 2×2 frequency-dependent propagator matrix $\mathbf{P}_1$. The expression of this matrix is derived in appendix A(a), by imposing the continuity of the pressure and velocity at $z=\pm d/2$. We state the result in the next lemma.

Lemma 2.1. *We have*

$$\begin{pmatrix} \widehat{a}\left(\omega,\frac{d}{2}\right) \\ \widehat{b}\left(\omega,\frac{d}{2}\right)\end{pmatrix} = \mathbf{P}_1(\omega)\begin{pmatrix} \widehat{a}\left(\omega,\frac{-d}{2}\right) \\ \widehat{b}\left(\omega,\frac{-d}{2}\right)\end{pmatrix} \quad \text{and} \quad \mathbf{P}_1(\omega) = \begin{pmatrix} \alpha(\omega) & \overline{\gamma(\omega)} \\ \gamma(\omega) & \overline{\alpha(\omega)}\end{pmatrix}, \tag{2.9}$$

where the bar denotes complex conjugate and

$$\alpha(\omega) = \left[\cos\left(\frac{\omega d}{c_1}\right) + \frac{i}{2}\left(\frac{\zeta_1}{\zeta_0}+\frac{\zeta_0}{\zeta_1}\right)\sin\left(\frac{\omega d}{c_1}\right)\right]\mathrm{e}^{-\mathrm{i}\omega d/c_0} \tag{2.10}$$

and

$$\gamma(\omega) = \frac{i}{2}\left(\frac{\zeta_0}{\zeta_1}-\frac{\zeta_1}{\zeta_0}\right)\sin\left(\frac{\omega d}{c_1}\right). \tag{2.11}$$

The scattering matrix $\mathbf{S}_1$ maps the wave mode amplitudes that impinge on the barrier to the outgoing wave mode amplitudes

$$\begin{pmatrix} \widehat{a}\left(\omega,\frac{d}{2}\right) \\ \widehat{b}\left(\omega,\frac{-d}{2}\right)\end{pmatrix} = \mathbf{S}_1(\omega)\begin{pmatrix} \widehat{a}\left(\omega,\frac{-d}{2}\right) \\ \widehat{b}\left(\omega,\frac{d}{2}\right)\end{pmatrix}. \tag{2.12}$$

Its expression follows from equation (2.9),

$$\mathbf{S}_1(\omega) = \begin{pmatrix} T_1(\omega) & R_1(\omega) \\ R_1(\omega) & T_1(\omega),\end{pmatrix} \quad \text{and} \quad R_1(\omega) = -\frac{\gamma(\omega)}{\overline{\alpha(\omega)}}, \quad T_1(\omega) = \frac{1}{\overline{\alpha(\omega)}}, \tag{2.13}$$

where T_1 and R_1 are the transmission and reflection coefficients of the barrier.

We introduce two asymptotic regimes that give an order one net effect of the barrier

1: The first regime is

$$\frac{\omega d}{c_j} \to 0 \quad \text{for } j=0,1 \quad \text{and} \quad \frac{\zeta_0}{\zeta_1} \to \infty \quad \text{such that } \frac{\zeta_0}{\zeta_1}\frac{\omega d}{2c_1} \to q(\omega), \tag{2.14}$$

with finite q. The asymptotic limit of the transmission and reflection coefficients is

$$T_1(\omega) = \frac{i}{i+q(\omega)} \quad \text{and} \quad R_1(\omega) = \frac{q(\omega)}{i+q(\omega)}. \tag{2.15}$$

2: The second regime is

$$\frac{\omega d}{c_j} \to 0 \quad \text{for } j=0,1 \quad \text{and} \quad \frac{\zeta_1}{\zeta_0} \to \infty \quad \text{such that } \frac{\zeta_1}{\zeta_0}\frac{\omega d}{2c_1} \to q(\omega), \tag{2.16}$$

and the transmission and reflection coefficients are

$$T_1(\omega) = \frac{i}{i+q(\omega)} \quad \text{and} \quad R_1(\omega) = -\frac{q(\omega)}{i+q(\omega)}. \tag{2.17}$$

These two regimes are similar. They both consider a thin barrier of width d that is much smaller than the wavelength, so there are no trapped propagating modes at $z \in (-d/2, d/2)$. They also assume a high contrast ratio of the wave impedance inside and outside the barrier, while ensuring that the coefficient q is $O(1)$. This is needed to have an $O(1)$ change of the transmission and reflection coefficients of the barrier. Indeed, equations (2.15) and (2.17) show that if q were negligible, then the barrier would have no effect because the transmission coefficient would be approximately 1 and the reflection coefficient would be approximately 0. The opposite case would be for a large q, where the barrier would be so strongly reflecting that no wave would go through. Since the regimes (2.14) and (2.16) are similar, we consider henceforth the first one, defined by equations (2.14)–(2.15).

(c) Reflection and transmission in the random medium

The propagation of waves in randomly layered media is studied in detail in [25]. We gather the relevant results from there and specialize them to the random medium with mirror symmetry in the next lemma, proved in appendix A(b). This lemma gives the expression of the complex 2×2 scattering matrices of the left and right random sections. They map the amplitudes of the waves impinging on the media to the amplitudes of the waves exiting the media and their entries are the transmission and reflection coefficients.

Lemma 2.2. *We have*

$$\begin{pmatrix} \widehat{a}\left(\omega, \frac{-d}{2}\right) \\ \widehat{b}(\omega, -L) \end{pmatrix} = \mathbf{S}_-(\omega) \begin{pmatrix} \widehat{a}(\omega, -L) \\ \widehat{b}\left(\omega, \frac{-d}{2}\right) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widehat{a}(\omega, L) \\ \widehat{b}\left(\omega, \frac{d}{2}\right) \end{pmatrix} = \mathbf{S}_+(\omega) \begin{pmatrix} \widehat{a}\left(\omega, \frac{d}{2}\right) \\ \widehat{b}(\omega, L) \end{pmatrix}, \tag{2.18}$$

where $\mathbf{S}_-$ *and* $\mathbf{S}_+$ *are the scattering matrices of the left and right random regions* $[-L, -d/2]$ *and* $[d/2, L]$, *respectively*

$$\mathbf{S}_-(\omega) = \begin{pmatrix} T_-(\omega) & \widetilde{R}_-(\omega) \\ R_-(\omega) & T_-(\omega) \end{pmatrix} \quad \text{and} \quad \mathbf{S}_+(\omega) = \begin{pmatrix} T_+(\omega) & \widetilde{R}_+(\omega) \\ R_+(\omega) & T_+(\omega) \end{pmatrix}. \tag{2.19}$$

Due to the symmetry, the transmission and reflection coefficients in these matrices satisfy

$$T_-(\omega) = T_+(\omega), \quad R_-(\omega) = \widetilde{R}_+(\omega), \quad \widetilde{R}_-(\omega) = R_+(\omega). \tag{2.20}$$

The moments of T_+ are studied in [25, Section 7.1.5] and the moments of R_+ and $\widetilde{R}_+$ are in [25, Section 9.2.1], in the so-called strongly heterogeneous white-noise regime defined by the scaling relations

$$\ell_c \ll \lambda \ll L, \quad \mathrm{Var}(\mu) = O(1), \tag{2.21}$$

where ℓ_c is the correlation length of the random fluctuations μ of the medium and $\lambda = 2\pi c_o/\omega$ is the wavelength. The first assumption in (2.21) says that the wavelength is long with respect to the correlation length, so the waves cannot probe the finely layered medium efficiently. However, the medium fluctuations are strong, with $O(1)$ variance, and because the waves propagate over many wavelengths, scattering in the random medium builds up to cause a significant effect at a distance L. In particular, if we have $\mathrm{Var}(\mu)\ell_c L/\lambda^2 = O(1)$, then the effect of the random medium on the transmittivity is of order one.

For quantifying the net mean intensity transmitted through the medium, we only need the expressions of the statistical moments of the square modulus of the transmission coefficient T_+,

$$\mathbb{E}[|T_+(\omega)|^{2n}] = \exp\left(-\frac{L}{4L_{\mathrm{loc}}(\omega)}\right)\int_0^\infty \mathrm{e}^{-Ls^2/L_{\mathrm{loc}}(\omega)}\frac{2\pi s\sinh(\pi s)}{\cosh^2(\pi s)}\phi_n(s)\,\mathrm{d}s, \tag{2.22}$$

for any positive integer n. Here, $\mathbb{E}$ is the expectation with respect to the law of the process μ, the functions ϕ_n are defined by

$$\phi_1(s) = 1, \quad \phi_n(s) = \prod_{j=1}^{n-1}\frac{s^2 + (j-(1/2))^2}{j^2}, \quad n \geq 2, \tag{2.23}$$

and L_{loc} is the localization length of the random medium, which depends on the frequency ω and the statistics of μ

$$\frac{1}{L_{\mathrm{loc}}(\omega)} = \frac{\omega^2}{4c_0^2}\int_{\mathbb{R}}\mathbb{E}[\mu(0)\mu(z)]\,\mathrm{d}z. \tag{2.24}$$

Note that $1/L_{\mathrm{loc}}$ is of the order of $\mathrm{Var}(\mu)\ell_c/\lambda^2$. If the wave travels deep in the medium, i.e. if $L \gg L_{\mathrm{loc}}$, then the moment formula (2.22) simplifies to

$$\mathbb{E}[|T_+(\omega)|^{2n}] \simeq \frac{\pi^{5/2}}{2[L/L_{\mathrm{loc}}(\omega)]^{3/2}}\phi_n(0)\exp\left[-\frac{L}{4L_{\mathrm{loc}}(\omega)}\right]. \tag{2.25}$$

It is shown in [25, Section 7.3] that the moment formulae (2.22) also hold in the asymptotic regime defined by the scaling relations

$$\ell_c \sim \lambda \ll L, \quad \mathrm{Var}(\mu) \ll 1. \tag{2.26}$$

In this so-called weakly heterogeneous regime, the waves are sensitive to the finely layered structure, because the wavelength is comparable to the correlation length. However, the fluctuations are small and cumulative scattering in the random medium becomes significant only at large distances L of propagation. In particular, the effect on the transmittivity is of order one when $\mathrm{Var}(\mu)\ell_c L/\lambda^2 = O(1)$. The moments of T_+ in the weakly scattering regime are given by the same equation (2.22), but the expression of the localization length is different than that in (2.21),

$$\frac{1}{L_{\mathrm{loc}}(\omega)} = \frac{\omega^2}{4c_0^2}\int_{\mathbb{R}}\mathbb{E}[\mu(0)\mu(z)]\cos\left(\frac{2\omega z}{c_0}\right)\mathrm{d}z.$$

(d) Transmission enhancement

We are interested in the transmission of the wave field through the medium, illustrated schematically in figure 1. The result is stated in the next theorem, proved in appendix A(c). In light of lemma 2.2, we simplify the notation in its statement using

$$T(\omega) = T_+(\omega) = T_-(\omega) \quad \text{and} \quad R(\omega) = R_+(\omega) = \widetilde{R}_-(\omega). \tag{2.27}$$

Theorem 2.3. *The transmission coefficient of the system is*

$$\mathcal{T}(\omega) = T^2(\omega)T_1(\omega)[1 - R(\omega)]^{-1}[1 - (2R_1(\omega) - 1)R(\omega)]^{-1}, \tag{2.28}$$

and the expression of the mean transmitted intensity is

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = \sum_{k=0}^{\infty} \tau_k(\omega)\mathbb{E}[|T(\omega)|^4(1 - |T(\omega)|^2)^k], \tag{2.29}$$

where the moments of T are given in equation (2.22) *and*

$$\tau_k(\omega) = \frac{1}{4}|1 - (2R_1(\omega) - 1)^{k+1}|^2. \tag{2.30}$$

Note that the coefficients (2.30) satisfy $\tau_k \leq 1$, because according to equation (2.15),

$$|2R_1(\omega) - 1| = \left|\frac{q(\omega) - i}{q(\omega) + i}\right| = 1. \tag{2.31}$$

This implies that the series in (2.29) is uniformly convergent and we have the bound

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \leq \mathbb{E}\left[|T(\omega)|^4 \sum_{k=0}^{\infty}(1 - |T(\omega)|^2)^k\right] = \mathbb{E}[|T(\omega)|^2]. \tag{2.32}$$

Thus, no matter how weak or strong the barrier is, $\mathbb{E}[|\mathcal{T}|^2]$ cannot exceed the mean intensity transmitted over half the distance, through one region of the random medium.

We analyse next the transmitted intensity in various scenarios. There are two extreme cases:

— The first extreme case is not interesting because it assumes no random fluctuations. We have $T = 1$ and the transmitted intensity is deterministic and equal to the squared modulus of the transmission coefficient of the barrier

$$|\mathcal{T}(\omega)|^2 \overset{(2.29)}{=} \tau_0(\omega) \overset{(2.30)}{=} |1 - R_1(\omega)|^2 \overset{(2.15)}{=} |T_1(\omega)|^2. \tag{2.33}$$

— The second extreme case assumes no barrier, i.e. $T_1 = 1$ and $R_1 = 0$. Then, coefficients (2.30) satisfy $\tau_k = 0$ for odd k and $\tau_k = 1$ for even k. The mean transmitted intensity is, from (2.29),

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = \mathbb{E}\left[|T(\omega)|^4 \sum_{k=0}^{\infty}(1 - |T(\omega)|^2)^{2k}\right] = \mathbb{E}\left[\frac{|T(\omega)|^2}{2 - |T(\omega)|^2}\right]. \tag{2.34}$$

If in addition, the random sections are strongly scattering, i.e. L is larger than L_{loc} so the approximation (2.25) holds, then we have

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \overset{(2.34)}{=} \sum_{k=1}^{\infty} 2^{-k}\mathbb{E}[|T(\omega)|^{2k}] \overset{(2.25)}{\simeq} C\mathbb{E}[|T(\omega)|^2], \tag{2.35}$$

where

$$C = \sum_{k=1}^{\infty} 2^{-k}\phi_k(0) \approx 0.59. \tag{2.36}$$

The result (2.34) says that, as expected from the estimate (2.32), the mean transmitted intensity through the symmetric random medium occupying the interval $[-L, L]$ is less than the intensity transmitted through the single region $z \in [0, L]$. However, the symmetry helps, because if the two random regions were independent, the mean transmitted intensity would be (appendix A(d))

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = \sum_{k=0}^{\infty}\{\mathbb{E}[|T(\omega)|^2(1 - |T(\omega)|^2)^k]\}^2. \tag{2.37}$$

This is smaller than (2.34), as illustrated in figure 2.

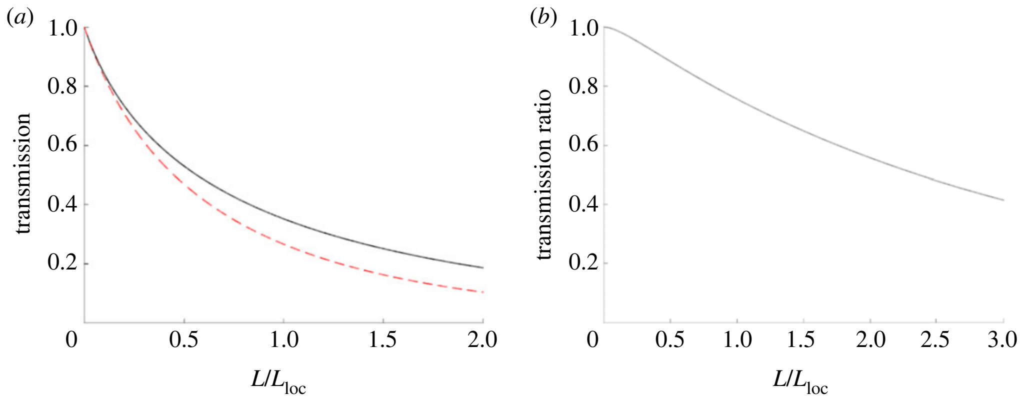

Figure 2. Mean transmitted intensity $\mathbb{E}[|\mathcal{T}|^2]$ of the system as a function of the strength L/L_{loc} of the randomly scattering medium in the absence of the barrier. (*a*) The black solid line is the result (2.34) for symmetric media and the red dashed line is the result (2.37) for independent media. (*b*) Ratio of the mean transmission of independent media and the mean transmission of symmetric media.

Physically, we can interpret the enhanced transmission due to symmetry as follows: it is known that the distribution of the random transmittivity has a small component close to one, that actually gives the value of the mean transmittivity [25, Section 7.1.6]. The medium configurations that give transmittivity close to one are called open channels in the physics literature [13,27]. Efficient transmission through two independent media of length L requires the lucky situation where both media are open channels. For the symmetric case, this requires only one medium of length L to be an open channel as the symmetric medium is then automatically an open channel.

Now we demonstrate the transmission enhancement in the presence of the barrier:

— First, we can see from equation (2.29) that if the random sections are weakly scattering, i.e. $L/L_{loc} \ll 1$, then $\mathbb{E}[|R|^2] = 1 - \mathbb{E}[|T|^2] \ll 1$ and we can approximate the mean transmitted intensity by

$$\begin{aligned}\mathbb{E}[|\mathcal{T}(\omega)|^2] &= \tau_0(\omega)\mathbb{E}[|T(\omega)|^4] + \tau_1(\omega)\mathbb{E}[[|T(\omega)|^4|R(\omega)|^2] + o(\mathbb{E}[|R(\omega)|^2]) \\ &= \tau_0(\omega) + (\tau_1(\omega) - 2\tau_0(\omega))\mathbb{E}[|R(\omega)|^2] + o(\mathbb{E}[|R(\omega)|^2]).\end{aligned}$$

Equation (2.30) gives

$$\tau_0(\omega) = |1 - R_1(\omega)|^2 \overset{(2.15)}{=} |T_1(\omega)|^2$$

and

$$\tau_1(\omega) = 4|R_1(\omega)|^2|1 - R_1(\omega)|^2 = 4|T_1(\omega)|^2(1 - |T_1(\omega)|^2),$$

so to leading order in the reflection coefficient, we have

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \approx |T_1(\omega)|^2\{1 + 2(1 - 2|T_1(\omega)|^2)\mathbb{E}[|R(\omega)|^2]\}. \tag{2.38}$$

This is larger than the transmission intensity of the barrier $|T_1(\omega)|^2$, as long as the barrier is reflecting enough i.e. for $|T_1(\omega)| < 1/\sqrt{2}$.

— If the random sections are more scattering, i.e. $L \gtrsim L_{loc}$, then we must consider the series in (2.29). We compare the result in figure 3 with the mean intensity calculated in the absence of symmetry, i.e. for two independent random media to the left and right of the

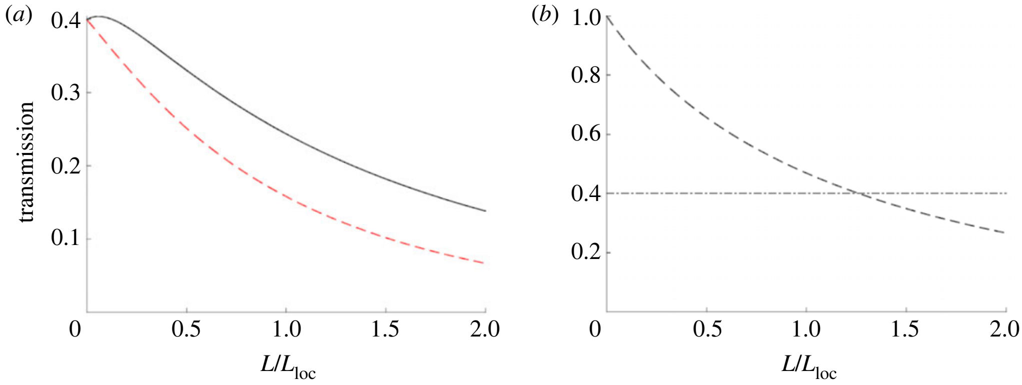

Figure 3. (*a*) Mean transmission $\mathbb{E}[|\mathcal{T}|^2]$ of the system as a function of the strength L/L_{loc} of the randomly scattering medium; the black solid line corresponds to the symmetric media and the red dashed line corresponds to the independent media. (*b*) the mean transmission $\mathbb{E}[|T|^2]$ of one random section (dashed) and the transmission $|T_1|^2$ of the barrier (dot-dashed). Here, $|T_1|^2 = 0.4$.

barrier. The expression of the latter is

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = |T_1(\omega)|^2 \sum_{k,k'=0}^{\infty} C_{k,k'}(\omega)\mathbb{E}[|T(\omega)|^2(1-|T(\omega)|^2)^k] \times \mathbb{E}[|T(\omega)|^2(1-|T(\omega)|^2)^{k'}], \tag{2.39}$$

with

$$C_{k,k'}(\omega) = \left| \sum_{j=\max(k,k')}^{k+k'} \frac{j!R_1^{2j-k-k'}(\omega)[1-2R_1(\omega)]^{k+k'-j}}{(k+k'-j)!(j-k)!(j-k')!} \right|^2.$$

Its derivation is given in appendix A(d).

Again, the transmission is enhanced by symmetry, and this is even more pronounced if the barrier is more reflecting, as shown in figure 4. See also the next case.

— If the barrier is strongly reflecting, i.e. $|T_1| \ll 1$, which is equivalent to having $q \gg 1$, we can use the identity

$$2R_1(\omega) - 1 \overset{(2.15)}{=} \frac{q(\omega)-i}{q(\omega)+i} = 1 - 2T_1(\omega),$$

in equation (2.30) to obtain

$$\tau_k(\omega) = (k+1)^2|T_1(\omega)|^2 + o(|T_1(\omega)|^2), \quad k \geq 0.$$

Substituting this into the expression (2.29) of the mean transmitted intensity, we have

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = |T_1(\omega)|^2\mathbb{E}\left[|T(\omega)|^4 \sum_{k=0}^{\infty}(k+1)^2(1-|T(\omega)|^2)^k\right]. \tag{2.40}$$

This expression can be simplified using the series $\sum_{k=0}^{\infty}(1+k)^2x^k = (1+x)/(1-x)^3$, $\forall x \in (0,1)$, and we obtain that

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = |T_1(\omega)|^2\{2\mathbb{E}[|T(\omega)|^{-2}] - 1\} + o(|T_1(\omega)|^2). \tag{2.41}$$

By solving the Kolmogorov equation $\partial_L U = L_{\text{loc}}^{-1}(2U-1)$ satisfied by $U(L) = \mathbb{E}[|T|^{-2}]$, derived using the expression of the infinitesimal generator of $|T|^2$ given in

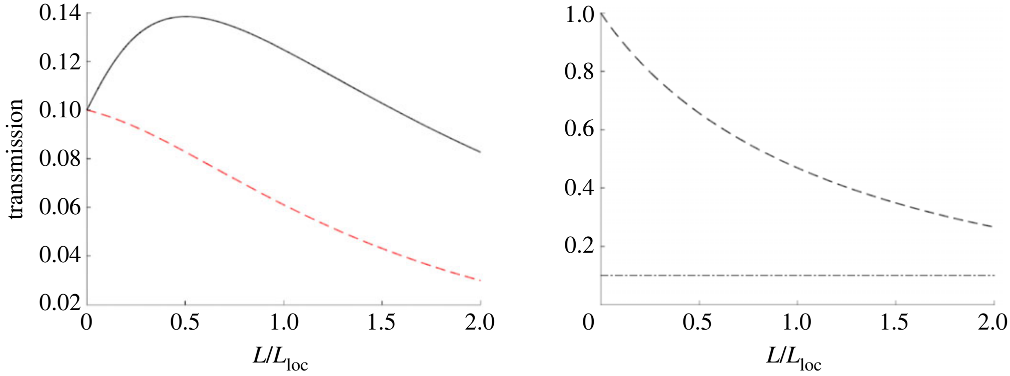

Figure 4. Same as in figure 3 but for a more reflecting barrier with $|T_1|^2 = 0.1$.

[25, Proposition 7.3], we get that

$$2\mathbb{E}[|T(\omega)|^{-2}] - 1 = \exp\left[\frac{2L}{L_{\mathrm{loc}}(\omega)}\right]. \tag{2.42}$$

This result and equation (2.41) show that the transmission enhancement by the random medium can be very large when the barrier is reflecting, as seen in figure 4.

3. Enhanced transmission in random waveguides

In this section, we study wave transmission in random waveguides. To simplify the analysis, we consider two-dimensional waveguides filled with a random medium and with straight, sound soft boundary, as described in §3a. However, the results extend qualitatively to waveguides with random fluctuations of the boundary, because as shown in [28,29], these fluctuations can be mapped mathematically to fluctuations of the coefficients of the wave equation.

The mathematical model is the scalar wave equation for the pressure field, with random wave speed. The decomposition of the wave into modes is in §3b. The interaction of these modes with the reflecting barrier is in §3c. The transmission and reflection of the modes through the random sections is described in §3d. The transmission through the whole system is analysed in §3e. We use the results in §3f to quantify the transmission enhancement induced by symmetry, in the case of weak random scattering.

(a) Setup

Consider a waveguide occupying the domain $\Omega = (-X/2, X/2) \times \mathbb{R}$ and introduce the system of coordinates $\boldsymbol{x} = (x, z)$, with $x \in (-X/2, X/2)$ and $z \in \mathbb{R}$. Assume, as illustrated in figure 5, that the waveguide contains a thin reflecting barrier at $|z| < d/2$, lying between two random sections at $|z| \in (d/2, L)$, which are mirror symmetric with respect to $z = 0$.

The wave at frequency ω is modelled by the Fourier transform $\widehat{p}$ of the pressure, the solution of the Helmholtz equation

$$\left[\frac{\omega^2}{c^2(x,z)} + \Delta\right]\widehat{p}(\omega, x, z) = 0, \quad (x,z) \in \Omega, \tag{3.1}$$

with Dirichlet boundary condition at the sound soft boundary $x = \pm X/2$,

$$\widehat{p}\left(\omega, \pm\frac{X}{2}, z\right) = 0, \quad z \in \mathbb{R}, \tag{3.2}$$

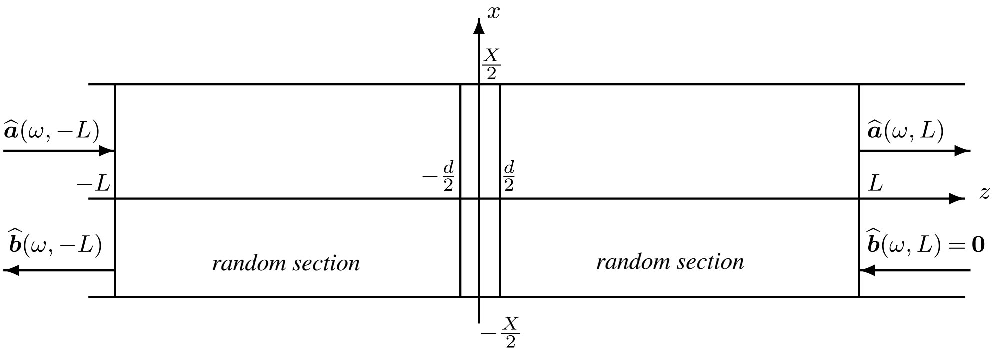

Figure 5. Waveguide occupying the domain $\Omega = (-X/2, X/2) \times \mathbb{R}$ filled at $|z| \in (d/2, L)$ with a random medium with mirror symmetry about the thin barrier located at $|z| < d/2$.

and outgoing boundary condition at $z \to +\infty$. The medium that fills the waveguide is heterogeneous, with wave speed c of the form

$$c^{-2}(x,z) = \begin{cases} c_0^{-2} & \text{if } |z| > L, \\ c_1^{-2} & \text{if } |z| < \dfrac{d}{2}, \\ c_0^{-2}[1+\mu(x,|z|)] & \text{if } \dfrac{d}{2} \le |z| \le L. \end{cases} \tag{3.3}$$

Here, c_0 and c_1 are constants satisfying $c_1 < c_0$, and μ is a zero mean, mixing random process, with the uniform bound $|\mu| < 1$.

The excitation is defined by a right going wave impinging on the random medium at $z = -L$ and our goal is to quantify the transmitted wave at $z = L$.

(b) Mode decomposition outside the barrier

We are interested in the case of small standard deviation of the fluctuations μ of c^{-2}, so we define the wave decomposition at $|z| > d/2$ in the reference medium with wave speed c_0.

The decomposition uses the spectrum of the self-adjoint, negative definite operator ∂_x^2 with Dirichlet boundary conditions at $x = \pm X/2$. The eigenvalues are given by $-\lambda_j$, where $\lambda_j = (j\pi/X)^2$ and the eigenfunctions are $\varphi_j(x) = \sqrt{2/X}\sin(j\pi x/X)$, for $j \ge 1$. These form an orthonormal basis of $L^2(-X/2, X/2)$.

Let $k(\omega) = \omega/c_0$ be the wavenumber and define the natural number

$$N(\omega) = \left\lfloor k(\omega)\frac{X}{\pi} \right\rfloor, \tag{3.4}$$

such that

$$\lambda_{N(\omega)} \le k^2(\omega) < \lambda_{N(\omega)+1}. \tag{3.5}$$

Here $\lfloor \cdot \rfloor$ denotes the integer part. The wave decomposition is

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{\infty} \varphi_j(x)\widehat{p}_j(\omega, z), \tag{3.6}$$

where $\widehat{p}_j$ are one dimensional, time-harmonic waves, called waveguide modes. The first N of them are propagating waves, with wavenumbers

$$\beta_j(\omega) = \sqrt{k^2(\omega) - \lambda_j}, \quad \text{if } j \le N(\omega), \tag{3.7}$$

and the remaining ones are evanescent waves. These decay exponentially in $|z|$ on the length scale β_j^{-1}, where

$$\beta_j(\omega) = \sqrt{\lambda_j - k^2(\omega)}, \quad \text{if } j > N(\omega). \tag{3.8}$$

Note that if $k^2 = \lambda_N$, the wave $\widehat{p}_N$ does not propagate. The analysis of waveguides with such standing modes is more involved than needed in this paper, so we assume that $\beta_N > 0$.

The propagating waves can be decomposed further into right (forward) and left (backward) going modes, using the following equations [25, ch. 20]

$$\widehat{p}_j(\omega, z) = \frac{1}{\sqrt{\beta_j(\omega)}} [\widehat{a}_j(\omega, z)\, \mathrm{e}^{\mathrm{i}\beta_j(\omega)z} + \widehat{b}_j(\omega, z)\, \mathrm{e}^{-\mathrm{i}\beta_j(\omega)z}] \tag{3.9}$$

and

$$\partial_z \widehat{p}_j(\omega, z) = i\sqrt{\beta_j(\omega)} [\widehat{a}_j(\omega, z)\, \mathrm{e}^{\mathrm{i}\beta_j(\omega)z} - \widehat{b}_j(\omega, z)\, \mathrm{e}^{-\mathrm{i}\beta_j(\omega)z}]. \tag{3.10}$$

The complex-valued amplitudes of these modes are gathered in the (column) vector fields

$$\widehat{\boldsymbol{a}}(\omega, z) = (\widehat{a}_j(\omega, z))_{j=1}^{N(\omega)} \quad \text{and} \quad \widehat{\boldsymbol{b}}(\omega, z) = (\widehat{b}_j(\omega, z))_{j=1}^{N(\omega)}, \tag{3.11}$$

and they satisfy the coupled system of equations

$$\partial_z \begin{pmatrix} \widehat{\boldsymbol{a}}(\omega, z) \\ \widehat{\boldsymbol{b}}(\omega, z) \end{pmatrix} = \begin{pmatrix} \mathbf{H}(\omega, z) & \mathbf{K}(\omega, z) \\ \overline{\mathbf{K}(\omega, z)} & \overline{\mathbf{H}(\omega, z)} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{a}}(\omega, z) \\ \widehat{\boldsymbol{b}}(\omega, z) \end{pmatrix}, \tag{3.12}$$

derived in [25, ch. 20]. The derivation involves substituting (3.6), (3.9)–(3.10) into (3.1), using the orthonormality of the eigenfunctions and also expressing the evanescent modes in terms of the propagating ones [25, Section 20.2.3]. The matrices $\mathbf{H}, \mathbf{K} \in \mathbb{C}^{N\times N}$ are given explicitly in [25, Section 20.2.4]. They depend on the mode wavenumbers (3.7-3.8) and the random process $\boldsymbol{\nu} = (\nu_{j,l})_{j,l\geq 1}$, with components

$$\nu_{j,l}(|z|) = \int_{-X/2}^{X/2} \mathrm{d}x\, \varphi_j(x)\varphi_l(x)\mu(x, |z|), \quad j, l \geq 1. \tag{3.13}$$

In the absence of fluctuations, the matrices $\mathbf{H}$ and $\mathbf{K}$ would be zero i.e. the mode amplitudes would be decoupled and constant. This is the case at $|z| > L$, where the wave speed equals the constant c_o.

The system of ODEs (3.12) is complemented with the excitation $\widehat{\boldsymbol{a}}(\omega, -L)$ that specifies the incoming wave impinging on the random medium and the outgoing boundary condition $\widehat{\boldsymbol{b}}(\omega, L) = \mathbf{0}$. Our goal is to characterize the transmitted mode amplitudes $\widehat{\boldsymbol{a}}(\omega, L)$. This requires the analysis of the transmission and reflection of the modes at the thin barrier, described next.

(c) Transmission and reflection at the barrier

The mode decomposition inside the barrier is similar to that in equations (3.6)–(3.10), except that the wave speed c_0 is replaced by c_1. Since we assume that $c_1 < c_0$, we deduce from equation (3.4) and its analogue inside the barrier that there are $N_1(\omega) > N(\omega)$ propagating modes at $|z| < d/2$. The modes are uncoupled, with constant amplitudes, because the wave speed is constant inside the barrier.

The x-profiles of the modes inside and outside the barrier are given by the same eigenfunctions φ_j for all $z \in \mathbb{R}$, so to analyse the wave reflection and transmission at the barrier, it is sufficient to match $\{\widehat{p}_j, \partial_z \widehat{p}_j\}_{j=1}^N$ at $z = \pm d/2$. For $j \geq N+1$, the modes impinging on the barrier are evanescent and their amplitude is negligible for large enough L.

The next lemma describes the propagator of the barrier. Its proof follows from the continuity of the first N modes, and it is similar to the proof of lemma 2.1 in appendix A(a).

Lemma 3.1. *We have*

$$\begin{pmatrix} \widehat{a}\left(\omega, \frac{d}{2}\right) \\ \widehat{b}\left(\omega, \frac{d}{2}\right) \end{pmatrix} = \mathbf{P}_1(\omega) \begin{pmatrix} \widehat{a}\left(\omega, \frac{-d}{2}\right) \\ \widehat{b}\left(\omega, \frac{-d}{2}\right) \end{pmatrix} \quad \text{and} \quad \mathbf{P}_1(\omega) = \begin{pmatrix} \mathbf{P}_1^{(a)}(\omega) & \overline{\mathbf{P}_1^{(b)}(\omega)} \\ \mathbf{P}_1^{(b)}(\omega) & \overline{\mathbf{P}_1^{(a)}(\omega)} \end{pmatrix}, \tag{3.14}$$

where $\mathbf{P_1}$ *is the* $2N \times 2N$ *propagator matrix of the barrier, with diagonal blocks*

$$\mathbf{P}_1^{(a)}(\omega) = diag(\alpha_j(\omega))_{j=1}^{N(\omega)} \quad and \quad \mathbf{P}_1^{(b)}(\omega) = diag(\gamma_j(\omega))_{j=1}^{N(\omega)}. \tag{3.15}$$

The entries of these blocks are

$$\alpha_j(\omega) = \cos(\beta_{1,j}(\omega)d) + \frac{i}{2}\left(\frac{\beta_{1,j}(\omega)}{\beta_j(\omega)} + \frac{\beta_j(\omega)}{\beta_{1,j}(\omega)}\right)\sin(\beta_{1,j}(\omega)d), \tag{3.16}$$

$$\gamma_j(\omega) = \frac{i}{2}\left(\frac{\beta_j(\omega)}{\beta_{1,j}(\omega)} - \frac{\beta_{1,j}(\omega)}{\beta_j(\omega)}\right)\sin(\beta_{1,j}(\omega)d) \tag{3.17}$$

and

$$\beta_{1,j}(\omega) = \sqrt{\frac{\omega^2}{c_1^2} - \lambda_j}, \quad j = 1, \ldots, N(\omega), \tag{3.18}$$

are the mode wavenumbers inside the barrier.

Remark 3.2. Although we focus attention on the case $c_1 < c_0$, one can also analyse the case $c_1 > c_0$ and obtain qualitatively similar results. The difference in the calculations is that when $c_1 > c_0$, there are fewer propagating modes inside the barrier than outside. The algebraic structure of the propagator matrices remains as in lemma 3.1, but the expression of the entries (3.16)–(3.17) involves the hyperbolic cosh and sinh for indexes $j = N_1 + 1, \ldots, N$, corresponding to the modes that transition from propagating outside the barrier to evanescent inside the barrier. Since the barrier is thin these entries remain $O(1)$.

As we have done in §2b, we derive from the propagator $\mathbf{P}_1$ the scattering matrix $\mathbf{S}_1 \in \mathbb{C}^{2N\times 2N}$ of the barrier. This relates the amplitudes of the modes impinging on the barrier to those leaving the barrier,

$$\begin{pmatrix} \widehat{a}\left(\omega, \frac{d}{2}\right) \\ \widehat{b}\left(\omega, \frac{-d}{2}\right) \end{pmatrix} = \mathbf{S}_1(\omega) \begin{pmatrix} \widehat{a}\left(\omega, \frac{-d}{2}\right) \\ \widehat{b}\left(\omega, \frac{d}{2}\right) \end{pmatrix}, \tag{3.19}$$

and has the block structure

$$\mathbf{S}_1(\omega) = \begin{pmatrix} \mathbf{T}_1(\omega) & \mathbf{R}_1(\omega) \\ \mathbf{R}_1(\omega) & \mathbf{T}_1(\omega) \end{pmatrix}, \tag{3.20}$$

with diagonal $N \times N$ blocks

$$\mathbf{T}_1(\omega) = \text{diag}(1/\overline{\alpha_j(\omega)})_{j=1}^{N(\omega)} \quad \text{and} \quad \mathbf{R}_1(\omega) = \text{diag}\left(\frac{-\gamma_j(\omega)}{\overline{\alpha_j(\omega)}}\right)_{j=1}^{N(\omega)}, \tag{3.21}$$

containing the mode-dependent transmission and reflection coefficients of the barrier.

Similar to the layered case, we are interested in the asymptotic regime

$$k(\omega)d \to 0, \quad \frac{c_0}{c_1} \to \infty, \quad \text{such that } \left(\frac{c_0}{c_1}\right)^2 k(\omega)d = O(1). \tag{3.22}$$

In this regime, we deduce from the expressions (3.16)–(3.17) of the coefficients that define the propagator that

$$\alpha_j(\omega) \approx 1 + iq_j(\omega) \quad \text{and} \quad \gamma_j(\omega) \approx -iq_j(\omega), \tag{3.23}$$

where

$$q_j(\omega) = \frac{\beta_{1,j}^2(\omega)d}{2\beta_j(\omega)} \overset{(3.22)}{=} O(1), \quad j = 1, \ldots, N. \tag{3.24}$$

The asymptotic approximation of the transmission and reflection coefficients is

$$T_{1,j}(\omega) \overset{(3.21)}{=} \frac{1}{\overline{\alpha_j(\omega)}} \overset{(3.23)}{\approx} \frac{1}{1 - iq_j(\omega)} \tag{3.25}$$

and

$$R_{1,j}(\omega) \overset{(3.21)}{=} \frac{-\overline{\gamma_j(\omega)}}{\overline{\alpha_j(\omega)}} \overset{(3.23)}{\approx} \frac{iq_j(\omega)}{1 - iq_j(\omega)}, \quad j = 1, \ldots, N(\omega). \tag{3.26}$$

Note that as in the layered case, the scaling relations (3.22) ensure that the barrier causes an opposition to transmission but it is not impenetrable.

(d) Transmission and reflection in the random sections

We collect here the relevant results from [25, ch. 20] and [26] on wave propagation in random waveguides. As stated at the beginning of §3b, we are interested in small random fluctuations μ of c^{-2}. These have a nontrivial effect at a long distance L of propagation with respect to the correlation length ℓ_c of the fluctuations and the wavelength λ. Thus, we consider the asymptotic regime

$$\ell_c \sim \lambda \sim X \ll L, \quad \mathrm{Var}(\mu) \ll 1, \tag{3.27}$$

where we deduce from equation (3.4) that the number N of propagative modes is of order one. The scattering effect of the random medium on the transmittivity is of order one when $\mathrm{Var}(\mu)\ell_c L/\lambda^2 = O(1)$ and it is smaller than one when $\mathrm{Var}(\mu)\ell_c L \ll \lambda^2$. The latter defines what we call the weak scattering regime and is of particular interest in this paper because it allows the explicit quantification of the mean transmittivity of the waveguide (see §3f).

The propagator matrix $\mathbf{P}_-$ for the left random section is the solution of

$$\partial_z \mathbf{P}_-(\omega, z) = \begin{pmatrix} \mathbf{H}(\omega, z) & \mathbf{K}(\omega, z) \\ \overline{\mathbf{K}(\omega, z)} & \overline{\mathbf{H}(\omega, z)} \end{pmatrix} \mathbf{P}_-(\omega, z), \quad z \in \left(-L, \frac{-d}{2}\right) \tag{3.28}$$

and

$$\mathbf{P}_-\left(\omega, \frac{-d}{2}\right) = \mathbf{I}_{2N} \tag{3.29}$$

where $\mathbf{I}_{2N}$ denotes the $2N \times 2N$ identity matrix. Given the algebraic form of the coupling matrix in the right-hand side of (3.28), one can deduce that the propagator has the block structure [25, Section 20.2.5]

$$\mathbf{P}_-(\omega, z) = \begin{pmatrix} \mathbf{P}_-^{(a)}(\omega, z) & \overline{\mathbf{P}_-^{(b)}(\omega, z)} \\ \mathbf{P}_-^{(b)}(\omega, z) & \overline{\mathbf{P}_-^{(a)}(\omega, z)} \end{pmatrix}, \tag{3.30}$$

with full blocks $\mathbf{P}_-^{(a)}, \mathbf{P}_-^{(b)} \in \mathbb{C}^{N \times N}$ that capture mode coupling induced by scattering in the random medium. We are interested in the propagator evaluated at $z = -L$, which defines the $N \times N$ transmission and reflection matrices of the left random section. These matrices are the analogues of the scalar-valued transmission and reflection coefficients in layered media, deduced from the propagator as explained in appendix A(b). We have

$$\begin{pmatrix} \mathbf{I} \\ \mathbf{R}_-(\omega) \end{pmatrix} = \mathbf{P}_-(\omega, -L) \begin{pmatrix} \mathbf{T}_-(\omega) \\ \mathbf{0} \end{pmatrix}, \tag{3.31}$$

which can be understood from the waveguide analogue of figure 8 and

$$\begin{pmatrix} \mathbf{0} \\ \widetilde{\mathbf{T}}_-(\omega) \end{pmatrix} = \mathbf{P}_-(\omega, -L) \begin{pmatrix} \widetilde{\mathbf{R}}_-(\omega) \\ \mathbf{I} \end{pmatrix}, \tag{3.32}$$

which corresponds to the analogue of figure 9. Here, $\mathbf{0}$ and $\mathbf{I}$ are the $N \times N$ zero and identity matrices, respectively.

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Similarly, the propagator $\mathbf{P}_+$ for the right random section is the solution of

$$\partial_z \mathbf{P}_+(\omega, z) = \begin{pmatrix} \mathbf{H}(\omega, z) & \overline{\mathbf{K}(\omega, z)} \\ \mathbf{K}(\omega, z) & \overline{\mathbf{H}(\omega, z)} \end{pmatrix} \mathbf{P}_+(\omega, z), \quad z \in \left(\frac{d}{2}, L\right) \tag{3.33}$$

and

$$\mathbf{P}_+\left(\omega, \frac{d}{2}\right) = \mathbf{I}_{2N}, \tag{3.34}$$

and its algebraic structure is like in equation (3.30), with $N \times N$ blocks $\mathbf{P}_+^{(a)}$ and $\mathbf{P}_+^{(b)}$. This propagator defines the $N \times N$ transmission and reflection matrices of the right random section according to equations

$$\begin{pmatrix} \mathbf{T}_+(\omega) \\ \mathbf{0} \end{pmatrix} = \mathbf{P}_+(\omega, L) \begin{pmatrix} \mathbf{I} \\ \mathbf{R}_+(\omega) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{\mathbf{R}}_+(\omega) \\ \mathbf{I} \end{pmatrix} = \mathbf{P}_+(\omega, L) \begin{pmatrix} \mathbf{0} \\ \widetilde{\mathbf{T}}_+(\omega) \end{pmatrix}. \tag{3.35}$$

These can be understood from the waveguide analogues of figures 6 and 7.

Note the symmetry of the definitions (3.28)–(3.29) and (3.33)–(3.34). Both propagators start as the identity $\mathbf{I}_{2N}$ at $z = \pm d/2$ and define the transmission and reflection matrices at $z = \pm L$. The expression of the coupling matrices $\mathbf{H}$ and $\mathbf{K}$ given in [25, Section 20.2.4] and the symmetry of the fluctuations about $z = 0$, give that

$$\mathbf{H}(\omega, z) = -\overline{\mathbf{H}(\omega, -z)} \quad \text{and} \quad \mathbf{K}(\omega, z) = -\overline{\mathbf{K}(\omega, -z)}. \tag{3.36}$$

This implies that

$$\mathbf{P}_-(\omega, -L) = \overline{\mathbf{P}_+(\omega, L)}, \tag{3.37}$$

and solving equations (3.31)–(3.32) and (3.35), we get: the transmission matrices satisfy

$$\left.\begin{aligned} &\mathbf{T}_+(\omega) = \widetilde{\mathbf{T}}_-(\omega) = \mathbf{P}_+^{(a)}(\omega, L) - \overline{\mathbf{P}_+^{(b)}(\omega, L)}[\overline{\mathbf{P}_+^{(a)}(\omega, L)}]^{-1}\mathbf{P}_+^{(b)}(\omega, L) \\ \text{and} \quad &\widetilde{\mathbf{T}}_+(\omega) = \mathbf{T}_-(\omega) = [\overline{\mathbf{P}_+^{(a)}(\omega, L)}]^{-1}, \end{aligned}\right\} \tag{3.38}$$

and the reflection matrices satisfy

$$\left.\begin{aligned} &\mathbf{R}_+(\omega) = \widetilde{\mathbf{R}}_-(\omega) = -[\overline{\mathbf{P}_+^{(a)}(\omega, L)}]^{-1}\mathbf{P}_+^{(b)}(\omega, L) \\ \text{and} \quad &\widetilde{\mathbf{R}}_+(\omega) = \mathbf{R}_-(\omega) = \overline{\mathbf{P}_+^{(b)}(\omega, L)}[\overline{\mathbf{P}_+^{(a)}(\omega, L)}]^{-1}. \end{aligned}\right\} \tag{3.39}$$

In addition, we have the energy conservation relation [25, eqn (20.41)]

$$\mathbf{R}_+^\star(\omega)\mathbf{R}_+(\omega) + \mathbf{T}_+^\star(\omega)\mathbf{T}_+(\omega) = \mathbf{I}, \tag{3.40}$$

and the reciprocity relations [26, p. 1582]

$$\mathbf{R}_+^T(\omega) \approx \mathbf{R}_+(\omega) \quad \text{and} \quad \widetilde{\mathbf{R}}_+^T(\omega) \approx \widetilde{\mathbf{R}}_+(\omega). \tag{3.41}$$

Here, the superscript T stands for transpose, the star $\star$ denotes the complex conjugate and transpose and the approximation in (3.41) means that reciprocity holds in the asymptotic regime (3.27).

(e) Transmission through the system

The propagator matrix $\mathcal{P}$ for the waveguide is defined by the equation

$$\begin{pmatrix} \widehat{a}(\omega, L) \\ \widehat{b}(\omega, L) \end{pmatrix} = \mathcal{P}(\omega) \begin{pmatrix} \widehat{a}(\omega, -L) \\ \widehat{b}(\omega, -L) \end{pmatrix}. \tag{3.42}$$

From the definitions (3.14), (3.28) and (3.33) of the propagators of the barrier and the random sections, and the identity (3.37), we deduce that

$$\mathcal{P}(\omega) = \mathbf{P}_+(\omega, L)\mathbf{P}_1(\omega)[\overline{\mathbf{P}_+(\omega, L)}]^{-1}. \tag{3.43}$$

It is more convenient for the calculations to consider first the adjoint configuration with an incoming left going wave impinging on the medium at $z = L$, stored in $\widehat{b}(\omega, L)$, and radiation condition at $z = -L$, expressed as $\hat{a}(\omega, L) = \mathbf{0}$. This is described by the following equation:

$$\begin{pmatrix} \widetilde{\mathcal{R}}(\omega) \\ \mathbf{I} \end{pmatrix} = \mathcal{P}(\omega) \begin{pmatrix} \mathbf{0} \\ \widetilde{\mathcal{T}}(\omega) \end{pmatrix}. \tag{3.44}$$

We are interested in the transmission matrix $\widetilde{\mathcal{T}}$. Its expression is given in the next theorem, proved in appendix B(a).

Theorem 3.3. *The $N \times N$ transmission matrix for the waveguide has the expression*

$$\begin{aligned} \widetilde{\mathcal{T}}(\omega) \approx \mathbf{T}_+(\omega)[\mathbf{T}_1^{-1}(\omega) - \mathbf{R}_+(\omega)\mathbf{T}_1^{-1}(\omega)\mathbf{R}_1(\omega) - \mathbf{T}_1^{-1}(\omega)\mathbf{R}_1(\omega)\mathbf{R}_+(\omega) \\ - \mathbf{R}_+(\omega)\overline{\mathbf{T}_1^{-1}(\omega)}\mathbf{R}_+(\omega)]^{-1}\mathbf{T}_+^T(\omega), \end{aligned} \tag{3.45}$$

where the approximation holds in the regime (3.27).

The transmissivity of the system is

$$\mathrm{Tr}[\widetilde{\mathcal{T}}^\star(\omega)\widetilde{\mathcal{T}}(\omega)] = \sum_{j,l=1}^{N(\omega)} |\widetilde{\mathcal{T}}_{jl}(\omega)|^2, \tag{3.46}$$

where 'Tr' denotes the trace. In the next section, we quantify the mean of (3.46) in the asymptotic regime (3.27). Note that, by the invariance of the wave system (3.1–3.3) to the change of direction $z \to -z$, the transmission problem corresponding to an incoming right-going wave impinging on the medium at $z \to -z$ and radiation condition at $z = L$ is equivalent to the adjoint transmission given above.

(f) Enhanced transmission

To quantify the effect of symmetry on the wave transmission through the waveguide, we derive next the expression of the mean transmissivity. This requires the statistical moments of the products of the entries of the transmission and reflection matrices $\mathbf{T}_+$ and $\mathbf{R}_+$. These moments are characterized in the regime (3.27) in [26, Propositions 3.1, 4.2]. Their expression is very complicated, so we do not repeat it here. However, the result simplifies in the case of weak scattering in the random medium:

Theorem 3.4. *When the random medium is weakly scattering, i.e. in the asymptotic regime* (3.27) *with* $\mathrm{Var}(\mu)\ell_c L \ll \lambda^2$, *the mean transmissivity is approximated by*

$$\mathbb{E}\left[\sum_{j,l=1}^{N(\omega)} |\widetilde{\mathcal{T}}_{jl}(\omega)|^2\right] \approx \mathbb{T}(\omega) = \sum_{l=1}^{N(\omega)} \left[|T_{1,l}(\omega)|^2 + \sum_{m=1}^{N(\omega)} \mathcal{M}_{lm}(\omega)\mathcal{B}_{lm}(\omega)\right], \tag{3.47}$$

where we introduced the positive coefficients

$$\mathcal{M}_{lm}(\omega) = \mathbb{E}[|R_{+,lm}(\omega)|^2] \approx \frac{k^4(\omega)L}{2\beta_l(\omega)\beta_m(\omega)} \int_0^\infty \cos[(\beta_j(\omega) + \beta_l(\omega))z]\mathbb{E}[\nu_{l,m}(0)\nu_{l,m}(z)]\, \mathrm{d}z, \tag{3.48}$$

and the factors

$$\mathcal{B}_{lm}(\omega) = |T_{1,l}(\omega) + T_{1,m}(\omega) - 2T_{1,l}(\omega)T_{1,m}(\omega)|^2 - |T_{1,l}(\omega)|^2 - |T_{1,m}(\omega)|^2, \tag{3.49}$$

depend only on the barrier.

The approximation in (3.48) is deduced from the second-order moments of $\mathbf{R}_+$ given in [26, Proposition 3.1]. It holds in the weakly scattering regime and shows that the coefficients $\mathcal{M}_{lm}$ are proportional to the power spectral density of the stationary process $\nu_{l,m}$, for $l, m = 1, \ldots, N$. Moreover, these coefficients are of the order of $\mathrm{Var}(\mu)\ell_c L/\lambda^2 \ll 1$, by the definition of the weak scattering regime.

The proof of this theorem is in appendix B(b). We conclude from its statement that if there is no random medium, the transmissivity equals that of the barrier, denoted by

$$\mathbb{T}_0(\omega) = \sum_{l=1}^{N(\omega)} |T_{1,l}(\omega)|^2. \tag{3.50}$$

If the random medium is present, its effect on the mean transmissivity depends on the strength of the barrier, which determines the sign of the factors (3.49). The moments $\mathcal{M}_{lm}$ are positive by definition, so if the factors $\mathcal{B}_{lm}$ are positive, we have transmission enhancement induced by the symmetry of the random medium.

Let us write more explicitly equation (3.49),

$$\begin{aligned}\mathcal{B}_{lm}(\omega) =& 4|T_{1,l}(\omega)|^2|T_{1,m}(\omega)|^2 - 4|T_{1,m}(\omega)|^2\mathrm{Re}[T_{1,l}(\omega)] - 4|T_{1,l}(\omega)|^2\mathrm{Re}[T_{1,m}(\omega)]\\ &+ 2\mathrm{Re}[T_{1,m}(\omega)\overline{T_{1,l}(\omega)}],\end{aligned}$$

and observe from equation (3.25) that $\mathrm{Re}(T_{1,l}) = |T_{1,l}|^2$. This gives that

$$\begin{aligned}\mathcal{B}_{lm}(\omega) &= -4|T_{1,l}(\omega)|^2|T_{1,m}(\omega)|^2 + 2\mathrm{Re}[T_{1,m}(\omega)\overline{T_{1,l}(\omega)}]\\ &\overset{(3.25)}{=} \frac{-2[1 - q_l(\omega)q_m(\omega)]}{[1 + q_l^2(\omega)][1 + q_m^2(\omega)]}, \quad l, m = 1, \ldots, N(\omega).\end{aligned} \tag{3.51}$$

Consequently, $\mathcal{B}_{lm} < 0$ if the barrier is weak i.e. the parameters $\{q_l\}_{l=1}^N$ are small, and the random medium has a negative effect on the transmissivity, because

$$\mathbb{T}(\omega) - \mathbb{T}_0(\omega) < 0. \tag{3.52}$$

However, if the barrier is strong enough to make the parameters $\{q_l\}_{l=1}^N$ larger than 1, the factors (3.51) are positive and we have transmission enhancement

$$\mathbb{T}(\omega) - \mathbb{T}_0(\omega) \approx \sum_{l=1}^{N(\omega)}\sum_{m=1}^{N(\omega)} \mathcal{M}_{lm}(\omega)\mathcal{B}_{lm}(\omega) > 0. \tag{3.53}$$

The enhancement is due to the symmetry of the random medium about the strong barrier. Without the symmetry, the mean transmissivity is reduced, as stated in the next proposition, proved in appendix B(c).

Proposition 3.5. *When the random medium is weakly scattering, i.e. in the asymptotic regime* (3.27) *with* $\mathrm{Var}(\mu)\ell_c L \ll \lambda^2$, *and the random media in the left and right sections of the waveguide are statistically independent, the mean transmissivity of the system is approximated by*

$$\mathbb{E}\left[\sum_{j,l=1}^{N(\omega)} |\widetilde{T}_{jl}(\omega)|^2\right] \approx \mathbb{T}_0(\omega) - 2\sum_{l=1}^{N(\omega)}\sum_{m=1}^{N(\omega)} \mathcal{M}_{lm}(\omega)|T_{1,l}(\omega)|^2|T_{1,m}(\omega)|^2, \tag{3.54}$$

and is therefore smaller than the transmissivity $\mathbb{T}_0(\omega)$ *of the barrier.*

4. Summary

We have introduced a detailed mathematical analysis of wave transmission enhancement in random systems with symmetry about a reflecting barrier. The analysis is motivated by recent experimental results reported in the physics literature, which observe such enhancement in symmetric cavities and in diffusive slabs. We consider acoustic waves for simplicity, although the methodology applies to any linear waves. The main result is the quantification of the mean transmissivity of two random systems with a preferred direction of propagation: plane waves in randomly layered media and waves in random waveguides. The first case is easier to analyse and we consider both weak and strongly scattering random media. The waveguide setting is significantly more complex, so we quantify the transmission enhancement only in the case of weakly scattering random media. The result is expected to extend to stronger scattering regimes, because the waveguide setting is somewhat close to that in diffusive slabs, where transmission enhancement has been observed experimentally in [22,23]. However, the explicit analysis and quantification of such enhancement is difficult, due to the very complicated expression of the statistical moments of the transmission and reflection coefficients in random waveguides.

The analysis that is carried out in this paper shows that the transmission enhancement in both layered media and in weakly scattering random waveguides is much more pronounced for large opacity of the barrier.

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. L.B.: conceptualization, formal analysis, funding acquisition, investigation, methodology, writing—review and editing; J.G.: conceptualization, formal analysis, funding acquisition, investigation, methodology, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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Appendix A. Derivation of the results for randomly layered media

In this appendix, we prove the results stated in §2. Since the frequency ω is fixed in the proofs, we simplify the notation throughout the appendix, by dropping the argument ω of the propagator and scattering matrices below.

(a) Proof of lemma 2.1

The statement of the lemma is derived from the continuity of the Fourier coefficients of the pressure and velocity fields. The decomposition of these fields is given in equations (2.7)–(2.8) outside the barrier and their analogues inside the barrier. The medium inside the barrier is homogeneous, so it follows from equation (2.1) that the right and left going mode amplitudes there, denoted by $\widehat{a}_1$ and $\widehat{b}_1$, satisfy

$$\partial_z \widehat{a}_1(z) = \partial_z \widehat{b}_1(z) = 0, \quad z \in \left(\frac{-d}{2}, \frac{d}{2}\right). \tag{A 1}$$

When imposing the continuity of the wave field at $z = -d/2$, we obtain that

$$\begin{pmatrix} \widehat{a}_1\left(-\frac{d}{2}\right) e^{-i\omega(d/2c_1)} \\ \widehat{b}_1\left(-\frac{d}{2}\right) e^{i\omega(d/2c_1)} \end{pmatrix} = \begin{pmatrix} r_+ & r_- \\ r_- & r_+ \end{pmatrix} \begin{pmatrix} \widehat{a}\left(-\frac{d}{2}\right) e^{-i\omega(d/2c_0)} \\ \widehat{b}\left(-\frac{d}{2}\right) e^{i\omega(d/2c_0)} \end{pmatrix}, \tag{A 2}$$

where

$$r_{\pm} = \frac{1}{2}\left(\sqrt{\frac{\zeta_1}{\zeta_0}} - \sqrt{\frac{\zeta_0}{\zeta_1}}\right). \tag{A 3}$$

The continuity at $z = d/2$ gives

$$\begin{pmatrix} \widehat{a}\left(\frac{d}{2}\right) \mathrm{e}^{i\omega(d/2c_0)} \\ \widehat{b}\left(\frac{d}{2}\right) \mathrm{e}^{-\mathrm{i}\omega(d/2c_0)} \end{pmatrix} = \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} \widehat{a}_1\left(\frac{d}{2}\right) \mathrm{e}^{\mathrm{i}\omega(d/2c_1)} \\ \widehat{b}_1\left(\frac{d}{2}\right) \mathrm{e}^{-\mathrm{i}\omega(d/2c_1)} \end{pmatrix}, \tag{A 4}$$

and from equation (A 1), we have

$$\widehat{a}_1\left(\frac{d}{2}\right) = \widehat{a}_1\left(-\frac{d}{2}\right) \quad \text{and} \quad \widehat{b}_1\left(\frac{d}{2}\right) = \widehat{b}_1\left(-\frac{d}{2}\right). \tag{A 5}$$

Combining these equations, we obtain

$$\begin{pmatrix} \widehat{a}\left(\frac{d}{2}\right) \\ \widehat{b}\left(\frac{d}{2}\right) \end{pmatrix} = \mathbf{P}_1 \begin{pmatrix} \widehat{a}\left(-\frac{d}{2}\right) \\ \widehat{b}\left(-\frac{d}{2}\right) \end{pmatrix}, \tag{A 6}$$

where

$$\mathbf{P}_1 = \begin{pmatrix} \mathrm{e}^{-i\omega(d/2c_0)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\omega(d/2c_0)} \end{pmatrix} \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega(d/c_1)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\omega(d/c_1)} \end{pmatrix}$$
$$\times \begin{pmatrix} r_+ & r_- \\ r_- & r_+ \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega(d/2c_0)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\omega(d/2c_0)} \end{pmatrix}. \tag{A 7}$$

Multiplying the matrices in (A 7), we get the algebraic form (2.9) of $\mathbf{P}_1$, with

$$\alpha = \left[(r_+^2 - r_-^2)\cos\left(\frac{\omega d}{c_1}\right) + i(r_+^2 + r_-^2)\sin\left(\frac{\omega d}{c_1}\right)\right] \mathrm{e}^{-\mathrm{i}\omega d/c_0} \tag{A 8}$$

and

$$\gamma = -2ir_+r_- \sin\left(\frac{\omega d}{c_1}\right). \tag{A 9}$$

Finally, definition (A 3) gives

$$r_+^2 - r_-^2 = 1 \quad \text{and} \quad r_+r_- = \frac{1}{4}\left(\frac{\zeta_1}{\zeta_0} - \frac{\zeta_0}{\zeta_1}\right), \tag{A 10}$$

and the statement of lemma 2.1 follows. ∎

(b) Proof of lemma 2.2

Consider first the random section $[d/2, L]$ and define the propagator $\mathbf{P}_+$ of the subsection $[d/2, z]$ by

$$\begin{pmatrix} \widehat{a}(z) \\ \widehat{b}(z) \end{pmatrix} = \mathbf{P}_+(z) \begin{pmatrix} \widehat{a}\left(\frac{d}{2}\right) \\ \widehat{b}\left(\frac{d}{2}\right) \end{pmatrix} \quad \text{and} \quad z \in \left(\frac{d}{2}, L\right]. \tag{A 11}$$

It is shown in [25, ch. 7 and §4.4.3] that

$$\mathbf{P}_+(z) = \begin{pmatrix} \alpha_+(z) & \overline{\gamma_+(z)} \\ \gamma_+(z) & \overline{\alpha_+(z)} \end{pmatrix}, \tag{A 12}$$

where α_+ and γ_+ satisfy the first-order system

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \alpha_+(z) \\ \gamma_+(z) \end{pmatrix} = \frac{i\omega}{2c_0}\mu(z) \begin{pmatrix} 1 & -\mathrm{e}^{-2\mathrm{i}\omega z/c_0} \\ \mathrm{e}^{2\mathrm{i}\omega z/c_0} & -1 \end{pmatrix} \begin{pmatrix} \alpha_+(z) \\ \gamma_+(z) \end{pmatrix}, \tag{A 13}$$

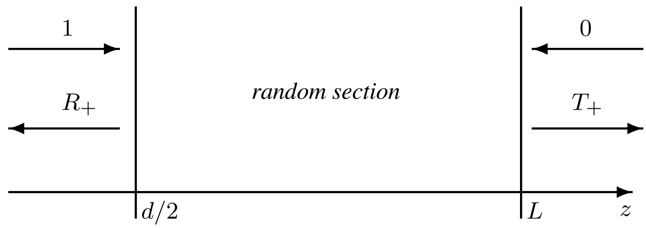

Figure 6. Reflection and transmission coefficients R_+ and T_+ for the random section $(d/2, L)$.

at $z \in (d/2, L)$, and the initial conditions

$$\alpha_+\left(\frac{d}{2}\right) = 1 \quad \text{and} \quad \gamma_+\left(\frac{d}{2}\right) = 0. \tag{A 14}$$

This is illustrated schematically in figure 6 and at $z = L$, we have

$$\begin{pmatrix} T_+ \\ 0 \end{pmatrix} = \mathbf{P}_+(L)\begin{pmatrix} 1 \\ R_+ \end{pmatrix}, \tag{A 15}$$

where T_+ and R_+ are the random transmission and reflection coefficients, defined by

$$T_+ = \frac{1}{\overline{\alpha_+(L)}} \quad \text{and} \quad R_+ = -\frac{\overline{\gamma_+(L)}}{\overline{\alpha_+(L)}}. \tag{A 16}$$

Since the matrix in equation (A 13) has trace zero, we have the conservation relation [25, Section 7.1.1]

$$\det[\mathbf{P}_+(L)] = |\alpha_+(L)|^2 - |\gamma_+(L)|^2 = 1, \tag{A 17}$$

which in light of definitions (A 16) is equivalent to $|R_+|^2 + |T_+|^2 = 1$. Because of this relation, the inverse of the propagator is

$$\mathbf{P}_+^{-1}(L) = \begin{pmatrix} \overline{\alpha_+(L)} & -\overline{\gamma_+(L)} \\ -\gamma_+(L) & \alpha_+(L) \end{pmatrix}, \tag{A 18}$$

and from (A 15), we obtain that

$$\begin{pmatrix} 1 \\ R_+ \end{pmatrix} = \begin{pmatrix} \overline{\alpha_+(L)} & -\overline{\gamma_+(L)} \\ -\gamma_+(L) & \alpha_+(L) \end{pmatrix}\begin{pmatrix} T_+ \\ 0 \end{pmatrix}. \tag{A 19}$$

Reordering these equations and defining

$$\widetilde{T}_+ = T_+ = \frac{1}{\overline{\alpha_+(L)}} \quad \text{and} \quad \widetilde{R}_+ = \frac{\overline{\gamma_+(L)}}{\overline{\alpha_+(L)}}, \tag{A 20}$$

we obtain the adjoint problem, illustrated schematically in figure 7 ,

$$\begin{pmatrix} \widetilde{R}_+ \\ 1 \end{pmatrix} = \mathbf{P}_+(L)\begin{pmatrix} 0 \\ \widetilde{T}_+(L) \end{pmatrix}. \tag{A 21}$$

Now we can obtain from equation (A 11) evaluated at $z = L$ and the definitions (A 16) and (A 20) of the transmission and reflection coefficients that

$$\begin{pmatrix} \widehat{a}(L) \\ \widehat{b}\left(\frac{d}{2}\right) \end{pmatrix} = \underbrace{\begin{pmatrix} T_+ & \widetilde{R}_+ \\ R_+ & T_+ \end{pmatrix}}_{\mathbf{S}_+}\begin{pmatrix} \widehat{a}\left(\frac{d}{2}\right) \\ \widehat{b}(L) \end{pmatrix}, \tag{A 22}$$

where $\mathbf{S}_+$ is the scattering matrix of the random section $[d/2, L]$.

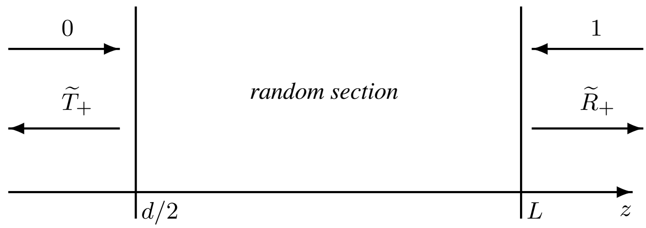

Figure 7. Adjoint reflection and transmission coefficients $\widetilde{R}_+$ and $\widetilde{T}_+$ for $z \in (d/2, L)$.

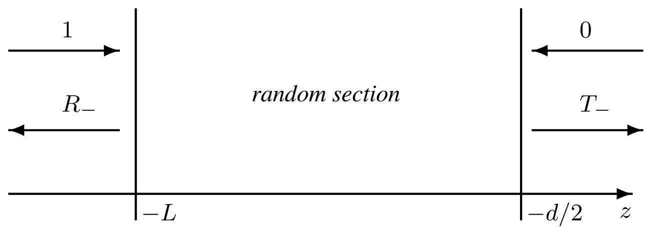

Figure 8. Reflection and transmission coefficients R_- and T_- for random section $(-L, -d/2)$.

Similarly, the propagator matrix for the left random section satisfies

$$\begin{pmatrix} \widehat{a}(z) \\ \widehat{b}(z) \end{pmatrix} = \mathbf{P}_-(z) \begin{pmatrix} \widehat{a}\left(-\frac{d}{2}\right) \\ \widehat{b}\left(-\frac{d}{2}\right) \end{pmatrix} \quad \text{and} \quad z \in \left[-L, -\frac{d}{2}\right), \tag{A 23}$$

where

$$\mathbf{P}_-(z) = \begin{pmatrix} \alpha_-(z) & \overline{\gamma_-(z)} \\ \gamma_-(z) & \overline{\alpha_-(z)} \end{pmatrix}, \tag{A 24}$$

and α_- and β_- satisfy

$$\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} \alpha_-(z) \\ \gamma_-(z) \end{pmatrix} = \frac{i\omega}{2c_0} \mu(-z) \begin{pmatrix} 1 & -\mathrm{e}^{-2\mathrm{i}\omega z/c_0} \\ \mathrm{e}^{2\mathrm{i}\omega z/c_0} & -1 \end{pmatrix} \begin{pmatrix} \alpha_-(z) \\ \gamma_-(z) \end{pmatrix}, \tag{A 25}$$

at $z \in (-L, -d/2)$, and the initial conditions

$$\alpha_-\left(-\frac{d}{2}\right) = 1 \quad \text{and} \quad \gamma_-\left(-\frac{d}{2}\right) = 0. \tag{A 26}$$

Note that due to the symmetry of the random medium, $(\overline{\alpha_-(-z)}, \overline{\gamma_-(-z)})$ satisfies the same equation and initial condition as $(\alpha_+(z), \gamma_+(z))$. Therefore,

$$\alpha_-(-L) = \overline{\alpha_+(L)} \quad \text{and} \quad \gamma_-(-L) = \overline{\gamma_+(L)}. \tag{A 27}$$

The reflection and transmission through the left random section is illustrated schematically in figures 8 and 9 and the transmission and reflection coefficients are defined by

$$\begin{pmatrix} 1 \\ R_- \end{pmatrix} = \mathbf{P}_-(-L) \begin{pmatrix} T_- \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \widetilde{T}_- \end{pmatrix} = \mathbf{P}_-(-L) \begin{pmatrix} \widetilde{R}_- \\ 1 \end{pmatrix}.$$

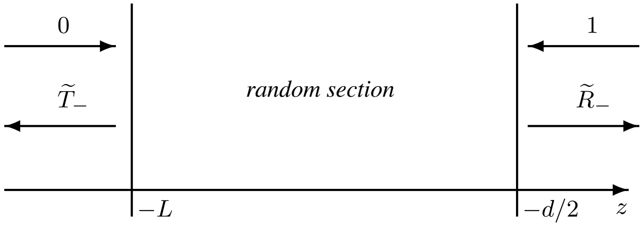

Figure 9. Adjoint reflection and transmission coefficients $\widetilde{R}_-$ and $\widetilde{T}_-$ for $z \in (-L, -d/2)$.

These equations and the relation (A 27) give

$$T_- = \widetilde{T}_- = \frac{1}{\alpha_-(-L)} = \frac{1}{\overline{\overline{\alpha_+(L)}}} \overset{(A.16)}{=} T_+,$$

$$R_- = \frac{\gamma_-(-L)}{\alpha_-(-L)} = \frac{\overline{\gamma_+(L)}}{\overline{\overline{\alpha_+(L)}}} \overset{(A.20)}{=} \widetilde{R}_+$$

and

$$\widetilde{R}_- = -\frac{\overline{\gamma_-(-L)}}{\alpha_-(-L)} = -\frac{\gamma_+(L)}{\overline{\overline{\alpha_+(L)}}} \overset{(A.16)}{=} R_+,$$

as stated in the lemma. ■

(c) Proof of theorem 2.3

Using the propagator matrices of the two random regions and the barrier, described in appendices A(a)–(b), we have

$$\begin{pmatrix} \widehat{a}(L) \\ \widehat{b}(L) \end{pmatrix} = \mathbf{P}_+(L)\mathbf{P}_1\mathbf{P}_-(-L) \begin{pmatrix} \widehat{a}(-L) \\ \widehat{b}(-L) \end{pmatrix}, \tag{A 28}$$

To calculate the scattering matrix, we need a basic lemma.

Lemma A.1. *Consider a system consisting of two successive sectors: the left one with propagator matrix* $\mathbf{P}_l$ *and the right one with propagator* $\mathbf{P}_r$,

$$\mathbf{P}_l = \begin{pmatrix} \alpha_l & \overline{\gamma_l} \\ \gamma_l & \overline{\alpha_l} \end{pmatrix} \quad \textit{and} \quad \mathbf{P}_r = \begin{pmatrix} \alpha_r & \overline{\gamma_r} \\ \gamma_r & \overline{\alpha_r} \end{pmatrix}. \tag{A 29}$$

The propagator matrix of the system is $\mathbf{P} = \mathbf{P}_r\mathbf{P}_l = \left[\begin{smallmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{smallmatrix}\right]$, *where*

$$\alpha = \alpha_l\alpha_r + \gamma_l\overline{\gamma_r}, \quad \gamma = \alpha_l\gamma_r + \gamma_l\overline{\alpha_r}. \tag{A 30}$$

The scattering matrix is $\mathbf{S} = \left[\begin{smallmatrix} T & \widetilde{R} \\ R & T \end{smallmatrix}\right]$, *with entries*

$$T = \frac{1}{\overline{\alpha}} = T_l T_r (1 - R_r \widetilde{R}_l)^{-1}, \tag{A 31}$$

$$R = -\frac{\gamma}{\overline{\alpha}} = R_l + T_l^2 R_r (1 - R_r \widetilde{R}_l)^{-1} \tag{A 32}$$

and

$$\widetilde{R} = \frac{\overline{\gamma}}{\overline{\alpha}} = \widetilde{R}_r + T_r^2 \widetilde{R}_l (1 - R_r \widetilde{R}_l)^{-1}. \tag{A 33}$$

Here, T_j, R_j *and* $\widetilde{R}_j$ *are the transmission and reflection coefficients of the two sectors, with* $j \in \{l, r\}$.

Proof. Equation (A 30) follows trivially from the multiplication of the matrices (A 29). The expression of the transmission and reflection coefficients in terms of α and γ is as in equations

(A 16) and (A 20). From definitions

$$T_j = \frac{1}{\alpha_j}, \quad R_j = -\frac{\gamma_j}{\alpha_j} \quad \text{and} \quad \widetilde{R}_j = \frac{\overline{\gamma_j}}{\alpha_j}, \ j \in \{l, r\}, \tag{A 34}$$

we get that the transmission coefficient satisfies

$$T = \frac{1}{\alpha} \overset{(\text{A 30})}{=} \frac{1}{\alpha_l \alpha_r}\left(1 + \frac{\overline{\gamma_l}}{\alpha_l}\frac{\gamma_r}{\alpha_r}\right)^{-1} \overset{(\text{A 34})}{=} T_l T_r (1 - R_l \widetilde{R}_r)^{-1}.$$

For the reflection coefficient, we have

$$\begin{aligned} R &= -\frac{\gamma}{\alpha} \overset{(\text{A 30})}{=} -\frac{(\alpha_l \gamma_r + \gamma_l \overline{\alpha_r})}{\alpha_l \alpha_r}\left(1 + \frac{\overline{\gamma_l}}{\alpha_l}\frac{\gamma_r}{\alpha_r}\right)^{-1} \overset{(\text{A 34})}{=} \left[\frac{|\alpha_l|^2}{\alpha_l{}^2} R_r + R_l\right](1 - \widetilde{R}_l R_r)^{-1} \\ &\overset{(\text{A 17})}{=} \left[\frac{(1 + |\gamma_l|^2)}{\alpha_l{}^2} R_r + R_l\right](1 - \widetilde{R}_l R_r)^{-1} \overset{(\text{A 34})}{=} (T_l^2 R_r - R_l \widetilde{R}_l R_r + R_l)(1 - \widetilde{R}_l R_r)^{-1} \\ &= T_l^2 R_r (1 - \widetilde{R}_l R_r)^{-1} + R_l. \end{aligned}$$

The derivation of the expression of the adjoint reflection coefficient is similar

$$\begin{aligned} \widetilde{R} &= \frac{\overline{\gamma}}{\alpha} \overset{(\text{A 30})}{=} \frac{(\overline{\alpha_l \gamma_r} + \overline{\gamma_l} \alpha_r)}{\alpha_l \alpha_r}\left(1 + \frac{\overline{\gamma_l}}{\alpha_l}\frac{\gamma_r}{\alpha_r}\right)^{-1} \overset{(\text{A 34})}{=} \left[\widetilde{R}_r + \frac{(1 + |\gamma_r|^2)}{\alpha_r{}^2}\widetilde{R}_l\right](1 - \widetilde{R}_l R_r)^{-1} \\ &\overset{(\text{A 34})}{=} (\widetilde{R}_r + T_r^2 \widetilde{R}_l - \widetilde{R}_r R_r \widetilde{R}_l)(1 - \widetilde{R}_l R_r)^{-1} = \widetilde{R}_r + T_r^2 \widetilde{R}_l (1 - \widetilde{R}_l R_r)^{-1}. \end{aligned}$$

The proof of the lemma is complete. ■

To derive the expression of the transmission coefficient stated in theorem 2.3, we apply lemma A.1 twice. The first time, we use the propagators $\mathbf{P}_l = \mathbf{P}_-(-L)$ and $\mathbf{P}_r = \mathbf{P}_1$ and obtain the transmission and reflection coefficients

$$T_{-,1} \overset{(\text{A 31})}{=} T_- T_1 (1 - R_1 \widetilde{R}_-)^{-1}, \tag{A 35}$$

$$R_{-,1} \overset{(\text{A 32})}{=} R_- + T_-^2 R_1 (1 - R_1 \widetilde{R}_-)^{-1} \tag{A 36}$$

and

$$\widetilde{R}_{-,1} \overset{(\text{A 33})}{=} R_1 + T_1^2 \widetilde{R}_- (1 - R_1 \widetilde{R}_-)^{-1}, \tag{A 37}$$

with $T_- = T_-(-L)$, $R_- = R_-(-L)$ and $\widetilde{R}_- = \widetilde{R}_-(-L)$. Here, we used that $R_1 = \widetilde{R}_1$, according to equation (2.13). The second time we apply lemma A.1, we use the propagators $\mathbf{P}_l = \mathbf{P}_{-,b}$ and $\mathbf{P}_r = \mathbf{P}_+(L)$. The transmission coefficient is

$$\begin{aligned} \mathcal{T} &\overset{(\text{A 31})}{=} T_{-,1} T_+ (1 - R_+ \widetilde{R}_{-,1})^{-1} \\ &\overset{(\text{A 35})}{=} T_- T_1 T_+ [1 - R_1 \widetilde{R}_- - R_+ R_1 (1 - R_1 \widetilde{R}_-) - R_+ T_1^2 \widetilde{R}_-]^{-1}, \end{aligned} \tag{A 38}$$

with $T_+ = T_+(L)$, $R_+ = R_+(L)$, and $\widetilde{R}_+ = \widetilde{R}_+(L)$. Now use the relations (2.27) in this equation to obtain

$$\mathcal{T} = T^2 T_1 [1 - 2RR_1 + (R_1^2 - T_1^2) R^2]^{-1}, \tag{A 39}$$

and deduce from equation (2.15) that

$$R_1^2 - T_1^2 = \frac{q^2 + 1}{(i + q)^2} = 2R_1 - 1. \tag{A 40}$$

The result (2.28) follows (A 39) and the identity

$$(1 - R)[1 - (2R_1 - 1)R] = 1 - 2RR_1 + (2R_1 - 1)R^2.$$

We are interested in the mean transmitted intensity. To derive its expression, we recall from [25, Section 7.1.1] that $|R| < 1$. Since R_1 satisfies equation (2.31) and R_1 satisfies equation (2.31), we can

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use the series expansions

$$(1-R)^{-1}=\sum_{k=0}^{\infty}R^k \quad \text{and} \quad [1-(2R_1-1)R]^{-1}=\sum_{k=0}^{\infty}(2R_1-1)^kR^k,$$

and rewrite equation (2.28) as

$$\mathbb{E}[|\mathcal{T}|^2]=|T_1|^2\sum_{k_1,k_2,k_3,k_4=0}^{\infty}(2R_1-1)^{k_2}(2\overline{R_1}-1)^{k_4}\mathbb{E}[|T|^4R^{k_1+k_2}\overline{R}^{k_3+k_4}]. \tag{A 41}$$

It is shown in [25, ch. 7 and 9] that

$$\mathbb{E}[|T|^2R^j\overline{R}^{j'}]=0, \quad \text{if } j\neq j',$$

so only the terms with $k_1+k_2=k_3+k_4$ contribute in (A 41). Moreover, since $|R|^2=1-|T|^2$, we obtain that

$$\mathbb{E}[|\mathcal{T}|^2]=|T_1|^2\sum_{k=0}^{\infty}\sum_{k_2,k_4=0}^{k}(2R_1-1)^{k_2}(2\overline{R_1}-1)^{k_4}\mathbb{E}[|T|^4(1-|T|^2)^k]. \tag{A 42}$$

Now, use the notation (2.30) and observe that

$$|T_1|^2\sum_{k_2,k_4=0}^{k}(2R_1-1)^{k_2}(2\overline{R_1}-1)^{k_4}=\left|T_1\sum_{k_2=0}^{k}(2R_1-1)^{k_2}\right|^2=\left|\frac{T_1}{2(1-R_1)}[1-(2R_1-1)^{k+1}]\right|^2,$$

where according to equation (2.15), we have

$$\left|\frac{T_1}{2(1-R_1)}\right|^2=\frac{1}{4}.$$

The result (2.29) follows, once we recall the definition (2.30) of τ_k.

(d) Transmission through two independent random sections

To derive the mean transmitted intensity in the absence of symmetry, we begin with the general formula (A 38), where now the transmission and reflection coefficients in the two random sections are statistically independent. Using equation (A 40) in (A 38) and writing the inverse of the curly bracket as power series, we get

$$\mathbb{E}[|\mathcal{T}|^2]=|T_1|^2\sum_{j,l=0}^{\infty}\mathbb{E}\Big[|T_-|^2|T_+|^2[R_1(R_++\widetilde{R}_-)+(1-2R_1)R_+\widetilde{R}_-]^j \\ \times\overline{[R_1(R_++\widetilde{R}_-)+(1-2R_1)R_+\widetilde{R}_-]}^l\Big].$$

Next, we expand the j and l powers using the binomial theorem and use the independence of $(T_-,\widetilde{R}_-)$ and $(T_+,\widetilde{R}_+)$. Using also that $\mathbb{E}[|T_+|^2R_+^n\overline{R_+}^m]=0$ unless $m=n$, and the same for $(T_-,\widetilde{R}_-)$, we get the result (2.39).

Appendix B. Derivation of the results for random waveguides

In this appendix, we prove the results stated in §3. The frequency ω is fixed, so we simplify notation as in the previous appendix, by droping the ω argument.

(a) Proof of theorem 3.3

We obtain from equations (3.38) to (3.39) that

$$\mathbf{P}_+^{(a)} = \mathbf{T}_+(\mathbf{I} - \overline{\mathbf{R}_+}\mathbf{R}_+)^{-1} \quad \text{and} \quad \mathbf{P}_+^{(b)} = -\overline{\mathbf{T}_+}(\mathbf{I} - \mathbf{R}_+\overline{\mathbf{R}_+})^{-1}\mathbf{R}_+. \tag{B 1}$$

Moreover, standard formulae for block matrix inversion give that

$$\mathbf{P}_-^{-1}(-L) \overset{(3.37)}{=} \overline{\mathbf{P}_+^{-1}(L)} = \begin{pmatrix} \overline{\mathbf{T}_+^{-1}} & \mathbf{R}_+\mathbf{T}_+^{-1} \\ \overline{\mathbf{R}_+\mathbf{T}_+^{-1}} & \mathbf{T}_+^{-1} \end{pmatrix}. \tag{B 2}$$

Then, using this result in (3.43) and recalling the block algebraic structure of $\mathbf{P}_+$ and $\mathbf{P}_1$, we get that the propagator of the system has the form

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}^{(a)} & \overline{\mathcal{P}^{(b)}} \\ \mathcal{P}^{(b)} & \overline{\mathcal{P}^{(a)}} \end{pmatrix}. \tag{B 3}$$

We are interested in the first block $\mathcal{P}^{(a)}$, which according to definition (3.44) defines the transmission matrix

$$\widetilde{\mathcal{T}} = [\overline{\mathcal{P}^{(a)}}]^{-1}. \tag{B 4}$$

The expression of this block follows by carrying out the multiplication in (3.43),

$$\mathcal{P}^{(a)} = \mathbf{T}_+(\mathbf{I} - \overline{\mathbf{R}_+}\mathbf{R}_+)^{-1}(\mathbf{P}_1^{(a)} - \overline{\mathbf{R}_+}\mathbf{P}_1^{(b)} + \overline{\mathbf{P}_1^{(b)}}\mathbf{R}_+ - \overline{\mathbf{R}_+\mathbf{P}_1^{(a)}}\mathbf{R}_+)(\overline{\mathbf{T}_+})^{-1}.$$

But we also have from the relations (3.40)–(3.41) that

$$\mathbf{I} - \overline{\mathbf{R}_+}\mathbf{R}_+ \approx \mathbf{I} - \mathbf{R}_+^\star\mathbf{R}_+ = \mathbf{T}_+^\star\mathbf{T}_+,$$

which simplifies the factor

$$\mathbf{T}_+(\mathbf{I} - \overline{\mathbf{R}_+}\mathbf{R}_+)^{-1} \approx \mathbf{T}_+(\mathbf{T}_+^\star\mathbf{T}_+)^{-1} = (\mathbf{T}_+^\star)^{-1}. \tag{B 5}$$

The statement of the theorem follows from (B 4) and the relations

$$\mathbf{P}_1^{(a)} = \overline{\mathbf{T}_1^{-1}} \quad \text{and} \quad \mathbf{P}_1^{(b)} = -\overline{\mathbf{P}_1^{(b)}} = -\mathbf{T}_1^{-1}\mathbf{R}_1,$$

deduced from equations (3.15) and (3.25)–(3.26). ■

(b) Proof of theorem 3.4

Weak scattering in the random medium means that the norm of the reflection matrix $\mathbf{R}_+$ is small. Thus, we can use Neumann series to approximate the square bracket in (3.45) by

$$\begin{aligned} \mathbf{Q} &= [\mathbf{T}_1^{-1} - \mathbf{R}_+\mathbf{T}_1^{-1}\mathbf{R}_1 - \mathbf{T}_1^{-1}\mathbf{R}_1\mathbf{R}_+ - \mathbf{R}_+\overline{\mathbf{T}_1^{-1}}\mathbf{R}_+]^{-1} \\ &= [\mathbf{I} - \mathbf{T}_1\mathbf{R}_+\mathbf{R}_1\mathbf{T}_1^{-1} - \mathbf{R}_1\mathbf{R}_+ - \mathbf{T}_1\mathbf{R}_+\overline{\mathbf{T}_1^{-1}}\mathbf{R}_+]^{-1}\mathbf{T}_1 \\ &\approx \mathbf{T}_1 + \mathbf{T}_1\mathbf{R}_+\mathbf{R}_1 + \mathbf{R}_1\mathbf{R}_+\mathbf{T}_1, \end{aligned} \tag{B 6}$$

where in the second equality we used that $\mathbf{R}_1$ and $\mathbf{T}_1^{-1}$ commute, because they are diagonal. This approximation is valid for weak scattering and neglects terms that contain a product involving two (or more) reflection matrices $\mathbf{R}_+$. Substituting (B 6) into (3.45), we get that

$$\widetilde{\mathcal{T}} \approx \mathbf{T}_+(\mathbf{T}_1 + \mathbf{T}_1\mathbf{R}_+\mathbf{R}_1 + \mathbf{R}_1\mathbf{R}_+\mathbf{T}_1)\mathbf{T}_+^T, \tag{B 7}$$

and the mean transmittivity is, from (3.46),

$$\mathbb{E}\left[\sum_{j,l=1}^{N}|\widetilde{\mathcal{T}}_{jl}|^2\right] \approx \mathrm{Tr}\{\mathbb{E}[\overline{(\mathbf{I}-\mathbf{R}_+^\star\mathbf{R}_+)}(\mathbf{T}_1 + \mathbf{T}_1\mathbf{R}_+\mathbf{R}_1 + \mathbf{R}_1\mathbf{R}_+\mathbf{T}_1)^\star \times (\mathbf{I}-\mathbf{R}_+^\star\mathbf{R}_+)(\mathbf{T}_1 + \mathbf{T}_1\mathbf{R}_+\mathbf{R}_1 + \mathbf{R}_1\mathbf{R}_+\mathbf{T}_1)]\}. \tag{B 8}$$

Here, we used the energy conservation relation (3.40) and the commutation property of the trace

$$\mathrm{Tr}[\overline{\mathbf{T}_+}\mathbf{A}\mathbf{T}_+^T] = \mathrm{Tr}[\mathbf{T}_+^T\overline{\mathbf{T}_+}\mathbf{A}] \overset{(3.40)}{=} \mathrm{Tr}[(\mathbf{I}-\mathbf{R}_+^T\overline{\mathbf{R}_+})\mathbf{A}], \quad \forall\, \mathbf{A} \in \mathbb{C}^{N\times N}.$$

The approximation (B 8) is consistent with (B 6) because, if $n \neq n'$, $n, n' \geq 0$, then

$$\mathbb{E}\left[\prod_{k=1}^{n} R_{+,j_k l_k}\prod_{k'=1}^{n'}\overline{R_{+,j'_{k'}l'_{k'}}}\right] = 0,$$

for any $j_k, l_k, j'_{k'}, l'_{k'} \in \{1, \ldots, N\}$, as shown by the analysis of the statistical moments of the transmission and reflection matrices of the random medium given in [26]. This is why we could neglect the quadratic terms in (B 6). Only the terms that do not involve $\mathbf{R}_+$ or that involve two reflection matrices, with one of them being complex-conjugated, contribute to the approximation of (B 8). Thus, the mean transmissivity is approximated by

$$\mathbb{T} = \mathrm{Tr}\{\mathbb{E}[\overline{(\mathbf{I}-\mathbf{R}_+^\star\mathbf{R}_+)}\mathbf{T}_1^\star\mathbf{T}_1 - \mathbf{T}_1^\star\mathbf{R}_+^\star\mathbf{R}_+\mathbf{T}_1 + \mathbf{R}_1^\star\mathbf{R}_+^\star\mathbf{T}_1^\star\mathbf{T}_1\mathbf{R}_+\mathbf{R}_1 + \mathbf{R}_1^\star\mathbf{R}_+^\star\mathbf{T}_1^\star\mathbf{R}_1\mathbf{R}_+\mathbf{T}_1 + \mathbf{T}_1^\star\mathbf{R}_+^\star\mathbf{R}_1^\star\mathbf{T}_1\mathbf{R}_+\mathbf{R}_1 + \mathbf{T}_1^\star\mathbf{R}_+^\star\mathbf{R}_1^\star\mathbf{R}_1\mathbf{R}_+\mathbf{T}_1]\}. \tag{B 9}$$

The statement of the theorem follows from this equation once we write explicitly the trace and use the expressions (3.25)–(3.26) of the entries of $\mathbf{T}_1$ and $\mathbf{R}_1$.

(c) Proof of proposition 3.5

The propagator matrix of the waveguide system with two independent random sections is

$$\mathcal{P} = \mathbf{P}_+(L)\mathbf{P}_1\check{\mathbf{P}}_+(L), \tag{B 10}$$

where $\check{\mathbf{P}}_+$ is an independent and identically distributed copy of $\mathbf{P}_+$. Given the algebraic structure of the propagator $\mathbf{P}_1$ of the barrier given in (3.14), and of the random medium propagator $\mathbf{P}_+$ given in equations (3.30) and (3.37), we conclude from (B 10) that $\mathcal{P}$ is of the form (B 3). We are interested in its first block $\mathcal{P}^{(a)}$ which determines the transmission matrix $\widetilde{\mathcal{T}}$, as in equation (B 4).

Using equation (B 1) and multiplying through in equation (B 10), we get that

$$\mathcal{P}^{(a)} = \mathbf{T}_+(\mathbf{I}-\overline{\mathbf{R}_+}\mathbf{R}_+)^{-1}[(\mathbf{P}_1^{(a)} - \overline{\mathbf{R}_+}\mathbf{P}_1^{(b)})\check{\mathbf{T}}_+(\mathbf{I}-\overline{\check{\mathbf{R}}_+}\check{\mathbf{R}}_+)^{-1} - \overline{(\mathbf{P}_1^{(b)} - \mathbf{R}_+\mathbf{P}_1^{(a)})\check{\mathbf{T}}_+}(\mathbf{I}-\check{\mathbf{R}}_+\overline{\check{\mathbf{R}}_+})^{-1}\check{\mathbf{R}}_+],$$

where the first factor is approximated in (B 5). This gives

$$\widetilde{\mathcal{T}} = (\overline{\mathcal{P}^{(a)}})^{-1} \approx [(\overline{\mathbf{P}_1^{(a)}} - \mathbf{R}_+\overline{\mathbf{P}_1^{(b)}})\overline{\check{\mathbf{T}}_+}(\mathbf{I}-\check{\mathbf{R}}_+\overline{\check{\mathbf{R}}_+})^{-1} - (\mathbf{P}_1^{(b)} - \mathbf{R}_+\mathbf{P}_1^{(a)})\check{\mathbf{T}}_+(\mathbf{I}-\overline{\check{\mathbf{R}}_+}\check{\mathbf{R}}_+)^{-1}\overline{\check{\mathbf{R}}_+}]^{-1}\mathbf{T}_+^T, \tag{B 11}$$

where the square bracket can be approximated with Neumann series for small reflection matrices. Such series are also used to expand $\mathrm{Tr}(\widetilde{\mathcal{T}}^\star\widetilde{\mathcal{T}})$ up to second order in terms of the reflection matrices of the random medium. The result stated in the proposition follows after we take the expectation and use that $(\mathbf{T}_+, \mathbf{R}_+)$ and $(\check{\mathbf{T}}_+, \check{\mathbf{R}}_+)$ are independent.

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