

# ENHANCED WAVE TRANSMISSION IN RANDOM MEDIA WITH MIRROR SYMMETRY

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**Abstract.** We present an analysis of enhanced wave transmission through random media with mirror symmetry about a reflecting barrier. The mathematical model is the acoustic wave equation and we consider two setups, where the wave propagation is along a preferred direction: in a randomly layered medium and in a randomly perturbed waveguide. We use the asymptotic stochastic theory of wave propagation in random media to characterize the statistical moments of the frequency-dependent random transmission and reflection coefficients, which are scalar-valued in layered media and matrix-valued in waveguides. With these moments, we can quantify explicitly the enhancement of the net mean transmitted intensity, induced by wave interference near the barrier.

**Key words.** Wave scattering, random media, enhanced transmission.

**AMS subject classifications.** 78A48, 35Q60, 35R60, 60H15

**1. Introduction.** Multiple scattering of waves traveling through disordered media is a serious impediment for applications like imaging and free space communications. This has motivated the pursuit of strategies for wave transmission enhancement and mitigation of scattering effects.

At propagation distances (depths) that do not exceed a few scattering mean free paths, the wave field retains some coherence. Mitigation strategies seek to enhance this coherence by: filtering the incoherent wave components, like in optical coherence tomography [17] and in imaging in waveguides with rough boundary [4]; correcting wavefront distortion in adaptive optics [16]; or using coherent interferometry [5, 3].

Beyond a few scattering mean free paths, the wave field is incoherent and it is typically described by the radiative transfer theory [7, 21, 2] or the diffusion theory [22]. These theories neglect wave interference effects that cause phenomena like coherent backscattering enhancement a.k.a. weak localization [25, 13] and Anderson localization [18]. Such interference effects can be exploited for enhancing transmission through a strongly scattering medium. In [10] it was shown, using random matrix theory, that in a disordered three-dimensional metallic body, some of the eigenvalues of the transmission matrix are close to one. The eigenvectors for such eigenvalues are known as open channels and their existence has been demonstrated in optics experiments in [23]. If the transmission matrix can be measured, the open channels can be determined and used for improved focusing and delivering waves deep inside disordered media [23, 20, 6].

Recent developments show that interesting wave interference phenomena can also be induced by mirror symmetry in chaotic cavities and in waveguides filled with disordered media. Large conductance enhancement through a reflecting barrier has been demonstrated in [24] for symmetric quantum dots and in [14, 15] for symmetric chaotic cavities. Experimental demonstration of broadband wave transmission enhancement through diffusive, symmetric slabs with a barrier in the middle is given in [8, 9]. Symmetric media are also encountered when studying waves propagating in a random

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half-space with Dirichlet boundary condition. The method of images replaces this half-space problem by a full-space problem with symmetric sources and media [19].

Our goal in this paper is to study mathematically wave transmission enhancement in disordered systems with mirror symmetry. The analysis can be carried out for any type of linear waves, but for simplicity we consider acoustic waves. We are interested in two setups, where the wave propagation is along a preferred direction: a randomly layered medium and a waveguide filled with a disordered medium. In both cases, the disordered medium is modeled by random fluctuations of the coefficients of the wave equation. These fluctuations are mirror symmetric with respect to a reflecting barrier. The interaction of the waves with the barrier and the random medium is described by frequency-dependent reflection and transmission coefficients, which are scalar-valued in the layered case and matrix-valued in waveguides. We use the stochastic asymptotic theory of wave propagation [11, 12] to write the statistical moments of these coefficients and thus quantify explicitly the net mean transmission intensity for various opacities of the barrier. In both settings we find that the mirror symmetry has a beneficial effect on the transmission. This effect is more striking when the obstruction at the barrier is strong.

We organize the analysis and results in two main sections: We begin in section 2 with the case of waves propagating at normal incidence through a randomly layered medium. This lets us introduce the main ideas in a simpler, one-dimensional setting, so we can analyze the enhanced transmission in great detail. Then, we study in section 3 transmission through random waveguides. The statistical moments of the reflection and transmission matrices in random waveguides are known, but their computation is much more complicated than in the one-dimensional case [12]. Thus, for waveguides, we consider a regime of weak scattering in the random medium, so we can get an explicit approximation of the net transmitted intensity. The involved calculations needed to derive the results in sections 2 and 3 are given in appendixes. We end with a summary in section 4.

**2. Enhanced transmission in randomly layered media.** We give here the analysis of wave transmission in one-dimensional (layered) random media with mirror symmetry. We begin in section 2.1 with the setup and the wave decomposition in forward and backward going modes. The analysis of the reflection and transmission of these modes at the reflecting barrier is in section 2.2 and in the random medium is in section 2.3. We gather the results in section 2.4 to quantify the transmission enhancement.

**2.1. Setup.** One-dimensional wave propagation along the  $z$ -axis is described by the first order system

$$\left[ \begin{pmatrix} \rho(z) & 0 \\ 0 & K^{-1}(z) \end{pmatrix} \partial_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_z \right] \begin{pmatrix} u(t, z) \\ p(t, z) \end{pmatrix} = \mathbf{0}, \quad t \in \mathbb{R}, \quad z \in \mathbb{R}, \quad (2.1)$$

where  $p$  is the acoustic pressure and  $u$  is the velocity. The medium is modeled by the variable density  $\rho$  and bulk modulus  $K$ , which determine the local wave speed  $c$  and impedance  $\zeta$ ,

$$c(z) = \sqrt{K(z)/\rho(z)}, \quad \zeta(z) = \sqrt{K(z)\rho(z)}. \quad (2.2)$$

The medium contains a thin barrier at  $z \in (-d/2, d/2)$ , sandwiched between two randomly perturbed, symmetric regions at  $d/2 \leq |z| \leq L$ . Assuming that the  $z$ -axis

is horizontal, we call the region  $z < -d/2$  the left side of the barrier and the region  $z > d/2$  the right side of the barrier. The medium is modeled by

$$\rho(z) = \begin{cases} \rho_0 & \text{if } |z| \geq d/2, \\ \rho_1 & \text{if } |z| < d/2, \end{cases} \quad \text{and} \quad \frac{1}{K(z)} = \begin{cases} \frac{1}{K_0} & \text{if } |z| > L, \\ \frac{1}{K_0} [1 + \mu(|z|)] & \text{if } |z| \in [d/2, L], \\ \frac{1}{K_1} & \text{if } |z| < d/2, \end{cases} \quad (2.3)$$

where  $\rho_j$  and  $K_j$  are positive constants, for  $j = 0, 1$  and  $\mu$  is a mean zero, mixing random process, satisfying the uniform bound  $|\mu| < 1$ , so that the bulk modulus is a positive function [11, Chapter 6]. Note that only the bulk modulus has random fluctuations in our model. This simplifies the presentation and unifies it with that in the next section, because (2.1) reduces to the standard second-order wave equation for the pressure at  $|z| > d/2$  and at  $|z| < d/2$ . Random fluctuations of the density can be included, and the results are qualitatively the same [11, Chapter 17].

The interaction of the waves with the medium depends on frequency, so we Fourier transform with respect to time,

$$\widehat{p}(\omega, z) = \int_{\mathbb{R}} dt e^{i\omega t} p(t, z), \quad \widehat{u}(\omega, z) = \int_{\mathbb{R}} dt e^{i\omega t} u(t, z), \quad (2.4)$$

and then decompose the wave field into right (forward) going and left (backward) going modes [11, Chapter 7]. The decomposition at  $|z| \geq d/2$  is

$$\widehat{a}(\omega, z) = \left[ \zeta_0^{-1/2} \widehat{p}(\omega, z) + \zeta_0^{1/2} \widehat{u}(\omega, z) \right] e^{-i\omega \frac{z}{c_0}}, \quad (2.5)$$

$$\widehat{b}(\omega, z) = \left[ -\zeta_0^{-1/2} \widehat{p}(\omega, z) + \zeta_0^{1/2} \widehat{u}(\omega, z) \right] e^{i\omega \frac{z}{c_0}}, \quad (2.6)$$

where  $c_0 = \sqrt{K_0/\rho_0}$  and  $\zeta_0 = \sqrt{K_0\rho_0}$ . The decomposition at  $|z| < d/2$  is similar, except that  $c_0$  and  $\zeta_0$  are replaced by  $c_1 = \sqrt{K_1/\rho_1}$  and  $\zeta_1 = \sqrt{K_1\rho_1}$ . Note that equations (2.4-2.6) give

$$p(t, z) = \frac{\zeta_0^{1/2}}{4\pi} \int_{\mathbb{R}} d\omega e^{-i\omega t} \left[ \widehat{a}(\omega, z) e^{i\omega \frac{z}{c_0}} - \widehat{b}(\omega, z) e^{-i\omega \frac{z}{c_0}} \right], \quad (2.7)$$

$$u(t, z) = \frac{\zeta_0^{-1/2}}{4\pi} \int_{\mathbb{R}} d\omega e^{-i\omega t} \left[ \widehat{a}(\omega, z) e^{i\omega \frac{z}{c_0}} + \widehat{b}(\omega, z) e^{-i\omega \frac{z}{c_0}} \right]. \quad (2.8)$$

This is a decomposition in monochromatic waves propagating along the  $z$ -axis in the right direction, with amplitude  $\widehat{a}$ , and the left direction, with amplitude  $\widehat{b}$ .

The wave excitation specifies  $\widehat{a}(\omega, -L)$ , and corresponds to a wave impinging on the heterogeneous medium at  $z = -L$ . The goal is to quantify the wave emerging at  $z = L$ , with amplitude  $\widehat{a}(\omega, L)$  (see Fig. 2.1). Since the medium is homogeneous at  $z > L$ , the wave is outgoing there i.e.,  $\widehat{b}(\omega, z) = \widehat{b}(\omega, L) = 0$  for  $z \geq L$ .

**2.2. Model of the barrier.** The mapping of the wave modes on the left of the barrier, at  $z = -d/2$ , to the modes on the right of the barrier, at  $z = d/2$ , is given by the  $2 \times 2$  frequency-dependent propagator matrix  $\mathbf{P}_1$ . The expression of this matrix is derived in Appendix A.1, by imposing the continuity of the pressure and velocity at  $z = \pm d/2$ . We state the result in the next lemma.

LEMMA 2.1. *We have*

$$\begin{pmatrix} \widehat{a}(\omega, d/2) \\ \widehat{b}(\omega, d/2) \end{pmatrix} = \mathbf{P}_1(\omega) \begin{pmatrix} \widehat{a}(\omega, -d/2) \\ \widehat{b}(\omega, -d/2) \end{pmatrix}, \quad \mathbf{P}_1(\omega) = \begin{pmatrix} \alpha(\omega) & \overline{\gamma(\omega)} \\ \gamma(\omega) & \alpha(\omega) \end{pmatrix}, \quad (2.9)$$

where the bar denotes complex conjugate and

$$\alpha(\omega) = \left[ \cos\left(\frac{\omega d}{c_1}\right) + \frac{i}{2} \left( \frac{\zeta_1}{\zeta_0} + \frac{\zeta_0}{\zeta_1} \right) \sin\left(\frac{\omega d}{c_1}\right) \right] e^{-i\omega d/c_0}, \quad (2.10)$$

$$\gamma(\omega) = \frac{i}{2} \left( \frac{\zeta_0}{\zeta_1} - \frac{\zeta_1}{\zeta_0} \right) \sin\left(\frac{\omega d}{c_1}\right). \quad (2.11)$$

The scattering matrix  $\mathbf{S}_1$  maps the wave mode amplitudes that impinge on the barrier to the outgoing wave mode amplitudes

$$\begin{pmatrix} \hat{a}(\omega, d/2) \\ \hat{b}(\omega, -d/2) \end{pmatrix} = \mathbf{S}_1(\omega) \begin{pmatrix} \hat{a}(\omega, -d/2) \\ \hat{b}(\omega, d/2) \end{pmatrix}. \quad (2.12)$$

Its expression follows from equation (2.9),

$$\mathbf{S}_1(\omega) = \begin{pmatrix} T_1(\omega) & R_1(\omega) \\ R_1(\omega) & T_1(\omega) \end{pmatrix}, \quad R_1(\omega) = -\frac{\gamma(\omega)}{\alpha(\omega)}, \quad T_1(\omega) = \frac{1}{\alpha(\omega)}, \quad (2.13)$$

where  $T_1$  and  $R_1$  are the transmission and reflection coefficients of the barrier.

We are interested in a thin barrier, with  $d$  much smaller than the wavelength, so there are no trapped propagating modes at  $z \in (-d/2, d/2)$ . There are two distinguished asymptotic regimes that give an order one net effect of the barrier:

**1:** The first regime is

$$\frac{\omega d}{c_j} \rightarrow 0 \text{ for } j = 0, 1 \text{ and } \frac{\zeta_0}{\zeta_1} \rightarrow \infty \text{ such that } \frac{\zeta_0}{\zeta_1} \frac{\omega d}{2c_1} \rightarrow q(\omega), \quad (2.14)$$

with finite  $q$ . The asymptotic limit of the transmission and reflection coefficients is

$$T_1(\omega) = \frac{i}{i + q(\omega)} \quad \text{and} \quad R_1(\omega) = \frac{q(\omega)}{i + q(\omega)}. \quad (2.15)$$

**2:** The second regime is

$$\frac{\omega d}{c_j} \rightarrow 0 \text{ for } j = 0, 1 \text{ and } \frac{\zeta_1}{\zeta_0} \rightarrow \infty \text{ such that } \frac{\zeta_1}{\zeta_0} \frac{\omega d}{2c_1} \rightarrow q(\omega), \quad (2.16)$$

and the transmission and reflection coefficients are

$$T_1(\omega) = \frac{i}{i + q(\omega)} \quad \text{and} \quad R_1(\omega) = -\frac{q(\omega)}{i + q(\omega)}. \quad (2.17)$$

The two cases are similar, so we consider henceforth the asymptotic regime (2.14).

**2.3. Reflection and transmission in the random medium.** The propagation of waves in randomly layered media is studied in detail in [11]. We gather the relevant results from there and characterize the transmission through the random medium with mirror symmetry in the next lemma, proved in Appendix A.2.

LEMMA 2.2. *We have*

$$\begin{pmatrix} \hat{a}(\omega, -d/2) \\ \hat{b}(\omega, -L) \end{pmatrix} = \mathbf{S}_-(\omega) \begin{pmatrix} \hat{a}(\omega, -L) \\ \hat{b}(\omega, -d/2) \end{pmatrix}, \quad \begin{pmatrix} \hat{a}(\omega, L) \\ \hat{b}(\omega, d/2) \end{pmatrix} = \mathbf{S}_+(\omega) \begin{pmatrix} \hat{a}(\omega, d/2) \\ \hat{b}(\omega, L) \end{pmatrix}, \quad (2.18)$$

where  $\mathbf{S}_-$  and  $\mathbf{S}_+$  are the scattering matrices of the first and second random regions  $[-L, -d/2]$  and  $[d/2, L]$ , respectively:

$$\mathbf{S}_-(\omega) = \begin{pmatrix} T_-(\omega) & \tilde{R}_-(\omega) \\ R_-(\omega) & T_-(\omega) \end{pmatrix}, \quad \mathbf{S}_+(\omega) = \begin{pmatrix} T_+(\omega) & \tilde{R}_+(\omega) \\ R_+(\omega) & T_+(\omega) \end{pmatrix}. \quad (2.19)$$

Due to the symmetry of the random medium, the transmission and reflection coefficients in these matrices satisfy

$$T_-(\omega) = T_+(\omega), \quad R_-(\omega) = \tilde{R}_+(\omega), \quad \tilde{R}_-(\omega) = R_+(\omega). \quad (2.20)$$

For quantifying the net mean intensity transmitted through the medium we only need the expressions of the statistical moments of the square modulus of the transmission coefficient of one random section. The moments of  $T_+$  are studied in [11, Section 7.1.5] and the moments of  $R_+$  and  $\tilde{R}_+$  are in [11, Section 9.2.1], in the so-called strongly heterogeneous white-noise regime defined by the scaling relations

$$\ell_c \ll \lambda \ll L, \quad \text{Var}(\mu) = O(1), \quad (2.21)$$

where  $\ell_c$  is the correlation length of the random fluctuations  $\mu$  of the medium and  $\lambda = 2\pi c_0/\omega$  is the wavelength. If in addition we have  $\text{Var}(\mu)\ell_c L/\lambda^2 = O(1)$ , then the effect of the random medium on the transmittivity is of order one. The expressions of the moments of  $T_+$  are

$$\mathbb{E}[|T_+(\omega)|^{2n}] = \exp\left(-\frac{L}{4L_{\text{loc}}(\omega)}\right) \int_0^\infty e^{-Ls^2/L_{\text{loc}}(\omega)} \frac{2\pi s \sinh(\pi s)}{\cosh^2(\pi s)} \phi_n(s) ds, \quad (2.22)$$

for any positive integer  $n$ . Here  $\mathbb{E}$  is the expectation with respect to the law of the process  $\mu$ , the functions  $\phi_n$  are defined by

$$\phi_1(s) = 1, \quad \phi_n(s) = \prod_{j=1}^{n-1} \frac{s^2 + (j - \frac{1}{2})^2}{j^2}, \quad n \geq 2, \quad (2.23)$$

and  $L_{\text{loc}}$  is the localization length of the random medium, which depends on the frequency  $\omega$  and the statistics of  $\mu$ :

$$\frac{1}{L_{\text{loc}}(\omega)} = \frac{\omega^2}{4c_0^2} \int_{\mathbb{R}} \mathbb{E}[\mu(0)\mu(z)] dz. \quad (2.24)$$

Note that  $1/L_{\text{loc}}$  is of the order of  $\text{Var}(\mu)\ell_c/\lambda^2$ . If the wave travels deep in the medium i.e., if  $L \gg L_{\text{loc}}$ , then the moment formula (2.22) simplifies to

$$\mathbb{E}[|T_+(\omega)|^{2n}] \simeq \frac{\pi^{5/2}}{2[L/L_{\text{loc}}(\omega)]^{3/2}} \phi_n(0) \exp\left[-\frac{L}{4L_{\text{loc}}(\omega)}\right]. \quad (2.25)$$

It is shown in [11, Section 7.3] that the moment formulas (2.22) also hold in the asymptotic regime where the correlation length of the medium is similar to the wavelength and smaller than the propagation distance, and the medium fluctuations have small relative amplitude,

$$\ell_c \sim \lambda \ll L, \quad \text{Var}(\mu) \ll 1. \quad (2.26)$$

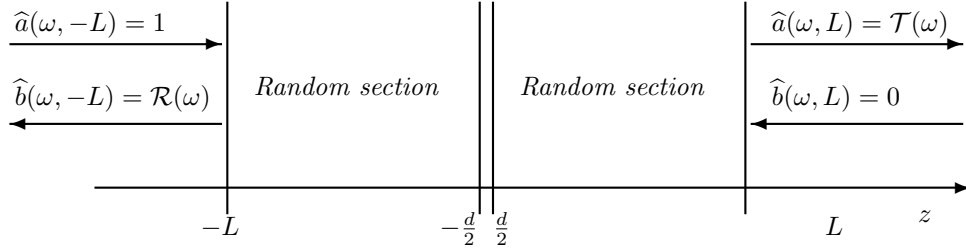


FIG. 2.1. Schematic of transmission and reflection in the random medium with mirror symmetry about the thin barrier located at  $z \in (-d/2, d/2)$ .

In this so-called weakly heterogeneous regime, the effect of the random medium fluctuations on the transmittivity is of order one when  $\text{Var}(\mu)\ell_c L/\lambda^2 = O(1)$ . The moments of  $T_+$  are given by (2.22) with the localization length

$$\frac{1}{L_{\text{loc}}(\omega)} = \frac{\omega^2}{4c_0^2} \int_{\mathbb{R}} \mathbb{E}[\mu(0)\mu(z)] \cos\left(\frac{2\omega z}{c_0}\right) dz.$$

This is slightly different from the expression (2.24) of  $L_{\text{loc}}$  in the strongly heterogeneous white-noise regime (2.21).

**2.4. Transmission enhancement.** We are interested in the transmission of the wave field through the medium, illustrated schematically in Fig. 2.1. The result is stated in the next theorem, proved in Appendix A.3. In light of Lemma 2.2, we simplify the notation in its statement using

$$T(\omega) = T_+(\omega) = T_-(\omega), \quad R(\omega) = R_+(\omega) = \tilde{R}_-(\omega). \quad (2.27)$$

**THEOREM 1.** *The transmission coefficient of the system is*

$$\mathcal{T}(\omega) = T^2(\omega)T_1(\omega)[1 - R(\omega)]^{-1}[1 - (2R_1(\omega) - 1)R(\omega)]^{-1}, \quad (2.28)$$

and the expression of the mean transmitted intensity is

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = \sum_{k=0}^{\infty} \tau_k(\omega) \mathbb{E}\left[|T(\omega)|^4(1 - |T(\omega)|^2)^k\right], \quad (2.29)$$

where the moments of  $T$  are given in equation (2.22) and

$$\tau_k(\omega) = \frac{1}{4} \left| 1 - (2R_1(\omega) - 1)^{k+1} \right|^2. \quad (2.30)$$

Note that the coefficients (2.30) satisfy  $\tau_k \leq 1$ , because according to equation (2.15),

$$|2R_1(\omega) - 1| = \left| \frac{q(\omega) - i}{q(\omega) + i} \right| = 1. \quad (2.31)$$

Using this inequality in equation (2.29), we deduce that\*

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \leq \mathbb{E}\left[|T(\omega)|^4 \sum_{k=0}^{\infty} (1 - |T(\omega)|^2)^k\right] = \mathbb{E}[|T(\omega)|^2]. \quad (2.32)$$

\*We can exchange the expectation with the sum because the series is uniformly convergent.

Thus, no matter how weak or strong the barrier is,  $\mathbb{E}[|\mathcal{T}|^2]$  cannot exceed the mean intensity transmitted over half the distance, through one region of the random medium.

We analyze next the transmitted intensity in various scenarios. There are two extreme cases:

- The first extreme case is not interesting because it assumes no random fluctuations. We have  $T = 1$  and the transmitted intensity is deterministic and equal to the squared modulus of the transmission coefficient of the barrier

$$|\mathcal{T}(\omega)|^2 \stackrel{(2.29)}{=} \tau_0(\omega) \stackrel{(2.30)}{=} |1 - R_1(\omega)|^2 \stackrel{(2.15)}{=} |T_1(\omega)|^2. \quad (2.33)$$

- The second extreme case assumes no barrier i.e.,  $T_1 = 1$  and  $R_1 = 0$ . Then, the coefficients (2.30) satisfy  $\tau_k = 0$  for odd  $k$  and  $\tau_k = 1$  for even  $k$ , and the mean transmitted intensity is, from (2.29),

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = \mathbb{E} \left[ |T(\omega)|^4 \sum_{k=0}^{\infty} (1 - |T(\omega)|^2)^{2k} \right] = \mathbb{E} \left[ \frac{|T(\omega)|^2}{2 - |T(\omega)|^2} \right]. \quad (2.34)$$

If in addition the random sections are strongly scattering, i.e.  $L$  is larger than  $L_{\text{loc}}$  so the approximation (2.25) holds, then we have

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \stackrel{(2.34)}{=} \sum_{k=1}^{\infty} 2^{-k} \mathbb{E}[|T(\omega)|^{2k}] \stackrel{(2.25)}{\simeq} C \mathbb{E}[|T(\omega)|^2], \quad (2.35)$$

where

$$C = \sum_{k=1}^{\infty} 2^{-k} \phi_k(0) \approx 0.59. \quad (2.36)$$

The result (2.34) says that, as expected from the estimate (2.32), the mean transmitted intensity through the symmetric random medium occupying the interval  $[-L, L]$  is less than the intensity transmitted through the single region  $z \in [0, L]$ . However, the symmetry helps, because if the two random regions were independent, the mean transmitted intensity would be (see Appendix A.4)

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = \sum_{k=0}^{\infty} \left\{ \mathbb{E} \left[ |T(\omega)|^2 (1 - |T(\omega)|^2)^k \right] \right\}^2. \quad (2.37)$$

This is smaller than (2.34), as illustrated in Fig. 2.2.

Physically, we can interpret the enhanced transmission due to symmetry as follows: It is known that the distribution of the random transmittivity has a small component close to one, that actually gives the value of the mean transmittivity [11, Section 7.1.6]. The medium configurations that give transmittivity close to one are called open channels in the physics literature [10, 1]. Efficient transmission through two independent media of length  $L$  requires the lucky situation where both media are open channels. For the symmetric case, this requires only one medium of length  $L$  to be an open channel as the symmetric medium is then automatically an open channel.

Now we demonstrate the transmission enhancement in the presence of the barrier:

- First, we can see from equation (2.29) that if the random sections are weakly scattering, i.e.  $L/L_{\text{loc}} \ll 1$ , then  $\mathbb{E}[|R|^2] = 1 - \mathbb{E}[|T|^2] \ll 1$  and we can approximate

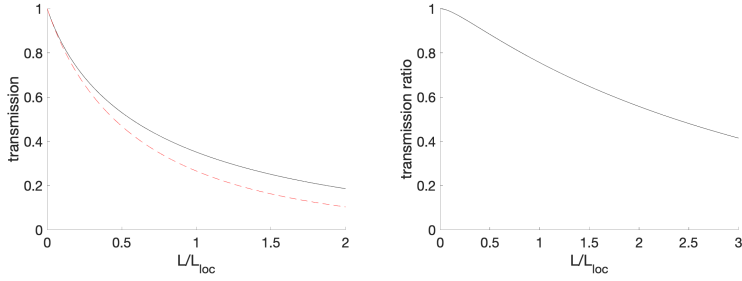


FIG. 2.2. Mean transmitted intensity  $\mathbb{E}[|\mathcal{T}|^2]$  of the system as a function of the strength  $L/L_{\text{loc}}$  of the randomly scattering medium in the absence of the barrier. Left: The black solid line is the result (2.34) for symmetric media and the red dashed line is the result (2.37) for independent media. Right: Ratio of the mean transmission of independent media and the mean transmission of symmetric media.

the mean transmitted intensity by

$$\begin{aligned} \mathbb{E}[|\mathcal{T}(\omega)|^2] &= \tau_0(\omega)\mathbb{E}[|T(\omega)|^4] + \tau_1(\omega)\mathbb{E}[|T(\omega)|^4|R(\omega)|^2] + o(\mathbb{E}[|R(\omega)|^2]) \\ &= \tau_0(\omega) + (\tau_1(\omega) - 2\tau_0(\omega))\mathbb{E}[|R(\omega)|^2] + o(\mathbb{E}[|R(\omega)|^2]). \end{aligned}$$

Equation (2.30) gives

$$\begin{aligned} \tau_0(\omega) &= |1 - R_1(\omega)|^2 \stackrel{(2.15)}{=} |T_1(\omega)|^2 \\ \tau_1(\omega) &= 4|R_1(\omega)|^2 |1 - R_1(\omega)|^2 = 4|T_1(\omega)|^2(1 - |T_1(\omega)|^2), \end{aligned}$$

so to leading order in the reflection coefficient we have

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] \approx |T_1(\omega)|^2 \{1 + 2(1 - 2|T_1(\omega)|^2)\mathbb{E}[|R(\omega)|^2]\}. \quad (2.38)$$

This is larger than the transmission intensity of the barrier  $|T_1(\omega)|^2$ , as long as the barrier is reflecting enough i.e., for  $|T_1(\omega)| < 1/\sqrt{2}$ .

• If the random sections are more scattering, i.e.  $L \gtrsim L_{\text{loc}}$ , then we must consider the series in (2.29). We compare the result in Fig. 2.3 with the mean intensity calculated in the absence of symmetry i.e., for two independent random media to the left and right of the barrier. The expression of the latter is

$$\begin{aligned} \mathbb{E}[|\mathcal{T}(\omega)|^2] &= |T_1(\omega)|^2 \sum_{k,k'=0}^{\infty} C_{k,k'}(\omega)\mathbb{E}[|T(\omega)|^2(1 - |T(\omega)|^2)^k] \\ &\quad \times \mathbb{E}[|T(\omega)|^2(1 - |T(\omega)|^2)^{k'}], \end{aligned} \quad (2.39)$$

with

$$C_{k,k'}(\omega) = \left| \sum_{j=\max(k,k')}^{k+k'} \frac{j!R_1^{2j-k-k'}(\omega)[1 - 2R_1(\omega)]^{k+k'-j}}{(k+k'-j)!(j-k)!(j-k')!} \right|^2.$$

Its derivation is given in Appendix A.4.

Again, the transmission is enhanced by symmetry, and this is even more pronounced if the barrier is more reflecting, as shown in Fig. 2.4. See also the next case.



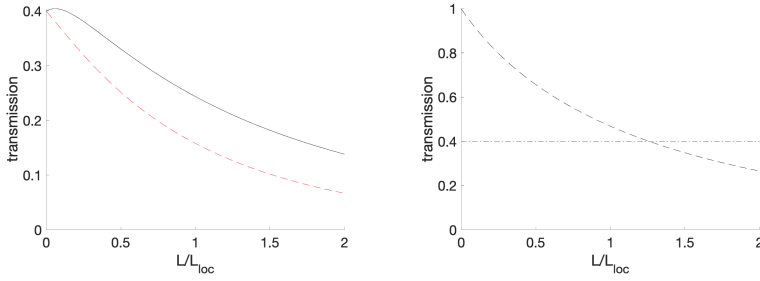


FIG. 2.3. *Left: Mean transmission  $\mathbb{E}[|\mathcal{T}|^2]$  of the system as a function of the strength  $L/L_{\text{loc}}$  of the randomly scattering medium; the black solid line corresponds to the symmetric media and the red dashed line corresponds to the independent media. Right: the mean transmission  $\mathbb{E}[|\mathcal{T}|^2]$  of one random section (dashed) and the transmission  $|T_1|^2$  of the barrier (dot-dashed). Here  $|T_1|^2 = 0.4$ .*

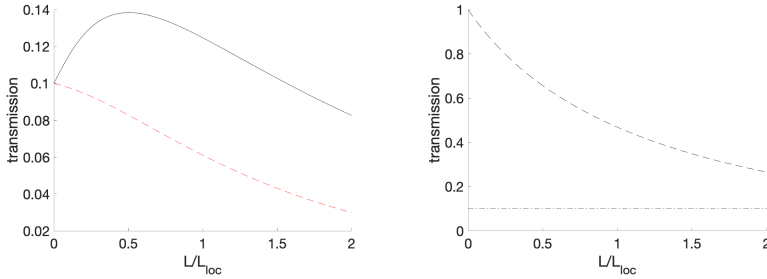


FIG. 2.4. *Same as in Figure 2.3 but but for a more reflecting barrier with  $|T_1|^2 = 0.1$ .*

- If the barrier is strongly reflecting, i.e.  $|T_1| \ll 1$ , which is equivalent to having  $q \gg 1$ , we can use the identity

$$2R_1(\omega) - 1 \stackrel{(2.15)}{=} \frac{q(\omega) - i}{q(\omega) + i} = 1 - 2T_1(\omega),$$

in equation (2.30) to obtain

$$\tau_k(\omega) = (k+1)^2 |T_1(\omega)|^2 + o(|T_1(\omega)|^2), \quad k \geq 0.$$

Substituting this into the expression (2.29) of the mean transmitted intensity we have

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = |T_1(\omega)|^2 \mathbb{E} \left[ |T(\omega)|^4 \sum_{k=0}^{\infty} (k+1)^2 (1 - |T(\omega)|^2)^k \right]. \quad (2.40)$$

This expression can be simplified using the series

$$\sum_{k=0}^{\infty} (1+k)^2 x^k = \frac{1+x}{(1-x)^3}, \quad \forall x \in (0, 1),$$

and we obtain that

$$\mathbb{E}[|\mathcal{T}(\omega)|^2] = |T_1(\omega)|^2 \{ 2\mathbb{E}[|T(\omega)|^{-2}] - 1 \} + o(|T_1(\omega)|^2). \quad (2.41)$$

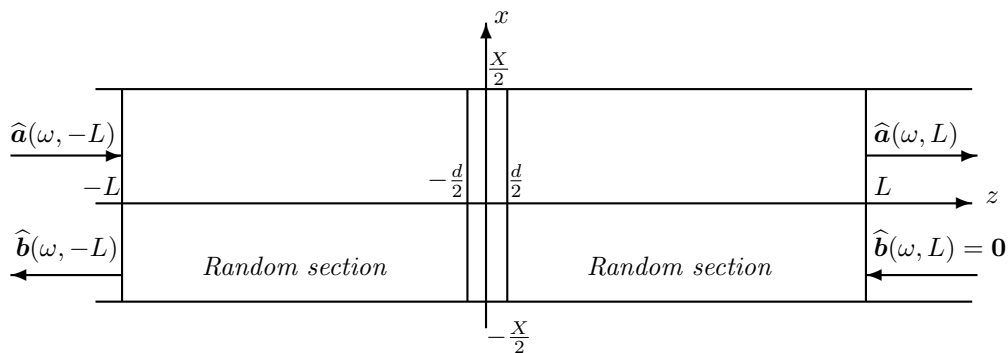


FIG. 3.1. Waveguide occupying the domain  $\Omega = (-X/2, X/2) \times \mathbb{R}$  filled at  $|z| \in (d/2, L)$  with a random medium with mirror symmetry about the thin barrier located at  $|z| < d/2$ .

By solving the Kolmogorov equation  $\partial_L U = L_{\text{loc}}^{-1}(2U - 1)$  satisfied by  $U(L) = \mathbb{E}[|T|^{-2}]$ , derived using the expression of the infinitesimal generator of  $|T|^2$  given in [11, Proposition 7.3], we get that

$$2\mathbb{E}[|T(\omega)|^{-2}] - 1 = \exp[2L/L_{\text{loc}}(\omega)]. \quad (2.42)$$

This result and equation (2.41) show that the transmission enhancement by the random medium can be very large when the barrier is reflecting, as seen in Fig. 2.4.

**3. Enhanced transmission in random waveguides.** In this section we study wave transmission in random waveguides. To simplify the analysis, we consider two-dimensional waveguides with straight, sound soft boundary, as described in section 3.1. The mathematical model is the scalar wave equation for the pressure field. The decomposition of the wave into modes is in section 3.2. The interaction of these modes with the reflecting barrier is in section 3.3. The transmission and reflection of the modes through the random sections is described in section 3.4. The transmission through the whole system is analyzed in section 3.5. We use the results in section 3.6 to quantify the transmission enhancement induced by symmetry, in the case of weak random scattering.

**3.1. Setup.** Consider a waveguide occupying the domain  $\Omega = (-X/2, X/2) \times \mathbb{R}$  and introduce the system of coordinates  $\mathbf{x} = (x, z)$ , with  $x \in (-X/2, X/2)$  and  $z \in \mathbb{R}$ . Assume, as illustrated in Fig. 3.1, that the waveguide contains a thin reflecting barrier at  $|z| < d/2$ , lying between two random sections at  $|z| \in (d/2, L)$ , which are mirror symmetric with respect to  $z = 0$ .

The wave at frequency  $\omega$  is modeled by the Fourier transform  $\hat{p}$  of the pressure, the solution of the Helmholtz equation

$$\left[ \frac{\omega^2}{c^2(x, z)} + \Delta \right] \hat{p}(\omega, x, z) = 0, \quad (x, z) \in \Omega, \quad (3.1)$$

with Dirichlet boundary condition at the sound soft boundary  $x = \pm X/2$ ,

$$\hat{p}(\omega, \pm X/2, z) = 0, \quad z \in \mathbb{R}, \quad (3.2)$$

and outgoing boundary condition at  $z \rightarrow +\infty$ . The medium that fills the waveguide is heterogeneous, with wave speed  $c$  of the form

$$c^{-2}(x, z) = \begin{cases} c_0^{-2} & \text{if } |z| > L, \\ c_1^{-2} & \text{if } |z| < d/2, \\ c_0^{-2} [1 + \mu(x, |z|)] & \text{if } d/2 \leq |z| \leq L. \end{cases} \quad (3.3)$$

Here  $c_0$  and  $c_1$  are constants satisfying  $c_1 < c_0$ , and  $\mu$  is a zero mean, mixing random process, with the uniform bound  $|\mu| < 1$ .

The excitation is defined by a right going wave impinging on the random medium at  $z = -L$  and our goal is to quantify the transmitted wave at  $z = L$ .

**3.2. Mode decomposition outside the barrier.** We are interested in the case of small standard deviation of the fluctuations  $\mu$  of  $c^{-2}$ , so we define the wave decomposition at  $|z| > d/2$  in the reference medium with wave speed  $c_0$ .

The decomposition uses the spectrum of the self-adjoint, negative definite operator  $\partial_x^2$  with Dirichlet boundary conditions at  $x = \pm X/2$ . The eigenvalues are given by  $-\lambda_j$ , where  $\lambda_j = (j\pi/X)^2$  and the eigenfunctions are  $\varphi_j(x) = \sqrt{2/X} \sin(j\pi x/X)$ , for  $j \geq 1$ . These form an orthonormal basis of  $L^2(-X/2, X/2)$ .

Let  $k(\omega) = \omega/c_0$  be the wavenumber and define the natural number

$$N(\omega) = \lfloor k(\omega)X/\pi \rfloor, \quad (3.4)$$

such that

$$\lambda_{N(\omega)} \leq k^2(\omega) < \lambda_{N(\omega)+1}. \quad (3.5)$$

Here  $\lfloor \cdot \rfloor$  denotes the integer part. The wave decomposition is

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{\infty} \varphi_j(x) \widehat{p}_j(\omega, z), \quad (3.6)$$

where  $\widehat{p}_j$  are one-dimensional, time-harmonic waves, called waveguide modes. The first  $N$  of them are propagating waves, with wavenumbers

$$\beta_j(\omega) = \sqrt{k^2(\omega) - \lambda_j}, \quad \text{if } j \leq N(\omega), \quad (3.7)$$

and the remaining ones are evanescent waves. These decay exponentially in  $|z|$  on the length scale  $\beta_j^{-1}$ , where

$$\beta_j(\omega) = \sqrt{\lambda_j - k^2(\omega)}, \quad \text{if } j > N(\omega). \quad (3.8)$$

Note that if  $k^2 = \lambda_N$ , the wave  $\widehat{p}_N$  does not propagate. The analysis of waveguides with such standing modes is more involved than needed in this paper, so we assume that  $\beta_N > 0$ .

The propagating waves can be decomposed further into right (forward) and left (backward) going modes, using the following equations [11, Chapter 20]

$$\widehat{p}_j(\omega, z) = \frac{1}{\sqrt{\beta_j(\omega)}} \left[ \widehat{a}_j(\omega, z) e^{i\beta_j(\omega)z} + \widehat{b}_j(\omega, z) e^{-i\beta_j(\omega)z} \right], \quad (3.9)$$

$$\partial_z \widehat{p}_j(\omega, z) = i\sqrt{\beta_j(\omega)} \left[ \widehat{a}_j(\omega, z) e^{i\beta_j(\omega)z} - \widehat{b}_j(\omega, z) e^{-i\beta_j(\omega)z} \right]. \quad (3.10)$$

The complex valued amplitudes of these modes are gathered in the vector fields

$$\widehat{\mathbf{a}}(\omega, z) = \begin{pmatrix} \widehat{a}_1(\omega, z) \\ \vdots \\ \widehat{a}_{N(\omega)}(\omega, z) \end{pmatrix}, \quad \widehat{\mathbf{b}}(\omega, z) = \begin{pmatrix} \widehat{b}_1(\omega, z) \\ \vdots \\ \widehat{b}_{N(\omega)}(\omega, z) \end{pmatrix}, \quad (3.11)$$

and they satisfy the coupled system of equations

$$\partial_z \begin{pmatrix} \widehat{\mathbf{a}}(\omega, z) \\ \widehat{\mathbf{b}}(\omega, z) \end{pmatrix} = \begin{pmatrix} \mathbf{H}(\omega, z) & \mathbf{K}(\omega, z) \\ \mathbf{K}(\omega, z) & \mathbf{H}(\omega, z) \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{a}}(\omega, z) \\ \widehat{\mathbf{b}}(\omega, z) \end{pmatrix}, \quad (3.12)$$

derived in [11, Chapter 20]. The derivation involves substituting (3.6), (3.9-3.10) into (3.1), using the orthonormality of the eigenfunctions and also expressing the evanescent modes in terms of the propagating ones [11, Section 20.2.3]. The matrices  $\mathbf{H}, \mathbf{K} \in \mathbb{C}^{N \times N}$  are given explicitly in [11, Section 20.2.4]. They depend on the mode wavenumbers (3.7-3.8) and the random process  $\boldsymbol{\nu} = (\nu_{j,l})_{j,l \geq 1}$ , with components

$$\nu_{j,l}(|z|) = \int_{-X/2}^{X/2} dx \varphi_j(x) \varphi_l(x) \mu(x, |z|), \quad j, l \geq 1. \quad (3.13)$$

In the absence of fluctuations, the matrices  $\mathbf{H}$  and  $\mathbf{K}$  would be zero i.e., the mode amplitudes would be decoupled and constant. This is the case at  $|z| > L$ , where the wave speed equals the constant  $c_0$ .

The system of ODEs (3.12) is complemented with the excitation  $\widehat{\mathbf{a}}(\omega, -L)$  that specifies the incoming wave impinging on the random medium and the outgoing boundary condition  $\widehat{\mathbf{b}}(\omega, L) = \mathbf{0}$ . Our goal is to characterize the transmitted mode amplitudes  $\widehat{\mathbf{a}}(\omega, L)$ . This requires the analysis of the transmission and reflection of the modes at the thin barrier, described next.

**3.3. Transmission and reflection at the barrier.** The mode decomposition inside the barrier is similar to that in equations (3.6-3.10), except that the wave speed  $c_0$  is replaced by  $c_1$ . Since we assume that  $c_1 < c_0$ , we deduce from equation (3.4) and its analogue inside the barrier that there are

$$N_1(\omega) > N(\omega) \quad (3.14)$$

propagating modes at  $|z| < d/2$ . The modes are uncoupled, with constant amplitudes, because the wave speed is constant inside the barrier.

The  $x$ -profiles of the modes inside and outside the barrier are given by the same eigenfunctions  $\varphi_j$  for all  $z \in \mathbb{R}$ , so to analyze the wave reflection and transmission at the barrier, it is sufficient to match  $\{\widehat{p}_j, \partial_z \widehat{p}_j\}_{j=1}^N$  at  $z = \pm d/2$ . For  $j \geq N + 1$  the modes impinging on the barrier are evanescent and their amplitude is negligible for large enough  $L$ .

The next lemma describes the propagator of the barrier. Its proof follows from the continuity of the first  $N$  modes, using a calculation that is similar to the proof of Lemma 2.1 in Appendix A.1.

LEMMA 1. *We have*

$$\begin{pmatrix} \widehat{\mathbf{a}}(\omega, d/2) \\ \widehat{\mathbf{b}}(\omega, d/2) \end{pmatrix} = \mathbf{P}_1(\omega) \begin{pmatrix} \widehat{\mathbf{a}}(\omega, -d/2) \\ \widehat{\mathbf{b}}(\omega, -d/2) \end{pmatrix}, \quad \mathbf{P}_1(\omega) = \begin{pmatrix} \mathbf{P}_1^{(a)}(\omega) & \overline{\mathbf{P}_1^{(b)}(\omega)} \\ \mathbf{P}_1^{(b)}(\omega) & \mathbf{P}_1^{(a)}(\omega) \end{pmatrix}, \quad (3.15)$$

where  $\mathbf{P}_1$  is the  $2N \times 2N$  propagator matrix of the barrier, with diagonal blocks

$$\mathbf{P}_1^{(a)}(\omega) = \text{diag}(\alpha_j(\omega))_{j=1}^{N(\omega)} \quad \text{and} \quad \mathbf{P}_1^{(b)}(\omega) = \text{diag}(\gamma_j(\omega))_{j=1}^{N(\omega)}. \quad (3.16)$$

The entries of these blocks are

$$\alpha_j(\omega) = \left[ \cos(\beta_{1,j}(\omega)d) + \frac{i}{2} \left( \frac{\beta_{1,j}(\omega)}{\beta_j(\omega)} + \frac{\beta_j(\omega)}{\beta_{1,j}(\omega)} \right) \sin(\beta_{1,j}(\omega)d) \right], \quad (3.17)$$

$$\gamma_j(\omega) = \frac{i}{2} \left( \frac{\beta_j(\omega)}{\beta_{1,j}(\omega)} - \frac{\beta_{1,j}(\omega)}{\beta_j(\omega)} \right) \sin(\beta_{1,j}(\omega)d), \quad (3.18)$$

and

$$\beta_{1,j}(\omega) = \sqrt{\left(\frac{\omega}{c_1}\right)^2 - \lambda_j}, \quad j = 1, \dots, N(\omega), \quad (3.19)$$

are the mode wavenumbers inside the barrier.

As we have done in section 2.2, we derive from the propagator  $\mathbf{P}_1$  the scattering matrix  $\mathbf{S}_1 \in \mathbb{C}^{2N \times 2N}$  of the barrier. This relates the amplitudes of the modes impinging on the barrier to those leaving the barrier,

$$\begin{pmatrix} \widehat{\mathbf{a}}(\omega, d/2) \\ \widehat{\mathbf{b}}(\omega, -d/2) \end{pmatrix} = \mathbf{S}_1(\omega) \begin{pmatrix} \widehat{\mathbf{a}}(\omega, -d/2) \\ \widehat{\mathbf{b}}(\omega, d/2) \end{pmatrix}, \quad (3.20)$$

and has the block structure

$$\mathbf{S}_1(\omega) = \begin{pmatrix} \mathbf{T}_1(\omega) & \mathbf{R}_1(\omega) \\ \mathbf{R}_1(\omega) & \mathbf{T}_1(\omega) \end{pmatrix} \quad (3.21)$$

with diagonal  $N \times N$  blocks

$$\mathbf{T}_1(\omega) = \text{diag} \left( 1/\overline{\alpha_j(\omega)} \right)_{j=1}^{N(\omega)} \quad \text{and} \quad \mathbf{R}_1(\omega) = \text{diag} \left( -\gamma_j(\omega)/\overline{\alpha_j(\omega)} \right)_{j=1}^{N(\omega)}, \quad (3.22)$$

containing the mode-dependent transmission and reflection coefficients of the barrier.

Similar to the layered case, we are interested in the asymptotic regime

$$k(\omega)d \rightarrow 0, \quad \frac{c_0}{c_1} \rightarrow \infty, \quad \text{such that} \quad \left( \frac{c_0}{c_1} \right)^2 k(\omega)d = O(1). \quad (3.23)$$

In this regime, we deduce from the expressions (3.17-3.18) of the coefficients that define the propagator that

$$\alpha_j(\omega) \approx 1 + iq_j(\omega) \quad \text{and} \quad \gamma_j(\omega) \approx -iq_j(\omega), \quad (3.24)$$

where

$$q_j(\omega) = \frac{\beta_{1,j}^2(\omega)d}{2\beta_j(\omega)} \stackrel{(3.23)}{=} O(1), \quad j = 1, \dots, N. \quad (3.25)$$

The asymptotic approximation of the transmission and reflection coefficients is

$$T_{1,j}(\omega) \stackrel{(3.22)}{=} \frac{1}{\overline{\alpha_j(\omega)}} \stackrel{(3.24)}{\approx} \frac{1}{1 - iq_j(\omega)}, \quad (3.26)$$

$$R_{1,j}(\omega) \stackrel{(3.22)}{=} \frac{-\gamma_j(\omega)}{\overline{\alpha_j(\omega)}} \stackrel{(3.24)}{\approx} \frac{iq_j(\omega)}{1 - iq_j(\omega)}, \quad j = 1, \dots, N(\omega). \quad (3.27)$$

**3.4. Transmission and reflection in the random sections.** We collect here the relevant results from [11, Chapter 20] and [12] on wave propagation in random waveguides. As stated at the beginning of section 3.2, we are interested in small random fluctuations  $\mu$  of  $c^{-2}$ . These have a nontrivial effect at a long distance  $L$  of propagation with respect to the correlation length  $\ell_c$  of the fluctuations and the wavelength  $\lambda$ . Thus, we consider the asymptotic regime

$$\ell_c \sim \lambda \sim X \ll L, \quad \text{Var}(\mu) \ll 1, \quad (3.28)$$

where we deduce from equation (3.4) that the number  $N$  of propagative modes is of order one. The scattering effect of the random medium on the transmittivity is of order one when  $\text{Var}(\mu)\ell_c L/\lambda^2 = O(1)$  and it is smaller than one when  $\text{Var}(\mu)\ell_c L \ll \lambda^2$ . The latter defines what we call the weak scattering regime and is of particular interest in this paper because it allows the explicit quantification of the mean transmittivity of the waveguide (see section 3.6).

The propagator matrix  $\mathbf{P}_-$  for the left random section is the solution of

$$\partial_z \mathbf{P}_-(\omega, z) = \begin{pmatrix} \mathbf{H}(\omega, z) & \mathbf{K}(\omega, z) \\ \overline{\mathbf{K}}(\omega, z) & \overline{\mathbf{H}}(\omega, z) \end{pmatrix} \mathbf{P}_-(\omega, z), \quad z \in (-L, -d/2), \quad (3.29)$$

$$\mathbf{P}_-(\omega, -d/2) = \mathbf{I}_{2N} \quad (3.30)$$

where  $\mathbf{I}_{2N}$  denotes the  $2N \times 2N$  identity matrix. Given the algebraic form of the coupling matrix in the right hand side of (3.29), one can deduce that the propagator has the block form [11, Section 20.2.5]

$$\mathbf{P}_-(\omega, z) = \begin{pmatrix} \mathbf{P}_-^{(a)}(\omega, z) & \overline{\mathbf{P}_-^{(b)}(\omega, z)} \\ \mathbf{P}_-^{(b)}(\omega, z) & \overline{\mathbf{P}_-^{(a)}(\omega, z)} \end{pmatrix}, \quad (3.31)$$

with full blocks  $\mathbf{P}_-^{(a)}, \mathbf{P}_-^{(b)} \in \mathbb{C}^{N \times N}$  that capture mode coupling induced by scattering in the random medium. We are interested in the propagator evaluated at  $z = -L$ , which defines the  $N \times N$  transmission and reflection matrices of the left random section. These matrices are the analogues of the scalar valued transmission and reflection coefficients in layered media, deduced from the propagator as explained in Appendix A.2. We have

$$\begin{pmatrix} \mathbf{I} \\ \mathbf{R}_-(\omega) \end{pmatrix} = \mathbf{P}_-(\omega, -L) \begin{pmatrix} \mathbf{T}_-(\omega) \\ \mathbf{0} \end{pmatrix}, \quad (3.32)$$

which can be understood from the waveguide analogue of Fig. A.3 and

$$\begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{T}}_-(\omega) \end{pmatrix} = \mathbf{P}_-(\omega, -L) \begin{pmatrix} \tilde{\mathbf{R}}_-(\omega) \\ \mathbf{I} \end{pmatrix}, \quad (3.33)$$

which corresponds to the analogue of Fig. A.4. Here  $\mathbf{0}$  and  $\mathbf{I}$  are the  $N \times N$  zero and identity matrices, respectively.

Similarly, the propagator  $\mathbf{P}_+$  for the right random section is the solution of

$$\partial_z \mathbf{P}_+(\omega, z) = \begin{pmatrix} \mathbf{H}(\omega, z) & \mathbf{K}(\omega, z) \\ \overline{\mathbf{K}}(\omega, z) & \overline{\mathbf{H}}(\omega, z) \end{pmatrix} \mathbf{P}_+(\omega, z), \quad z \in (d/2, L), \quad (3.34)$$

$$\mathbf{P}_+(\omega, d/2) = \mathbf{I}_{2N}, \quad (3.35)$$

and its algebraic structure is like in equation (3.31), with  $N \times N$  blocks  $\mathbf{P}_+^{(a)}$  and  $\mathbf{P}_+^{(b)}$ . This propagator defines the  $N \times N$  transmission and reflection matrices of the right random section according to equations

$$\begin{pmatrix} \mathbf{T}_+(\omega) \\ \mathbf{0} \end{pmatrix} = \mathbf{P}_+(\omega, L) \begin{pmatrix} \mathbf{I} \\ \mathbf{R}_+(\omega) \end{pmatrix}, \quad (3.36)$$

and

$$\begin{pmatrix} \tilde{\mathbf{R}}_+(\omega) \\ \mathbf{I} \end{pmatrix} = \mathbf{P}_+(\omega, L) \begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{T}}_+(\omega) \end{pmatrix}. \quad (3.37)$$

These can be understood from the waveguide analogues of Fig. A.1-A.2.

Note the symmetry of the definitions (3.29-3.30) and (3.34-3.35). Both propagators start as the identity  $\mathbf{I}_{2N}$  at  $z = \pm d/2$  and define the transmission and reflection matrices at  $z = \pm L$ . The expression of the coupling matrices  $\mathbf{H}$  and  $\mathbf{K}$  given in [11, Section 20.2.4] and the symmetry of the fluctuations about  $z = 0$ , give that

$$\mathbf{H}(\omega, z) = -\overline{\mathbf{H}(\omega, -z)} \quad \text{and} \quad \mathbf{K}(\omega, z) = -\overline{\mathbf{K}(\omega, -z)}. \quad (3.38)$$

This implies that

$$\mathbf{P}_-(\omega, -L) = \overline{\mathbf{P}_+(\omega, L)}, \quad (3.39)$$

and solving equations (3.32-3.33) and (3.36-3.37), we get: The transmission matrices satisfy

$$\begin{aligned} \mathbf{T}_+(\omega) &= \tilde{\mathbf{T}}_-(\omega) = \mathbf{P}_+^{(a)}(\omega, L) - \overline{\mathbf{P}_+^{(b)}(\omega, L)} \left[ \overline{\mathbf{P}_+^{(a)}(\omega, L)} \right]^{-1} \mathbf{P}_+^{(b)}(\omega, L), \\ \tilde{\mathbf{T}}_+(\omega) &= \mathbf{T}_-(\omega) = \left[ \overline{\mathbf{P}_+^{(a)}(\omega, L)} \right]^{-1}, \end{aligned} \quad (3.40)$$

and the reflection matrices satisfy

$$\begin{aligned} \mathbf{R}_+(\omega) &= \tilde{\mathbf{R}}_-(\omega) = - \left[ \overline{\mathbf{P}_+^{(a)}(\omega, L)} \right]^{-1} \mathbf{P}_+^{(b)}(\omega, L), \\ \tilde{\mathbf{R}}_+(\omega) &= \mathbf{R}_-(\omega) = \overline{\mathbf{P}_+^{(b)}(\omega, L)} \left[ \overline{\mathbf{P}_+^{(a)}(\omega, L)} \right]^{-1}. \end{aligned} \quad (3.41)$$

In addition, we have the energy conservation relation [11, Eq. (20.41)]

$$\mathbf{R}_+^*(\omega) \mathbf{R}_+(\omega) + \mathbf{T}_+^*(\omega) \mathbf{T}_+(\omega) = \mathbf{I}, \quad (3.42)$$

and the reciprocity relations [12, Page 1582]

$$\mathbf{R}_+^T(\omega) \approx \mathbf{R}_+(\omega) \quad \text{and} \quad \tilde{\mathbf{R}}_+^T(\omega) \approx \tilde{\mathbf{R}}_+(\omega). \quad (3.43)$$

Here the index  $T$  stands for transpose, the star  $\star$  denotes the complex conjugate and transpose and the approximation in (3.43) means that reciprocity holds in the asymptotic regime (3.28).

**3.5. Transmission through the system.** The propagator matrix  $\mathcal{P}$  for the waveguide is defined by the equation

$$\begin{pmatrix} \widehat{\mathbf{a}}(\omega, L) \\ \widehat{\mathbf{b}}(\omega, L) \end{pmatrix} = \mathcal{P}(\omega) \begin{pmatrix} \widehat{\mathbf{a}}(\omega, -L) \\ \widehat{\mathbf{b}}(\omega, -L) \end{pmatrix}. \quad (3.44)$$

From the definitions (3.15), (3.29) and (3.34) of the propagators of the barrier and the random sections, and the identity (3.39), we deduce that

$$\mathcal{P}(\omega) = \mathbf{P}_+(\omega, L) \mathbf{P}_1(\omega) \left[ \overline{\mathbf{P}_+(\omega, L)} \right]^{-1}. \quad (3.45)$$

Recalling that  $\widehat{\mathbf{b}}(\omega, L) = \mathbf{0}$  and that the excitation specifies the incoming mode amplitudes stored in  $\widehat{\mathbf{a}}(\omega, -L)$ , we can define the transmission and reflection matrices  $\mathcal{T}, \mathcal{R} \in \mathbb{C}^{N \times N}$  of the system by the following equation

$$\begin{pmatrix} \mathbf{I} \\ \mathcal{R}(\omega) \end{pmatrix} = \mathcal{P}(\omega) \begin{pmatrix} \mathcal{T}(\omega) \\ \mathbf{0} \end{pmatrix}. \quad (3.46)$$

The expression of the transmission matrix  $\mathcal{T}$  is given in the next theorem, proved in Appendix B.1.

**THEOREM 2.** *The  $N \times N$  transmission matrix for the waveguide has the expression*

$$\mathcal{T}(\omega) \approx \mathbf{T}_+(\omega) \left[ \mathbf{T}_1^{-1}(\omega) - \mathbf{R}_+(\omega) \mathbf{T}_1^{-1}(\omega) \mathbf{R}_1(\omega) - \mathbf{T}_1^{-1}(\omega) \mathbf{R}_1(\omega) \mathbf{R}_+(\omega) \right. \quad (3.47)$$

$$\left. - \mathbf{R}_+(\omega) \overline{\mathbf{T}_1^{-1}(\omega)} \mathbf{R}_+(\omega) \right]^{-1} \mathbf{T}_+^T(\omega), \quad (3.48)$$

where the approximation holds in the regime (3.28).

The transmissivity of the system is

$$\mathrm{Tr} [\mathcal{T}^*(\omega) \mathcal{T}(\omega)] = \sum_{j,l=1}^{N(\omega)} |\mathcal{T}_{jl}(\omega)|^2, \quad (3.49)$$

where “Tr” denotes the trace. In the next section we quantify the mean of (3.49) in the asymptotic regime (3.28).

**3.6. Enhanced transmission.** To quantify the effect of symmetry on the wave transmission through the waveguide, we derive next the expression of the mean transmissivity. This requires the statistical moments of the products of the entries of the transmission and reflection matrices  $\mathbf{T}_+$  and  $\mathbf{R}_+$ . These moments are characterized in the regime (3.28) in [12, Propositions 3.1, 4.2]. Their expression is very complicated, so we do not repeat it here. However, the result simplifies in the case of weak scattering in the random medium:

**THEOREM 3.** *When the random medium is weakly scattering i.e., in the asymptotic regime (3.28) with  $\mathrm{Var}(\mu) \ell_c L \ll \lambda^2$ , the mean transmissivity is approximated by*

$$\mathbb{E} \left[ \sum_{j,l=1}^{N(\omega)} |\mathcal{T}_{jl}(\omega)|^2 \right] \approx \mathbb{T}(\omega) = \sum_{l=1}^{N(\omega)} \left[ |T_{1,l}(\omega)|^2 + \sum_{m=1}^{N(\omega)} \mathcal{M}_{lm}(\omega) \mathcal{B}_{lm}(\omega) \right], \quad (3.50)$$



where we introduced the moments

$$\mathcal{M}_{lm}(\omega) = \mathbb{E}[|R_{+,lm}(\omega)|^2], \quad (3.51)$$

and the factors

$$\mathcal{B}_{lm}(\omega) = |T_{1,l}(\omega) + T_{1,m}(\omega) - 2T_{1,l}(\omega)T_{1,m}(\omega)|^2 - |T_{1,l}(\omega)|^2 - |T_{1,m}(\omega)|^2, \quad (3.52)$$

that depend only on the barrier.

The proof of this theorem is in Appendix B.2. We conclude from its statement that if there is no random medium, the transmissivity equals that of the barrier, denoted by

$$\mathbb{T}_0(\omega) = \sum_{l=1}^{N(\omega)} |T_{1,l}(\omega)|^2. \quad (3.53)$$

If the random medium is present, its effect on the mean transmissivity depends on the strength of the barrier, which determines the sign of the factors (3.52). The moments  $\mathcal{M}_{lm}$  are positive by definition, so if the factors  $\mathcal{B}_{lm}$  are positive, we have transmission enhancement induced by the symmetry of the random medium.

Let us write more explicitly equation (3.52),

$$\begin{aligned} \mathcal{B}_{lm}(\omega) &= 4|T_{1,l}(\omega)|^2|T_{1,m}(\omega)|^2 - 4|T_{1,m}(\omega)|^2\text{Re}[T_{1,l}(\omega)] - 4|T_{1,l}(\omega)|^2\text{Re}[T_{1,m}(\omega)] \\ &\quad + 2\text{Re}[T_{1,m}(\omega)\overline{T_{1,l}(\omega)}], \end{aligned}$$

and observe from equation (3.26) that  $\text{Re}(T_{1,l}) = |T_{1,l}|^2$ . This gives that

$$\begin{aligned} \mathcal{B}_{lm}(\omega) &= -4|T_{1,l}(\omega)|^2|T_{1,m}(\omega)|^2 + 2\text{Re}[T_{1,m}(\omega)\overline{T_{1,l}(\omega)}] \\ &\stackrel{(3.26)}{=} \frac{-2[1 - q_l(\omega)q_m(\omega)]}{[1 + q_l^2(\omega)][1 + q_m^2(\omega)]}, \quad l, m = 1, \dots, N(\omega). \end{aligned} \quad (3.54)$$

Consequently,  $\mathcal{B}_{lm} < 0$  if the barrier is weak i.e., the parameters  $\{q_l\}_{l=1}^N$  are small, and the random medium has a negative effect on the transmissivity, because

$$\mathbb{T}(\omega) - \mathbb{T}_0(\omega) < 0. \quad (3.55)$$

However, if the barrier is strong enough to make the parameters  $\{q_l\}_{l=1}^N$  larger than 1, the factors (3.54) are positive and we have transmission enhancement

$$\mathbb{T}(\omega) - \mathbb{T}_0(\omega) \approx \sum_{l=1}^{N(\omega)} \sum_{m=1}^{N(\omega)} \mathcal{M}_{lm}(\omega)\mathcal{B}_{lm}(\omega) > 0. \quad (3.56)$$

The enhancement is due to the symmetry of the random medium about the strong barrier. Without the symmetry, the mean transmissivity is reduced, as stated in the next proposition, proved in Appendix B.3.

**PROPOSITION 1.** *When the random medium is weakly scattering i.e., in the asymptotic regime (3.28) with  $\text{Var}(\mu)\ell_c L \ll \lambda^2$ , and the random media in the left and right sections of the waveguide are statistically independent, the mean transmissivity of the system is approximated by*

$$\mathbb{E} \left[ \sum_{j,l=1}^{N(\omega)} |\mathcal{T}_{jl}(\omega)|^2 \right] \approx \mathbb{T}_0(\omega) - 2 \sum_{l=1}^{N(\omega)} \sum_{m=1}^{N(\omega)} \mathcal{M}_{lm}(\omega) |T_{1,l}(\omega)|^2 |T_{1,m}(\omega)|^2, \quad (3.57)$$

and is therefore smaller than the transmissivity  $\mathbb{T}_0(\omega)$  of the barrier.

**4. Summary.** We have introduced a detailed mathematical analysis of wave transmission enhancement in random systems with symmetry about a reflecting barrier. The analysis is motivated by recent experimental results reported in the physics literature, which observe such enhancement in symmetric cavities and in diffusive slabs. We consider acoustic waves for simplicity, although the methodology applies to any linear waves. The main result is the quantification of the mean transmissivity of two random systems with a preferred direction of propagation: plane waves in randomly layered media and waves in random waveguides. The first case is easier to analyze and we consider both weak and strongly scattering random media. The waveguide setting is significantly more complex, so we quantify the transmission enhancement only in the case of weakly scattering random media. The transmission enhancement induced by symmetry is shown in both settings and it is much more pronounced for large opacity of the barrier.

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**Appendix A. Derivation of the results for randomly layered media.** In this appendix we prove the results stated in section 2. Since the frequency  $\omega$  is fixed in the proofs, we simplify the notation throughout the appendix, by dropping the argument  $\omega$  of the propagator and scattering matrices below.

**A.1. Proof of Lemma 2.1.** The statement of the lemma is derived from the continuity of the Fourier coefficients of the pressure and velocity fields. The decomposition of these fields is given in equations (2.7-2.8) outside the barrier and their analogues inside the barrier. The medium inside the barrier is homogenous, so it follows from equation (2.1) that the right and left going mode amplitudes there, denoted by  $\hat{a}_1$  and  $\hat{b}_1$ , satisfy

$$\partial_z \hat{a}_1(z) = \partial_z \hat{b}_1(z) = 0, \quad z \in (-d/2, d/2). \quad (\text{A.1})$$

When imposing the continuity of the wave field at  $z = -d/2$ , we obtain that

$$\begin{pmatrix} \hat{a}_1(-\frac{d}{2})e^{-i\omega\frac{d}{2c_1}} \\ \hat{b}_1(-\frac{d}{2})e^{i\omega\frac{d}{2c_1}} \end{pmatrix} = \begin{pmatrix} r_+ & r_- \\ r_- & r_+ \end{pmatrix} \begin{pmatrix} \hat{a}(-\frac{d}{2})e^{-i\omega\frac{d}{2c_0}} \\ \hat{b}(-\frac{d}{2})e^{i\omega\frac{d}{2c_0}} \end{pmatrix}, \quad (\text{A.2})$$

where

$$r_{\pm} = \frac{1}{2} \left( \sqrt{\frac{\zeta_1}{\zeta_0}} - \sqrt{\frac{\zeta_0}{\zeta_1}} \right). \quad (\text{A.3})$$

The continuity at  $z = d/2$  gives

$$\begin{pmatrix} \hat{a}(\frac{d}{2})e^{i\omega\frac{d}{2c_0}} \\ \hat{b}(\frac{d}{2})e^{-i\omega\frac{d}{2c_0}} \end{pmatrix} = \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} \hat{a}_1(\frac{d}{2})e^{i\omega\frac{d}{2c_1}} \\ \hat{b}_1(\frac{d}{2})e^{-i\omega\frac{d}{2c_1}} \end{pmatrix}, \quad (\text{A.4})$$

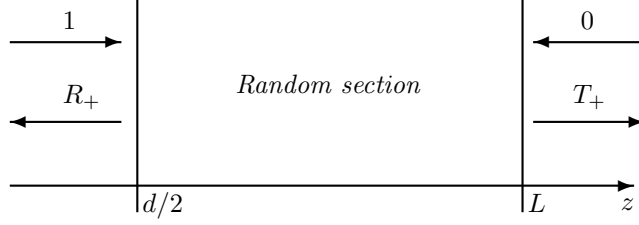


FIG. A.1. Reflection and transmission coefficients  $R_+$  and  $T_+$  for the random section  $(d/2, L)$ .

and from equation (A.1) we have

$$\widehat{a}_1\left(\frac{d}{2}\right) = \widehat{a}_1\left(-\frac{d}{2}\right), \quad \widehat{b}_1\left(\frac{d}{2}\right) = \widehat{b}_1\left(-\frac{d}{2}\right). \quad (\text{A.5})$$

Combining these equations we obtain

$$\begin{pmatrix} \widehat{a}\left(\frac{d}{2}\right) \\ \widehat{b}\left(\frac{d}{2}\right) \end{pmatrix} = \mathbf{P}_1 \begin{pmatrix} \widehat{a}\left(-\frac{d}{2}\right) \\ \widehat{b}\left(-\frac{d}{2}\right) \end{pmatrix}, \quad (\text{A.6})$$

where

$$\begin{aligned} \mathbf{P}_1 = & \begin{pmatrix} e^{-i\omega\frac{d}{2c_0}} & 0 \\ 0 & e^{i\omega\frac{d}{2c_0}} \end{pmatrix} \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} e^{i\omega\frac{d}{c_1}} & 0 \\ 0 & e^{-i\omega\frac{d}{c_1}} \end{pmatrix} \\ & \times \begin{pmatrix} r_+ & r_- \\ r_- & r_+ \end{pmatrix} \begin{pmatrix} e^{-i\omega\frac{d}{2c_0}} & 0 \\ 0 & e^{i\omega\frac{d}{2c_0}} \end{pmatrix}. \end{aligned} \quad (\text{A.7})$$

Multiplying the matrices in (A.7) we get the algebraic form (2.9) of  $\mathbf{P}_1$ , with

$$\alpha = \left[ (r_+^2 - r_-^2) \cos\left(\frac{\omega d}{c_1}\right) + i(r_+^2 + r_-^2) \sin\left(\frac{\omega d}{c_1}\right) \right] e^{-i\omega d/c_0}, \quad (\text{A.8})$$

$$\gamma = -2ir_+r_- \sin\left(\frac{\omega d}{c_1}\right). \quad (\text{A.9})$$

Finally, definition (A.3) gives

$$r_+^2 - r_-^2 = 1 \quad \text{and} \quad r_+r_- = \frac{1}{4} \left( \frac{\zeta_1}{\zeta_0} - \frac{\zeta_0}{\zeta_1} \right), \quad (\text{A.10})$$

and the statement of Lemma 2.1 follows.  $\square$

**A.2. Proof of Lemma 2.2.** Consider first the random section  $[d/2, L]$  and define the propagator  $\mathbf{P}_+$  of the subsection  $[d/2, z]$  by

$$\begin{pmatrix} \widehat{a}(z) \\ \widehat{b}(z) \end{pmatrix} = \mathbf{P}_+(z) \begin{pmatrix} \widehat{a}\left(\frac{d}{2}\right) \\ \widehat{b}\left(\frac{d}{2}\right) \end{pmatrix}, \quad z \in \left(\frac{d}{2}, L\right]. \quad (\text{A.11})$$

It is shown in [11, Chapter 7 and Section 4.4.3] that

$$\mathbf{P}_+(z) = \begin{pmatrix} \alpha_+(z) & \overline{\gamma_+(z)} \\ \gamma_+(z) & \alpha_+(z) \end{pmatrix}, \quad (\text{A.12})$$

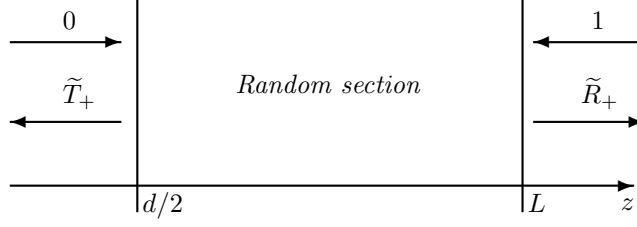


FIG. A.2. Adjoint reflection and transmission coefficients  $\tilde{R}_+$  and  $\tilde{T}_+$  for  $z \in (d/2, L)$ .

where  $\alpha_+$  and  $\gamma_+$  satisfy the first order system

$$\frac{d}{dz} \begin{pmatrix} \alpha_+(z) \\ \gamma_+(z) \end{pmatrix} = \frac{i\omega}{2c_0} \mu(z) \begin{pmatrix} 1 & -e^{-2i\omega z/c_0} \\ e^{2i\omega z/c_0} & -1 \end{pmatrix} \begin{pmatrix} \alpha_+(z) \\ \gamma_+(z) \end{pmatrix}, \quad (\text{A.13})$$

at  $z \in (d/2, L)$ , and the initial conditions

$$\alpha_+\left(\frac{d}{2}\right) = 1, \quad \gamma_+\left(\frac{d}{2}\right) = 0. \quad (\text{A.14})$$

This is illustrated schematically in Fig. A.1 and at  $z = L$  we have

$$\begin{pmatrix} T_+ \\ 0 \end{pmatrix} = \mathbf{P}_+(L) \begin{pmatrix} 1 \\ R_+ \end{pmatrix}, \quad (\text{A.15})$$

where  $T_+$  and  $R_+$  are the random transmission and reflection coefficients, defined by

$$T_+ = \frac{1}{\alpha_+(L)}, \quad R_+ = -\frac{\gamma_+(L)}{\alpha_+(L)}. \quad (\text{A.16})$$

Since the matrix in equation (A.13) has trace zero, we have the conservation relation [11, Section 7.1.1]

$$\det[\mathbf{P}_+(L)] = |\alpha_+(L)|^2 - |\gamma_+(L)|^2 = 1, \quad (\text{A.17})$$

which in light of definitions (A.16) is equivalent to  $|R_+|^2 + |T_+|^2 = 1$ . Because of this relation, the inverse of the propagator is

$$\mathbf{P}_+^{-1}(L) = \begin{pmatrix} \overline{\alpha_+(L)} & -\overline{\gamma_+(L)} \\ -\gamma_+(L) & \alpha_+(L) \end{pmatrix}, \quad (\text{A.18})$$

and from (A.15) we obtain that

$$\begin{pmatrix} 1 \\ R_+ \end{pmatrix} = \begin{pmatrix} \overline{\alpha_+(L)} & -\overline{\gamma_+(L)} \\ -\gamma_+(L) & \alpha_+(L) \end{pmatrix} \begin{pmatrix} T_+ \\ 0 \end{pmatrix}. \quad (\text{A.19})$$

Reordering these equations and defining

$$\tilde{T}_+ = T_+ = \frac{1}{\alpha_+(L)}, \quad \tilde{R}_+ = \frac{\overline{\gamma_+(L)}}{\alpha_+(L)}, \quad (\text{A.20})$$

we obtain the adjoint problem, illustrated schematically in Fig. A.2 ,

$$\begin{pmatrix} \tilde{R}_+ \\ 1 \end{pmatrix} = \mathbf{P}_+(L) \begin{pmatrix} 0 \\ \tilde{T}_+(L) \end{pmatrix}. \quad (\text{A.21})$$

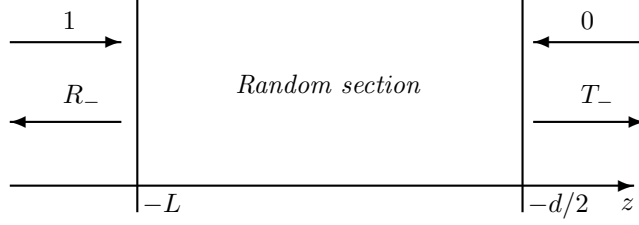


FIG. A.3. Reflection and transmission coefficients  $R_-$  and  $T_-$  for random section  $(-L, -d/2)$ .

Now we can obtain from equation (A.11) evaluated at  $z = L$  and the definitions (A.16) and (A.20) of the transmission and reflection coefficients that

$$\begin{pmatrix} \widehat{a}(L) \\ \widehat{b}(L) \end{pmatrix} = \underbrace{\begin{pmatrix} T_+ & \widetilde{R}_+ \\ R_+ & T_+ \end{pmatrix}}_{\mathbf{S}_+} \begin{pmatrix} \widehat{a}(d/2) \\ \widehat{b}(L) \end{pmatrix}, \quad (\text{A.22})$$

where  $\mathbf{S}_+$  is the scattering matrix of the random section  $[d/2, L]$ .

Similarly, the propagator matrix for the left random section satisfies

$$\begin{pmatrix} \widehat{a}(z) \\ \widehat{b}(z) \end{pmatrix} = \mathbf{P}_-(z) \begin{pmatrix} \widehat{a}(-\frac{d}{2}) \\ \widehat{b}(-\frac{d}{2}) \end{pmatrix}, \quad z \in \left[-L, -\frac{d}{2}\right), \quad (\text{A.23})$$

where

$$\mathbf{P}_-(z) = \begin{pmatrix} \alpha_-(z) & \overline{\gamma_-(z)} \\ \gamma_-(z) & \alpha_-(z) \end{pmatrix}, \quad (\text{A.24})$$

and  $\alpha_-$  and  $\beta_-$  satisfy

$$\frac{d}{dz} \begin{pmatrix} \alpha_-(z) \\ \gamma_-(z) \end{pmatrix} = \frac{i\omega}{2c_0} \mu(-z) \begin{pmatrix} 1 & -e^{-2i\omega z/c_0} \\ e^{2i\omega z/c_0} & -1 \end{pmatrix} \begin{pmatrix} \alpha_-(z) \\ \gamma_-(z) \end{pmatrix}, \quad (\text{A.25})$$

at  $z \in (-L, -d/2)$ , and the initial conditions

$$\alpha_-\left(-\frac{d}{2}\right) = 1, \quad \gamma_-\left(-\frac{d}{2}\right) = 0. \quad (\text{A.26})$$

Note that due to the symmetry of the random medium,  $(\overline{\alpha_-(-z)}, \overline{\gamma_-(-z)})$  satisfies the same equation and initial condition as  $(\alpha_+(z), \gamma_+(z))$ . Therefore,

$$\alpha_-(-L) = \overline{\alpha_+(L)}, \quad \gamma_-(-L) = \overline{\gamma_+(L)}. \quad (\text{A.27})$$

The reflection and transmission through the left random section is illustrated schematically in Figs. A.3-A.4 and the transmission and reflection coefficients are defined by

$$\begin{pmatrix} 1 \\ R_- \end{pmatrix} = \mathbf{P}_-(-L) \begin{pmatrix} T_- \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \widetilde{T}_- \end{pmatrix} = \mathbf{P}_-(-L) \begin{pmatrix} \widetilde{R}_- \\ 1 \end{pmatrix}.$$

These equations and the relation (A.27) give

$$T_- = \widetilde{T}_- = \frac{1}{\alpha_-(-L)} = \frac{1}{\alpha_+(L)} \stackrel{(\text{A.16})}{=} T_+,$$

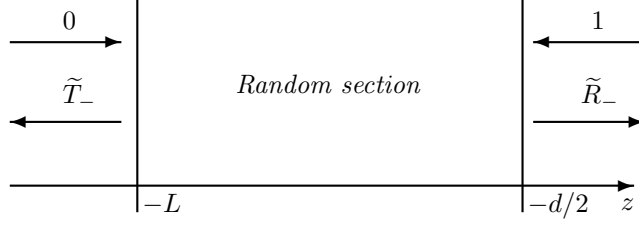


FIG. A.4. Adjoint reflection and transmission coefficients  $\tilde{R}_-$  and  $\tilde{T}_-$  for  $z \in (-L, -d/2)$ .

and

$$R_- = \frac{\gamma_-(-L)}{\alpha_-(-L)} = \frac{\overline{\gamma_+(L)}}{\alpha_+(L)} \stackrel{\text{(A.20)}}{=} \tilde{R}_+,$$

and

$$\tilde{R}_- = -\frac{\overline{\gamma_-(-L)}}{\alpha_-(-L)} = -\frac{\gamma_+(L)}{\alpha_+(L)} \stackrel{\text{(A.16)}}{=} R_+,$$

as stated in the lemma.  $\square$

**A.3. Proof of Theorem 1.** Using the propagator matrices of the two random regions and the barrier, described in Appendices A.1-A.2, we have

$$\begin{pmatrix} \hat{a}(L) \\ \hat{b}(L) \end{pmatrix} = \mathbf{P}_+(L) \mathbf{P}_1 \mathbf{P}_-(-L) \begin{pmatrix} \hat{a}(-L) \\ \hat{b}(-L) \end{pmatrix}, \quad (\text{A.28})$$

To calculate the scattering matrix, we need a basic lemma.

LEMMA 2. Consider a system consisting of two successive sectors: The left one with propagator matrix  $\mathbf{P}_l$  and the right one with propagator  $\mathbf{P}_r$ ,

$$\mathbf{P}_l = \begin{pmatrix} \alpha_l & \bar{\gamma}_l \\ \gamma_l & \bar{\alpha}_l \end{pmatrix} \quad \text{and} \quad \mathbf{P}_r = \begin{pmatrix} \alpha_r & \bar{\gamma}_r \\ \gamma_r & \bar{\alpha}_r \end{pmatrix}. \quad (\text{A.29})$$

The propagator matrix of the system is  $\mathbf{P} = \mathbf{P}_r \mathbf{P}_l = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$ , where

$$\alpha = \alpha_l \alpha_r + \gamma_l \bar{\gamma}_r, \quad \gamma = \alpha_l \gamma_r + \gamma_l \bar{\alpha}_r. \quad (\text{A.30})$$

The scattering matrix is  $\mathbf{S} = \begin{pmatrix} T & \tilde{R} \\ R & T \end{pmatrix}$ , with entries

$$T = \frac{1}{\bar{\alpha}} = T_l T_r (1 - R_r \tilde{R}_l)^{-1}, \quad (\text{A.31})$$

$$R = -\frac{\gamma}{\bar{\alpha}} = R_l + T_l^2 R_r (1 - R_r \tilde{R}_l)^{-1}, \quad (\text{A.32})$$

$$\tilde{R} = \frac{\bar{\gamma}}{\bar{\alpha}} = \tilde{R}_r + T_r^2 \tilde{R}_l (1 - R_r \tilde{R}_l)^{-1}. \quad (\text{A.33})$$

Here  $T_j$ ,  $R_j$  and  $\tilde{R}_j$  are the transmission and reflection coefficients of the two sectors, with  $j \in \{l, r\}$ .

**Proof:** Equation (A.30) follows trivially from the multiplication of the matrices (A.29). The expression of the transmission and reflection coefficients in terms of  $\alpha$  and  $\beta$  is as in equations (A.16) and (A.20). From definitions

$$T_j = \frac{1}{\alpha_j}, \quad R_j = -\frac{\gamma_j}{\alpha_j}, \quad \tilde{R}_j = \frac{\bar{\gamma}_j}{\alpha_j}, \quad j \in \{l, r\}, \quad (\text{A.34})$$

we get that the transmission coefficient satisfies

$$T = \frac{1}{\alpha} \stackrel{(\text{A.30})}{=} \frac{1}{\alpha_l \alpha_r} \left( 1 + \frac{\bar{\gamma}_l \gamma_r}{\alpha_l \alpha_r} \right)^{-1} \stackrel{(\text{A.34})}{=} T_l T_r \left( 1 - R_l \tilde{R}_r \right)^{-1}.$$

For the reflection coefficient we have

$$\begin{aligned} R &= -\frac{\beta}{\alpha} \stackrel{(\text{A.30})}{=} -\frac{(\alpha_l \gamma_r + \gamma_l \bar{\alpha}_r)}{\alpha_l \alpha_r} \left( 1 + \frac{\bar{\gamma}_l \gamma_r}{\alpha_l \alpha_r} \right)^{-1} \\ &\stackrel{(\text{A.34})}{=} \left( \frac{|\alpha_l|^2}{\alpha_l^2} R_r + R_l \right) \left( 1 - \tilde{R}_l R_r \right)^{-1} \\ &\stackrel{(\text{A.17})}{=} \left[ \frac{(1 + |\gamma_l|^2)}{\alpha_l^2} R_r + R_l \right] \left( 1 - \tilde{R}_l R_r \right)^{-1} \\ &\stackrel{(\text{A.34})}{=} \left( T_l^2 R_r - R_l \tilde{R}_l R_r + R_l \right) \left( 1 - \tilde{R}_l R_r \right)^{-1} \\ &= T_l^2 R_r \left( 1 - \tilde{R}_l R_r \right)^{-1} + R_l. \end{aligned}$$

The derivation of the expression of the adjoint reflection coefficient is similar

$$\begin{aligned} \tilde{R} &= \frac{\bar{\beta}}{\bar{\alpha}} \stackrel{(\text{A.30})}{=} \frac{(\bar{\alpha}_l \bar{\gamma}_r + \bar{\gamma}_l \alpha_r)}{\alpha_l \alpha_r} \left( 1 + \frac{\bar{\gamma}_l \gamma_r}{\alpha_l \alpha_r} \right)^{-1} \\ &\stackrel{(\text{A.34})}{=} \left[ \tilde{R}_r + \frac{(1 + |\gamma_r|^2)}{\alpha_r^2} \tilde{R}_l \right] \left( 1 - \tilde{R}_l R_r \right)^{-1} \\ &\stackrel{(\text{A.34})}{=} \left( \tilde{R}_r + T_r^2 \tilde{R}_l - \tilde{R}_r R_r \tilde{R}_l \right) \left( 1 - \tilde{R}_l R_r \right)^{-1} \\ &= \tilde{R}_r + T_r^2 \tilde{R}_l \left( 1 - \tilde{R}_l R_r \right)^{-1}. \end{aligned}$$

The proof of the lemma is complete.  $\square$

To derive the expression of the transmission coefficient stated in Theorem 1, we apply Lemma 2 twice. The first time, we use the propagators  $\mathbf{P}_l = \mathbf{P}_-(-L)$  and  $\mathbf{P}_r = \mathbf{P}_1$  and obtain the transmission and reflection coefficients

$$T_{-,1} \stackrel{(\text{A.31})}{=} T_- T_1 (1 - R_1 \tilde{R}_-)^{-1}, \quad (\text{A.35})$$

$$R_{-,1} \stackrel{(\text{A.32})}{=} R_- + T_-^2 R_1 (1 - R_1 \tilde{R}_-)^{-1}, \quad (\text{A.36})$$

$$\tilde{R}_{-,1} \stackrel{(\text{A.33})}{=} R_1 + T_1^2 \tilde{R}_- (1 - R_1 \tilde{R}_-)^{-1}, \quad (\text{A.37})$$

with  $T_- = T_-(-L)$ ,  $R_- = R_-(-L)$ , and  $\tilde{R}_- = \tilde{R}_-(-L)$ . Here we used that  $R_1 = \tilde{R}_1$ , according to equation (2.13). The second time we apply Lemma 2, we use the propagators  $\mathbf{P}_l = \mathbf{P}_{-,b}$  and  $\mathbf{P}_r = \mathbf{P}_+(L)$ . The transmission coefficient is

$$\begin{aligned} \mathcal{T} &\stackrel{(\text{A.31})}{=} T_{-,1} T_+ (1 - R_+ \tilde{R}_{-,1})^{-1} \\ &\stackrel{(\text{A.35})}{=} T_- T_1 T_+ \left[ 1 - R_1 \tilde{R}_- - R_+ R_1 (1 - R_1 \tilde{R}_-) - R_+ T_1^2 \tilde{R}_- \right]^{-1}, \quad (\text{A.38}) \end{aligned}$$

with  $T_+ = T_+(L)$ ,  $R_+ = R_+(L)$ , and  $\widetilde{R}_+ = \widetilde{R}_+(L)$ . Now use the relations (2.27) in this equation to obtain

$$\mathcal{T} = T^2 T_1 [1 - 2RR_1 + (R_1^2 - T_1^2)R^2]^{-1} \quad (\text{A.39})$$

and deduce from equation (2.15) that

$$R_1^2 - T_1^2 = \frac{q^2 + 1}{(i + q)^2} = 2R_1 - 1. \quad (\text{A.40})$$

The result (2.28) follows (A.39) and the identity

$$(1 - R)[1 - (2R_1 - 1)R] = 1 - 2RR_1 + (2R_1 - 1)R^2.$$

We are interested in the mean transmitted intensity. To derive its expression, we recall from [11, Section 7.1.1] that  $|R| < 1$ . Since  $R_1$  satisfies equation (2.31) and  $R_1$  satisfies equation (2.31), we can use the series expansions

$$(1 - R)^{-1} = \sum_{k=0}^{\infty} R^k \quad \text{and} \quad [1 - (2R_1 - 1)R]^{-1} = \sum_{k=0}^{\infty} (2R_1 - 1)^k R^k,$$

and rewrite equation (2.28) as

$$\mathbb{E}[|\mathcal{T}|^2] = |T_1|^2 \sum_{k_1, k_2, k_3, k_4=0}^{\infty} (2R_1 - 1)^{k_2} (2\overline{R}_1 - 1)^{k_4} \mathbb{E}[|T|^{4k_1 + k_2} \overline{R}^{k_3 + k_4}]. \quad (\text{A.41})$$

It is shown in [11, Chapters 7 and 9] that

$$\mathbb{E}[|T|^2 R^j \overline{R}^{j'}] = 0, \quad \text{if } j \neq j',$$

so only the terms with  $k_1 + k_2 = k_3 + k_4$  contribute in (A.41). Moreover, since  $|R|^2 = 1 - |T|^2$ , we obtain that

$$\mathbb{E}[|\mathcal{T}|^2] = |T_1|^2 \sum_{k=0}^{\infty} \sum_{k_2, k_4=0}^k (2R_1 - 1)^{k_2} (2\overline{R}_1 - 1)^{k_4} \mathbb{E}[|T|^{4k} (1 - |T|^2)^k]. \quad (\text{A.42})$$

Now, use the notation (2.30) and observe that

$$\begin{aligned} |T_1|^2 \sum_{k_2, k_4=0}^k (2R_1 - 1)^{k_2} (2\overline{R}_1 - 1)^{k_4} &= \left| T_1 \sum_{k_2=0}^k (2R_1 - 1)^{k_2} \right|^2 \\ &= \left| \frac{T_1}{2(1 - R_1)} [1 - (2R_1 - 1)^{k+1}] \right|^2, \end{aligned}$$

where according to equation (2.15) we have

$$\left| \frac{T_1}{2(1 - R_1)} \right|^2 = \frac{1}{4}.$$

The result (2.29) follows, once we recall the definition (2.30) of  $\tau_k$ .  $\square$



**A.4. Transmission through two independent random sections.** To derive the mean transmitted intensity in the absence of symmetry, we begin with the general formula (A.38), where now the transmission and reflection coefficients in the two random sections are statistically independent. Using equation (A.40) in (A.38) and writing the inverse of the curly bracket as power series, we get

$$\mathbb{E}[|\mathcal{T}|^2] = |T_1|^2 \sum_{j,l=0}^{\infty} \mathbb{E}\left[|T_-|^2 |T_+|^2 [R_1(R_+ + \tilde{R}_-) + (1 - 2R_1)R_+ \tilde{R}_-]^j \times \overline{[R_1(R_+ + \tilde{R}_-) + (1 - 2R_1)R_+ \tilde{R}_-]^l}\right].$$

Next, we expand the  $j$  and  $l$  powers using the binomial theorem and use the independence of  $(T_-, \tilde{R}_-)$  and  $(T_+, \tilde{R}_+)$ . Using also that  $\mathbb{E}[|T_+|^2 R_+^n \tilde{R}_+^m] = 0$  unless  $m = n$ , and the same for  $(T_-, \tilde{R}_-)$ , we get the result (2.39).

**Appendix B. Derivation of the results for random waveguides.** In this appendix we prove the results stated in section 3. The frequency  $\omega$  is fixed, so we simplify notation as in the previous appendix, by dropping the  $\omega$  argument.

**B.1. Proof of Theorem 2.** We obtain from equations (3.40-3.41) that

$$\mathbf{P}_+^{(a)} = \mathbf{T}_+ (\mathbf{I} - \overline{\mathbf{R}_+ \mathbf{R}_+})^{-1} \quad \text{and} \quad \mathbf{P}_+^{(b)} = -\overline{\mathbf{T}_+} (\mathbf{I} - \mathbf{R}_+ \overline{\mathbf{R}_+})^{-1} \mathbf{R}_+. \quad (\text{B.1})$$

Moreover, standard formulas for block matrix inversion give that

$$\mathbf{P}_-^{-1}(-L) \stackrel{(3.39)}{=} \overline{\mathbf{P}_+^{-1}(L)} = \begin{pmatrix} \overline{\mathbf{T}_+^{-1}} & \mathbf{R}_+ \mathbf{T}_+^{-1} \\ \overline{\mathbf{R}_+ \mathbf{T}_+^{-1}} & \mathbf{T}_+^{-1} \end{pmatrix}. \quad (\text{B.2})$$

Then, using this result in (3.45) and recalling the block algebraic structure of  $\mathbf{P}_+$  and  $\mathbf{P}_-$ , we get that the propagator of the system has the form

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}^{(a)} & \overline{\mathcal{P}^{(b)}} \\ \mathcal{P}^{(b)} & \mathcal{P}^{(a)} \end{pmatrix}. \quad (\text{B.3})$$

We are interested in the first block  $\mathcal{P}^{(a)}$ , which according to definition (3.46) defines the transmission matrix

$$\mathcal{T} = [\overline{\mathcal{P}^{(a)}}]^{-1}. \quad (\text{B.4})$$

The expression of this block follows by carrying out the multiplication in (3.45),

$$\mathcal{P}^{(a)} = \mathbf{T}_+ (\mathbf{I} - \overline{\mathbf{R}_+ \mathbf{R}_+})^{-1} \left( \mathbf{P}_1^{(a)} - \overline{\mathbf{R}_+ \mathbf{P}_1^{(b)}} + \overline{\mathbf{P}_1^{(b)} \mathbf{R}_+} - \overline{\mathbf{R}_+ \mathbf{P}_1^{(a)} \mathbf{R}_+} \right) (\overline{\mathbf{T}_+})^{-1}.$$

But we also have from the relations (3.42-3.43) that

$$\mathbf{I} - \overline{\mathbf{R}_+ \mathbf{R}_+} \approx \mathbf{I} - \mathbf{R}_+^* \mathbf{R}_+ = \mathbf{T}_+^* \mathbf{T}_+,$$

which simplifies the factor

$$\mathbf{T}_+ (\mathbf{I} - \overline{\mathbf{R}_+ \mathbf{R}_+})^{-1} \approx \mathbf{T}_+ (\mathbf{T}_+^* \mathbf{T}_+)^{-1} = (\mathbf{T}_+^*)^{-1}. \quad (\text{B.5})$$

The statement of the theorem follows from (B.4) and the relations

$$\mathbf{P}_1^{(a)} = \overline{\mathbf{T}_1^{-1}}, \quad \mathbf{P}_1^{(b)} = -\overline{\mathbf{P}_1^{(b)}} = -\mathbf{T}_1^{-1} \mathbf{R}_1,$$

deduced from equations (3.16) and (3.26-3.27).  $\square$

**B.2. Proof of Theorem 3.** Weak scattering in the random medium means that the norm of the reflection matrix  $\mathbf{R}_+$  is small. Thus, we can use Neumann series to approximate the square bracket in (3.48) by

$$\begin{aligned}\mathbf{Q} &= \left[ \mathbf{T}_1^{-1} - \mathbf{R}_+ \mathbf{T}_1^{-1} \mathbf{R}_1 - \mathbf{T}_1^{-1} \mathbf{R}_1 \mathbf{R}_+ - \mathbf{R}_+ \overline{\mathbf{T}_1^{-1}} \mathbf{R}_+ \right]^{-1} \\ &= \left[ \mathbf{I} - \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 \mathbf{T}_1^{-1} - \mathbf{R}_1 \mathbf{R}_+ - \mathbf{T}_1 \mathbf{R}_+ \overline{\mathbf{T}_1^{-1}} \mathbf{R}_+ \right]^{-1} \mathbf{T}_1 \\ &\approx \mathbf{T}_1 + \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 + \mathbf{R}_1 \mathbf{R}_+ \mathbf{T}_1,\end{aligned}\tag{B.6}$$

where in the second equality we used that  $\mathbf{R}_1$  and  $\mathbf{T}_1^{-1}$  commute, because they are diagonal. This approximation is valid for weak scattering and neglects terms that contain a product involving two (or more) reflection matrices  $\mathbf{R}_+$ . Substituting (B.6) into (3.48), we get that

$$\mathcal{T} \approx \mathbf{T}_+ (\mathbf{T}_1 + \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 + \mathbf{R}_1 \mathbf{R}_+ \mathbf{T}_1) \mathbf{T}_+^T,\tag{B.7}$$

and the mean transmittivity is, from (3.49),

$$\begin{aligned}\mathbb{E} \left[ \sum_{j,l=1}^N |\mathcal{T}_{jl}|^2 \right] &\approx \text{Tr} \left\{ \mathbb{E} \left[ \overline{(\mathbf{I} - \mathbf{R}_+^* \mathbf{R}_+)} (\mathbf{T}_1 + \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 + \mathbf{R}_1 \mathbf{R}_+ \mathbf{T}_1)^* \right. \right. \\ &\quad \left. \left. \times (\mathbf{I} - \mathbf{R}_+^* \mathbf{R}_+) (\mathbf{T}_1 + \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 + \mathbf{R}_1 \mathbf{R}_+ \mathbf{T}_1) \right] \right\}.\end{aligned}\tag{B.8}$$

Here we used the energy conservation relation (3.42) and the commutation property of the trace

$$\text{Tr} \left[ \overline{\mathbf{T}_+} \mathbf{A} \mathbf{T}_+^T \right] = \text{Tr} \left[ \mathbf{T}_+^T \overline{\mathbf{T}_+} \mathbf{A} \right] \stackrel{(3.42)}{=} \text{Tr} \left[ (\mathbf{I} - \mathbf{R}_+^T \overline{\mathbf{R}_+}) \mathbf{A} \right], \quad \forall \mathbf{A} \in \mathbb{C}^{N \times N}.$$

The approximation (B.8) is consistent with (B.6) because, if  $n \neq n'$ ,  $n, n' \geq 0$ , then

$$\mathbb{E} \left[ \prod_{k=1}^n R_{+,j_k l_k} \prod_{k'=1}^{n'} \overline{R_{+,j'_{k'} l'_{k'}}} \right] = 0,$$

for any  $j_k, l_k, j'_{k'}, l'_{k'} \in \{1, \dots, N\}$ , as shown by the analysis of the statistical moments of the transmission and reflection matrices of the random medium given in [12]. This is why we could neglect the quadratic terms in (B.6). Only the terms that do not involve  $\mathbf{R}_+$  or that involve two reflection matrices, with one of them being complex-conjugated, contribute to the approximation of (B.8). Thus, the mean transmittivity is approximated by

$$\begin{aligned}\mathbb{T} = \text{Tr} \left\{ \mathbb{E} \left[ \overline{(\mathbf{I} - \mathbf{R}_+^* \mathbf{R}_+)} \mathbf{T}_1^* \mathbf{T}_1 - \mathbf{T}_1^* \mathbf{R}_+^* \mathbf{R}_+ \mathbf{T}_1 + \mathbf{R}_1^* \mathbf{R}_+^* \mathbf{T}_1^* \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 \right. \right. \\ \left. \left. + \mathbf{R}_1^* \mathbf{R}_+^* \mathbf{T}_1^* \mathbf{R}_1 \mathbf{R}_+ \mathbf{T}_1 + \mathbf{T}_1^* \mathbf{R}_+^* \mathbf{R}_1^* \mathbf{T}_1 \mathbf{R}_+ \mathbf{R}_1 + \mathbf{T}_1^* \mathbf{R}_+^* \mathbf{R}_1^* \mathbf{R}_1 \mathbf{R}_+ \mathbf{T}_1 \right] \right\}.\end{aligned}\tag{B.9}$$

The statement of the theorem follows from this equation once we write explicitly the trace and use the expressions (3.26-3.27) of the entries of  $\mathbf{T}_1$  and  $\mathbf{R}_1$ .

**B.3. Proof of Proposition 1.** The propagator matrix of the waveguide system with two independent random sections is

$$\mathcal{P} = \mathbf{P}_+(L) \mathbf{P}_1 \check{\mathbf{P}}_+(L),\tag{B.10}$$

where  $\check{\mathbf{P}}_+$  is an independent and identically distributed copy of  $\mathbf{P}_+$ . Given the algebraic structure of the propagator  $\mathbf{P}_1$  of the barrier given in (3.15), and of the random medium propagator  $\mathbf{P}_+$  given in equations (3.31) and (3.39), we conclude from (B.10) that  $\mathcal{P}$  is of the form (B.3). We are interested in its first block  $\mathcal{P}^{(a)}$  which determines the transmission matrix  $\mathcal{T}$ , as in equation (B.4).

Using equation (B.1) and multiplying through in equation (B.10) we get that

$$\mathcal{P}^{(a)} = \mathbf{T}_+ (\mathbf{I} - \overline{\mathbf{R}}_+ \mathbf{R}_+)^{-1} \left[ \left( \mathbf{P}_1^{(a)} - \overline{\mathbf{R}}_+ \mathbf{P}_1^{(b)} \right) \check{\mathbf{T}}_+ \left( \mathbf{I} - \overline{\check{\mathbf{R}}}_+ \check{\mathbf{R}}_+ \right)^{-1} - \overline{\left( \mathbf{P}_1^{(b)} - \mathbf{R}_+ \mathbf{P}_1^{(a)} \right)} \check{\mathbf{T}}_+ \left( \mathbf{I} - \check{\mathbf{R}}_+ \overline{\check{\mathbf{R}}}_+ \right)^{-1} \check{\mathbf{R}}_+ \right],$$

where the first factor is approximated in (B.5). This gives

$$\mathcal{T} = \left( \overline{\mathcal{P}^{(a)}} \right)^{-1} \approx \left[ \overline{\left( \mathbf{P}_1^{(a)} - \mathbf{R}_+ \mathbf{P}_1^{(b)} \right)} \overline{\check{\mathbf{T}}}_+ \left( \mathbf{I} - \check{\mathbf{R}}_+ \overline{\check{\mathbf{R}}}_+ \right)^{-1} - \left( \mathbf{P}_1^{(b)} - \mathbf{R}_+ \mathbf{P}_1^{(a)} \right) \check{\mathbf{T}}_+ \left( \mathbf{I} - \overline{\check{\mathbf{R}}}_+ \check{\mathbf{R}}_+ \right)^{-1} \check{\mathbf{R}}_+ \right]^{-1} \mathbf{T}_+^T, \quad (\text{B.11})$$

where the square bracket can be approximated with Neumann series for small reflection matrices. Such series are also used to expand  $\text{Tr}(\mathcal{T}^* \mathcal{T})$  up to second order in terms of the reflection matrices of the random medium. The result stated in the proposition follows after we take the expectation and use that  $(\mathbf{T}_+, \mathbf{R}_+)$  and  $(\check{\mathbf{T}}_+, \check{\mathbf{R}}_+)$  are independent.

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