

Inverse Problems and Imaging

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First lecture: friday, january 12, 2024, 9:00-12:00 (ENS, room 2E30).

Material on the course website.

Validation: project (notebook jupyter + oral presentation).

Sensor array imaging

- Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, radar, etc) has two steps:
 - data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver array.
 - data processing: the recorded signals are processed to identify the quantities of interest (reflector locations, etc).

- Example:

Ultrasound echography

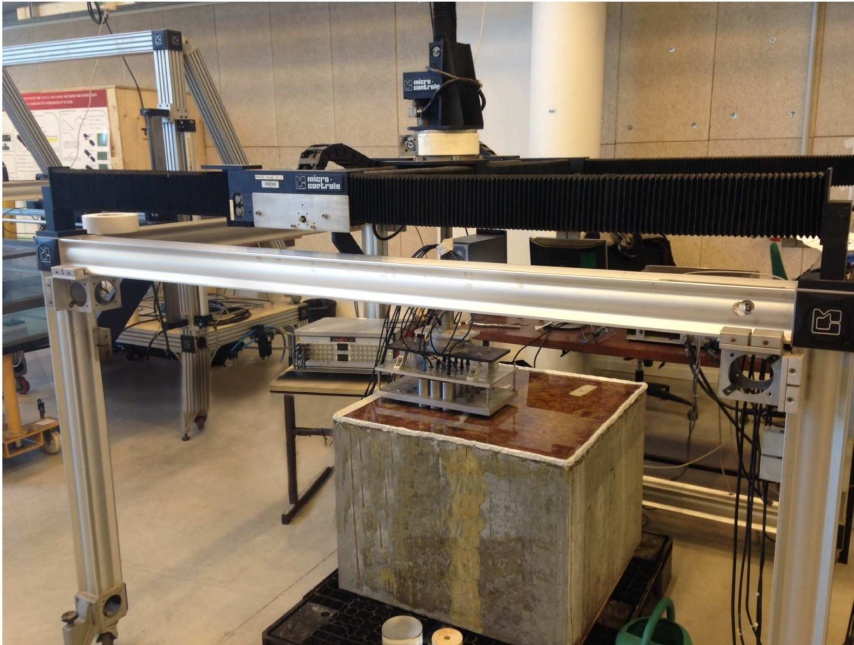


- Standard imaging techniques require:
 - good receivers,
 - suitable conditions for wave propagation (ideally, the “target” is embedded in a homogeneous medium),
 - controlled and known sources.

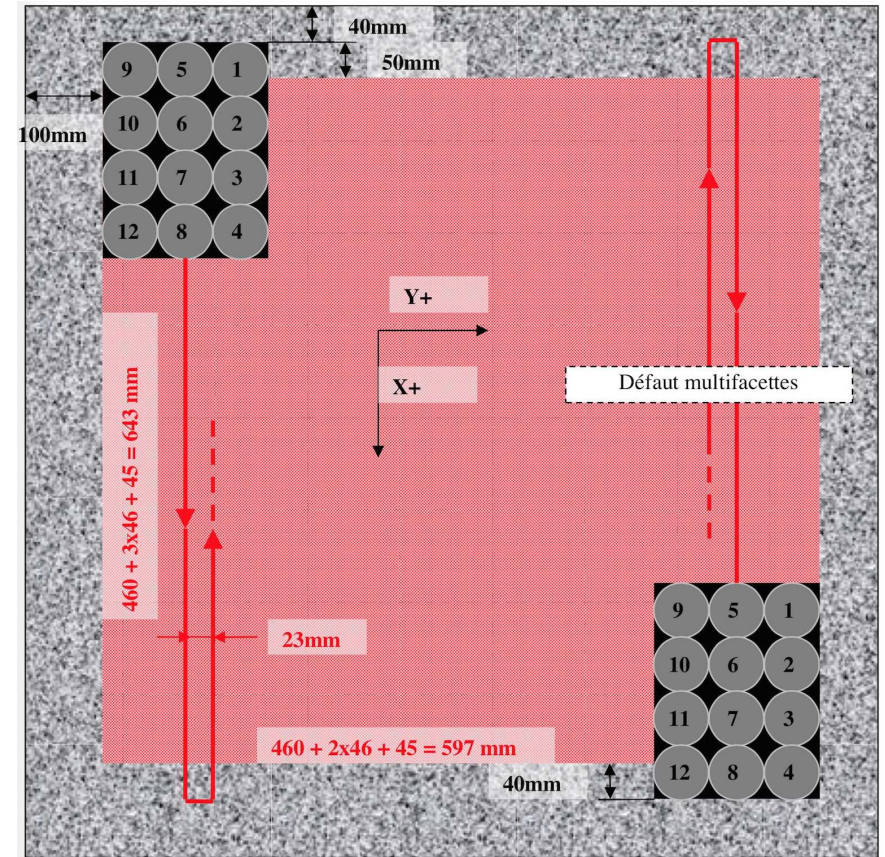
Sensor array imaging

- Goal: Propose and study imaging techniques that are robust with respect to:
 - measurement noise,
 - the complexity of the medium (heterogeneous medium),
 - the control and the knowledge of the sources.
- More generally: resolution of ill-posed inverse problems.
 - ↪ Introduce probabilistic and statistical techniques:
 - Bayesian analysis,
 - Random matrix theory,
 - Spectral theory for stationary processes,
 - Gaussian processes.

Application 1: Ultrasound echography in concrete

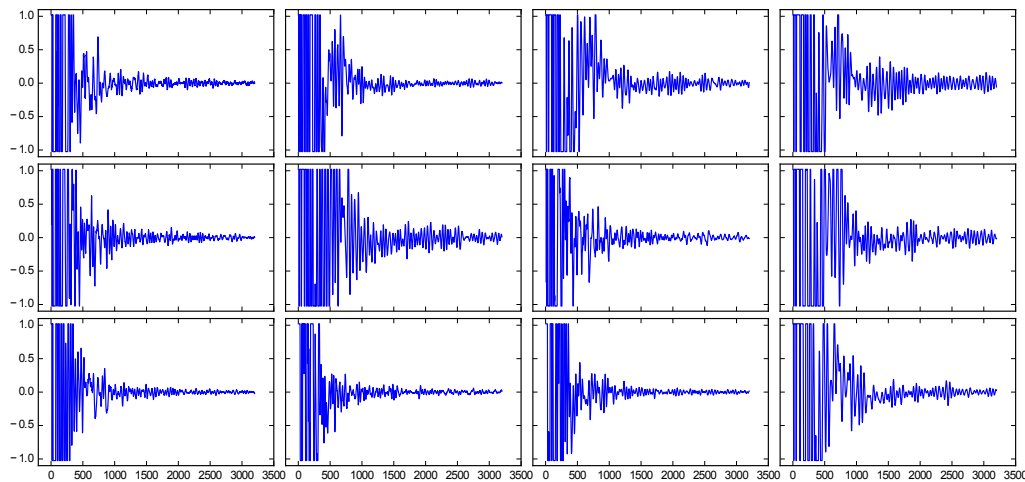


Experimental configuration

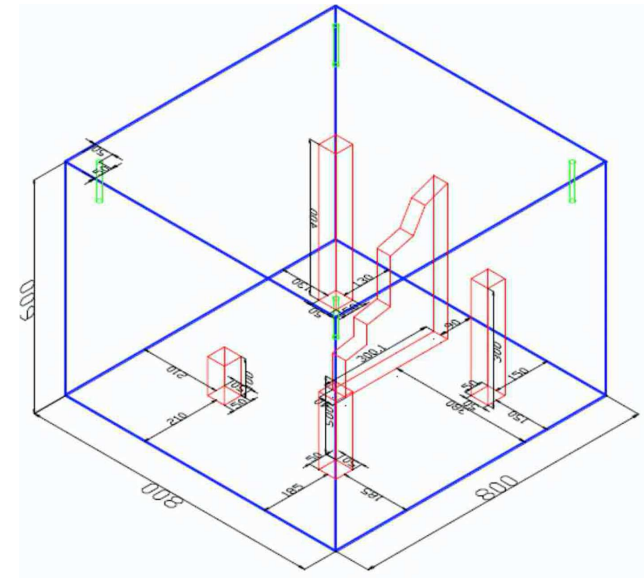


Top view of the acquisition geometry

Application 1: Ultrasound echography in concrete

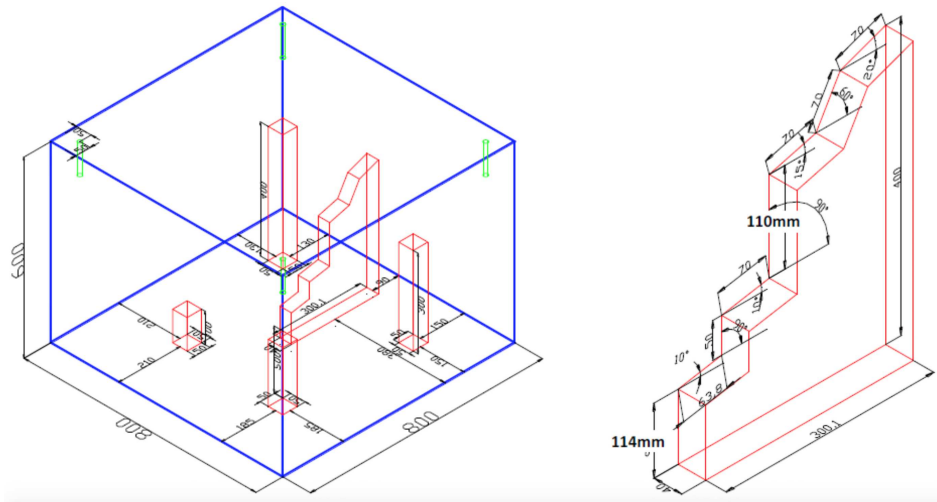


Data

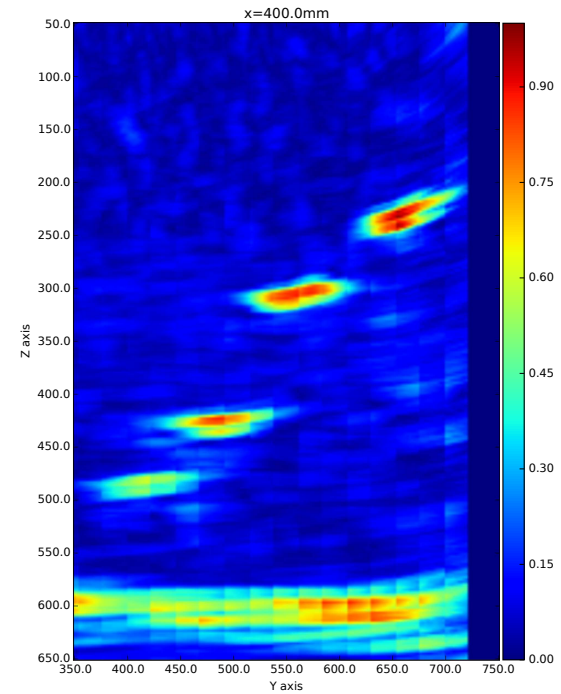


Real configuration

Application 1: Ultrasound echography in concrete

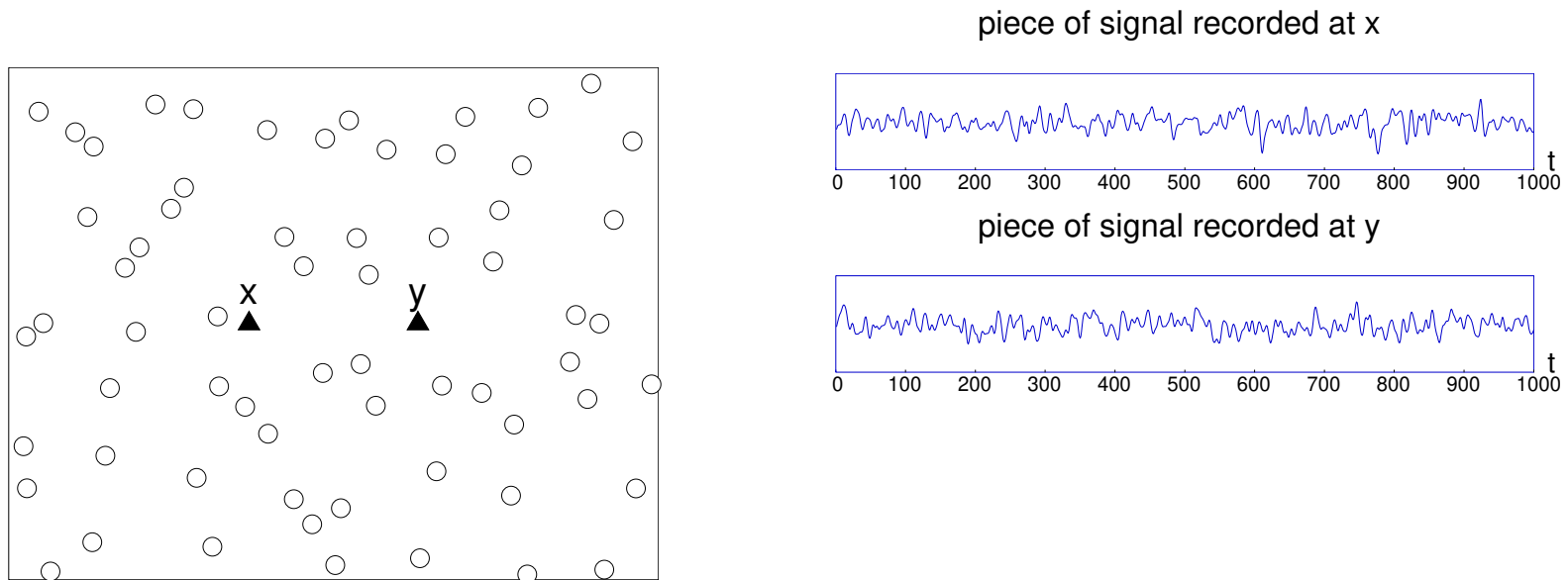


Real configuration



2D Image (along the complex defect plane)

Theory: Cross correlation of signals transmitted by noise sources



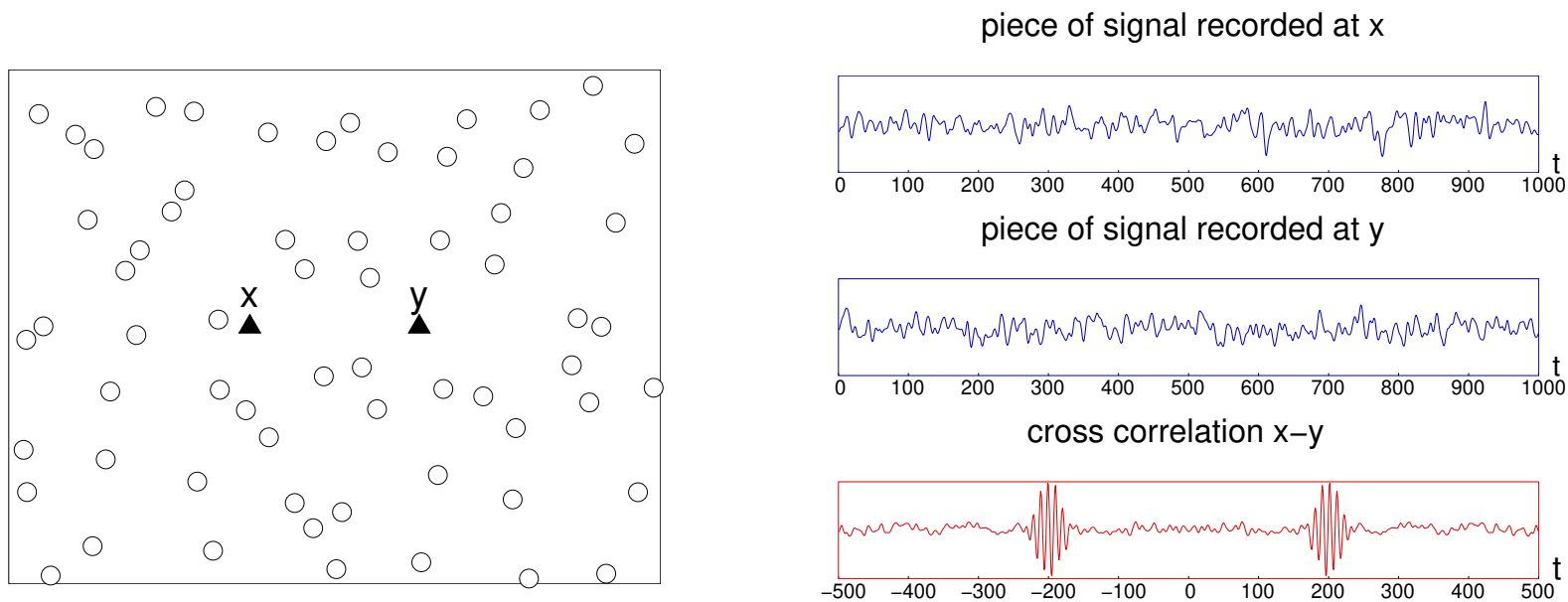
Numerical simulation of wave propagation

with many noise sources (\circ) and two receivers at x and y (\blacktriangle)

How to extract information from the recorded signals $u_x(t)$ and $u_y(t)$?

These signals are just noise !

Theory: Cross correlation of signals transmitted by noise sources



Numerical simulation of wave propagation

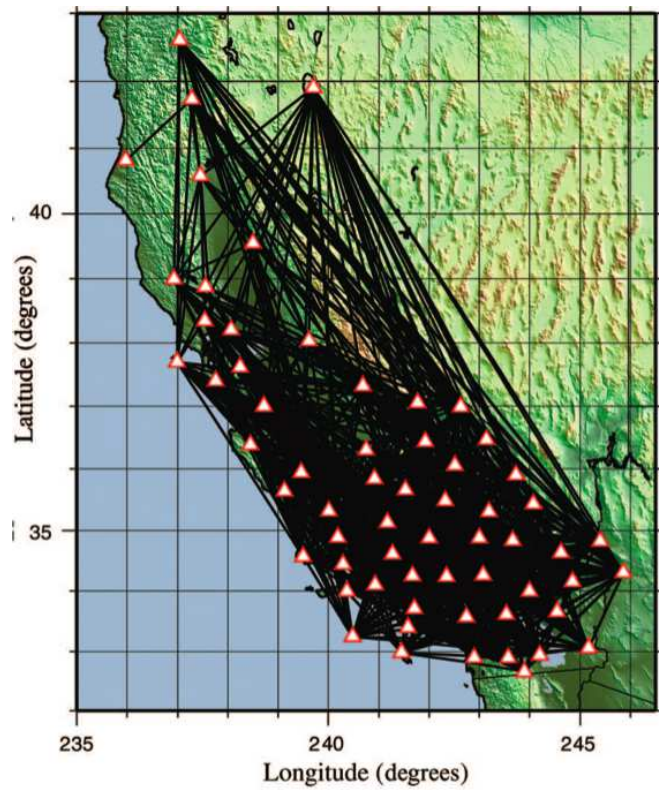
with many noise sources (\circ) and two receivers at x and y (\blacktriangle)

\hookrightarrow Compute the cross correlation of the recorded signals

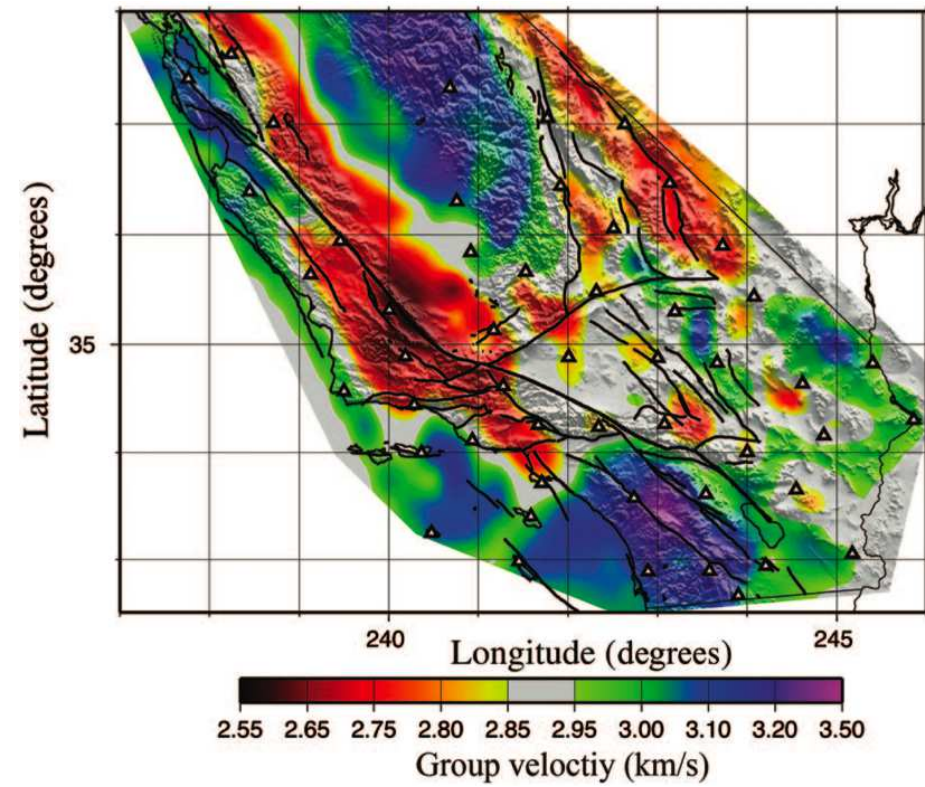
$$C_{x,y}^T(t) = \frac{1}{T} \int_0^T u_x(s) u_y(s+t) ds$$

and extract the travel time between the receivers at x and y .

Application 2: Seismic interferometry

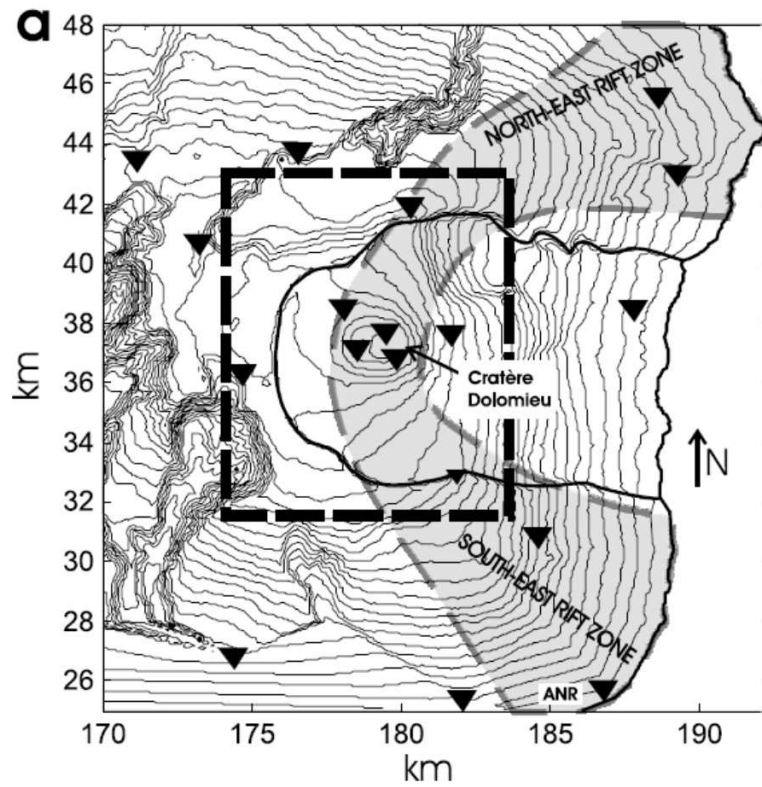


Travel time estimation

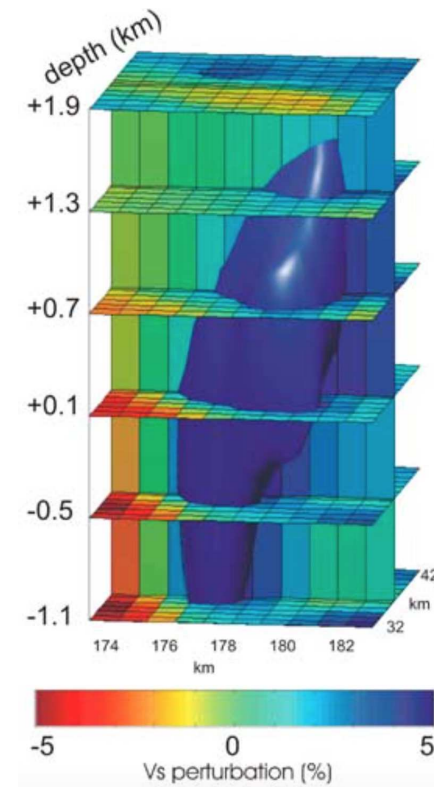


Background velocity estimation

Application 2: Seismic interferometry



Piton de la Fournaise



Eruption warning system

An inverse problem: Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x \in \Omega}$ of the medium, compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t, x) = f(t)\delta(x - x_s), \quad t \in \mathbb{R}, x \in \Omega,$$

starting from $p^{(s)}(t, x) = 0, t \ll 0$.

At the locations of the receivers:

$$d_{r,s}(t) = p^{(s)}(t, x_r), \quad r, s = 1, \dots, N$$

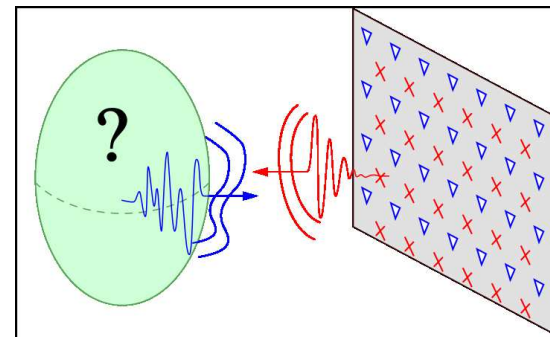
\hookrightarrow forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t \in [t_{\min}, t_{\max}]}$, is the array response matrix.

- *Inverse problem:*

Given the time-resolved measurements \mathbf{d} , determine the velocity map c .



Full Waveform Inversion (FWI)

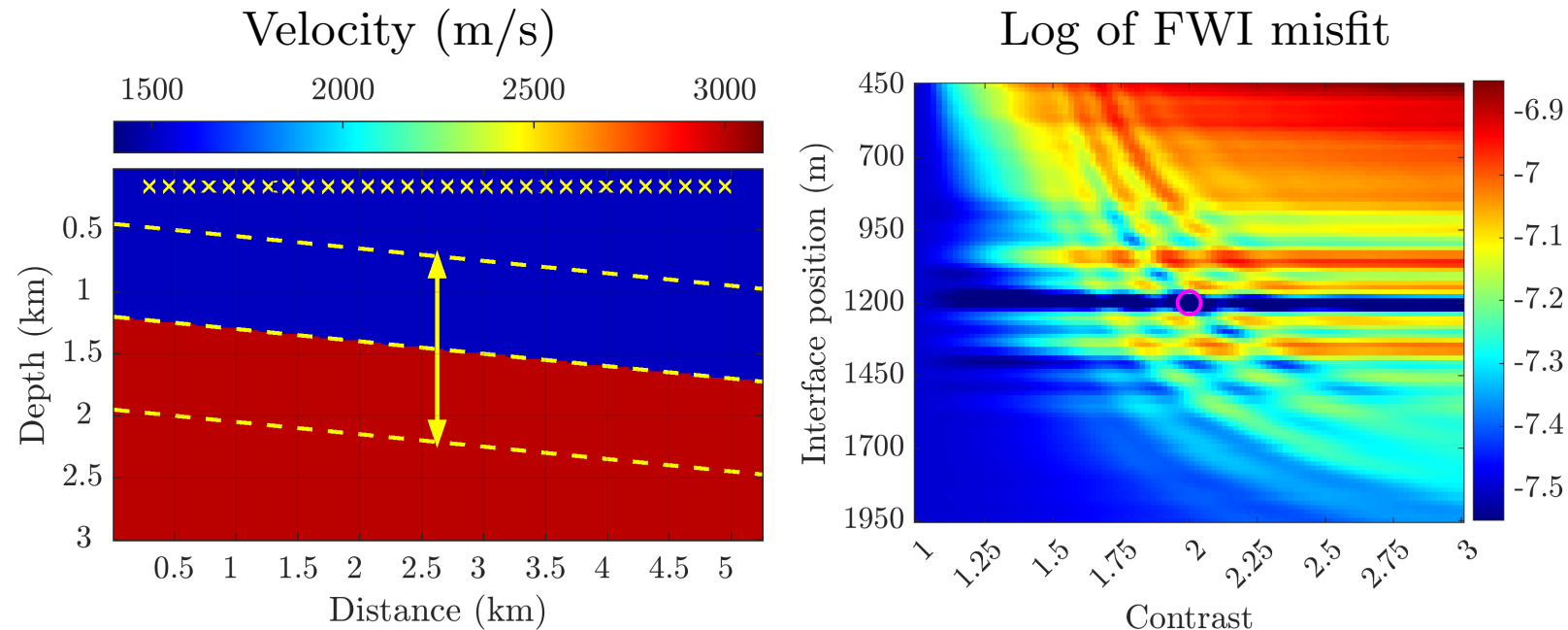
- FWI fits data with its model prediction in L^2 -norm (least-square minimization):

$$\hat{c} = \underset{c}{\operatorname{argmin}} \{ \mathcal{O}_{FWI}[c] + \operatorname{Reg}[c] \},$$

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2 = \sum_{r,s=1}^N \int_{t_{\min}}^{t_{\max}} |d_{meas}(t)_{r,s} - \mathcal{D}[c](t)_{r,s}|^2 dt$$

- The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c .
 \hookrightarrow optimization needs hard to get good initial guess.

Topography of the FWI objective function



- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$.
- $N = 30$ sensors and $N_t = 39$ time samples at interval $\tau = 0.0435s$.
- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2$$

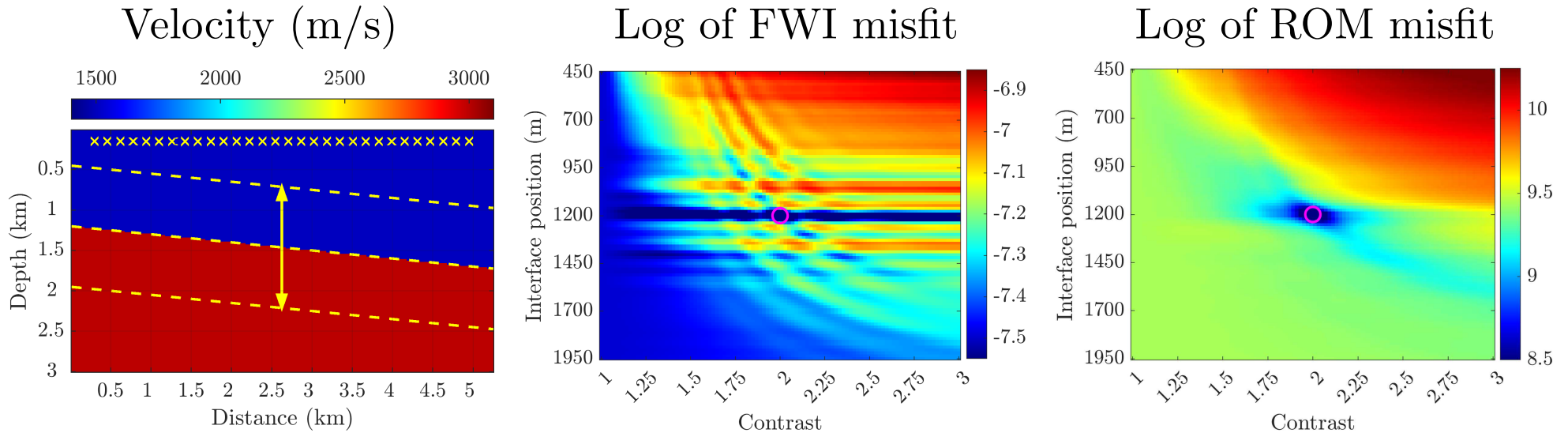
Main objective

- Our main objective: find a convex formulation of FWI.
- Proposed approach: Find a nonlinear mapping \mathcal{R} : data $\mathbf{d} \mapsto$ reduced order model (ROM) \mathbf{A}^{rom} such that minimization of ROM misfit is better for velocity estimation.
- We can think of the data to ROM mapping \mathcal{R} as a preconditioner of the forward mapping \mathcal{D}

$$c \xrightarrow{\mathcal{D}} \mathbf{d} \xrightarrow{\mathcal{R}} \mathbf{A}^{rom}$$

because the composition $\mathcal{R} \circ \mathcal{D}$, which gives $\mathbf{A}^{rom} = \mathcal{R}(\mathcal{D}[c])$, is easier to “invert”.

Topographies of the FWI and ROM objective functions



- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).
- FWI objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}^{meas}\|_2^2$$

- ROM objective function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$