Some useful tools

Fourier identities

- Let $f(t)$ be a "nice" real-valued function. Its Fourier transform is

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{i\omega t} dt$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{-i\omega t} d\omega$$

Parseval's identity:

$$\int_{\mathbb{R}} f(t)^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega, \qquad \int_{\mathbb{R}} f(t) g(t) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\hat{f}(\omega)} \hat{g}(\omega) d\omega,$$

	$f(t)$	$\hat{f}(\omega)$
	$\frac{d^n f}{dt^n}$	$(-i\omega)^n \hat{f}(\omega)$
	$f * g(t) = \int f(s) g(t-s) ds$	$2\pi \hat{f}(\omega) \hat{g}(\omega)$
time reversal	$f(-t)$	$\overline{\hat{f}(\omega)}$
cross correlation	$\int f(s) g(t+s) ds$	$2\pi \, \overline{\hat{f}(\omega)} \hat{g}(\omega)$

Fourier identities

• Extension of Fourier transform to functions in L^2 and to distributions.

• A distribution g is a continuous linear functional on the Schwartz class $\mathcal{S}$. It is characterized by $\int_{\mathbb{R}} f(t)g(t)dt$ for any test function $f \in \mathcal{S}$.

• The Fourier transform of a distribution g is the distribution $\hat{g}$ such that

$$\frac{1}{2\pi}\int \overline{\hat{f}(\omega)}\hat{g}(\omega)d\omega = \int f(t)g(t)dt$$

• Here: we will deal with the Dirac distribution δ, such that $\int f(t)\delta(t)dt = f(0)$ for all test functions.

• The Fourier transform of δ is $\hat{\delta} = 1$.

Proof 1. $\hat{\delta}(\omega)$ is such that $\int \overline{\hat{f}(\omega)}\hat{\delta}(\omega)d\omega = 2\pi \int f(t)\delta(t)dt = 2\pi f(0)$.
However we do know that $f(0) = \frac{1}{2\pi}\int \hat{f}(\omega)d\omega = \frac{1}{2\pi}\int \hat{f}(-\omega)d\omega = \frac{1}{2\pi}\int \overline{\hat{f}(\omega)}d\omega$.

Proof 2. δ =limit of $f_a(t) = \frac{1}{\sqrt{2\pi}a}\exp(-\frac{t^2}{2a^2})$ as $a \to 0^+$.
$\hat{\delta}$ =limit of $\hat{f}_a(\omega) = \exp(-\frac{a^2\omega^2}{2})$ as $a \to 0^+$.

Remark. If $g(t) = 1$, then we have $\hat{g}(\omega) = 2\pi\delta(\omega)$.

Divergence theorem

- Fundamental theorem of analysis:

$$\int_a^b f'(t)dt = f(b) - f(a)$$

- Divergence theorem: Let D be a smooth bounded domain in $\mathbb{R}^d$ and $\boldsymbol{f}$ be a smooth function $\mathbb{R}^d \to \mathbb{R}^d$:

$$\int_D \nabla \cdot \boldsymbol{f}(\boldsymbol{x}) d\boldsymbol{x} = \int_{\partial D} \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) d\sigma(\boldsymbol{x})$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit outgoing normal vector.

- If $D = B(\mathbf{0}, R)$ and $d = 3$, we have $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{x}/|\boldsymbol{x}|$ and in spherical coordinates:

$$\int_D \nabla \cdot \boldsymbol{f}(\boldsymbol{x}) d\boldsymbol{x} = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \big[\partial_x f_x + \partial_y f_y + \partial_z f_z\big](r\boldsymbol{e}_{\theta,\phi})$$

$$\int_{\partial D} \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) d\sigma(\boldsymbol{x}) = R^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \boldsymbol{e}_{\theta,\phi} \cdot \boldsymbol{f}(R\boldsymbol{e}_{\theta,\phi})$$

where $\boldsymbol{e}_{\theta,\phi} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$.

Wave equation (1D)

$$\frac{1}{c_o^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad t > 0,\, x \in \mathbb{R}$$

with the initial conditions: $u(t=0,x) = u_0(x)$, $\partial_t u(t=0,x) = u_1(x)$.

Proposition. If $u_0 \in \mathcal{C}^2$ and $u_1 \in \mathcal{C}^1$, then $\exists ! u \in \mathcal{C}^2((0,+\infty)\times\mathbb{R})$ solution of the wave equation.

Proof of uniqueness: Let $u \in \mathcal{C}^2$ be a solution. Make the change of variables $\alpha = x - c_o t$ and $\beta = x + c_o t$. The function $\tilde{u}(\alpha,\beta) = u(\frac{\beta-\alpha}{2c_o}, \frac{\beta+\alpha}{2})$ satisfies

$$\frac{\partial^2 \tilde{u}}{\partial\alpha\partial\beta} = \frac{1}{4}\Big(-\frac{1}{c_o}\partial_t + \partial_x\Big)\Big(\frac{1}{c_o}\partial_t + \partial_x\Big)u(\frac{\beta-\alpha}{2c_o}, \frac{\beta+\alpha}{2}) = 0$$

Therefore $\tilde{u}$ is of the form

$$\tilde{u}(\alpha,\beta) = f(\alpha) + g(\beta)$$

and u is of the form

$$u(t,x) = f(x - c_o t) + g(x + c_o t)$$

f and g can be determined by the initial conditions:

$$u(t,x) = \frac{1}{2}\left[u_0(x+c_o t) + u_0(x-c_o t)\right] + \frac{1}{2c_o}\int_{x-c_o t}^{x+c_o t} u_1(x')dx'$$

(D'Alembert formula)

One can check directly that u is solution.

- The initial conditions split into two contributions, one going to the left and one going to the right at speed c_o.

- The smoothness of the solution is determined by the smoothness of the initial conditions (no gain, no loss of regularity)

- The solution at (t,x) depends only on the initial conditions in $[x-c_o t, x+c_o t]$ (hyperbolic system).

Wave equation (1D)

$$\frac{1}{c_o^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad t > 0,\ x \in (0, +\infty)$$

with the initial conditions: $u(t=0,x) = u_0(x)$, $\partial_t u(t=0,x) = u_1(x)$ and the Dirichlet condition $u(t, x=0) = 0$.

Proposition. If $u_0 \in \mathcal{C}^2$ and $u_1 \in \mathcal{C}^1$, with $u_0(0) = u_1(0) = 0$, then $\exists! u \in \mathcal{C}^2((0,+\infty)\times(0,+\infty))$ solution of the wave equation.

Proof of uniqueness: Let $u \in \mathcal{C}^2$ be a solution. Use method of images. The function $\tilde{u}(t,x) = \mathrm{sgn}(x)u(t,|x|)$ satisfies

$$\frac{1}{c_o^2}\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0, \qquad t > 0,\ x \in \mathbb{R}$$

with the initial conditions: $\tilde{u}(t=0,x) = \tilde{u}_0(x)$, $\partial_t \tilde{u}(t=0,x) = \tilde{u}_1(x)$ with $\tilde{u}_j(x) = \mathrm{sgn}(x)u_j(|x|)$.
(A "mirror" initial condition has been created on $\mathbb{R}^-$).

Therefore (D'Alembert)

$$\tilde{u}(t,x) = \frac{1}{2}\left[\tilde{u}_0(x+c_ot) + \tilde{u}_0(x-c_ot)\right] + \frac{1}{2c_o}\int_{x-c_ot}^{x+c_ot} \tilde{u}_1(x')dx'$$

If $x \geq c_o t \geq 0$:

$$u(t,x) = \frac{1}{2}\left[u_0(x + c_o t) + u_0(x - c_o t)\right] + \frac{1}{2c_o}\int_{x-c_o t}^{x+c_o t} u_1(x')dx'$$

If $c_o t \geq x \geq 0$:

$$u(t,x) = \frac{1}{2}\left[u_0(x + c_o t) - u_0(c_o t - x)\right] + \frac{1}{2c_o}\int_{c_o t-x}^{c_o t+x} u_1(x')dx'$$

Wave equation (3D)

$$\frac{1}{c_o^2}\frac{\partial^2 u}{\partial t^2} - \Delta_{\boldsymbol{x}} u = 0, \qquad t > 0,\ \boldsymbol{x} \in \mathbb{R}^3$$

with the initial conditions: $u(t = 0, \boldsymbol{x}) = u_0(\boldsymbol{x})$, $\partial_t u(t = 0, \boldsymbol{x}) = u_1(\boldsymbol{x})$.

Proposition. If $u_0 \in \mathcal{C}^2$ and $u_1 \in \mathcal{C}^1$, then $\exists! u \in \mathcal{C}^2((0, +\infty) \times \mathbb{R}^3)$ solution of the wave equation.

We can reduce the problem to a 1D wave equation. Let $u \in \mathcal{C}^2$ be a solution. For $t, s > 0$:

$$\begin{aligned} \tilde{u}_{\boldsymbol{x}}(t, s) &:= \frac{1}{4\pi s}\int_{\partial B(\boldsymbol{x},s)} u(t, \boldsymbol{x}')d\sigma(\boldsymbol{x}') \\ &= \frac{1}{4\pi s}s^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi u(t, \boldsymbol{x} + \boldsymbol{e}_{\theta,\phi} s) \\ &= \frac{s}{4\pi}\int_{\partial B(\boldsymbol{0},1)} u(t, \boldsymbol{x} + s\boldsymbol{y})d\sigma(\boldsymbol{y}) \end{aligned}$$

We have

$$\tilde{u}_{\boldsymbol{x}}(t, s = 0) = 0, \qquad \partial_s \tilde{u}_{\boldsymbol{x}}(t, s = 0) = u(t, \boldsymbol{x})$$

$$\tilde{u}_{\boldsymbol{x}}(t,s) = \frac{s}{4\pi}\int_{\partial B(\mathbf{0},1)} u(t,\boldsymbol{x}+s\boldsymbol{y})d\sigma(\boldsymbol{y})$$

We also have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\tilde{u}_{\boldsymbol{x}}}{s} &= \frac{1}{4\pi}\int_{\partial B(\mathbf{0},1)} \boldsymbol{y}\cdot\nabla u(t,\boldsymbol{x}+s\boldsymbol{y})d\sigma(\boldsymbol{y}) \\
&= \frac{1}{4\pi s^2}\int_{\partial B(\mathbf{0},s)} \frac{\boldsymbol{y}-\boldsymbol{x}}{s}\cdot\nabla u(t,\boldsymbol{y})d\sigma(\boldsymbol{y}) \\
&= \frac{1}{4\pi s^2}\int_{B(\mathbf{0},s)} \Delta u(t,\boldsymbol{y})d\boldsymbol{y} \\
&= \frac{1}{4\pi c_o^2 s^2}\frac{\partial^2}{\partial t^2}\int_{B(\mathbf{0},s)} u(t,\boldsymbol{y})d\boldsymbol{y}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial s}s^2\frac{\partial}{\partial s}\frac{\tilde{u}_{\boldsymbol{x}}}{s} &= \frac{1}{4\pi c_o^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\int_{B(\mathbf{0},s)} u(t,\boldsymbol{y})d\boldsymbol{y} \\
&= \frac{1}{4\pi c_o^2}\frac{\partial^2}{\partial t^2}\lim_{\delta s\to 0}\frac{1}{\delta s}\int_{B(\mathbf{0},s+\delta s)\setminus B(\mathbf{0},s)} u(t,\boldsymbol{y})d\boldsymbol{y} \\
&= \frac{1}{4\pi c_o^2}\frac{\partial^2}{\partial t^2}\int_{\partial B(\mathbf{0},s)} u(t,\boldsymbol{y})d\sigma(\boldsymbol{y}) = \frac{s}{c_o^2}\frac{\partial^2}{\partial t^2}\tilde{u}_{\boldsymbol{x}}(t,s)
\end{aligned}$$

We can check

$$\frac{1}{s}\frac{\partial}{\partial s}s^2\frac{\partial}{\partial s}\frac{\tilde{u}_{\boldsymbol{x}}}{s} = \frac{\partial^2}{\partial s^2}\tilde{u}_{\boldsymbol{x}}$$

which shows that $\tilde{u}_{\boldsymbol{x}}$ is solution of

$$\frac{1}{c_o^2}\frac{\partial^2 \tilde{u}_{\boldsymbol{x}}}{\partial t^2} - \frac{\partial^2 \tilde{u}_{\boldsymbol{x}}}{\partial s^2} = 0, \quad t > 0,\ s \in (0, +\infty)$$

with the initial conditions $\tilde{u}_{\boldsymbol{x}}(t=0,s) = \tilde{u}_{\boldsymbol{x},0}(s)$ and $\partial_t \tilde{u}_{\boldsymbol{x}}(t=0,s) = \tilde{u}_{\boldsymbol{x},1}(s)$ and the Dirichlet condition $\tilde{u}_{\boldsymbol{x}}(t, s=0) = 0$,

$$\tilde{u}_{\boldsymbol{x},j}(s) = \frac{1}{4\pi s}\int_{\partial B(\boldsymbol{x},s)} u_j(\boldsymbol{x}')d\sigma(\boldsymbol{x}'), \quad j = 0,1$$

Therefore, for $c_o t \geq s \geq 0$,

$$\tilde{u}_{\boldsymbol{x}}(t,s) = \frac{1}{2}\big[\tilde{u}_{\boldsymbol{x},0}(s + c_o t) - \tilde{u}_{\boldsymbol{x},0}(c_o t - s)\big] + \frac{1}{2c_o}\int_{c_o t - s}^{c_o t + s} \tilde{u}_{\boldsymbol{x},1}(s')ds'$$

We have

$$\begin{aligned}
u(t,\boldsymbol{x}) &= \partial_s \tilde{u}_{\boldsymbol{x}}(t, s=0) \\
&= \partial_s \tilde{u}_{\boldsymbol{x},0}(c_o t) + \frac{1}{c_o} \tilde{u}_{\boldsymbol{x},1}(c_o t) \\
&= \frac{\partial}{\partial t} \Big[\frac{1}{4\pi c_o^2 t} \int_{\partial B(\boldsymbol{x}, c_o t)} u_0(\boldsymbol{x}') d\sigma(\boldsymbol{x}') \Big] + \frac{1}{4\pi c_o^2 t} \int_{\partial B(\boldsymbol{x}, c_o t)} u_1(\boldsymbol{x}') d\sigma(\boldsymbol{x}')
\end{aligned}$$

(Kirchhoff formula)

Wave equation with source (3D)

$$\frac{1}{c_o^2}\frac{\partial^2 u}{\partial t^2} - \Delta_{\boldsymbol{x}} u = n(t,\boldsymbol{x}), \qquad t>0,\ \boldsymbol{x}\in\mathbb{R}^3$$

with the initial conditions: $u(t=0,\boldsymbol{x})=0$, $\partial_t u(t=0,\boldsymbol{x})=0$.

Use Duhamel principle. Let $s>0$. Denote $\tilde{u}_s(t,\boldsymbol{x})$ the unique solution of

$$\frac{1}{c_o^2}\frac{\partial^2 \tilde{u}_s}{\partial t^2} - \Delta_{\boldsymbol{x}} \tilde{u}_s = 0, \qquad t>0,\ \boldsymbol{x}\in\mathbb{R}^3$$

with the initial conditions: $\tilde{u}_s(t=s,\boldsymbol{x})=0$, $\partial_t \tilde{u}_s(t=s,\boldsymbol{x})=c_o^2 n(s,\boldsymbol{x})$.
The function

$$u(t,\boldsymbol{x}) := \int_0^t \tilde{u}_s(t,\boldsymbol{x})ds$$

satisfies

$$\frac{\partial u}{\partial t}(t,\boldsymbol{x}) = \tilde{u}_{s=t}(t,\boldsymbol{x}) + \int_0^t \frac{\partial \tilde{u}_s}{\partial t}(t,\boldsymbol{x})ds = \int_0^t \frac{\partial \tilde{u}_s}{\partial t}(t,\boldsymbol{x})ds$$

The function u satisfies

$$\frac{\partial^2 u}{\partial t^2}(t,\boldsymbol{x}) = \frac{\partial \tilde{u}_s}{\partial t}(t,\boldsymbol{x}) \mid_{s=t} + \int_0^t \frac{\partial^2 \tilde{u}_s}{\partial t^2}(t,\boldsymbol{x}) ds$$
$$= c_o^2 n(t,\boldsymbol{x}) + \int_0^t c_o^2 \Delta_{\boldsymbol{x}} \tilde{u}_s(t,\boldsymbol{x}) ds$$
$$= c_o^2 n(t,\boldsymbol{x}) + c_o^2 \Delta_{\boldsymbol{x}} u(t,\boldsymbol{x})$$

with the initial conditions $u(t=0,\boldsymbol{x}) = 0$ and $\partial u(t=0,\boldsymbol{x}) = 0$.

We have

$$\tilde{u}_s(t,\boldsymbol{x}) = \frac{1}{4\pi c_o^2 (t-s)} \int_{\partial B(\boldsymbol{x}, c_o(t-s))} c_o^2 n(s,\boldsymbol{x}') d\sigma(\boldsymbol{x}')$$

and

$$u(t,\boldsymbol{x}) = \int_0^t \tilde{u}_s(t,\boldsymbol{x}) ds$$

Therefore

$$
\begin{aligned}
u(t,\boldsymbol{x}) &= \int_0^t ds \frac{1}{4\pi(t-s)} \int_{\partial B(\boldsymbol{x},c_o(t-s))} c_o^2 n(s,\boldsymbol{x}') d\sigma(\boldsymbol{x}') \\
&= \frac{1}{4\pi} \int_0^t ds \int_{\partial B(\boldsymbol{x},c_o s)} \frac{1}{s} n(t-s,\boldsymbol{x}') d\sigma(\boldsymbol{x}') \\
&= \frac{1}{4\pi} \int_0^t ds \int_{\partial B(\boldsymbol{x},c_o s)} \frac{c_o}{|\boldsymbol{x}-\boldsymbol{x}'|} n\Big(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c_o}, \boldsymbol{x}'\Big) d\sigma(\boldsymbol{x}') \\
&= \frac{1}{4\pi} \int_0^{c_o t} ds \int_{\partial B(\boldsymbol{x},s)} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} n\Big(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c_o}, \boldsymbol{x}'\Big) d\sigma(\boldsymbol{x}') \\
&= \frac{1}{4\pi} \int_{B(\boldsymbol{x},c_o t)} \frac{c_o}{|\boldsymbol{x}-\boldsymbol{x}'|} n\Big(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c_o}, \boldsymbol{x}'\Big) d\boldsymbol{x}'
\end{aligned}
$$

which we can write

$$
u(t,\boldsymbol{x}) = \iint G(t-s,\boldsymbol{x},\boldsymbol{y}) n(s,\boldsymbol{y}) ds d\boldsymbol{y}
$$

with

$$
G(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \delta\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_o} - t\Big)
$$

The (time-dependent) Green's function

$$G(t, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_o} - t\Big)$$

is the fundamental solution of the 3D wave equation. It is the unique solution with $n(t, \boldsymbol{x}) = \delta(s)\delta(\boldsymbol{x}-\boldsymbol{y})$.

The unique solution with the source term $n(t, \boldsymbol{x})$ is

$$u(t, \boldsymbol{x}) = \iint G(t-s, \boldsymbol{x}, \boldsymbol{y}) n(s, \boldsymbol{y}) ds d\boldsymbol{y}$$