A PIECEWISE DETERMINISTIC MARKOV PROCESS APPROACH MODELING A DRY FRICTION PROBLEM WITH NOISE

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Abstract. Understanding and predicting the dynamical properties of systems involving dry friction is a major concern in physics and engineering. It abounds in many mechanical processes, from the sound produced by a violin to the screeching of chalk on a blackboard to human infant crawling dynamics and friction-based locomotion of a multitude of living organisms (snakes, bacteria, scallops..) to the displacement of mechanical structures (building, bridges, nuclear plants, massive industrial infrastructures) under earthquakes and beyond. Surprisingly, even for low-dimensional systems, the modeling of dry friction in the presence of random forcing has not been elucidated. In this paper, we propose a piecewise deterministic Markov process approach modeling a system with dry friction including different coefficients for the static and dynamic forces. In this mathematical framework, we derive the corresponding Kolmogorov equations and related tools to compute statistical quantities of interest related to the distributions of the static (sticked) and dynamic phases. We show ergodicity and provide a representation formula of the stationary measure using independent identically distributed portions of the trajectory (excursions). We also obtain deterministic characterizations of the Laplace transforms of the probability density functions of the durations of the static and dynamic phases.

1. Introduction. Modeling dry friction is a major concern in physics and engineering. Indeed, it is estimated that 20% of the world's total energy consumption is used to overcome friction [19]. The present work is motivated by the study of the probability distribution of the response of a dry friction model subjected to a certain type of random forces. To understand the problem, the simplest way is to consider the one-dimensional displacement x of an object (with unit mass) lying on a motionless surface, see Figure 1.1. The velocity is denoted by v and thus $\dot{x} = v$. Newton's law implies $\dot{v} + \mathbb{F} = b$ where $\mathbb{F}$ is the force of dry friction and b represents all the other external and internal forces. It is important to emphasize that the force $\mathbb{F}$ cannot be expressed in terms of a standard function. Below a certain threshold for the applied forces $|b| \leqslant \mu_{\mathrm{s}}$ and when $v = 0$, the object remains at rest so that we may have $v = 0$ in a non-empty time interval (static phase). Otherwise when $|b| > \mu_{\mathrm{s}}$ or $v \neq 0$ it moves (dynamic phase). Here $\mu_{\mathrm{s}} > 0$ is called static friction coefficient. *In static phase*, a necessary condition for equilibrium is therefore $\mathbb{F} = b$. *In dynamic phase*, the force $\mathbb{F}$ opposes the motion and Coulomb's law implies $|\mathbb{F}| = \mu_{\mathrm{d}}$ where $\mu_{\mathrm{d}} > 0$ is called dynamic friction coefficient. We will assume that b has the form of an internal forcing described by a well behaved function $b(\eta, v)$, where η is an external forcing that can be random. In this way, the equation of motion becomes

$$\dot{v} + \mathbb{F} = b(\eta, v). \tag{1.1}$$

The predictive power of dry friction models that appear in the science literature (engineering or physics) is generally not supported by a mathematical analysis justifying the well-posedness of the models. Surprisingly, there is no general mathematical framework for modelling dry friction where $\mu_{\mathrm{d}} \leqslant \mu_{\mathrm{s}}$.

Nonetheless, in some cases, it is possible to justify the well-posedness of the model with an ad-hoc mathematical analysis. We have in mind the case where $\mu_{\mathrm{d}} = \mu_{\mathrm{s}} \triangleq \mu$ (in this case we drop the subscript notation "s" or "d") and η is a deterministic continuous function or η is the continuous solution of a stochastic differential equation. Under such circumstances, the model is well-posed in terms of a differential inclusion (also called multivalued differential equation) [6, 22, 24] as follows

$$\dot{v} + \partial\varphi(v) \ni b(\eta, v), \tag{1.2}$$

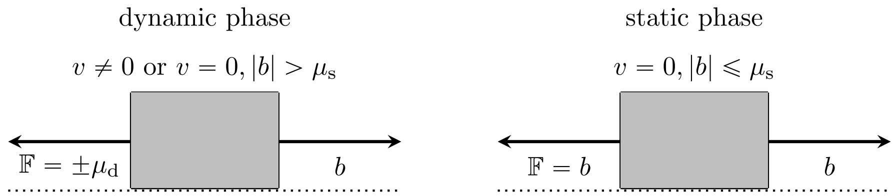

FIG. 1.1. *Dynamic (left) and static (right) phases. $\mathbb{F}$ represents the friction force in response to the applied forces b to the object (shaded area) lying on a motionless surface (dotted line).*

where $b(\eta, v)$ is a Lipschitz function, $\varphi(v) \triangleq \mu|v|$ and $\partial\varphi(v)$ is the subdifferential operator (in the sense of Moreau)

$$\partial\varphi(v) = \begin{cases} \{\mu \text{sign}(v)\} & \text{if } v \neq 0, \\ [-\mu, \mu] & \text{if } v = 0. \end{cases} \tag{1.3}$$

When $b(\eta, v) = b(v) + \eta$ and η is a white noise (formal time derivative of a Wiener process) the framework of stochastic differential inclusions can be used to define the solution [23]. Numerical techniques to simulate such dry friction systems are proposed in [3, 4, 8]. Moreover, we also have in mind the case where $\mu_d < \mu_s$ and η is a deterministic continuous function for which the framework of differential inclusions does not allow us to formulate a well-posed problem [2], but an extended variational inequality approach can then be used to resolve this issue [1].

In [18], the authors consider the case of pure dry friction (1.2) with $\mu_d = \mu_s = 1$ and $b(\eta, v) = \eta$ and replace the term $\partial\varphi(v)$ by a smoother term $\sigma_\epsilon(v) \triangleq \tanh(v/\epsilon)$. In this way, they investigate the equation $\dot{v} + \sigma_\epsilon(v) = \eta$, where η is a Gaussian process with mean zero and covariance function $\mathbb{E}[\eta(t)\eta(s)] = \frac{1}{2\tau}\exp(-\frac{|t-s|}{\tau})$. Then, they apply the unified colored noise approximation (UCNA), previously developed by Jung and Hänggi [20], to obtain an approximate expression of the stationary probability density function (pdf) of the process v for any fixed $\epsilon > 0$. They then take a formal limit as $\epsilon \to 0$ to obtain a formula for the probability of sticking (the mass of the singular part of the pdf at $v = 0$). This analytic approximation works rather well for small values of τ, but fails for values of order one. It provides, however, valuable insights into the underlying stochastic dynamics. The approach that we propose in this paper is different and has more rigorous theoretical foundations.

In this paper, we propose a piecewise deterministic Markov process (PDMP) approach to model dry friction as informally presented in (1.1). In this approach 1) the external forcing η takes discrete values and it is assumed to be a Markov jump process; 2) given the step-wise constant trajectory η, the velocity v satisfies (1.1). In this way, the process satisfies a well-posed problem. In this regard, we obtain a solid mathematical framework for deriving the Kolmogorov equations, shown in section 2, and related tools to compute statistical quantities of interest.

In the case where $\mu_d = \mu_s = \mu$, we show in Proposition 2.1 that the aforementioned process converges in distribution towards the solution of the differential inclusion (1.2) driven by the continuous solution of a stochastic differential equation as the step size in η goes to 0.

The introduction of the PDMP framework makes it possible to obtain relevant results about the dry friction problem with noise. We obtain the general representation formulas (3.18) and (4.1) for the stationary distribution of the dry friction process. The first one makes it possible to compute relevant quantities by solving

Kolmogorov equations, while the second one makes it possible to estimate the same quantities by an efficient Monte Carlo method. We compute dynamically properties in Section 4, such as the power spectral density of the velocity and the distributions of the durations of the sticking and sliding periods.

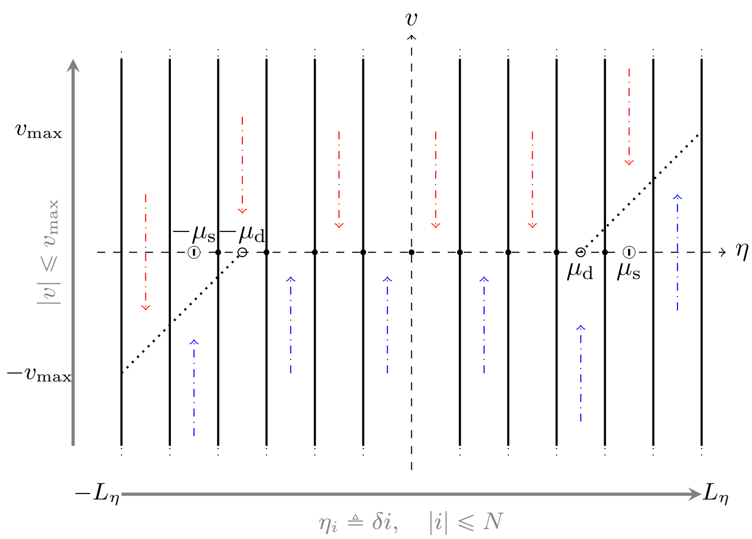

FIG. 1.2. *Moving directions of* (η^δ, v^δ) *when* $\eta_N > \mu_s > \mu_d$ *and* $b(\eta, v) = \eta - v$ *(so that* $v_{\max} = \eta_N - \mu_d$*). **Dynamic phase:** away from the black points* $\{(\eta_i, 0), i = -k_{\mu_s}, \dots, k_{\mu_s}\}$, (η^δ, v^δ) *can move continuously upward and downward respectively and by jumps along the* η*-axis. **Static phase:** at the black points* $\{(\eta_i, 0), i = -k_{\mu_s}, \dots, k_{\mu_s}\}$, (η^δ, v^δ) *moves only by jumps along the* η*-axis.*

2. A semi-discrete Markov process approach model for dry friction. Ideally we would like to consider an external forcing that is a colored noise η_t, that is itself solution of a stochastic differential equation

$$\dot{\eta} = -\tau^{-1}\eta + \sqrt{2\tau^{-1}}\dot{w} \tag{2.1}$$

where $\dot{w}_t$ is a white noise and $\tau > 0$ is the noise correlation time. The infinitesimal generator Q of the continuous Markov process η_t has the form

$$\forall \varphi \text{ twice differentiable function, } Q\varphi = \tau^{-1}(\varphi'' - \eta\varphi'). \tag{2.2}$$

The process η_t is stationary and ergodic and its invariant probability distribution is the standard normal distribution with density $g(\eta) = e^{-\eta^2/2}/\sqrt{2\pi}$.

In this section, we propose an approximation of η by a pure jump process η^δ where $\delta > 0$ is a small number. The state space of η^δ is denoted by $S^\delta = \delta\mathbb{Z} \cap [-L^\delta_\eta, L^\delta_\eta]$, with $L^\delta_\eta \to +\infty$ as $\delta \to 0$. Thus, for any $\delta > 0$, it is a finite set of equally spaced points. We denote the cardinality of S^δ by $2N+1$, $N = [L^\delta_\eta \delta^{-1}]$. We denote by k_{μ_s} the index such that $\eta_{k_{\mu_s}} \leqslant \mu_s$ and $\eta_{k_{\mu_s}+1} > \mu_s$. Therefore we have $S^\delta = \{\eta_{-N}, \dots, \eta_N\}$. We also have $\{\eta_i,\ |i| \leqslant k_{\mu_s}\} \subset [-\mu_s, \mu_s]$ and if $|i| > k_{\mu_s}$ then $|\eta_i| > \mu_s$. The process η^δ is a jump Markov process with the infinitesimal generator

$$Q^\delta\varphi(\eta) \triangleq 2\tau^{-1}\delta^{-2}\left(\alpha(\eta)\varphi(\eta+\delta) - \varphi(\eta) + (1-\alpha(\eta))\varphi(\eta-\delta)\right), \tag{2.3}$$

where

$$\alpha(\eta) = \begin{cases} \frac{1}{2}\big(1 - \frac{\eta\delta}{2}\big) & \text{if } |\eta| < N\delta, \\ 0 & \text{if } \eta = N\delta, \\ 1 & \text{if } \eta = -N\delta. \end{cases} \tag{2.4}$$

In this context, replacing η_t by η_t^δ, the pure dry friction model is replaced by

$$\dot{v}^\delta + \partial\varphi(v^\delta) \ni b(\eta^\delta, v^\delta), \tag{2.5}$$

where $b(\eta, v)$ is a Lipschitz function. For simplicity we take $b(\eta, v) = \eta - v$ in this paper.

The process $(\eta_t^\delta, v_t^\delta)$ is well defined as a càdlàg process on $S^\delta \times \mathbb{R}$. It is also possible to interpret the semi-discrete Markov process $(\eta_t^\delta, v_t^\delta)$ in terms of a Piecewise Deterministic Markov Process (PDMP) and this is the main idea of this paper. The theory developed for PDMPs then makes it possible to write Kolmogorov equations and use dedicated tools and results. We introduce the process $\boldsymbol{Z}_t^\delta = (\eta_t^\delta, \nu_t^\delta, v_t^\delta)$, where $(\eta_t^\delta, v_t^\delta)$ is the process defined here above by (2.3-2.5) and we have added the marker $\nu_t^\delta = \Theta(\eta_t^\delta, v_t^\delta)$, with

$$\Theta(\eta, v) = \begin{cases} 1 & \text{if } v > 0 \text{ or if } v = 0,\ \eta > \mu_{\rm s}, \\ -1 & \text{if } v < 0 \text{ or if } v = 0,\ \eta < -\mu_{\rm s}, \\ 0 & \text{if } v = 0,\ \eta \in [-\mu_{\rm s}, \mu_{\rm s}]. \end{cases} \tag{2.6}$$

Then $(\eta_t^\delta, \nu_t^\delta)$ is a jump Markov process which takes values in the finite space $S^\delta \times \{-1, 0, 1\}$ and which has càdlàg trajectories, v_t^δ is a real-valued continuous process, and $(\eta_t^\delta, \nu_t^\delta, v_t^\delta)$ is a Markov process, more exactly a PDMP, whose infinitesimal generator is given below. The introduction of the marker ν_t^δ makes it possible to adopt the formalism of PDMPs, with smooth flows for the continuous process v_t^δ and jumps of the mode $(\eta_t^\delta, \nu_t^\delta)$ that occur at random times (when η_t^δ jumps) and at deterministic times when the process hits the boundaries of the state space described below (when v_t^δ reaches 0 the dynamics for v_t^δ changes).

When $\mu_{\rm d} = \mu_{\rm s} = \mu$ we can establish the connection between this semi-discrete Markov process and the continuous process solution of (2.1)-(1.2). Such a result in the case $\mu_{\rm s} > \mu_{\rm d}$ is beyond the scope of this paper as the limit system is not clear in this case.

Proposition 2.1. *When $\mu_{\rm d} = \mu_{\rm s} = \mu$, the random processes $(\eta_t^\delta, v_t^\delta)$ converge in distribution in the space of the càdlàg functions to the Markov process (η_t, v_t) which is solution of* (2.1)-(1.2).

Proof. This can proved in two steps: one first shows that (η_t^δ) converges to (η_t) as $\delta \to 0$ by standard diffusion approximation theory, and then one shows that the mapping from (η_t^δ) to (v_t^δ) through (2.5) is continuous. The detailed proof is in Appendix A. □

The state space of the process $\boldsymbol{Z}^\delta$ is

$$E = \bigcup_{(\eta,\nu)\in S^\delta\times\{-1,0,1\}} E_{\eta,\nu}, \qquad E_{\eta,\nu} = \{(\eta,\nu)\} \times V_{\eta,\nu}, \tag{2.7}$$

where $V_{\eta,\nu} = (-\infty, 0)$ if $(\eta, \nu) \in \{\eta_{-k_{\mu_{\rm s}}}, \ldots, \eta_N\} \times \{-1\}$, $V_{\eta,\nu} = (0, +\infty)$ if $(\eta, \nu) \in \{\eta_{-N}, \ldots, \eta_{k_{\mu_{\rm s}}}\} \times \{1\}$, and $V_{\eta,\nu} = \mathbb{R}$ otherwise. Let $\mathcal{E}$ denote the class of measurable sets in E:

$$\mathcal{E} = \sigma\big(A_{\eta,\nu},\ A_{\eta,\nu} \in \mathcal{E}_{\eta,\nu}, (\eta, \nu) \in S^\delta \times \{-1, 0, 1\}\big), \tag{2.8}$$

where $\mathcal{E}_{\eta,\nu}$ denotes the Borel sets of $E_{\eta,\nu}$.

The Markov evolution of $\boldsymbol{Z}^\delta$ is determined by the following objects:
- the real-valued and smooth vectors fields $B(\boldsymbol{z})$, $\boldsymbol{z} = (\eta, \nu, v)$, given by

$$B(\eta,-1,v) = \mu_{\rm d} + b(\eta,v), \qquad B(\eta,0,v) = 0, \qquad B(\eta,1,v) = -\mu_{\rm d} + b(\eta,v), \quad (2.9)$$

- the constant rate function $\Lambda = 2\tau^{-1}\delta^{-2}$,
- the probability transition measure $\mathcal{Q} : \mathcal{E} \times \overline{E} \to [0,1]$ which is discrete because v does not jump and which is given by

$$\forall \nu \in \{-1,0,1\},\ \forall \eta \in \{\eta_{-N},\dots,\eta_{-k_{\mu_{\rm s}}-1}\},\ \mathcal{Q}\big((\eta,-1,0);(\eta,\nu,0)\big) = 1, \quad (2.10a)$$

$$\forall \nu \in \{-1,0,1\},\ \forall \eta \in \{\eta_{k_{\mu_{\rm s}}+1},\dots,\eta_N\},\ \mathcal{Q}\big((\eta,1,0);(\eta,\nu,0)\big) = 1, \quad (2.10b)$$

$$\forall \nu \in \{-1,1\},\ \forall \eta \in \{\eta_{-k_{\mu_{\rm s}}},\dots,\eta_{k_{\mu_{\rm s}}}\},\ \mathcal{Q}\big((\eta,0,0);(\eta,\nu,0)\big) = 1, \quad (2.10c)$$

$$\forall \eta \in \{\eta_{-k_{\mu_{\rm s}}+1},\dots,\eta_{k_{\mu_{\rm s}}-1}\},\ \mathcal{Q}\big((\eta+\delta,0,0);(\eta,0,0)\big) = \alpha(\eta), \quad (2.10d)$$

$$\forall \eta \in \{\eta_{-k_{\mu_{\rm s}}+1},\dots,\eta_{k_{\mu_{\rm s}}-1}\},\ \mathcal{Q}\big((\eta-\delta,0,0);(\eta,0,0)\big) = 1-\alpha(\eta), \quad (2.10e)$$

$$\mathcal{Q}\big((\eta_{-k_{\mu_{\rm s}}}+\delta,0,0);(\eta_{-k_{\mu_{\rm s}}},0,0)\big) = \alpha(\eta), \quad (2.10f)$$

$$\mathcal{Q}\big((\eta_{k_{\mu_{\rm s}}}-\delta,0,0);(\eta_{k_{\mu_{\rm s}}},0,0)\big) = 1-\alpha(\eta), \quad (2.10g)$$

$$\mathcal{Q}\big((\eta_{-k_{\mu_{\rm s}}-1},-1,0);(\eta_{-k_{\mu_{\rm s}}},0,0)\big) = 1-\alpha(\eta), \quad (2.10h)$$

$$\mathcal{Q}\big((\eta_{k_{\mu_{\rm s}}+1},1,0);(\eta_{k_{\mu_{\rm s}}},0,0)\big) = \alpha(\eta), \quad (2.10i)$$

$$\forall v \neq 0,\ \forall \nu \in \{-1,0,1\},\ \forall \eta \in \{\eta_{-N},\dots,\eta_N\},\ \mathcal{Q}\big((\eta+\delta,\nu,v);(\eta,\nu,v)\big) = \alpha(\eta), \quad (2.10j)$$

$$\forall v \neq 0,\ \forall \nu \in \{-1,0,1\},\ \forall \eta \in \{\eta_{-N},\dots,\eta_N\},\ \mathcal{Q}\big((\eta-\delta,\nu,v);(\eta,\nu,v)\big) = 1-\alpha(\eta). \quad (2.10k)$$

In Eq. (2.10):
- The first three lines (*a-c*) describe the jumps of the modes when the process $\boldsymbol{Z}^\delta$ reaches the boundaries of the domain ∂E:

$$\partial E = \{\eta_{-k_{\mu_{\rm s}}},\dots,\eta_N\} \times \{-1\} \times \{0\} \bigcup \{\eta_{-N},\dots,\eta_{k_{\mu_{\rm s}}}\} \times \{1\} \times \{0\}. \quad (2.11)$$

When the process $\boldsymbol{Z}^\delta$ reaches $(\eta_k,-1,0)$ for $k \in \{k_{\mu_{\rm s}}+1,\dots,N\}$, it jumps to $(\eta_k,1,0)$. When the process $\boldsymbol{Z}^\delta$ reaches $(\eta_k,-1,0)$ for $k \in \{-k_{\mu_{\rm s}},\dots,k_{\mu_{\rm s}}\}$, it jumps to $(\eta_k,0,0)$. These jumps represent the transitions from the dynamic phase with negative velocity (mode $\nu = -1$) to the dynamic phase with positive velocity (mode $\nu = 1$) and to the static phase (mode $\nu = 0$). Similarly, when the process $\boldsymbol{Z}^\delta$ reaches $(\eta_k,1,0)$ for $k \in \{-N,\dots,-k_{\mu_{\rm s}}-1\}$, it jumps to $(\eta_k,-1,0)$. When the process $\boldsymbol{Z}^\delta$ reaches $(\eta_k,1,0)$ for $k \in \{-k_{\mu_{\rm s}},\dots,k_{\mu_{\rm s}}\}$, it jumps to $(\eta_k,0,0)$. These jumps represent the transitions from the dynamic phase with positive velocity (mode $\nu = 1$) to the dynamic phase with negative velocity (mode $\nu = -1$) and to the static phase (mode $\nu = 0$).
- The following lines (*d-k*) describe the jumps of the modes that are triggerered by the random clock of the driving noise η_t^δ. The lines (*d-g*) describe the jumps from the static phase to itself, the lines (*h-i*) describe the jumps from the static phase to the dynamic phase and the lines (*j-k*) describe the jumps from the dynamic phase to itself. Note in particular that lines (*h-i*) describe how the process at the border of the static domain at $(\eta_{\pm k_{\mu_{\rm s}}},0,0)$ can escape the static domain by a jump of η_t^δ which allows the process to pull itself out of the sticked phase.

We denote by $\Phi_{\eta,\nu}(t,v)$ the flow solution of

$$\partial_t \Phi_{\eta,\nu}(t,v) = B(\eta,\nu,\Phi_{\eta,\nu}(t,v)), \qquad \Phi_{\eta,\nu}(t=0,v) = v. \tag{2.12}$$

For $\boldsymbol{z} = (\eta,\nu,v) \in E$, we denote by $T^*(\boldsymbol{z})$ the hitting time of the boundary $\partial V_{\eta,\nu}$ by $\Phi_{\eta,\nu}(t,v)$:

$$T^*(\boldsymbol{z}) = \inf\big\{t > 0, \Phi_{\eta,\nu}(t,v) = 0\big\}, \tag{2.13}$$

with the convention $\inf \varnothing = +\infty$. Since $b(\eta,v) = \eta - v$, we have here:

$$T^*(\boldsymbol{z}) = \begin{cases} -\log \frac{\eta-\mu_{\rm d}}{\eta-\mu_{\rm d}-v} & \text{if } v > 0,\ \eta - \mu_{\rm d} < 0, \\ -\log \frac{\eta+\mu_{\rm d}}{\eta+\mu_{\rm d}-v} & \text{if } v < 0,\ \eta + \mu_{\rm d} > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $\boldsymbol{z} \in E$, we define the survivor function $F_{\boldsymbol{z}}$:

$$F_{\boldsymbol{z}}(t) = \begin{cases} \exp(-\Lambda t) & \text{if } t < T^*(\boldsymbol{z}), \\ 0 & \text{if } t \geqslant T^*(\boldsymbol{z}). \end{cases} \tag{2.14}$$

The Markov process $\boldsymbol{Z}^\delta$ starting from $\boldsymbol{z}_0 = (\eta_0,\nu_0,v_0) \in E$ is defined as follows.
1) Generate a random variable T_1 such that $\mathbb{P}(T_1 > t) = F_{\boldsymbol{z}_0}(t)$. Generate a random variable $\boldsymbol{Z}_1 = (\eta_1,\nu_1,V_1)$ with distribution $\mathcal{Q}(\cdot;\eta_0,\nu_0,\Phi_{\eta_0,\nu_0}(T_1,v_0))$. The trajectory of $\boldsymbol{Z}^\delta_t$ for $t \in [0,T_1]$ is given by

$$\boldsymbol{Z}^\delta_t = \begin{cases} (\eta_0,\nu_0,\Phi_{\eta_0,\nu_0}(t,v_0)) & \text{if } 0 \leqslant t < T_1, \\ (\eta_1,\nu_1,V_1) & \text{if } t = T_1. \end{cases} \tag{2.15}$$

2) Starting from $\boldsymbol{Z}_{T_1}$, generate the next inter-jump time $T_2 - T_1$ such that $\mathbb{P}(T_2 - T_1 > t) = F_{\boldsymbol{Z}_1}(t)$ and the post-jump location $\boldsymbol{Z}_2 = (\eta_2,\nu_2,V_2)$ with distribution $\mathcal{Q}(\cdot;\eta_1,\nu_1,\Phi_{\eta_1,\nu_1}(T_2-T_1,V_1))$. The trajectory of $\boldsymbol{Z}^\delta_t$ for $t \in [T_1,T_2]$ is given by

$$\boldsymbol{Z}^\delta_t = \begin{cases} (\eta_1,\nu_1,\Phi_{\eta_1,\nu_1}(t-T_1,V_1)) & \text{if } T_1 \leqslant t < T_2, \\ (\eta_2,\nu_2,V_2) & \text{if } t = T_2. \end{cases} \tag{2.16}$$

3) Iterate. This gives a piecewise deterministic trajectory $\boldsymbol{Z}^\delta_t$ with jump times T_1, T_2, ...

The process $\boldsymbol{Z}^\delta_t = (\eta^\delta_t,\nu^\delta_t,v^\delta_t)$ is a PDMP as introduced by [7] and $(\eta^\delta_t,v^\delta_t)$ follows exactly the random dynamics (2.3-2.5). We can then use the theory and simulation methods developed for PDMPs described for instance in [7,9].

The Markov process $\boldsymbol{Z}^\delta_t$ is irreducible on E', with

$$E' = S^\delta \times \{-1\} \times (-v_{\max},0) \bigcup S^\delta \times \{1\} \times (0,+v_{\max}) \bigcup \{\eta_{-k_{\mu_{\rm s}}},\dots,\eta_{k_{\mu_{\rm s}}}\} \times \{0\} \times \{0\}, \tag{2.17}$$

with $v_{\max} = \eta_N - \mu_{\rm d}$. More exactly, starting from any $(\eta,\nu,v) \in E$, the Markov process $\boldsymbol{Z}^\delta_t$ reaches E' in finite time and it remains in E' after that time.

The domain of the generator $\mathcal{L}$ of the process $\boldsymbol{Z}^\delta_t$ contains the functions f that are smooth and bounded in v and that satisfy the boundary condition:

$$\forall \boldsymbol{z} = (\eta,\nu,v) \in \partial E, \quad f(\boldsymbol{z}) = \sum_{(\eta',\nu') \in S^\delta \times \{-1,0,1\}} f(\eta',\nu',v)\mathcal{Q}((\eta',\nu',v);\boldsymbol{z}). \tag{2.18}$$

For those functions we have [7, Theorem 5.5]

$$\mathcal{L}f(\boldsymbol{z}) = B(\boldsymbol{z})\partial_v f(\boldsymbol{z}) + \Lambda \sum_{(\eta',\nu')\in S^\delta\times\{-1,0,1\}} [f(\eta',\nu',v) - f(\boldsymbol{z})]\mathcal{Q}((\eta',\nu',v);\boldsymbol{z}). \quad (2.19)$$

More exactly, the domain of the generator consists of the functions f that satisfy the continuity condition: for all $\boldsymbol{z} = (\eta,\nu,v) \in E$,

$$t \mapsto f(\eta,\nu,\Phi_{\eta,\nu}(t,v)) \text{ is absolutely continuous for } t \in [0, T^*(\boldsymbol{z})),$$

an integrability condition (fulfilled when f is bounded), and the boundary condition (2.18) [7, Theorem 5.5]. The boundary condition (2.18) can be written more explicitely as

$$f(\eta,-1,0) = f(\eta,0,0) = f(\eta,1,0), \quad \forall \eta \in S^\delta. \quad (2.20)$$

As a consequence, for a smooth and bounded function $\phi(\boldsymbol{z})$, we have

$$\mathbb{E}[\phi(\boldsymbol{Z}_t^\delta)|\boldsymbol{Z}_0^\delta = \boldsymbol{z}] = F(0,\boldsymbol{z};t), \quad (2.21)$$

where $(s,\boldsymbol{z}) \mapsto F(s,\boldsymbol{z};t)$ is the solution of the backward Kolmogorov equation

$$\partial_s F + \mathcal{L}F = 0 \text{ in } E \text{ for } s \in (0,t), \quad (2.22)$$

with the boundary condition (2.18) on ∂E for $s \in (0,t)$, and the terminal condition $F(s = t,\boldsymbol{z};t) = \phi(\boldsymbol{z})$. One can also obtain forward Kolmogorov equations. The transition probability is such that

$$\mathbb{E}[\phi(\boldsymbol{Z}_t^\delta)|\boldsymbol{Z}_s^\delta = \boldsymbol{z}] = \mathbb{E}[\phi(\boldsymbol{Z}_{t-s}^\delta)|\boldsymbol{Z}_0^\delta = \boldsymbol{z}] = \int_E \phi(\boldsymbol{z}')p_{t-s}(\mathrm{d}\boldsymbol{z}';\boldsymbol{z}), \quad s \leqslant t \quad (2.23)$$

for any test function ϕ in the domain of the generator $\mathcal{L}$. It has the form

$$\begin{aligned} p_t(\eta',\nu',\mathrm{d}v';\boldsymbol{z}) =& p_{t,0}(\eta';\boldsymbol{z})\mathbf{1}_0(\nu')\delta(\mathrm{d}v') + p_{t,+}(\eta',v';\boldsymbol{z})\mathbf{1}_1(\nu')\mathbf{1}_{(0,+\infty)}(v')\mathrm{d}v' \\ &+ p_{t,-}(\eta',v';\boldsymbol{z})\mathbf{1}_{-1}(\nu')\mathbf{1}_{(-\infty,0)}(v')\mathrm{d}v', \end{aligned} \quad (2.24)$$

where p_t satisfies for any $\eta' \in S^\delta$:

$$\begin{aligned} \partial_t p_{t,+}(\eta',v';\boldsymbol{z}) &= \big[\partial_{v'}B(\eta',1,v') + \mathcal{Q}^*\big]p_{t,+}(\eta',v';\boldsymbol{z}), \quad v' > 0, \\ \partial_t p_{t,-}(\eta',v';\boldsymbol{z}) &= \big[\partial_{v'}B(\eta',-1,v') + \mathcal{Q}^*\big]p_{t,-}(\eta',v';\boldsymbol{z}), \quad v' < 0, \\ \partial_t p_{t,0}(\eta';\boldsymbol{z}) &= -B(\eta',1,0)p_{t,+}(\eta',0^+;\boldsymbol{z}) + B(\eta',-1,0)p_{t,-}(\eta',0^-;\boldsymbol{z}) + \mathcal{Q}^* p_{t,0}(\eta';\boldsymbol{z}), \end{aligned} \quad (2.25)$$

and $\mathcal{Q}^*$ is the adjoint (transpose) of $\mathcal{Q}$.

Proof. Let p_t be given by (2.24). For any test function ϕ in the domain of the generator $\mathcal{L}$, we have

$$\int_E \phi(\boldsymbol{z}')p_t(\mathrm{d}\boldsymbol{z}';\boldsymbol{z}) = \sum_{(\eta',\nu')\in S^\delta\times\{-1,0,1\}} \int_{V_{(\eta',\nu')}} \phi(\eta',\nu',v')p_t(\eta',\nu',\mathrm{d}v';\boldsymbol{z}).$$

By (2.25) we get

$$\sum_{\eta'}\int_0^{\infty}\phi(\eta',1,v')\big[-\partial_t-\partial_{v'}B(\eta',1,v')+\mathcal{Q}^*\big]p_{t,+}(\eta',v';\boldsymbol{z})\mathrm{d}v'$$
$$+\sum_{\eta'}\int_{-\infty}^{0}\phi(\eta',-1,v')\big[-\partial_t-\partial_{v'}B(\eta',-1,v')+\mathcal{Q}^*\big]p_{t,-}(\eta',v';\boldsymbol{z})\mathrm{d}v'$$
$$+\sum_{\eta'}-\phi(\eta',1,0)B(\eta',1,0)p_{t,+}(\eta',0^+;\boldsymbol{z})+\phi(\eta',-1,0)B(\eta',-1,0)p_{t,-}(\eta',0^-;\boldsymbol{z})$$
$$+\sum_{\eta'}\phi(\eta',0,0)(-\partial_t+\mathcal{Q}^*)p_{t,0}(\eta';\boldsymbol{z})=0.$$

After integration by parts, we find that $\int_E\phi(\boldsymbol{z}')p_{t-s}(d\boldsymbol{z}';\boldsymbol{z})$ satisfies (2.22). Since this holds true for any smooth, bounded test functions ϕ satisfying the boundary conditions (2.20), we get that p_t is the transition probability. □

Note that $p_{t,0}$ is supported on $\{\eta_{-k_{\mu_s}},\dots,\eta_{k_{\mu_s}}\}$ and satisfies

$$\begin{aligned}
&\partial_t p_{t,0}(\eta';\boldsymbol{z})=\mathfrak{T}_t(\eta',z)+\mathcal{Q}^*p_{t,0}(\eta';\boldsymbol{z}),\quad |\eta'|\leqslant\eta_{k_\mu},\\
&0=\mathfrak{T}_t(\eta',z)+\mathcal{Q}^*p_{t,0}(\eta';\boldsymbol{z}),\quad |\eta'|=\eta_{k_\mu+1},\\
&0=\mathfrak{T}_t(\eta',z),\quad |\eta'|>\eta_{k_\mu+1},\\
&\mathfrak{T}_t(\eta',z)\triangleq B(\eta',-1,0)p_{t,-}(\eta',0^-;\boldsymbol{z})-B(\eta',1,0)p_{t,+}(\eta',0^+;\boldsymbol{z}).
\end{aligned}$$

3. Ergodicity and stationary state. Given f a function in the domain of $\mathcal{L}$, we consider the function

$$u_\lambda(\eta,\nu,v;f)\triangleq\mathbb{E}_{(\eta,\nu,v)}\Big[\int_0^{\infty}e^{-\lambda s}f(\eta_s^\delta,\nu_s^\delta,v_s^\delta)\mathrm{d}s\Big].\tag{3.1}$$

From the theory of Markov processes [7,10], it satisfies the equation

$$\lambda u_\lambda-\mathcal{L}u_\lambda=f\ \text{ in }\ E.\tag{3.2}$$

Let us introduce the stopping times

$$\tau_0\triangleq\inf\big\{t\geqslant0,\,(\eta_t^\delta,\nu_t^\delta,v_t^\delta)\in\{\mathfrak{s}_-,\mathfrak{s}_+\}\big\},\tag{3.3}$$
$$\hat\tau_n\triangleq\inf\big\{t\geqslant\tau_{n-1},\,v_t^\delta=0\text{ and }|\eta_t^\delta|\leqslant\mu_s\big\},\quad n\geqslant1,\tag{3.4}$$
$$\tau_n\triangleq\inf\big\{t\geqslant\hat\tau_n,\,(\eta_t^\delta,\nu_t^\delta,v_t^\delta)\in\{\mathfrak{s}_-,\mathfrak{s}_+\}\big\},\quad n\geqslant1,\tag{3.5}$$

where $\mathfrak{s}_\pm\triangleq(\pm\eta_{k_{\mu_s}+1},\pm1,0)$. The two points $\mathfrak{s}_\pm$ are the two possible exit points of the static phase. The time τ_0 is introduced in order to deal with a well-defined starting point. For $n\geqslant1$, the stopping times $\hat\tau_n$ and τ_n represent respectively the entry and exit times of the n-th static phases which are defined as the time intervals when $(\eta_t^\delta,\nu_t^\delta,v_t^\delta)\in D^0$, $D^0=\{(k,0,0),k=-k_{\mu_s},\dots,k_{\mu_s}\}$. The recurrence and ergodicity of the Markov process is a consequence of the following proposition (see Appendix B for the proof).

Proposition 3.1. *We have* $\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big]<\infty$.

We propose below a representation formula for the stationary measure of the process $(\eta_t^\delta,\nu_t^\delta,v_t^\delta)$ that is based on the functions $h_\lambda^\pm$ and w_λ that are defined by

$$h_\lambda^\pm(\eta,\nu,v)\triangleq\mathbb{E}_{(\eta,\nu,v)}\big[e^{-\lambda\tau_1}\mathbf{1}_{(\eta_{\tau_1}^\delta,\nu_{\tau_1}^\delta,v_{\tau_1}^\delta)=\mathfrak{s}_\pm}\big],\tag{3.6}$$
$$w_\lambda(\eta,\nu,v;f)\triangleq\mathbb{E}_{(\eta,\nu,v)}\Big[\int_0^{\tau_1}e^{-\lambda s}f(\eta_s^\delta,\nu_s^\delta,v_s^\delta)\mathrm{d}s\Big],\tag{3.7}$$

where f is a bounded function. The functions $h_\lambda^\pm$ and w_λ can be computed as explained in the following proposition.

PROPOSITION 3.2. *Let us introduce two absorbing states $\mathfrak{s}'_\pm$ and a modified kernel $\mathcal{L}'$ such that*

$$\mathcal{L}'f(\boldsymbol{z}) = B(\boldsymbol{z})\partial_v f(\boldsymbol{z}) + \lambda \sum_{(\eta',\nu')\in S^\delta\times\{-1,0,1\}} [f(\eta',\nu',v) - f(\boldsymbol{z})]\mathcal{Q}((\eta',\nu',v);\boldsymbol{z}) \quad (3.8)$$

for all $\boldsymbol{z} \in \overline{E}\backslash\{(\eta_{-k_{\mu_s}},0,0),(\eta_{k_{\mu_s}},0,0)\}$, $\mathcal{L}'f(\mathfrak{s}'_\pm) = 0$, and

$$\begin{aligned}\mathcal{L}'f\big((\eta_{k_{\mu_s}},0,0)\big) =&\alpha(\eta_{k_{\mu_s}})\big[f(\mathfrak{s}'_+) - f\big((\eta_{k_{\mu_s}},0,0)\big)\big]\\ &+ \big(1-\alpha(\eta_{k_{\mu_s}})\big)\big[f\big((\eta_{k_{\mu_s}-1},0,0)\big) - f\big((\eta_{k_{\mu_s}},0,0)\big)\big], \quad (3.9)\\ \mathcal{L}'f\big((\eta_{-k_{\mu_s}},0,0)\big) =&\alpha(\eta_{-k_{\mu_s}})\big[f\big((\eta_{-k_{\mu_s}+1},0,0)\big) - f\big((\eta_{-k_{\mu_s}},0,0)\big)\big]\\ &+ \big(1-\alpha(\eta_{-k_{\mu_s}})\big)\big[f(\mathfrak{s}'_-) - f\big((\eta_{-k_{\mu_s}},0,0)\big)\big]. \quad (3.10)\end{aligned}$$

The functions $h_\lambda^\pm$ and $w_\lambda(\cdot;f)$ defined by (3.6) and (3.7) satisfy

$$\lambda h_\lambda^+ - \mathcal{L}'h_\lambda^+ = 0 \ \text{ in } E, \qquad h_\lambda^+(\mathfrak{s}'_+) = 1,\ h_\lambda^+(\mathfrak{s}'_-) = 0, \quad (3.11)$$

$$\lambda h_\lambda^- - \mathcal{L}'h_\lambda^- = 0 \ \text{ in } E, \qquad h_\lambda^-(\mathfrak{s}'_+) = 0,\ h_\lambda^-(\mathfrak{s}'_-) = 1, \quad (3.12)$$

and

$$\lambda w_\lambda(\cdot;f) - \mathcal{L}'w_\lambda(\cdot;f) = f \ \text{ in } E, \qquad w_\lambda(\mathfrak{s}'_\pm;f)) = 0. \quad (3.13)$$

Proof. A trajectory of $(\eta^\delta,\nu^\delta,v^\delta)$ can reach $\mathfrak{s}_+$ by different ways. Indeed, just before the time at which $\mathfrak{s}_+$ is reached, the process can be in a static phase at the point $(\eta_{k_{\mu_s}},0,0)$ (and it jumps from the static phase to the dynamic phase) or else it can be in a dynamic phase at $(\eta_{k_{\mu_s}+1},-1,v)$, $v<0$, and it jumps when v_t^δ reaches 0 to $\mathfrak{s}_+$. This comment motivates the introduction of the two absorbing states $\mathfrak{s}'_\pm$. The Markov process $(\eta^{\delta,\prime},\nu^{\delta,\prime},v^{\delta,\prime})$ with the modified kernel $\mathcal{L}'$ follows the same dynamics as the original process $(\eta^\delta,\nu^\delta,v^\delta)$, except for one point: when the original process is at $(\eta_{k_{\mu_s}},0,0)$ (resp. $(\eta_{-k_{\mu_s}},0,0)$) and jumps to the east (resp. to the west), it jumps to $\mathfrak{s}_+$ (resp. $\mathfrak{s}_-$); when the modified process is at $(\eta_{k_{\mu_s}},0,0)$ (resp. $(\eta_{-k_{\mu_s}},0,0)$) and jumps to the east (resp. to the west), it jumps to $\mathfrak{s}'_+$ (resp. $\mathfrak{s}'_-$) and does not move anymore. As a consequence, we have for any $(\eta,\nu,v) \in E$:

$$h_\lambda^\pm(\eta,\nu,v) \triangleq \mathbb{E}_{(\eta,\nu,v)}\big[e^{-\lambda\tau'_1}\mathbf{1}_{(\eta^{\delta,\prime}_{\tau'_1},\nu^{\delta,\prime}_{\tau'_1},v^{\delta,\prime}_{\tau'_1})=\mathfrak{s}'_\pm}\big], \quad (3.14)$$

$$w_\lambda(\eta,\nu,v;f) \triangleq \mathbb{E}_{(\eta,\nu,v)}\Big[\int_0^{\tau'_1} e^{-\lambda s} f(\eta_s^{\delta,\prime},\nu_s^{\delta,\prime},v_s^{\delta,\prime})\mathrm{d}s\Big], \quad (3.15)$$

where

$$\tau'_1 = \inf\big\{t \geqslant 0, (\eta_t^{\delta,\prime},\nu_t^{\delta,\prime},v_t^{\delta,\prime}) \in \{\mathfrak{s}'_-,\mathfrak{s}'_+\}\big\},$$

and the statement of the proposition follows immediately. □

The function $w_\lambda(\cdot;1)$ is $w_\lambda(\cdot;f)$ when $f=1$. The following proposition is inspired from [21] and is proved in Appendix C.

Proposition 3.3. *We have the representation formula*

$$u_\lambda(\eta,\nu,v;f) = w_\lambda(\eta,\nu,v;f) - \pi_\lambda(f) w_\lambda(\eta,\nu,v;1) + \mu_\lambda(f)\big(h_\lambda^+(\eta,\nu,v) - h_\lambda^-(\eta,\nu,v)\big) + \frac{\pi_\lambda(f)}{\lambda}, \tag{3.16}$$

for all $(\eta,\nu,v)\in E$*, where*

$$\pi_\lambda(f) \triangleq \frac{w_\lambda(\mathfrak{s}_+;f)+w_\lambda(\mathfrak{s}_-;f)}{2w_\lambda(\mathfrak{s}_+;1)} \quad \text{and} \quad \mu_\lambda(f) \triangleq \frac{w_\lambda(\mathfrak{s}_+;f)-w_\lambda(\mathfrak{s}_-;f)}{2(1-h_\lambda^+(\mathfrak{s}_+)+h_\lambda^-(\mathfrak{s}_+))}. \tag{3.17}$$

We have the following characterization for the stationary distribution of the process $(\eta_t^\delta,\nu_t^\delta,v_t^\delta)$*:*

$$\pi(f) = \lim_{\lambda\to 0} \lambda u_\lambda(\eta,\nu,v;f) = \frac{w_0(\mathfrak{s}_+;f)+w_0(\mathfrak{s}_-;f)}{2w_0(\mathfrak{s}_+;1)}, \tag{3.18}$$

for any bounded function f*.*

4. Dynamical properties.

4.1. Excursions. The random variables $(\tau_{n+1}-\tau_n)_{n\geqslant 0}$ are integrable, independent and identically distributed. This follows from the strong Markov property, from the fact that τ_1 is integrable under $\mathbb{P}_{\mathfrak{s}_+}$ (we have $\tau_0 = 0$ a.s. under $\mathbb{P}_{\mathfrak{s}_+}$), and from the symmetry of the system which implies that the distributions of τ_1 starting from $\mathfrak{s}_+$ and from $\mathfrak{s}_-$ are identical.

For any bounded (in fact, π-integrable) function f, $\pi(f)$ can be computed by (3.18). By ergodicity we also have

$$\pi(f) = \frac{1}{\mathbb{E}_{\mathfrak{s}_+}[\tau_1]}\mathbb{E}_{\mathfrak{s}_+}\Big[\int_0^{\tau_1} f_{\text{even}}(\eta_s^\delta,\nu_s^\delta,v_s^\delta)\mathrm{d}s\Big], \tag{4.1}$$

with $f_{\text{even}}(\eta,\nu,v) \triangleq (f(\eta,\nu,v)+f(-\eta,-\nu,-v))/2$. Indeed, on the one hand, by symmetry of the system $(-\eta_t^\delta,-\nu_t^\delta,-v_t^\delta)$ has the same stationary distribution as $(\eta_t^\delta,\nu_t^\delta,v_t^\delta)$. Therefore, if f is odd, then $\pi(f)=0$ [this follows also from (3.18) since $w_0(\mathfrak{s}_-,f) = w_0(\mathfrak{s}_+,f(-\cdot)) = w_0(\mathfrak{s}_+,-f) = -w_0(\mathfrak{s}_+,f)$ when f is odd]. On the other hand, denoting $\varepsilon_n = \text{sgn}(\eta_{\tau_n}^\delta)$, the strong Markov property implies that the excursions $(\varepsilon_n\eta_{\tau_n+t}^\delta,\varepsilon_n\nu_{\tau_n+t}^\delta,\varepsilon_n v_{\tau_n+t}^\delta)_{t\in[0,\tau_{n+1}-\tau_n]}$ are independent and identically distributed with the distribution of $(\eta_t^\delta,\nu_t^\delta,v_t^\delta)_{t\in[0,\tau_1]}$ starting from $\mathfrak{s}_+$. Therefore, if f is even, then $\pi(f) = \frac{1}{\mathbb{E}_{\mathfrak{s}_+}[\tau_1]}\mathbb{E}_{\mathfrak{s}_+}\big[\int_0^{\tau_1} f(\eta_s^\delta,\nu_s^\delta,v_s^\delta)\mathrm{d}s\big]$.

4.2. Power spectral density. The power spectral density of the process v_t^δ can be defined as [17]

$$S_v(\omega) \triangleq \lim_{T\to\infty}\frac{1}{T}S_{v,T}(\omega) \quad \text{where} \quad S_{v,T}(\omega) \triangleq \mathbb{E}\Big[\Big|\int_0^T v_t^\delta \exp(-\boldsymbol{i}\omega t)\mathrm{d}t\Big|^2\Big]. \tag{4.2}$$

We have

$$S_v(\omega) = 2\text{Re}\big(\pi\big(v\hat{\phi}(\omega,\eta,v)\big)\big), \tag{4.3}$$

where $\hat{\phi}$ is the solution of

$$(\boldsymbol{i}\omega - \mathcal{L})\hat{\phi} = v \text{ in } E. \tag{4.4}$$

Remark: For $\omega = 0$ the solution is unique up to an additive constant which does not play any role in the evaluation of $S_v(\omega)$.

Proof. We have

$$S_v(\omega) = 2\int_0^{+\infty} \mathbb{E}_\pi[v_0^\delta v_t^\delta]\cos(\omega t)\mathrm{d}t, \qquad \mathbb{E}_\pi[v_0^\delta v_t^\delta] = \pi\big(v\phi(t,\eta,\nu,v)\big),$$

where $\phi(t,\eta,\nu,v) = \mathbb{E}[v_t^\delta | \eta_0^\delta = \eta, \nu_0^\delta = \nu, v_0^\delta = v]$ is the solution of

$$(\partial_t - \mathcal{L})\phi = 0 \text{ in } E,\ t > 0, \qquad \phi(t=0,v,\nu,\eta) = v.$$

The function $\hat\phi(\omega,\eta,\nu,v) = \int_0^\infty \phi(t,\eta,\nu,v)\exp(-\boldsymbol{i}\omega t)\mathrm{d}t$ satisfies (4.4) because $\int_0^{+\infty} \partial_t\phi(t,\eta,\nu,v)\exp(-\boldsymbol{i}\omega t)\mathrm{d}t = -\phi(t=0,\eta,\nu,v) + \boldsymbol{i}\omega\hat\phi(\omega,\eta,\nu,v)$. We then get (4.3). □

4.3. Probability of sticking. We are interested in the probability of sticking, that is the empirical proportion of time spent in the sticking phase:

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathbf{1}_{(\eta_t^\delta,\nu_t^\delta,v_t^\delta)\in D^0}\mathrm{d}t, \tag{4.5}$$

where $D^0 = \{(\eta_k,0,0), k = -k_{\mu_s},\dots,k_{\mu_s}\}$. By ergodicity the limit (4.5) exists almost surely, is deterministic and its value is independent of the starting point $(\eta_0^\delta,\nu_0^\delta,v_0^\delta)$ and given by

$$P_{\text{stick}} = \frac{\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1 - \hat\tau_1\big]}{\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big]} = 1 - \frac{\mathbb{E}_{\mathfrak{s}_+}\big[\hat\tau_1\big]}{\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big]}. \tag{4.6}$$

This number can be evaluated as follows.

Proposition 4.1. *Let $\hat W(\eta,\nu,v)$, $\check W(\eta,\nu,v)$, $(\eta,\nu,v)\in E$, and $W(\eta)$, $\eta\in\{\eta_{-k_{\mu_s}-1},\dots,\eta_{k_{\mu_s}+1}\}$, be the solutions of*

$$\mathcal{L}\hat W = -1 \ \text{ in } \ E\backslash D^0, \qquad \hat W = 0 \ \text{ in } \ D^0, \tag{4.7}$$

$$\mathcal{L}\check W = -1 \ \text{ in } \ E\backslash\{(\eta_{\pm k_{\mu_s}},0,0)\}, \qquad \check W(\eta_{\pm k_{\mu_s}},0,0) = 0, \tag{4.8}$$

$$Q^\delta W = -1 \ \text{ in } \ \{\eta_{-k_{\mu_s}},\dots,\eta_{k_{\mu_s}}\}, \qquad W(\eta_{\pm(k_\mu+1)}) = 0. \tag{4.9}$$

Then we have $\mathbb{E}_{\mathfrak{s}_+}\big[\hat\tau_1\big] = \hat W(\mathfrak{s}_+)$, $\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big] = \check W(\mathfrak{s}_+) + W(\eta_{k_{\mu_s}})$, and

$$P_{\text{stick}} = 1 - \frac{\hat W(\mathfrak{s}_+)}{\check W(\mathfrak{s}_+) + W(\eta_{k_{\mu_s}})}. \tag{4.10}$$

Note that we also have $\mathbb{E}_{\mathfrak{s}_+}\big[\hat\tau_1\big] = w_0(\mathfrak{s}_+,\mathbf{1}_{E\backslash D^0})$ and $\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big] = w_0(\mathfrak{s}_+,1) = w_0(\mathfrak{s}_+,\mathbf{1}_{E\backslash D^0}) + w_0(\mathfrak{s}_+,\mathbf{1}_{D^0})$, where w_0 has been introduced in (3.7), so that we can also write

$$P_{\text{stick}} = 1 - \frac{w_0(\mathfrak{s}_+,\mathbf{1}_{E\backslash D^0})}{w_0(\mathfrak{s}_+,\mathbf{1}_{E\backslash D^0}) + w_0(\mathfrak{s}_+,\mathbf{1}_{D^0})} = \frac{w_0(\mathfrak{s}_+,\mathbf{1}_{D^0})}{w_0(\mathfrak{s}_+,\mathbf{1}_{E\backslash D^0}) + w_0(\mathfrak{s}_+,\mathbf{1}_{D^0})}. \tag{4.11}$$

Proof. We introduce

$$\check\tau_1 = \inf\big\{t \geqslant 0,\ v_t^\delta = 0 \text{ and } \eta_t^\delta \in \{-\eta_{k_{\mu_s}},\eta_{k_{\mu_s}}\}\big\}.$$

We have $\mathbb{E}_{(\eta,\nu,v)}\big[\hat{\tau}_1\big] = \hat{W}(\eta,\nu,v)$ for any $(\eta,\nu,v) \in E\backslash D^0$, $\mathbb{E}_{(\eta,\nu,v)}\big[\check{\tau}_1\big] = \check{W}(\eta,\nu,v)$ for any $(\eta,\nu,v) \in E$, and $\mathbb{E}_{(\eta,0,0)}[\tau_1] = W(\eta)$ for any $\eta \in \{\eta_{-k_{\mu_s}},\dots,\eta_{k_{\mu_s}}\}$. Moreover, by the strong Markov property,

$$\begin{aligned}\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big] &= \mathbb{E}_{\mathfrak{s}_+}\big[\check{\tau}_1\big] + \mathbb{E}_{\mathfrak{s}_+}\big[\mathbb{E}[\tau_1 - \check{\tau}_1 | \check{\tau}_1]\big] \\ &= \mathbb{E}_{\mathfrak{s}_+}\big[\check{\tau}_1\big] + \mathbb{E}_{(\eta_{k_{\mu_s}},0,0)}[\tau_1]\mathbb{P}_{\mathfrak{s}_+}\big(\eta^{\delta}_{\check{\tau}_1} = \eta_{k_{\mu_s}}\big) + \mathbb{E}_{(\eta_{-k_{\mu_s}},0,0)}[\tau_1]\mathbb{P}_{\mathfrak{s}_+}\big(\eta^{\delta}_{\check{\tau}_1} = \eta_{-k_{\mu_s}}\big).\end{aligned}$$

By symmetry of the system we have $\mathbb{E}_{(\eta_{-k_{\mu_s}},0,0)}[\tau_1] = \mathbb{E}_{(\eta_{k_{\mu_s}},0,0)}[\tau_1]$, so that $\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big] = \mathbb{E}_{\mathfrak{s}_+}\big[\check{\tau}_1\big] + \mathbb{E}_{(\eta_{k_{\mu_s}},0,0)}\big[\tau_1\big]$ and we get

$$\mathbb{E}_{\mathfrak{s}_+}\big[\tau_1\big] = \check{W}(\mathfrak{s}_+) + W(\eta_{k_{\mu_s}}),$$

which completes the proof of the proposition. □

4.4. Distributions of sticking and sliding periods. The dynamics of the system consists of an alternate sequence of sticking periods and sliding (dynamic) periods. Each sliding period starts from $\mathfrak{s}_\pm$ and the system is symmetric for the transform $(\eta,\nu,v) \to (-\eta,-\nu,-v)$. As a result the Laplace transform of the distribution of the duration of a sticking period is

$$F_{\rm stick}(\lambda) = \mathbb{E}_{\mathfrak{s}_+}\big[e^{-\lambda(\tau_1 - \hat{\tau}_1)}\big]. \tag{4.12}$$

This Laplace transform can be evaluated as follows.

Proposition 4.2. *For $|k| \leqslant k_{\mu_s}$ and $\lambda > 0$, let $\hat{P}_k(\eta,\nu,v)$, $(\eta,\nu,v) \in E$, and $F_\lambda(\eta)$, $\eta \in \{\eta_{-k_{\mu_s}-1},\dots,\eta_{k_{\mu_s}+1}\}$ be the solutions of*

$$\mathcal{L}\hat{P}_k = 0 \ \ in \ \ E\backslash D^0, \qquad \hat{P}_k(\eta_k,0,0) = 1, \qquad \hat{P}_k = 0 \ \ in \ \ D^0\backslash\{(\eta_k,0,0)\}, \tag{4.13}$$

$$\lambda F_\lambda - Q^\delta F_\lambda = 0 \ \ in \ \ \{\eta_{-k_{\mu_s}},\dots,\eta_{k_{\mu_s}}\}, \qquad F_\lambda(\eta_{\pm(k_{\mu_s}+1)}) = 1. \tag{4.14}$$

Then we have $\mathbb{P}_{\mathfrak{s}_+}\big(\eta^\delta_{\hat{\tau}_1} = \eta_k\big) = \hat{P}_k(\mathfrak{s}_+)$ for any $k \in \{-k_{\mu_s},\dots,k_{\mu_s}\}$, $\mathbb{E}_{(\eta,0,0)}\big[e^{-\lambda\tau_1}\big] = F_\lambda(\eta)$ for any $\eta \in \{\eta_{-k_{\mu_s}},\dots,\eta_{k_{\mu_s}}\}$ and $\lambda > 0$, and

$$F_{\rm stick}(\lambda) = \sum_{k=-k_{\mu_s}}^{k_{\mu_s}} F_\lambda(\eta_k)\hat{P}_k(\mathfrak{s}_+). \tag{4.15}$$

Proof. This is a consequence of the strong Markov property:

$$F_{\rm stick}(\lambda) = \mathbb{E}_{\mathfrak{s}_+}\big[\mathbb{E}[e^{-\lambda(\tau_1-\hat{\tau}_1)}|\hat{\tau}_1,\eta^\delta_{\hat{\tau}_1}]\big] = \sum_{k=-k_{\mu_s}}^{k_{\mu_s}} \mathbb{P}_{\mathfrak{s}_+}\big(\eta^\delta_{\hat{\tau}_1} = \eta_k\big)\mathbb{E}_{(\eta_k,0,0)}\big[e^{-\lambda\tau_1}\big].$$

□

Similarly the Laplace transform of the distribution of the duration of a sliding period is

$$F_{\rm slid}(\lambda) = \mathbb{E}_{\mathfrak{s}_+}\big[e^{-\lambda\hat{\tau}_1}\big], \tag{4.16}$$

and it can be expressed as follows.

Proposition 4.3. *For $\lambda > 0$, let $G_\lambda(\eta,\nu,v)$, $(\eta,\nu,v) \in E$ be the solution of*

$$\lambda G_\lambda - \mathcal{L}G_\lambda = 0 \ \ in \ \ E\backslash D^0\,, \qquad G_\lambda = 1 \ \ in \ \ D^0\,. \tag{4.17}$$

Then we have

$$F_{\text{slid}}(\lambda) = G_\lambda(\mathfrak{s}_+). \tag{4.18}$$

From (3.7) we also have

$$\begin{aligned} w_\lambda(\mathfrak{s}_+, \mathbf{1}_{E\setminus D^0}) &= \mathbb{E}_{\mathfrak{s}_+}\Big[\int_0^{\tau_1} e^{-\lambda s}\mathbf{1}_{(\eta_s^\delta,\nu_s^\delta,v_s^\delta)\in E\setminus D^0}\mathrm{d}s\Big] = \mathbb{E}_{\mathfrak{s}_+}\Big[\int_0^{\hat\tau_1} e^{-\lambda s}\mathrm{d}s\Big] \\ &= \frac{1}{\lambda}\big(1-\mathbb{E}_{\mathfrak{s}_+}\big[e^{-\lambda\hat\tau_1}\big]\big), \end{aligned} \tag{4.19}$$

so that we get $F_{\text{slid}}(\lambda) = 1 - \lambda w_\lambda(\mathfrak{s}_+, \mathbf{1}_{E\setminus D^0})$.

5. Numerics. In this section, we are interested in the numerical computation of the following statistics under the stationary measure:

$$S^1 \triangleq P_{\text{stick}}, \qquad S^2 \triangleq \mathbb{E}_\pi[(v_t^\delta)^2], \qquad S^3 \triangleq \mathbb{P}_\pi\left(|\eta_t^\delta| \leqslant \mu_{\text{s}}\right), \qquad S^4 \triangleq \mathbb{E}_\pi[(\eta_t^\delta)^2]. \tag{5.1}$$

We use two different methods. The first method is probabilistic and relies on the representation formula (4.1) of S^i in terms of the excursions $\{(\eta_s^\delta,\nu_s^\delta,v_s^\delta),\ s\in[0,\tau_1]\}$ starting from $\mathfrak{s}_+$. For instance, P_{stick} is estimated with $f(\eta,v) = \mathbf{1}_{\{v=0,|\eta|\leqslant\mu_{\text{s}}\}}$, for the other statistics we replace the integrand f by the function v^2,$\mathbf{1}_{\{|\eta|\leqslant\mu_{\text{s}}\}}$ and η^2. The second method is deterministic and consists in solving the equation (3.2) with f in the right-hand side and we look for $\lambda u_\lambda(f)$ for small λ.

5.1. The probabilistic method: simulation of the excursions of $(\eta_s^\delta,\nu_s^\delta,v_s^\delta)$ on $[0,\tau_1]$. We generate a large number, say N, of independent and identically distributed (i.i.d.) versions of the excursions $\{(\eta_s^\delta,\nu_s^\delta,v_s^\delta),\, s\in[0,\tau_1]\}$ where $(\eta_0^\delta,\nu_0^\delta,v_0^\delta) = \mathfrak{s}_+$. From this family of excursions, we construct N i.i.d. versions $(X_i,Y_i)_{i=1}^N$ of

$$(X,Y) \triangleq \Big(\int_0^{\tau_1} f(\eta_s^\delta,v_s^\delta)\mathrm{d}s, \tau_1\Big), \tag{5.2}$$

whose empirical mean and covariance are denoted by $(\hat X_N,\hat Y_N)$ and $\hat{\mathbf{C}}_N$ respectively:

$$\hat X_N = \frac{1}{N}\sum_{i=1}^N X_i, \quad \hat Y_N = \frac{1}{N}\sum_{i=1}^N Y_i, \quad \hat{\mathbf{C}}_N = \frac{1}{N}\sum_{i=1}^N \begin{pmatrix} X_i^2 & X_iY_i \\ X_iY_i & Y_i^2\end{pmatrix} - \begin{pmatrix} \hat X_N^2 & \hat X_N\hat Y_N \\ \hat X_N\hat Y_N & \hat Y_N^2\end{pmatrix}. \tag{5.3}$$

Here X is one of the four random variables

$$\int_0^{\tau_1}\mathbf{1}_{\{v_s^\delta=0,\,|\eta_s^\delta|\leqslant\mu_{\text{s}}\}}\mathrm{d}s, \quad \int_0^{\tau_1}\mathbf{1}_{\{|\eta_s^\delta|\leqslant\mu_{\text{s}}\}}\mathrm{d}s, \quad \int_0^{\tau_1}(v_s^\delta)^2\mathrm{d}s, \quad \int_0^{\tau_1}(\eta_s^\delta)^2\mathrm{d}s. \tag{5.4}$$

From the central limit theorem, we have the convergence in distribution of the pair $(\hat X_N,\hat Y_N)$:

$$\sqrt{N}\Big(\begin{pmatrix}\hat X_N\\ \hat Y_N\end{pmatrix} - \begin{pmatrix}\mathbb{E}[X]\\ \mathbb{E}[Y]\end{pmatrix}\Big) \overset{N\to+\infty}{\longrightarrow} \mathcal{N}\Big(\begin{pmatrix}0\\0\end{pmatrix},\mathbf{C}\Big), \tag{5.5}$$

where

$$\mathbf{C} = \begin{pmatrix}\mathbb{E}[X^2]-\mathbb{E}[X]^2 & \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]\\ \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y] & \mathbb{E}[Y^2]-\mathbb{E}[Y]^2\end{pmatrix}. \tag{5.6}$$

By the delta method, we get

$$\sqrt{N}\Big(\frac{\hat{X}_N}{\hat{Y}_N}-\frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\Big)\overset{N\to+\infty}{\longrightarrow}\mathcal{N}(0,\sigma^2),\qquad \sigma^2=\begin{pmatrix}1/\mathbb{E}[Y]\\ -\mathbb{E}[X]/\mathbb{E}[Y]^2\end{pmatrix}^T\mathbf{C}\begin{pmatrix}1/\mathbb{E}[Y]\\ -\mathbb{E}[X]/\mathbb{E}[Y]^2\end{pmatrix}. \tag{5.7}$$

By Slutsky's theorem,

$$\sqrt{N}\hat{\sigma}_N^{-1}\Big(\frac{\hat{X}_N}{\hat{Y}_N}-\frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\Big)\overset{N\to+\infty}{\longrightarrow}\mathcal{N}(0,1),\qquad \hat{\sigma}_N^2=\begin{pmatrix}1/\hat{Y}_N\\ -\hat{X}_N/\hat{Y}_N^2\end{pmatrix}^T\hat{\mathbf{C}}_N\begin{pmatrix}1/\hat{Y}_N\\ -\hat{X}_N/\hat{Y}_N^2\end{pmatrix}. \tag{5.8}$$

We can deduce from this convergence in distribution an asymptotic 95 % confidence interval for $\mathbb{E}[X]/\mathbb{E}[Y]$ (which is the quantity of interest by (4.1)):

$$\mathbb{P}\left(\frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\in\Big(\frac{\hat{X}_N}{\hat{Y}_N}-1.96\hat{\sigma}_N N^{-\frac{1}{2}},\frac{\hat{X}_N}{\hat{Y}_N}+1.96\hat{\sigma}_N N^{-\frac{1}{2}}\Big)\right)\overset{N\to+\infty}{\longrightarrow}0.95. \tag{5.9}$$

Note that

$$\begin{aligned}X=&\sum_{0\leqslant i\leqslant i_{\tau_1}-1}\int_{T_i}^{T_{i+1}}f(\eta_{T_i}^\delta,\Phi_{\eta_{T_i}^\delta,\nu_{T_i}^\delta}(s-T_i,v_{T_i}^\delta))\mathrm{d}s\\&+\int_{T_{i_{\tau_1}}}^{\tau_1}f(\eta_{T_{i_{\tau_1}}}^\delta,\Phi_{\eta_{T_{i_{\tau_1}}}^\delta,\nu_{T_{i_{\tau_1}}}^\delta}(s-T_{i_{\tau_1}},v_{T_{i_{\tau_1}}}^\delta))\mathrm{d}s,\end{aligned} \tag{5.10}$$

where $i_{\tau_1}\triangleq\max\{i,\ T_i\leqslant\tau_1\}$. If f does not depend on v then the formula above becomes simpler

$$\int_0^{\tau_1}f(\eta_s^\delta)\mathrm{d}s=\sum_{0\leqslant i\leqslant i_{\tau_1}-1}f(\eta_{T_i}^\delta)(T_{i+1}-T_i)+f(\eta_{T_{i_{\tau_1}}}^\delta)(\tau_1-T_{i_{\tau_1}}). \tag{5.11}$$

It includes the cases where $f(\eta)=\mathbf{1}_{\{|\eta|\leqslant\mu_s\}}$ and $f(\eta)=\eta^2$. With the particular choice of $f(\eta,v)=\mathbf{1}_{\{v=0,\ |\eta|\leqslant\mu_s\}}$, the formula remains simple as well

$$\int_0^{\tau_1}f(\eta_s^\delta,v_s^\delta)\mathrm{d}s=\sum_{0\leqslant i\leqslant i_{\tau_1}-1}f(\eta_{T_i}^\delta,v_{T_i}^\delta)(T_{i+1}-T_i)+f(\eta_{T_{i_{\tau_1}}}^\delta,v_{T_{i_{\tau_1}}}^\delta)(\tau_1-T_{i_{\tau_1}}). \tag{5.12}$$

When $f(\eta,v)=v^2$ then it becomes slightly more complicated

$$\begin{aligned}\int_0^{\tau_1}f(\eta_s^\delta,v_s^\delta)\mathrm{d}s=&\sum_{0\leqslant i\leqslant i_{\tau_1}-1}\int_{T_i}^{T_{i+1}}\Big(\Phi_{\eta_{T_i}^\delta,\nu_{T_i}^\delta}(s-T_i,v_{T_i}^\delta)\Big)^2\mathrm{d}s\mathbf{1}_{\{|\nu_{T_i}^\delta|=1\}}\\&+\int_{T_{i_{\tau_1}}}^{\tau_1}\Big(\Phi_{\eta_{T_{i_{\tau_1}}}^\delta,\nu_{T_{i_{\tau_1}}}^\delta}(s-T_{i_{\tau_1}},v_{T_{i_{\tau_1}}}^\delta)\Big)^2\mathrm{d}s\mathbf{1}_{\{|\nu_{T_{i_{\tau_1}}}^\delta|=1\}},\end{aligned} \tag{5.13}$$

here each of the integrals in the right hand side can be computed explicitly. In this way,

$$\begin{aligned}\int_0^{\tau_1}f(\eta_s^\delta,v_s^\delta)\mathrm{d}s=&\sum_{0\leqslant i\leqslant i_{\tau_1}-1}\Psi(T_i,T_{i+1};\eta_{T_i}^\delta,\nu_{T_i}^\delta,v_{T_i}^\delta)\mathbf{1}_{\{|\nu_{T_i}^\delta|=1\}}\\&+\Psi(T_{i_{\tau_1}},\tau_1;\eta_{T_{i_{\tau_1}}}^\delta,\nu_{T_{i_{\tau_1}}}^\delta,v_{T_{i_{\tau_1}}}^\delta)\mathbf{1}_{\{|\nu_{T_{i_{\tau_1}}}^\delta|=1\}},\end{aligned} \tag{5.14}$$

with $\Psi(\theta,\theta';\eta,\nu,v) \triangleq (\theta'-\theta)(\eta-\nu\mu_{\mathrm{d}})^2 + 0.5(v-\eta+\nu\mu_{\mathrm{d}})^2(1-e^{-2(\theta'-\theta)}) + 2(\eta-\nu\mu_{\mathrm{d}})(v-\eta+\nu\mu_{\mathrm{d}})(1-e^{-(\theta'-\theta)})$. In Figure 5.1, an estimation of the four statistics $\{S^i\}$ as a function of δ is presented together with the error bars using the probabilistic method and the formulas above.

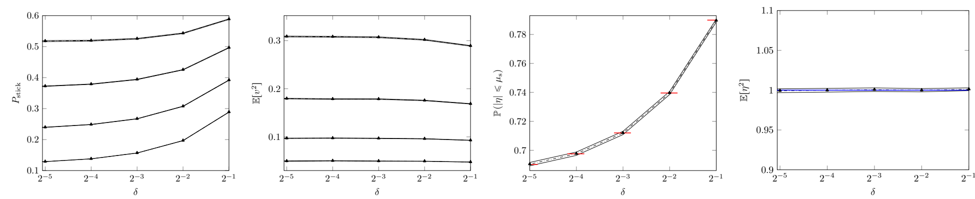

FIG. 5.1. *Approximation of $P_{\mathrm{stick}}, \mathbb{E}[v^2], \mathbb{P}(|\eta| \leqslant 1), \mathbb{E}[\eta^2]$ versus δ (taking values 2^{-j}, $j = 1,\dots,5$) based on $N = 10^6$ simulated excursions of $(\eta^\delta,\nu^\delta,v^\delta)$. Here $\mu_{\mathrm{s}} = 1$. In the subfigure for P_{stick}, the four curves from bottom to top correspond to $\mu_{\mathrm{d}} = \frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 respectively whereas in the subfigure for $\mathbb{E}[v^2]$ the order is reversed. In the subfigure for $\mathbb{P}(|\eta| \leqslant \mu_{\mathrm{s}})$, the red segments represent the numerical results associated with finding the invariant measure of η in the left kernel of Q and then computing the targeted statistics. In the subfigure for $\mathbb{E}[\eta^2]$, the blue line represent the theoretical value which is one whatever the value of δ. The asymptotic 95 % confidence intervals are represented by two solid lines around the estimated results.*

5.2. The deterministic method : discretization in the v-axis of the λ-problem. To numerically approximate the solution of (3.2), we use a finite difference scheme where only the v-axis is discretized. We consider a two-dimensional grid, for any $p \in \mathbb{N}^\star$, with $I \triangleq 2N+1$ and $J \triangleq 2Np+1$,

$$\mathcal{G}_p \triangleq \left\{(\eta_i, v_j) \triangleq \left((i-N-1)\delta, (j-Np-1)\frac{\delta}{p}\right),\ 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant J\right\}. \qquad (5.15)$$

The number of points in the grid $\mathcal{G}_p$ is $N_p \triangleq (2N+1)(2Np+1) \sim 4N^2p$ as $N \to \infty$. The numerical approximation of $u_\lambda(\eta_i, \Theta(\eta_i, v_j), v_j)$ is denoted by u_{ij} and the corresponding vector collecting the unknowns is $\boldsymbol{u}$. We also use the notation f_{ij} for $f(\eta_i, \Theta(\eta_i, v_j), v_j)$ and $\boldsymbol{f}$ for the corresponding vector. We use a standard finite difference scheme in the v direction:

$$\lambda u_{ij} - (\mathbf{K}\boldsymbol{u})_{ij} - (\mathbf{J}\boldsymbol{u})_{ij} = f_{ij} \ \text{ on } \ 1 \leqslant i \leqslant I,\ 1 \leqslant j \neq Np+1 \leqslant J, \qquad (5.16)$$

otherwise when $j = Np+1$ ($v_j = 0$)

$$\lambda u_{ij} - (\mathbf{K}\boldsymbol{u})_{ij} = f_{ij} \text{ on } |i-N-1| > k_{\mu_{\mathrm{s}}} \text{ and } \lambda u_{ij} - (\mathbf{J}\boldsymbol{u})_{ij} = f_{ij} \text{ on } |i-N-1| \leqslant k_{\mu_{\mathrm{s}}}, \qquad (5.17)$$

with $(\mathbf{J}\boldsymbol{u})_{ij} \triangleq 2\tau^{-1}\delta^{-2}\left(\alpha_i u_{i+1j} - u_{ij} + \alpha_i^\star u_{i-1j}\right)$ and

$$(\mathbf{K}\boldsymbol{u})_{ij} \triangleq p\max(0, B_{ij})\left(\frac{u_{ij+1}-u_{ij}}{\delta}\right) + p\min(0, B_{ij})\left(\frac{u_{ij}-u_{ij-1}}{\delta}\right), \qquad (5.18)$$

with $B_{ij} = B(\eta_i, \mathrm{sign}(v_j), v_j)$. This results in a linear system to be solved of the form $(\lambda\mathbf{I} - \mathbf{M})\boldsymbol{u} = \boldsymbol{f}$ where both $\mathbf{I}$ and $\mathbf{M}$ are $N_p \times N_p$ sparse matrices, $\mathbf{I}$ is the identity matrix and $\mathbf{M}$ is a sparse matrix with at most five nonzero entries per row.

Empirical convergence rate w.r.t p. If the finite difference scheme (5.16)-(5.17) is of order κ then $\|u^p - u\| \leqslant Cp^{-\kappa}$ where C is independent of p. Moreover, if there exists an $\epsilon > 0$ such that $\|u^p - u\| = Cp^{-\kappa} + O(p^{-\kappa-\epsilon})$ then $\|u^{2p} - u^p\|\|u^p - u^{\frac{p}{2}}\|^{-1} \approx$

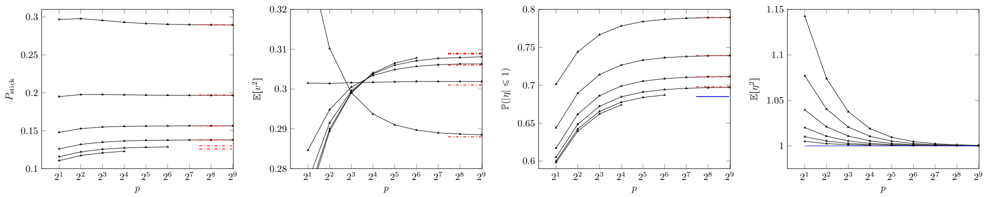

FIG. 5.2. *Case $\mu_{\rm d} = 0.25\mu_{\rm s}$. Approximation of $P_{\rm stick}, \mathbb{E}[v^2], \mathbb{P}(|\eta| \leqslant 1), \mathbb{E}[\eta^2]$ versus $2 \leqslant p \leqslant 2^9$ for several values of δ (from 2^{-1} halved successively until 2^{-6}). The p-axis is represented in the scale of log base 2. The time spent to find $\mathbf{u}$ corresponds essentially to the LU factorization of the matrix $\lambda\mathbf{I} - \mathbf{M}$ associated with the system (5.16)-(5.17). We employ the MATLAB procedure lu(.) which seeks five invertible matrices $\mathbf{L}, \mathbf{U}, \mathbf{P}, \mathbf{Q}, \mathbf{D}$ where $\mathbf{L}$ aud $\mathbf{U}$ are resp. lower and upper triangular such that $\lambda\mathbf{I} - \mathbf{M} = \mathbf{D}\mathbf{P}^{-1}\mathbf{L}\mathbf{U}\mathbf{Q}^{-1}$. In red (the limit case in p), the Monte Carlo result is shown for comparison. In blue (the limit case in p), the theoretical value (when available).*

$2^\kappa + O(p^{-\epsilon})$. With such a relation in mind, we test the convergence of the finite difference scheme by considering $\kappa(p) \triangleq \log_2\left(\|u^{2p} - u^p\|\|u^p - u^{\frac{p}{2}}\|^{-1}\right)$. In Table 5.1, we present a set of empirical estimations of $\kappa(p)$ in two cases. The data indicates that $\kappa(p) \sim 1$.

$\delta = 2^{-1}$	$\kappa(64)$	$\kappa(128)$	$\kappa(256)$	$\kappa(512)$
S^1	0.881	0.939	0.976	0.997
S^2	0.997	0.999	0.996	0.986
S^3	0.997	0.999	1.002	1.010
S^4	1.007	1.004	1.005	1.012

$\delta = 2^{-2}$	$\kappa(64)$	$\kappa(128)$	$\kappa(256)$	$\kappa(512)$
S^1	0.865	0.941	0.995	0.988
S^2	0.995	0.996	0.992	0.998
S^3	0.994	1.000	1.010	1.001
S^4	0.954	0.941	0.825	0.974

TABLE 5.1
Computation of $\kappa(p)$ when $\delta = 2^{-1}$ on the left, $\delta = 2^{-2}$ on the right. The method is empirically of order 1.

5.3. Discussion.

Durations of excursions, static and dynamique phases. As shown in Figure 5.3, we observe two different behaviors for the pdf of the dynamic phase duration, say $f_{\rm slide}$. When $\mu_{\rm d} < \mu_{\rm s}$, $f_{\rm slide}$ vanishes at 0 and is very close to 0 in its neighborhood. This indicates the absence of small dynamic phases. When $\mu_{\rm d} = \mu_{\rm s}$, $f_{\rm slide}$ vanishes at 0 but increases very fast. This indicates the presence of dramatically short dynamic phases. In all cases, the pdf of the static phase duration, say $f_{\rm stick}$, is positive around 0 and $f_{\rm stick}(0^+)$ is a finite positive number. This indicates the presence of short static phases. Finally, the behavior of the excursion is essentially inherited from the behavior of the dynamic phase.

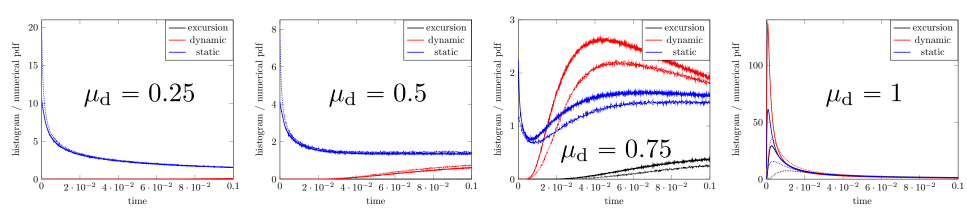

FIG. 5.3. *Numerical pdf associated with (red) duration of the static phase $f_{\rm stick}$, (blue) duration of the dynamic phase $f_{\rm slide}$ and (black) duration of an excursion. Each curve corresponds to a δ which takes the following values 2^{-4} (dotted line) and 2^{-5} (solid line). The number of excursions in 10^7. In this way, the size of the 95 % confidence interval is $10^{-7/2}$.*

$\mu_d {\mu_s}^{-1}$ \ δ	2^{-3}	2^{-4}	2^{-5}
0.25	5.32/6.73	10.83/10.51	21.81/20.94
0.50	2.20/2.78	4.26/4.10	8.36/7.98
0.75	0.68/0.89	1.24/1.17	2.36/2.27
1.00	0.17/0.18	0.41/0.76	1.12/9.20

TABLE 5.2
Estimation of $f^\delta_{\text{stick}}(0^+)$ using Kolmogorov (left numbers) and Monte Carlo (right numbers) methods.

Comments on the cases $\mu_d < \mu_s$. Using Monte Carlo and Kolmogorov methods, we can estimate $f^\delta_{\text{stick}}(0^+)$ (shown in Table 5.2) and $f^\delta_{\text{slide}}(0^+)$. For $\mu_d = 0.25, 0.5$ and 0.75 and for $\delta = 2^{-k}, k = 3, \ldots, 6$, the computed numbers are positive and finite. Both methods agree qualitatively. In addition, we observe that the calculated values are multiplied by two when the parameter δ is divided by two. It seems to indicate that $f^\delta_{\text{stick}}(0^+) \uparrow \infty$ as $\delta \downarrow 0$. Besides, both methods indicate that $f^\delta_{\text{slide}}(0^+) = 0$. Furthermore the empirical histograms from the MC method reveal that f^δ_{slide} vanishes in the neighborhood of 0.

Comments on the case $\mu_d = \mu_s$. Using the Kolmogorov method, we can estimate $f^\delta_{\text{stick}}(0^+)$ (shown in the last row of Table 5.2) for $\delta = 2^{-k}, k = 3, \ldots, 6$, the computed numbers are positive and finite but they do not agree very well with the Monte Carlo (right) method. This may be due to the fact that the slope of f^δ_{stick} is significantly steep in the neighborhood of 0. We can notice that the MC value overestimates the value given by the discretized Kolmogorov equations. With the Kolmogorov method, we also show that $f^\delta_{\text{slide}}(0^+) = 0$ but the convergence is rather slow in finding the limit of $\lambda F^\delta_{\text{slide}}(\lambda)$ as $\lambda \to 0$. With the Monte Carlo method, it is difficult to capture this value directly. For each $\delta = 2^{-k}, k = 3, \ldots, 6$, the empirical histograms from the MC method indicate that $f^\delta_{\text{slide}}(t) > 0$ in the neighborhood of 0 while $f^\delta_{\text{slide}}(t) \downarrow 0$ as $t \downarrow 0^+$.

Comment on our memory limit. The numerical results reported on this work have been performed on a MacBook Air (13-inch, Mid 2013) with the following characteristics: Processor 1,3 GHz Intel Core i5, Memory 8 GB 1600 MHz DDR3, Graphics Intel HD Graphics 5000 1536 MB. With such a memory limit, the size of the matrix $\mathbf{M}$ must remain below $10^7 \times 10^7$.

A data-learning heuristic to go beyond our memory limit: extrapolation of our results. Below, we use the notation $S^{[i]}(k,l)$ for the estimation of S^i defined by (5.1) using the deterministic method where $p = 2^k$ and $\delta = 2^{-l}$. As shown in Figure 5.2, due to memory limit, we cannot evaluate $S^{[i]}(k,l)$ when $(k,l) \in \bar{\mathfrak{U}} \triangleq \{7 \leqslant k \leqslant 9\} \times \{5\} \cup \{5 \leqslant k \leqslant 9\} \times \{6\}$. We can only evaluate them when $(k,l) \in \mathfrak{U} \triangleq \{1 \leqslant k \leqslant 9\} \times \{1 \leqslant l \leqslant 4\} \cup \{1 \leqslant k \leqslant 6\} \times \{5\} \cup \{1 \leqslant k \leqslant 4\} \times \{6\}$. Nonetheless, we can cook up an extrapolation approach to estimate the missing data on $\bar{\mathfrak{U}}$ where we keep the notation $S^{[i]}(k,l)$. To compute $S^{[i]}(k,l)$ on $\bar{\mathfrak{U}}$, we assume that the error trends observed when δ is large or p small remain the same as when δ is small and p large. Our *heuristics* starts from this observation

$$S^{[i]}(k,l) = S^{[i]}(k,l-1) + R_V(k,l)\left(S^{[i]}(k,l-1) - S^{[i]}(k,l-2)\right), \tag{5.19}$$

$$S^{[i]}(k,l) = S^{[i]}(k-1,l) + R_H(k,l)\left(S^{[i]}(k-1,l) - S^{[i]}(k-2,l)\right), \tag{5.20}$$

with

$$R_V(k,l) \triangleq \frac{S^{[i]}(k,l) - S^{[i]}(k,l-1)}{S^{[i]}(k,l-1) - S^{[i]}(k,l-2)} \text{ and } R_H(k,l) \triangleq \frac{S^{[i]}(k,l) - S^{[i]}(k-1,l)}{S^{[i]}(k-1,l) - S^{[i]}(k-2,l)}. \tag{5.21}$$

Clearly, $R_V(k,l)$ and $R_H(k,l)$ are unknown since they depend on $S^{[i]}(k,l)$, the targeted unknown quantity. However, we have an idea of the error trend and then our heuristics consists in the following natural approximation $R_V(k,l) \approx R_V(k,l-1)$ and $R_V(k,l) \approx R_V(k-1,l)$. Then, we define

$$S_V^{[i]}(k,l) \triangleq S^{[i]}(k,l-1) + R_V(k,l-1)\left(S^{[i]}(k,l-1) - S^{[i]}(k,l-2)\right), \tag{5.22}$$

$$S_H^{[i]}(k,l) \triangleq S^{[i]}(k-1,l) + R_H(k-1,l)\left(S^{[i]}(k-1,l) - S^{[i]}(k-2,l)\right), \tag{5.23}$$

and finally

$$S^{[i]}(k,l) \triangleq \frac{1}{2}\left(S_V^{[i]}(k,l) + S_H^{[i]}(k,l)\right). \tag{5.24}$$

In this way, this heuristical definition of $S^{[i]}(k,l)$ requires six values:

$$S^{[i]}(k-1,l),\ S^{[i]}(k-2,l),\ S^{[i]}(k-3,l) \text{ and } S^{[i]}(k,l-1),\ S^{[i]}(k,l-2),\ S^{[i]}(k,l-3). \tag{5.25}$$

Starting from the data on $\mathfrak{U}$, we can propagate this data-learning scheme on $\bar{\mathfrak{U}}$. In Table 5.3, we present our results which show the stability of the extrapolation procedure.

	(7,5)	(8,5)	(9,5)	(5,6)	(6,6)	(7,6)	(8,6)	(9,6)
$S^{[1]}$	0.1291	0.1294	0.1295 (0.1302)	0.1239	0.1244	0.1249	0.1254	0.1258 (0.1257)
$S^{[2]}$	0.6886	0.6893	0.6896 (0.6980)	0.6803	0.6835	0.6850	0.6855	0.6857 (0.6850)

Table 5.3

Computation of $S^{[i]}(k,l)$ with our heuristics on $\bar{\mathfrak{U}}$. The first row lists the elements of $\bar{\mathfrak{U}}$. The values in parenthesis are shown for comparison and have been obtained with different methods, Monte Carlo in the $S^{[1]}$ row and (stationary) Kolmogorov equation for η in the $S^{[2]}$ row.

6. Conclusions and perspectives. In this work, we tackle the problem of modeling stochastic dry friction including different coefficients for the static and dynamic forces by proposing a PDMP approach. Here the external forcing takes discrete values and it is assumed to be a Markov jump process depending on a small parameter. We show ergodicity and provide a representation formula of the stationary measure. We also obtain a characterization of the Laplace transforms of the probability density functions of the durations of the static and dynamic phases. Moreover, when the aforementioned parameter vanishes and when the two coefficients of static and dynamic forces are identical, we show that the PDMP converges in distribution to the solution of a well-known dry friction model. This model is subjected to a colored noise (an Ornstein-Uhlenbeck process) and its definition involves a differential inclusion formalism. This bridges the gap between our approach and existing well-posed continuous models when the coefficients for the static and dynamic forces are identical. As a future work, it should be possible to consider the extension of the PDMP approach to higher dimensions and more realistic systems such as randomly driven moveable rigid bodies in frictional contact with rigid obstacles or mechanical systems of rigid bodies as inspired by [11,15]. It would also be of interest to develop numerical simulation and stochastic control methods for these systems by combining existing techniques such as Lagrange multipliers in the same spirit as [16] and [14].

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Appendix A. Proof of Proposition 2.1. This is a diffusion approximation result.

First, the process η_t^δ is Markov with the generator $Q^\delta\psi(\eta) = 2\delta^{-2}\tau^{-1}\big(\alpha(\eta)\psi(\eta+\delta)+\alpha^\star(\eta)\psi(\eta-\delta)-\psi(\eta)\big)$ and it converges in distribution in the space of the càdlàg functions to the diffusion process with generator Q. Indeed $Q^\delta\varphi(\eta) = Q\varphi(\eta)+o(1)$ for any smooth test function φ and $Q = \tau^{-1}\big(\partial_\eta^2 - \eta\partial_\eta\big)$ is the infinitesimal generator of η_t [10, Chapter 12].

Second the map $\eta \mapsto v$ from the space of the càdlàg functions to the space of the continuous functions, with v solution of $\dot v + \partial\varphi(v) \ni b(\eta, v)$, is continuous. We now present the proof of this statement (formulated in Proposition A.2).

Notation and assumption. The set of (real valued) right continous left limit (càdlàg) functions on $[0,T]$ is denoted by $D[0,T]$. The set of continuous functions on $[0,T]$ is denoted by $C[0,T]$. Clearly $C[0,T]\subset D[0,T]$. We consider $b:\mathbb{R}\to\mathbb{R}$ a Lipschitz function of Lipschitz constant $L_b>0$, $\xi\in\mathbb{R}$ and $\varphi(v)=\mu|v|, \mu>0$.

Converging sequence in the J_1 topology [5]. We say that a sequence of functions $\{w_n\}\in D[0,T]$ converges towards a function $w\in D[0,T]$ in the sense of J_1 topology if there exists a sequence of increasing homeomorphims $\{\lambda_n\}$ on $[0,T]$ such that $\lambda_n(0)=0,\ \lambda_n(T)=T$ and

$$\textbf{(a)} \lim_{n\to\infty}\sup_{0\leqslant t\leqslant T}|\lambda_n(t)-t| = 0 \quad\text{and}\quad \textbf{(b)} \lim_{n\to\infty}\sup_{0\leqslant t\leqslant T}|w_n(\lambda_n(t))-w(t)| = 0. \tag{A.1}$$

Preliminary : case of a differential equation with a càdlàg function at the rhs. Let $w\in D[0,T]$ and $\xi\in\mathbb{R}$. Consider the following problem:

$$\begin{cases} \text{find a function } \ x(w)\in C[0,T] \ \text{ satisfying} \\ \forall t\geqslant 0,\ x_t(w) = \xi + \int_0^t b(x_s(w))\mathrm{d}s + \int_0^t w(s)\mathrm{d}s. \end{cases} \tag{A.2}$$

Proposition A.1. *There exists a unique solution to the problem (A.2). As a consequence, the mapping x which associates w to $x(w)$ from $D[0,T]$ to $C[0,T]$ is well defined. Moreover, x is continous with respect to the J_1 topology on $D[0,T]$ in the sense that if a sequence of functions $w_n\in D[0,T]$ converges to a function $w\in D[0,T]$ as $n\to\infty$ then $x(w_n)$ converges to $x(w)$ as $n\to\infty$ in $C[0,T]$.*

Proof. Part 1. The existence of a solution can be obtained by Picard's iteration. First define $\forall t\geqslant 0,\ x_t^0(w)\equiv\xi$ and then

$$\forall n\geqslant 0,\ \forall t\geqslant 0,\ x_t^{n+1}(w) = \xi + \int_0^t b(x_s^n(w))\mathrm{d}s + \int_0^t w(s)\mathrm{d}s.$$

The sequence $\{x^n(w)\}$ is composed of continuous functions. Since b is Lipschitz it converges uniformly on $[0,T]$. The limit is denoted by $x(w)$ and it satisfies (A.2).
Part 2. If $x(w)$ and $\tilde x(w)$ are two solutions of (A.2) then we must have

$$\sup_{0\leqslant r\leqslant t}|x_r(w)-\tilde x_r(w)| \leqslant L_b\int_0^t \sup_{0\leqslant r\leqslant s}|x_r(w)-\tilde x_r(w)|\mathrm{d}s,$$

which implies, by Gronwall's lemma, that $x(w)=\tilde x(w)$ in $C[0,T]$.
Part 3. Let $\{w_n\}$ be a sequence of functions in $D[0,T]$ converging towards a function $w\in D[0,T]$ in the J_1 topology. Since $\forall t\geqslant 0$,

$$x_t(w_n) = \xi+\int_0^t b(x_s(w_n))\mathrm{d}s+\int_0^t w_n(s)\mathrm{d}s \text{ and } x_t(w) = \xi+\int_0^t b(x_s(w))\mathrm{d}s+\int_0^t w(s)\mathrm{d}s,$$

we have

$$\sup_{0\leqslant r\leqslant t}|x_r(w)-x_r(w_n)|\leqslant L_b\int_0^t\sup_{0\leqslant r\leqslant s}|x_r(w)-x_r(w_n)|\mathrm{d}s+\int_0^t|w(s)-w_n(s)|\mathrm{d}s.$$

The latter implies using Gronwall's lemma that

$$\sup_{0\leqslant r\leqslant T}|x_r(w)-x_r(w_n)|\leqslant \exp(L_bT)\int_0^T|w(s)-w_n(s)|\mathrm{d}s.$$

Finally, we verity that

$$\int_0^T|w(s)-w_n(s)|\mathrm{d}s\to 0\ \text{ as }\ n\to\infty.$$

Indeed $\int_0^T|w(s)-w_n(s)|\mathrm{d}s\leqslant A_n+B_n$, where $A_n=\int_0^T|w(s)-w(\lambda_n^{-1}(s))|\mathrm{d}s$, $B_n=\int_0^T|w(\lambda_n^{-1}(s))-w_n(s)|\mathrm{d}s$, and λ_n is a sequence of increasing homeomorphims associated to the convergence of w_n in the J_1 sense. We have $B_n\leqslant\|w-w_n\circ\lambda_n\|_\infty T$ which shows that $\lim_{n\to\infty}B_n=0$ by (A.1b). We also have $w(\lambda_n^{-1}(s))\to w(s)$ at any point of continuity of w, that is to say, almost surely (with respect to the Lebesgue measure over $[0,T]$), and $w(s)-w(\lambda_n^{-1}(s))$ is bounded by $2\|w\|_\infty$, so that $A_n\to 0$ by dominated convergence. □

Case of a differential inclusion with a càdlàg function at the rhs. Let $w\in D[0,T]$ and $\xi\in\mathbb{R}$. Consider the following problem

$$\begin{cases}\text{find a function } v(w)\in C[0,T]\ \text{ satisfying}\\ \forall t\geqslant 0,\ v_t(w)+\Delta_t(w)=\xi+\int_0^t b(v_s(w))\mathrm{d}s+\int_0^t w(s)\mathrm{d}s,\end{cases}\tag{A.3}$$

where $\Delta(w)\in H^1(0,T)$ and with the notation $\delta(w)=\dot\Delta(w)$

$$\forall\zeta\in C[0,T],\forall 0\leqslant t<t+h\leqslant T,$$
$$\int_t^{t+h}\left(\delta_s(w)(\zeta(s)-v_s(w))+\varphi(v_s(w))\right)\mathrm{d}s\leqslant\int_t^{t+h}\varphi(\zeta(s))\mathrm{d}s.$$

The conditions in (A.3) are encoded in the differential inclusion notation

$$\dot v+\partial\varphi(v)\ni b(v)+w.$$

Proposition A.2. *There exists a unique solution to the problem (A.3). As a consequence, the mapping v which associates w to $v(w)$ from $D[0,T]$ to $C[0,T]$ is well defined. Moreover, v is continous with respect to the J_1 topology on $D[0,T]$.*

Proof. Part 1 The proof follows the steps of the one of [13, proposition C.1] which addresses the same problem when $w\in C[0,T]$. We recall the essential steps. For any p we denote by φ_p the Moreau-Yosida regularization of φ,

$$\varphi_p(v)=\begin{cases}|v|-\frac{1}{2p} & |v|>\frac{1}{p}\\ p\frac{v^2}{2} & |v|\leqslant\frac{1}{p}\end{cases}$$

and we consider the penalized problem

$$\forall t\geqslant 0,\ v_t^p(w)+\int_0^t\varphi_p'(v_s^p(w))\mathrm{d}s=\xi+\int_0^t b(v_s^p(w))\mathrm{d}s+\int_0^t w(s)\mathrm{d}s.$$

From Proposition A.1, this ODE has a unique solution $v^p(w) \in C[0,T]$. It can be shown that $\{v^p(w)\}$ is a Cauchy sequence in $C[0,T]$ and

$$\sup_{0\leqslant t\leqslant T} |v_t^p(w) - v_t^q(w)|^2 \leqslant \left(\frac{1}{p} + \frac{1}{q}\right) C_T,$$

where the constant C_T depends only on the Lipschitz constant of b, T and μ. It is a consequence of the property $\sup\limits_{p\geqslant 1}\sup\limits_{v\in\mathbb{R}} |\varphi_p'(v)| = \mu$. Thus the limit $v(w) \in C[0,T]$ exists and satisfies

$$\sup_{0\leqslant t\leqslant T} |v_t^p(w) - v_t(w)| \leqslant \sqrt{\frac{C_T}{p}}.$$

It can then be shown that $v(w)$ satisfies the conditions in (A.3).
<u>Part 2.</u> Assume $w_n \in D[0,T]$ converges to a function $w \in D[0,T]$ as $n \to \infty$. We want to show that then $v(w_n)$ converges to $v(w)$ as $n \to \infty$ in $C[0,T]$. We can write

$$\sup_{0\leqslant t\leqslant T} |v_t(w_n) - v_t(w)| \leqslant \sup_{0\leqslant t\leqslant T} |v_t(w_n) - v_t^p(w_n)| + \sup_{0\leqslant t\leqslant T} |v_t^p(w_n) - v_t^p(w)| + \sup_{0\leqslant t\leqslant T} |v_t^p(w) - v_t(w)|.$$

Let $\varepsilon > 0$. For p large enough we have

$$\sup_{0\leqslant t\leqslant T} |v_t^p(w) - v_t(w)| \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \sup_n \sup_{0\leqslant t\leqslant T} |v_t(w_n) - v_t^p(w_n)| \leqslant \frac{\varepsilon}{3}.$$

Finally for $n \geqslant n_p$ large enough

$$\sup_{0\leqslant t\leqslant T} |v_t^p(w_n) - v_t^p(w)| \leqslant \frac{\varepsilon}{3},$$

which completes the proof of the proposition. □

Appendix B. Proof of Proposition 3.1. We want to establish that τ_1 is integrable, $\mathbb{E}_{\mathfrak{s}_+}[\tau_1] < +\infty$ (which also proves by symmetry that $\mathbb{E}_{\mathfrak{s}_-}[\tau_1] = \mathbb{E}_{\mathfrak{s}_+}[\tau_1] < +\infty$).

We have $b(\eta, v) = \eta - v$, so that the process v_t^δ is bounded by $\max(|v_0^\delta|, \eta_N - \mu_{\mathrm{d}})$.

Step 1. Let $\tilde{\tau}_1 = \inf\{t > 0,\, v_t^\delta = 0\}$. We have

$$C_{\tilde{\tau}} := \sup_{\eta\in\{\eta_{k_{\mu_{\mathrm{s}}}+1},\dots,\eta_N\}} \mathbb{E}_{(\eta,1,0)}[\tilde{\tau}_1] < +\infty.$$

By symmetry we have $\mathbb{E}_{(\eta,-1,0)}[\tilde{\tau}_1] = \mathbb{E}_{(-\eta,1,0)}[\tilde{\tau}_1]$ for $\eta \in \{\eta_{-N},\dots,\eta_{-k_{\mu_{\mathrm{s}}}-1}\}$, and therefore $\sup_{\eta\in\{\eta_{-N},\dots,\eta_{-k_{\mu_{\mathrm{s}}}-1}\}} \mathbb{E}_{(\eta,-1,0)}[\tilde{\tau}_1] = C_{\tilde{\tau}}$.

Proof. If $\eta_0^\delta \in \{\eta_{k_{\mu_{\mathrm{s}}}+1},\dots,\eta_N\}$, $v_0^\delta = 0$ and $t < \tilde{\tau}_1$, then $0 \leqslant v_t^\delta = \int_0^t (-\mu_{\mathrm{d}} + \eta_s^\delta - v_s^\delta)\mathrm{d}s \leqslant -\mu_{\mathrm{d}} t + \int_0^t \eta_s^\delta \mathrm{d}s$. Therefore, for any $\eta \in \{\eta_{k_{\mu_{\mathrm{s}}}+1},\dots,\eta_N\}$ and $t > 0$, we have

$$\mathbb{P}_{(\eta,1,0)}\big(\tilde{\tau}_1 > t\big) = \mathbb{P}_{(\eta,1,0)}\big(\tilde{\tau}_1 > t,\, v_t^\delta \geqslant 0\big) \leqslant \mathbb{P}_{(\eta,1,0)}\big(\int_0^t \eta_s^\delta \mathrm{d}s \geqslant \mu_{\mathrm{d}} t\big) \leqslant {\mu_{\mathrm{d}}}^{-4} t^{-4} \mathbb{E}_\eta\big[\big(\int_0^t \eta_s^\delta \mathrm{d}s\big)^4\big].$$

By the ergodic properties of (η_t^δ), we have $t^{-2}\mathbb{E}_\eta\big[\big(\int_0^t \eta_s^\delta \mathrm{d}s\big)^4\big] \xrightarrow{t\to+\infty} 6\big(\int_0^{+\infty} \mathbb{E}_s[\eta_0^\delta \eta_s^\delta]\mathrm{d}s\big)^2$ which is finite (where $\mathbb{E}_s$ is the expectation under the stationary distribution of the process η_t^δ). This shows that there exists $C_{\mu_\mathrm{d}} > 0$ such that, for all $t > 0$,

$$\sup_{\eta\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{P}_{(\eta,1,0)}\big(\tilde{\tau}_1 > t\big) \leqslant \frac{C_{\mu_\mathrm{d}}}{1+t^2},$$

which gives the desired result. □

Step 2. We have

$$C_p := \inf_{\eta\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{P}_{(\eta,1,0)}\big(\eta_{\tilde{\tau}_1}^\delta \in \{-\eta_{k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}\big) > 0.$$

Proof. Let k_d be the largest index such that $\eta_{k_\mathrm{d}} < \mu_\mathrm{d}$. We here denote by θ_j the times between two random jumps of the process η^δ and by $X_j \in \{-\delta,\delta\}$ the jump amplitudes. For $k \in \{k_{\mu_\mathrm{s}}+1,\dots,N\}$, we consider $B_k = \{\theta_1+\dots+\theta_{k-k_\mathrm{d}} < 1,\ X_{k-k_\mathrm{d}} = \dots = X_1 = -\delta, \theta_{k-k_\mathrm{d}+1} > (\eta_k-\mu_\mathrm{d})/(\mu_\mathrm{d}-\eta_{k_\mathrm{d}})\}$. This corresponds to a trajectory that goes northwest from $(\eta_k,1,0)$ up to the line $(\eta_{k_\mathrm{d}},1,*)$ in time less than 1, and then goes south until reaching $(\eta_{k_\mathrm{d}},1,0)$ which triggers a deterministic jump to $(\eta_{k_\mathrm{d}},0,0)$. We have $\mathbb{P}_{(\eta_k,1,0)}(B_k) > 0$ and $\mathbb{P}_{(\eta_k,1,0)}\big(\eta_{\tilde{\tau}_1}^\delta \in \{\eta_{-k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}\big) \geqslant \mathbb{P}_{(\eta_k,1,0)}(B_k) > 0$, which gives the desired result after taking $\inf_{k\in\{k_{\mu_\mathrm{s}}+1,\dots,N\}}$. □

Step 3. We have

$$C_{\hat{\tau}} := \sup_{\eta\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{E}_{(\eta,1,0)}[\hat{\tau}_1] < +\infty.$$

By symmetry we have $\mathbb{E}_{(\eta,-1,0)}[\hat{\tau}_1] = \mathbb{E}_{(-\eta,1,0)}[\hat{\tau}_1]$ for $\eta \in \{\eta_{-N},\dots,\eta_{-k_{\mu_\mathrm{s}}-1}\}$, and therefore $\sup_{\eta\in\{\eta_{-N},\dots,\eta_{-k_{\mu_\mathrm{s}}-1}\}} \mathbb{E}_{(\eta,-1,0)}[\hat{\tau}_1] = C_{\hat{\tau}}$.

Proof. Using the strong Markov property, we have for $\eta \in \{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}$:

$$\begin{aligned}
\mathbb{E}_{(\eta,1,0)}[\hat{\tau}_1] &= \mathbb{E}_{(\eta,1,0)}\big[\hat{\tau}_1 \mathbf{1}_{\eta_{\tilde{\tau}_1}^\delta \in\{-\eta_{k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}}\big] + \mathbb{E}_{(\eta,1,0)}\big[\hat{\tau}_1 \mathbf{1}_{\eta_{\tilde{\tau}_1}^\delta \notin\{-\eta_{k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}}\big] \\
&= \mathbb{E}_{(\eta,1,0)}\big[\tilde{\tau}_1 \mathbf{1}_{\eta_{\tilde{\tau}_1}^\delta \in\{-\eta_{k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}}\big] + \mathbb{E}_{(\eta,1,0)}\big[(\hat{\tau}_1-\tilde{\tau}_1+\tilde{\tau}_1) \mathbf{1}_{\eta_{\tilde{\tau}_1}^\delta \notin\{-\eta_{k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}}\big] \\
&= \mathbb{E}_{(\eta,1,0)}\big[\tilde{\tau}_1\big] + \mathbb{E}_{(\eta,1,0)}\big[(\hat{\tau}_1-\tilde{\tau}_1) \mathbf{1}_{\eta_{\tilde{\tau}_1}^\delta \notin\{-\eta_{k_{\mu_\mathrm{s}}},\dots,\eta_{k_{\mu_\mathrm{s}}}\}}\big] \\
&= \mathbb{E}_{(\eta,1,0)}\big[\tilde{\tau}_1\big] + \sum_{\tilde{\eta}\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{E}_{(\tilde{\eta},1,0)}\big[\hat{\tau}_1\big]\mathbb{P}_{(\eta,1,0)}\big(\eta_{\tilde{\tau}_1}^\delta = \tilde{\eta}\big) \\
&\quad + \sum_{\tilde{\eta}\in\{\eta_{-N},\dots,\eta_{-k_{\mu_\mathrm{s}}-1}\}} \mathbb{E}_{(\tilde{\eta},-1,0)}\big[\hat{\tau}_1\big]\mathbb{P}_{(\eta,1,0)}\big(\eta_{\tilde{\tau}_1}^\delta = \tilde{\eta}\big) \\
&\leqslant C_{\tilde{\tau}} + \sup_{\tilde{\eta}\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{E}_{(\tilde{\eta},1,0)}\big[\hat{\tau}_1\big](1-C_p),
\end{aligned}$$

hence $\sup_{\eta\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{E}_{(\eta,1,0)}\big[\hat{\tau}_1\big] \leqslant C_{\tilde{\tau}}/C_p$. □

Step 4. We have

$$C_\tau := \sup_{\eta\in\{\eta_{k_{\mu_\mathrm{s}}+1},\dots,\eta_N\}} \mathbb{E}_{(\eta,1,0)}[\tau_1] < +\infty.$$

By symmetry we have $\mathbb{E}_{(\eta,-1,0)}[\tau_1] = \mathbb{E}_{(-\eta,1,0)}[\tau_1]$ for $\eta \in \{\eta_{-N},\dots,\eta_{-k_{\mu_\mathrm{s}}-1}\}$, and therefore $\sup_{\eta\in\{\eta_{-N},\dots,\eta_{-k_{\mu_\mathrm{s}}-1}\}} \mathbb{E}_{(\eta,-1,0)}[\tau_1] = C_\tau$.

Proof. Using the strong Markov property, we have for $\eta \in \{\eta_{k_{\mu_s}+1}, \ldots, \eta_N\}$:

$$\mathbb{E}_{(\eta,1,0)}[\tau_1] = \mathbb{E}_{(\eta,1,0)}[\hat{\tau}_1] + \mathbb{E}_{(\eta,1,0)}[\tau_1 - \hat{\tau}_1] \leqslant C_{\hat{\tau}} + \sup_{\hat{\eta} \in \{-\eta_{k_{\mu_s}}, \ldots, \eta_{k_{\mu_s}}\}} \mathbb{E}_{(\hat{\eta},0,0)}[\check{\tau}_1],$$

where $\check{\tau}_1 = \inf\{t > 0,\ \eta_t^\delta \in \{\eta_{-k_{\mu_s}-1}, \eta_{k_{\mu_s}+1}\}\}$. Since the process η_t^δ is ergodic, we have $\mathbb{E}_{(\hat{\eta},0,0)}[\check{\tau}_1] < +\infty$ for all $\hat{\eta} \in \{-\eta_{k_{\mu_s}}, \ldots, \eta_{k_{\mu_s}}\}$. □

Appendix C. Proof of Proposition 3.3. We introduce an auxiliary Markov process $(\eta_t^\epsilon, \nu_t^\epsilon, v_t^\epsilon)$ that depends on an additional time parameter $\epsilon > 0$:

- From $\mathfrak{s}'_\pm$ the process $(\eta_t^\epsilon, \nu_t^\epsilon, v_t^\epsilon)$ moves to $\mathfrak{s}_\pm = \pm(\eta_{k_{\mu_s}+1}, 1, 0)$ with probability one. The exponential time of jump has mean $\epsilon\tau^2\delta^2$.
- From $(\eta_{k_{\mu_s}}, 0, 0)$ the process $(\eta_t^\epsilon, \nu_t^\epsilon, v_t^\epsilon)$ moves to $\mathfrak{s}'_+$ with probability $\alpha(k_{\mu_s})$ and to $(\eta_{k_{\mu_s}-1}, 0, 0)$ with probability $1 - \alpha(k_{\mu_s})$. The exponential time of jump has mean $\tau^2\delta^2$.
- From $(\eta_{-k_{\mu_s}}, 0, 0)$ the process $(\eta_t^\epsilon, \nu_t^\epsilon, v_t^\epsilon)$ moves to $\mathfrak{s}'_-$ with probability $1 - \alpha(-k_{\mu_s})$ and to $(\eta_{-k_{\mu_s}+1}, 0, 0)$ with probability $\alpha(-k_{\mu_s})$. The exponential time of jump has mean $\tau^2\delta^2$.
- Otherwise the random dynamics of $(\eta_t^\epsilon, \nu_t^\epsilon, v_t^\epsilon)$ is the one of $(\eta_t^\delta, \nu_t^\delta, v_t^\delta)$.

In this context, the generator $\mathcal{L}^\epsilon$ of $(\eta_t^\epsilon, \nu_t^\epsilon, v_t^\epsilon)$ is

$$(\mathcal{L}^\epsilon\varphi)(\boldsymbol{z}) = (\mathcal{L}'\varphi)(\boldsymbol{z}) \text{ for } \boldsymbol{z} \in E, \qquad (\mathcal{L}^\epsilon\varphi)(\mathfrak{s}'_\pm) = \frac{\varphi(\mathfrak{s}_\pm) - \varphi(\mathfrak{s}'_\pm)}{\epsilon\tau^2\delta^2}.$$

Given f a bounded function, we consider the function

$$u_\lambda^\epsilon(\eta, \nu, v; f) \triangleq \mathbb{E}_{(\eta,\nu,v)}\Big[\int_0^\infty e^{-\lambda s} f(\eta_s^\epsilon, \nu_s^\epsilon, v_s^\epsilon)\mathrm{d}s\Big]$$

which satisfies the equation

$$\lambda u_\lambda^\epsilon - \mathcal{L}^\epsilon u_\lambda^\epsilon = f \ \text{ in } \ E \cup \{\mathfrak{s}'_\pm\}. \tag{C.1}$$

We want to establish the representation formula (3.16). The function f is arbitrary and can be decomposed as a sum of two functions: one symmetric $f_s \triangleq \frac{1}{2}(f + f \circ \gamma)$ and one antisymmetric $f_a \triangleq \frac{1}{2}(f - f \circ \gamma)$ where $\forall(\eta, \nu, v) \in E$, $\gamma(\eta, \nu, v) \triangleq -(\eta, \nu, v)$ and $\gamma(\mathfrak{s}'_\pm) = \mathfrak{s}'_\mp$. We first show that we have the representation formula

$$u_\lambda^\epsilon(\eta, \nu, v; f) = w_\lambda(\eta, \nu, v; f) - \nu_\lambda^\epsilon(f) w_\lambda(\eta, \nu, v; 1) + \mu_\lambda^\epsilon(f)\big((h_\lambda^+(\eta, \nu, v) - h_\lambda^-\eta, \nu, v)\big) + \frac{\pi_\lambda^\epsilon(f)}{\lambda},$$

where

$$\pi_\lambda^\epsilon(f) \triangleq \frac{w_\lambda(\mathfrak{s}_+; f) + w_\lambda(\mathfrak{s}_-; f) + \epsilon\tau^2\delta^2(f(\mathfrak{s}_+) + f(\mathfrak{s}_-))}{2w_\lambda(\mathfrak{s}_+; 1) + 2\epsilon\tau^2\delta^2},$$

$$\mu_\lambda^\epsilon(f) \triangleq \frac{w_\lambda(\mathfrak{s}_+; f) - w_\lambda(\mathfrak{s}_-; f) + \epsilon\tau^2\delta^2(f(\mathfrak{s}_+) - f(\mathfrak{s}_-))}{2(1 - h_\lambda^+(\mathfrak{s}_+) + h_\lambda^-(\mathfrak{s}_+) + \epsilon\tau^2\delta^2)}.$$

We split the proof into two parts.

Step 1. Assume f is symmetric. We have

$$\pi_\lambda^\epsilon(f) = \frac{w_\lambda(\mathfrak{s}_+, f) + \epsilon\tau^2\delta^2 f(\mathfrak{s}'_+)}{w_\lambda(\mathfrak{s}_+, 1) + \epsilon\tau^2\delta^2}, \tag{C.2}$$

$$u_\lambda^\epsilon(\eta, \nu, v; f) = w_\lambda(\eta, \nu, v; f) + \pi_\lambda^\epsilon(f)\Big(\frac{1}{\lambda} - w_\lambda(\eta, \nu, v; 1)\Big). \tag{C.3}$$

Proof of step1. By linearity of the operator $\mathcal{L}'$, it is clear that $w_\lambda(\cdot;f)+\pi^\epsilon_\lambda(f)\left(\frac{1}{\lambda}-w_\lambda(\cdot;1)\right)$ satisfies (C.1) in E. Moreover, by definition of the constant $\pi^\epsilon_\lambda(f)$ and by linearity of the operator $\mathcal{L}^\epsilon$, the equation is also satisfied in $\{\mathfrak{s}'_\pm\}$. Since $\lambda>0$, the solution of (C.1) is unique and thus (C.3) is shown.

Step 2. Assume f is antisymmetric. We have

$$\mu^\epsilon_\lambda(f) = \frac{w_\lambda(\mathfrak{s}_+,f)+\epsilon\tau^2\delta^2 f(\mathfrak{s}'_+)}{1-h^+_\lambda(\mathfrak{s}_+)+h^-_\lambda(\mathfrak{s}_+)+\epsilon\tau^2\delta^2},$$
$$u^\epsilon_\lambda(\eta,\nu,v;f) = w_\lambda(\eta,\nu,v;f)+\mu^\epsilon_\lambda(f)\big(h^+_\lambda(\eta,\nu,v)-h^-_\lambda(\eta,\nu,v)\big).$$

Proof of step 2. The proof follows the same logic to what is done in step 1 except that we replace the function $w_\lambda(\cdot;f)+\pi^\epsilon_\lambda(f)\left(\frac{1}{\lambda}-w_\lambda(\cdot;1)\right)$ by $w_\lambda(\cdot;f)+\mu^\epsilon_\lambda(f)\left(h^+_\lambda-h^-_\lambda\right)$ and the constant $\pi^\epsilon_\lambda(f)$ by $\mu^\epsilon_\lambda(f)$.

Step 3. To treat the general case of f, we collect what was done in the two previous steps:

$$\begin{aligned}
u^\epsilon_\lambda(\cdot;f) &= u^\epsilon_\lambda(\cdot;f_s)+u^\epsilon_\lambda(\cdot;f_a)\\
&= w_\lambda(\cdot;f_s)+\pi^\epsilon_\lambda(f_s)\Big(\frac{1}{\lambda}-w_\lambda(\cdot;1)\Big)+w_\lambda(\cdot;f_a)+\mu^\epsilon_\lambda(f_a)\left(h^+_\lambda-h^-_\lambda\right)\\
&= w_\lambda(\cdot;f)+\pi^\epsilon_\lambda(f)\Big(\frac{1}{\lambda}-w_\lambda(\cdot;1)\Big)+\mu^\epsilon_\lambda(f)\big(h^+_\lambda-h^-_\lambda\big).
\end{aligned}$$

We finally get the representation formula (3.16) from the fact that

$$u^\epsilon_\lambda(\eta,\nu,v;f)\to u_\lambda(\eta,\nu,v;f)=\mathbb{E}_{(\eta,\nu,v)}\Big[\int_0^\infty e^{-\lambda s}f(\eta^\delta_s,\nu^\delta_s,v^\delta_s)\mathrm{d}s\Big]\text{ as }\epsilon\to 0,$$

and $\lim_{\lambda\downarrow 0}\lambda u_\lambda(f)=\lim_{\lambda\downarrow 0}\lim_{\epsilon\downarrow 0}\lambda u^\epsilon_\lambda(f)$.