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A two-step procedure for time-dependent reliability-based design optimization involving piece-wise stationary Gaussian processes

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Abstract We consider in this paper a time-dependent reliability-based design optimization (RBDO) problem with constraints involving the maximum and/or the integral of a random process over a time interval. We focus especially on problems where the process is a stationary or a piece-wise stationary Gaussian process. A two-step procedure is proposed to solve the problem. First, we use ergodic theory and extreme value theory to reformulate the original constraints into time-independent ones. We obtain an equivalent RBDO problem for which classical algorithms perform poorly. The second step of the procedure is to solve the reformulated problem with a new method introduced in this paper and based on an adaptive kriging strategy well suited to the reformulated constraints called AK-ECO for Adaptive Kriging for Expectation Constraints Optimization. The procedure is first applied to a toy example involving a harmonic oscillator subjected to random forces. It is then applied to an optimal design problem for a floating offshore wind turbine.

Keywords Reliability-based design optimization (RBDO) · time-dependent reliability · extreme value theory · adaptive kriging · active learning · Monte-Carlo

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Introduction

To ensure the reliability of a structure, it is important to take into account the different sources of uncertainty in the modelling of physical phenomena. For this purpose, uncertainties are usually represented by random variables. In the context of Reliability Analysis (RA) (Lemaire et al. (2009)), failure probability is defined as:

$$\mathbb{P}\left(g\left(X_d, X_p, X_r\right) < 0\right), \tag{1}$$

where X_d, X_p, X_r are the random variables representing respectively the uncertainties of the design variables d, the model parameters, and the safety thresholds. The function g is called performance function and failure occurs when the function is negative. One evaluation of the performance function often requires a call to a time-consuming simulator. Therefore, the methods proposed in RA aim to evaluate the failure probability accurately and with as few calls to g as possible.

Optimization with a deterministic cost function and constraints expressed as failure probabilities is called Reliability-Based Design Optimization (RBDO). The intuitive way to solve a RBDO problem is to couple a classical optimization algorithm and a method to estimate the failure probabilities. This approach is called double-loop since two loops are nested: one estimates the failure probabilities, the other one updates the design. The Reliability Index Approach (RIA) and the Performance Measure Approach (PMA) (Tu et al. (1999)) are among the most popular double-loop approaches. For RIA, the First-Order Reliability Method (FORM) (Madsen et al. (2006)) is used to estimate the failure probabilities. In PMA, the initial constraints are replaced by equivalent ones which are evaluated with an inverse reliability analysis. A double loop strategy can also use a sampling method such as the Monte Carlo approach or one of the variance reduction techniques that require a smaller sample size (El Hami et al. (2017); Bourinet (2018)). In practice, the nesting of the two loops turns out to be very expensive in terms of number of calls to the performance functions.

The Single-Loop Approach (SLA) (Liang et al. (2008); Yang et al. (2020)) and the decoupled approaches have been proposed to overcome this drawback. The decoupled approaches consist in solving a sequence of deterministic optimization problems. The final design of an optimization cycle is the starting point of the next one. The Sequential Optimization and Reliability Assessment method (SORA) (Du and Chen (2004)) separates optimization and reliability loops. A deterministic optimization cycle is carried out for fixed uncertainty values. The latter are then shifted at the next cycle to make the minimum more reliable. Different methods (Torii et al. (2016); Jiang et al. (2020); Wang et al. (2020); Zhang et al. (2021)) use the same approach as SORA but propose a different shifting strategy. Benchmarks of the classical reliability approaches have been carried out in Aoues and Chateauneuf (2010) and Lopez and Beck (2012).

In recent years, in the context of RA, many articles have adopted a strategy where the costly performance function is replaced by an approximation model, called metamodel, that is fast to evaluate. Therefore, the failure probability can be estimated with sampling methods since the sample size is no longer an issue. The estimation of failure probabilities with this approach can lead to large errors if the metamodel fitting is of poor quality. Thus, an initial calibration of the metamodel is usually followed by an adaptive enrichment procedure (also called active learning). A learning function makes it possible to select optimal points in the input space of the performance function where the fitting of the metamodel must be improved. In RA, many adaptive procedures have been proposed and vary according to the choice of metamodel, learning function and enrichment stopping criterion. An extensive review and a comparison of these adaptive approaches are detailed in Teixeira et al. (2021).

Metamodels have also been applied to solve RBDO problems. In Dubourg (2011), the performance functions are replaced by kriging models (Krige (1951); Rasmussen (2004)) which are enriched during the optimization algorithm. Kriging models with different enrichment strategies are also used in Wu et al. (2021); Li et al. (2020). In Shang et al. (2021), a combination of the Polynomial Chaos Expansion (PCE) (Wiener (1938)) and Radial Basis Functions (RBF) (Buhmann (2003)) is preferred and is enriched before performing a gradient-based optimization to solve the RBDO problem. A general modular framework is proposed in Moustapha and Sudret (2019) to solve RBDO problem with metamodels: the user can choose the adaptive metamodel, the reliability analysis method, and the optimization algorithm.

Other approaches combine metamodels with PMA, SORA (Zhang et al. (2020b)) or SLA (Zhang et al. (2020a)). The method proposed in Stieng and Muskulus (2020) that we will call the Stieng method can deal with general RBDO problems under the assumption that the performance function can be approximated by the product of two functions: one depending only on the design variables which is the performance function evaluated at the mean values of the uncertainties and the other one depending on the uncertain variables. A metamodel is fitted on the second function. The RBDO problem is then solved with sequential cycles of optimization. Each cycle is composed of the update of the metamodels and a resolution with PMA of the problem.

In time-dependent Reliability Analysis (t-RA), the performance function involves a time-dependent stochastic process denoted $\mathcal{Y}$. The failure probability is usually written as follows:

$$\mathbb{P}\left(\exists t \in [0,T],\, g\left(X_d, X_p, X_r, \mathcal{Y}(t), t\right) < 0\right). \tag{2}$$

The methods estimating this quantity can be classified into the out-crossing approach and the extreme performance approach. The first one provides an upper bound of the failure probability that requires the estimation of an outcrossing rate which can be obtained in several ways (Hawchar (2017)). In the PHI2 method (Andrieu-Renaud et al. (2004)), the outcrossing rate for a given time t is obtained by performing two reliability analyses with FORM at t and $t + \Delta t$. When $\mathcal{Y}$ is stationary, the outcrossing rate needs to be estimated at only one time t whereas multiple evaluations are required for the non-stationary case. On the other hand, the extreme performance approaches directly provide an estimation of the failure probability. A common way to proceed is to consider the random variable $G_{min} = \min_{[0,T]} g\left(X_d, X_p, X_r, \mathcal{Y}(t), t\right)$ and the probability (2) can then be obtained with a RA method (Hawchar (2017); Hu and Du (2015)). Others methods based on adaptive kriging have been proposed and rely on different metamodel strategies (Hu et al. (2020a); Wang and Chen (2016); Jiang et al. (2019); Hu et al. (2020b)). For all of these methods, a sequential enrichment strategy is usually performed to improve the accuracy of the metamodel. The failure probability is then computed with Monte Carlo based on the metamodel or with other sampling methods (Ling et al. (2019)).

The time-variant Reliability-Based Design Optimization (t-RBDO) methods seek to solve optimization problems with constraints involving failure probabilities expressed as in (2). The most straightforward approach to solve a t-RBDO problem is to couple an optimization algorithm with a t-RA method to estimate the constraints at each iteration of the optimization problem (Hawchar (2017)). In TROSK (Hawchar et al. (2018)) and in PSO-t-IRS (Li and Chen (2019)), a kriging model of the performance function is built and enriched over all the design space and then the resolution of the optimization is done using Monte Carlo with the metamodel. Finally, t-SORA and t-SLA are introduced in Shi et al. (2020) and represent the time-variant versions of SORA and SLA.

This paper is motivated by an optimal design problem that aims at minimizing the material cost of the mooring system of a floating offshore wind turbine under Fatigue Limit State reliability constraints. The structure (the wind turbine) is subjected to random loads due to environmental conditions (wind and waves) which leads to constraints expressed as time-dependent failure probabilities (Cousin et al. (2021)). The methods cited above in t-RA and t-RBDO do not suppose any assumption about the process $\mathcal{Y}$. However, in some applications, such as the one that motivates our work, for fixed values of X_d, X_p and X_r the performance function is a time-dependent process with known distribution. In offshore engineering, the wind speed and sea elevation are usually represented as stationary or piece-wise stationary Gaussian processes (Vorpahl et al. (2013)). When the linearization of the movement equation is a reasonable approximation, quantities of interest such as the displacement of the structure inherit the stationary and Gaussian properties of the input processes.

Motivated by this problem, we focus in this paper on t-RBDO with stationary and piece-wise stationary Gaussian processes appearing in two types of constraints: one involving the extreme value of a process and another one, not usually addressed in t-RA and t-RBDO, which involves the integral over time of a process. Instead of following the classical approaches in t-RBDO, we propose a two-step procedure better suited to the characteristics of the studied problem. The first contribution of this article is to use ergodic theory and extreme value theory to reformulate the initial constraints into time-independent ones that are much easier to evaluate. This first part of the procedure is described in section 1 for the stationary case and in section 2 when a piece-wise stationary process is involved. At this stage, we obtain a RBDO problem with specific properties that make existing RBDO approaches suboptimal. Hence, we propose in section 3, a new adaptive kriging strategy called Adaptive Kriging for Expectation Constraints Optimization (AK-ECO) to solve the reformulated problem efficiently. To illustrate the methodology introduced in this paper, we study the academic case of a harmonic oscillator presenting the same characteristics as the industrial offshore wind turbine optimization problem. The numerical results obtained with AK-ECO on the oscillator case as well as a comparison with state-of-the-art algorithms are presented in section 4. Our approach is then applied in section 5 to the offshore wind turbine optimization problem.

1 Constraints defined in terms of a stationary Gaussian process

Note 1 In this article, the notations $\mathbb{P}_X$ and $\mathbb{E}_X$ mean that the probability and the expectation are considered with respect to the distribution of X (X can be a random variable, a random vector or a random process).

1.1 Definition of the extreme-based and integral-based constraints

Throughout this paper, we consider a t-RBDO problem of the following form:

$$\begin{aligned}
&\min_{d\in\Omega_d} \; cost(d) \quad \text{such that} \\
&\mathbb{P}_{X_d,X_p,X_{r_{\mathrm{E}}},\mathcal{Y}}\left(\min_{t\in[0,T]} g_{\mathrm{E}}\left(X_d,X_p,X_{r_{\mathrm{E}}},\mathcal{Y}(t)\right)<0\right)<p_s \\
&\mathbb{P}_{X_d,X_p,X_{r_{\mathrm{I}}},\mathcal{Y}}\left(\int_0^T g_{\mathrm{I}}\left(X_d,X_p,X_{r_{\mathrm{I}}},\mathcal{Y}(t)\right)dt<0\right)<p_s.
\end{aligned} \tag{3}$$

In the above equation, the cost function is deterministic and depends on design variables gathered in the vector d. The design space is denoted Ω_d and is a subset of $\mathbb{R}^{n_d}$. The uncertainties on the design variables, on the model and on the resistance thresholds are respectively represented by the random vectors X_d, X_p, $X_{r_{\mathrm{E}}}$, and $X_{r_{\mathrm{I}}}$ which are assumed to have known probability density functions. The performance functions are defined by:

$$g_{\mathrm{E}}(X_d,X_p,X_{r_{\mathrm{E}}},\mathcal{Y}(t)) = X_{r_{\mathrm{E}}} - \mathcal{Y}(X_d,X_p;t), \tag{4}$$

$$g_{\mathrm{I}}(X_d,X_p,X_{r_{\mathrm{I}}},\mathcal{Y}(t)) = X_{r_{\mathrm{I}}} - f\left(\mathcal{Y}\left(X_d,X_p;t\right)\right), \tag{5}$$

where $f:\mathbb{R}\to\mathbb{R}$ is measurable. The time-dependent process is denoted $\mathcal{Y}(x_d,x_p;.)$ and its distribution depends on the outcomes of the random vectors X_d and

X_p. The constraints are met if the failure probabilities do not exceed the failure probability threshold p_s.

We will call extreme-based (resp. integral-based) constraint, constraints expressed as the first (resp. second) constraint of problem (3). The failure probability involved in an extreme-based (resp. integral-based) constraint will be denoted $p_{\mathrm{E}}(d)$ (resp. $p_{\mathrm{I}}(d)$) and will be called extreme-based (resp. integral-based) failure probability. Extreme-based failure probability refers to the usual failure probability in t-RBDO except that the process distribution of $\mathcal{Y}$ depends on the random vectors X_d and X_p. The integral-based failure probability is less studied in t-RBDO and represents the probability that the accumulation of a quantity depending on $\mathcal{Y}$ exceeds some threshold over the time interval $[0, T]$. In the industrial applications are concerned with (i.e. offshore wind turbines), the evaluation of the functions g_{E} and g_{I} is costly, the time T is large and the probability threshold p_s is small.

In t-RBDO, methods used to estimate failure probabilities are often time-consuming as they require numerous evaluations of the performance functions. The first contribution of this article is to show that, if for fixed values x_d, x_p, the process $\mathcal{Y}(x_d, x_p; .)$ is a stationary or a piece-wise stationary Gaussian process, it is possible to reformulate $p_{\mathrm{E}}(d)$ and $p_{\mathrm{I}}(d)$ in expectations depending only on X_d, X_p, $X_{r_{\mathrm{E}}}$, and $X_{r_{\mathrm{I}}}$. Therefore, we obtain an optimization problem much easier to solve. Indeed, the extreme-based failure probability can be written:

$$p_{\mathrm{E}}(d) = \mathbb{E}_{X_d, X_p, X_{r_{\mathrm{E}}}} \Bigg[\mathbb{P}_{\mathcal{Y}|X_d, X_p, X_{r_{\mathrm{E}}}} \left(\min_{t\in[0,T]} g_{\mathrm{E}} \left(X_d, X_p, X_{r_{\mathrm{E}}}, \mathcal{Y}(t)\right) < 0 \right) \Bigg] \tag{6}$$

and we will show that, for fixed values of X_d, X_p, and $X_{r_{\mathrm{E}}}$ and when T is large, limit theorems for functionals of stationary or piece-wise stationary processes provide a good approximation of the conditional probability $\mathbb{P}_{\mathcal{Y}}\big(\min_{t\in[0,T]} g_{\mathrm{E}}(x_d, x_p, x_{r_{\mathrm{E}}}, \mathcal{Y}(t)) < 0\big)$. Furthermore, this approximation only uses the spectral properties of $\mathcal{Y}(x_d, x_p; .)$ and the resulting constraint is much easier to evaluate since it does not require any evaluation of $\mathcal{Y}$ and only depends on the random vectors X_d, X_p, and $X_{r_{\mathrm{E}}}$. The same reasoning will be applied to give an approximation of the integral-based failure probability.

For fixed values x_d and x_p, we consider in this section that the process $\mathcal{Y}(x_d, x_p; .)$ is a stationary Gaussian process with zero mean. Its distribution is defined by its spectral density $K_{\mathcal{Y}}(x_d, x_p; .)$ which is the Fourier transform of its autocorrelation function $k_{\mathcal{Y}}(x_d, x_p; .)$ that depends on x_d and x_p:

$$K_{\mathcal{Y}}(x_d, x_p; \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} k_{\mathcal{Y}}(x_d, x_p; t)\, e^{-i\omega t} dt. \tag{7}$$

The spectral moment of order n of $\mathcal{Y}(x_d, x_p; .)$ is defined as:

$$m_{\mathcal{Y},n}(x_d, x_p) = \int_{\mathbb{R}} \omega^n K_{\mathcal{Y}}(x_d, x_p; \omega) d\omega. \tag{8}$$

1.2 Approximation of the extreme-based failure probability

For fixed values of X_d, X_p, and $X_{r_{\mathrm{E}}}$, the probability $\mathbb{P}_{\mathcal{Y}}\left(\min_{t\in[0,T]} g_{\mathrm{E}}(x_d, x_p, x_{r_{\mathrm{E}}}, \mathcal{Y}(t)) < 0\right)$ involves the maximum of a stationary Gaussian process. Thus, the extreme value theory (Leadbetter et al. (1983)) and especially the following theorem are well suited to provide a reformulation of $p_{\mathrm{E}}(d)$. We present the theorem 8.2.7 of Leadbetter et al. (1983) for a stationary Gaussian process $\{\xi(t); t \geq 0\}$ with zero mean, autocorrelation function k_ξ, spectral density K_ξ and spectral moment of order n denoted $m_{\xi,n}$.

Theorem 1 *Suppose that the Gaussian stationary process ξ has non-zero spectral moments $m_{\xi,0}$ and $m_{\xi,2}$ and satisfies the following conditions:*

$$k_\xi(\tau) = m_{\xi,0} - \frac{m_{\xi,2}\tau^2}{2} + o(\tau^2) \text{ as } \tau \to 0, \tag{9}$$

$$k_\xi(\tau)\log(\tau) \to 0 \text{ as } \tau \to \infty, \tag{10}$$

then, as $T \to +\infty$,

$$\mathbb{P}\left(a_T \left(\max_{t\in[0,T]} \frac{\xi(t)}{\sqrt{m_{\xi,0}}} - a_T \right) \leq x \right) \to \exp\left(-e^{-x}\right), \tag{11}$$

with $a_T = \sqrt{2\log(T/T_c)}$ and $T_c = 2\pi\sqrt{m_{\xi,0}/m_{\xi,2}}$.

The formulation of theorem 1 of this paper is obtained by applying theorem 8.2.7 of Leadbetter et al. (1983) to the process $\xi(tT_c)$.

Remark 1 It is possible to give sufficient conditions for theorem 1 that are explicit in K_ξ. Denoting K'_ξ the derivative of K_ξ, conditions (9) and (10) are met if we have:

$$m_{\xi,0} < \infty, m_{\xi,2} < \infty, \tag{12}$$

$$K_\xi \in C^1, K_\xi \text{ and } K'_\xi \text{ are integrable.} \tag{13}$$

For condition (13), we use that if $K_\xi \in C^1$ and K_ξ and K'_ξ are integrable then $\exists c > 0$ such that $|k_\xi(\tau)| \leq \frac{c}{|\tau|}$ and therefore, condition (10) is met. Other sufficient conditions on the spectral density are discussed in Berman (1991).

For fixed values x_d and x_p, if the process $\mathcal{Y}(x_d, x_p; .)$ introduced in section 1.1 meets conditions (12) and (13), we can apply theorem 1 and obtain $\forall x$, as $T \to +\infty$:

$$\mathbb{P}_{\mathcal{Y}}\left(a_T(x_d, x_p) \left(\max_{t\in[0,T]} \frac{\mathcal{Y}(x_d, x_p; t)}{\sqrt{m_{\mathcal{Y},0}(x_d, x_p)}} - a_T(x_d, x_p) \right) \leq x \right) \to \exp\left(-e^{-x}\right), \quad (14)$$

with $a_T(x_d, x_p) = \sqrt{2 \log \left(\frac{T}{2\pi} \sqrt{\frac{m_{\mathcal{Y},2}(x_d,x_p)}{m_{\mathcal{Y},0}(x_d,x_p)}} \right)}$. Therefore, for T large enough, it is reasonable to make the following approximation:

$$p_{\mathrm{E}}(d) \simeq \mathbb{E}_{X_d, X_p, X_{r_{\mathrm{E}}}} \left[F_\epsilon \left(\exp\left(a_T(X_d, X_p)^2 - \frac{a_T(X_d, X_p)\, X_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(X_d, X_p)}} \right) \right) \right], \quad (15)$$

with $F_\epsilon(x) = 1 - \exp(-x)$. The approximation error made in equation (15) can be bounded with classical results (Kratz and Rootzén (1997)) and is discussed in section A.1 of appendix A.

Remark 2 The initial failure probability that depends on a random process has been approximated by an expectation which only depends on random vectors. Furthermore, to compute the quantity within the square brackets in (15) only two spectral moments of $\mathcal{Y}(x_d, x_p; .)$ need to be evaluated (for each outcome of X_d and X_p).

1.3 Approximation of the integral-based failure probability

We focus now on $p_{\mathrm{I}}(d)$. For fixed values x_d, x_p, the process $f(\mathcal{Y}(x_d, x_p; .))$ is denoted by $\mathcal{F}(x_d, x_p; .)$. Since $\mathcal{Y}(x_d, x_p; .)$ is stationary, $\mathcal{F}(x_d, x_p; .)$ is also stationary and we denote by $k_{\mathcal{F}}(x_d, x_p; .)$ its autocovariance function. In the following we will use the definition of ergodicity below.

Definition 1 The process $\mathcal{F}(x_d, x_p; .)$ is said to be ergodic if:

$$\frac{1}{T}\int_0^T \mathcal{F}(x_d, x_p; t)\, dt \xrightarrow[T\to+\infty]{\mathbb{P}} \mathbb{E}_{\mathcal{F}}\left[\mathcal{F}(x_d, x_p; 0)\right], \quad (16)$$

where $\xrightarrow{\mathbb{P}}$ refers to the convergence in probability. A sufficient condition for the stationary process $\mathcal{F}(x_d, x_p; .)$ to be ergodic (cf section 13.1 of Papoulis (1991)) is that $k_{\mathcal{F}}(x_d, x_p; .)$ is integrable.

Suppose that $\mathcal{F}(x_d, x_p; .)$ is ergodic. Then, for almost every x (more exactly $\forall x \neq \mathbb{E}_{\mathcal{F}}[\mathcal{F}(x_d, x_p; 0)]$):

$$\mathbb{P}_{\mathcal{Y}}\left(\frac{1}{T}\int_0^T \mathcal{F}(x_d, x_p; t)\, dt > x \right) \to \mathbb{1}_{\mathbb{E}_{\mathcal{F}}[\mathcal{F}(x_d,x_p;0)]>x}, \quad (17)$$

as $T \to +\infty$. Thus, for T large enough, it is reasonable to approximate the integral-based failure probability as follows:

$$p_{\mathrm{I}}(d) \simeq \mathbb{E}_{X_d, X_p}\left[F_{r_{\mathrm{I}}}\left(T\mathbb{E}_{\mathcal{F}}\left[\mathcal{F}(X_d, X_p; 0)\right]\right)\right], \quad (18)$$

with $F_{r_{\mathrm{I}}}$ the cumulative distribution function of $X_{r_{\mathrm{I}}}$ (which is continuous). The approximation error made in equation (18) is discussed in section A.2 of appendix A.

Remark 3 To compute $\mathbb{E}_{\mathcal{F}}[\mathcal{F}(x_d, x_p; 0)]$, it is necessary to know the distribution of $\mathcal{Y}(x_d, x_p; 0)$. Since the process $\mathcal{Y}(x_d, x_p; .)$ is Gaussian with zero mean, the variance of $\mathcal{Y}(x_d, x_p; 0)$ determines its distribution. Hence, to compute the quantity within the square brackets in (18), we only need to know the variance of $\mathcal{Y}(x_d, x_p; 0)$ for each outcome of X_d and X_p.

1.4 Optimization problem involving a stationary harmonic oscillator

We present in this section a concrete optimization problem with constraints involving extreme-based and integral-based failure probabilities and we apply the reformulation procedure described in sections 1.2 and 1.3.

Let us consider a harmonic oscillator on an interval of time $[0, T]$: a spring/mass system. We denote respectively by x_{d_1} the mass of the object, x_{d_2} the spring stiffness, and x_p the damping coefficient. An external force is exerted on the system. To account for all the sources of uncertainty of the experiment, the values of x_{d_1}, x_{d_2}, x_p, and the external force are considered random. These uncertainties are respectively represented by the random variables X_{d_1}, X_{d_2}, X_p, and the stochastic process $\eta(t)$. We denote X_d the random vector (X_{d_1}, X_{d_2}) of outcome $x_d = (x_{d_1}, x_{d_2})$ whose distribution depends on the design variable $d = (d_1, d_2)$.

Consequently, for fixed values x_d and x_p, the displacement of the mass with respect to the equilibrium position is represented by a stochastic process denoted $\mathcal{D}(x_d, x_p; .)$ which is solution of the harmonic oscillator equation:

$$x_{d_1}\mathcal{D}^{(2)}(x_d, x_p; t) + x_p\mathcal{D}^{(1)}(x_d, x_p; t) + x_{d_2}\mathcal{D}(x_d, x_p; t) = \eta(t)\,,\ t \in [0, T], \quad (19)$$

where $\mathcal{D}^{(1)}\left(x_d, x_p;.\right)$ and $\mathcal{D}^{(2)}\left(x_d, x_p;.\right)$ are respectively the velocity and acceleration processes whose sample paths are the first and second time derivatives of the sample path of $\mathcal{D}(x_d, x_p;.)$.

The optimization problem consists in minimizing a linear function $cost(d)$ while constraints are imposed on the design variable such that:

- the velocity and the acceleration of the oscillator must stay below given thresholds x_{r_1} and x_{r_2} respectively (this is a simplified model for the extreme constraints that we have in mind in our industrial application);
- the accumulated amount of acceleration of the object exceeding the threshold ρ must remain under a resistance threshold x_{r_3} (this is a simplified model for the fatigue constraint that we have in mind in our industrial application).

The thresholds $x_{r_1}, x_{r_2}, x_{r_3}$ are also random and therefore outcomes of random variables denoted X_{r_1}, X_{r_2}, and X_{r_3}. The optimization problem is formulated as follows:

$$\begin{aligned}
&\min_{d\in\Omega_d}\ cost(d) \quad \text{such that} \\
&\mathbb{P}_{X_d,X_p,X_{r_k},\eta}\left(\max_{t\in[0,T]} \mathcal{D}^{(k)}\left(X_d, X_p; t\right) > X_{r_k}\right) < p_s, \\
&\hfill k = 1,2 \\
&\mathbb{P}_{X_d,X_p,X_{r_3},\eta}\left(\int_0^T \left(\left|\mathcal{D}^{(2)}\left(X_d, X_p; t\right)\right| - \rho\right)^+ dt > X_{r_3}\right) \\
&\hfill < p_s
\end{aligned} \tag{20}$$

with $x^+ = \max(0, x)$. The design space is defined as $\Omega_d = [d_1^-, d_1^+] \times [d_2^-, d_2^+] \subset \mathbb{R}^2$. The distributions of X_d and X_p are chosen such that the oscillator is underdamped for almost all realizations i.e. $X_p^2 - 4X_{d_1}X_{d_2} < 0$ almost surely. Moreover, we assume that the process η is stationary, Gaussian with spectral density $K_\eta(\omega) = \frac{\theta}{\sqrt{2\pi}} \exp\left(-\frac{(\theta\omega)^2}{2}\right)$ with $\theta > 0$. Finally all the sources of uncertainty $(X_{d_1}, X_{d_2}, X_p, X_{r_1}, X_{r_2}, X_{r_3}, \eta)$ are independent.

For fixed values of X_d, X_p, it follows from equation (19) and the stationarity of η (see Lindgren (2012)) that the process $\mathcal{D}\left(x_d, x_p;.\right)$ is stationary and can be written as the output of a linear filter $\mathcal{D}\left(x_d, x_p; t\right) = \left(h_\mathcal{D}\left(x_d, x_p,.\right) * \eta\right)(t)$, with $h_\mathcal{D}$ defined by:

$$\begin{aligned}
H_\mathcal{D}\left(x_d, x_p; \omega\right) &= \mathrm{FT}\left(h_\mathcal{D}\left(x_d, x_p;.\right)\right)(\omega) \\
&= \frac{1}{-\omega^2 x_{d_1} + i\omega x_p + x_{d_2}}
\end{aligned} \tag{21}$$

where FT refers to the Fourier transformation. It is shown (see Lindgren (2012)) that the process $\mathcal{D}\left(x_d, x_p;.\right)$ is then also Gaussian with zero mean and its spectral density is given by:

$$K_\mathcal{D}\left(x_d, x_p; \omega\right) = \left|H_\mathcal{D}\left(x_d, x_p; \omega\right)\right|^2 K_\eta(\omega). \tag{22}$$

Furthermore, if the process $\mathcal{D}\left(x_d, x_p;.\right)$ has finite spectral moment of order 2 and 4 (see theorem 2.2 of Lindgren (2012)), the processes $\mathcal{D}^{(1)}\left(x_d, x_p;.\right)$ and $\mathcal{D}^{(2)}\left(x_d, x_p;.\right)$ are also zero-mean stationary Gaussian processes with spectral densities, denoted respectively $K_{\mathcal{D}^{(1)}}\left(x_d, x_p,.\right)$ and $K_{\mathcal{D}^{(2)}}\left(x_d, x_p,.\right)$, given by:

$$K_{\mathcal{D}^{(k)}}\left(x_d, x_p; \omega\right) = \omega^{2k} K_\mathcal{D}\left(x_d, x_p; \omega\right) \quad k = 1,2. \tag{23}$$

Remark 4 In fact, the result of theorem 2.2 of Lindgren (2012) holds for the first and second order derivatives of $\mathcal{D}\left(x_d, x_p;.\right)$ in the quadratic mean sense. But under the supplementary condition that the spectral moment of order 6 of $\mathcal{D}\left(x_d, x_p;.\right)$ is also finite, the result holds for sample path derivatives too (see Lindgren (2012)).

The spectral moments of order n of $\mathcal{D}^{(k)}(x_d, x_p;.)$ $k = 1,2$, are denoted $m_{\mathcal{D}^{(k)},n}\left(x_d, x_p\right)$ $k = 1,2$. It follows from the properties of the processes $\mathcal{D}^{(k)}\left(x_d, x_p;.\right)$, $k = 1,2$ that the two first constraints of problem (20) are extreme-based constraints whereas the third constraint is integral-based. We show in proof A.3 in appendix A that, for all x_d, x_p, the processes $\mathcal{D}^{(k)}\left(x_d, x_p;.\right)$, $k = 1,2$, meet the sufficient conditions (12,13) which allow the reformulation of the two first constraints of the problem under study. Since the process $\mathcal{D}^{(2)}\left(x_d, x_p;.\right)$ is Gaussian and its autocovariance function converges to 0 at infinity and is integrable, it is easy to show that the process $\mathcal{F}\left(x_d, x_p;.\right) = \left(\left|\mathcal{D}^{(2)}\left(x_d, x_p;.\right)\right| - \rho\right)^+$ has an integrable autocovariance function and thus, $\mathcal{F}\left(x_d, x_p;.\right)$ is ergodic.

Since all the required conditions are met, for T large enough, we can apply the reformulation steps described in sections 1.2 and 1.3 to the constraints of problem (20). The two first constraints are replaced by the approximation given by (15) where a_T, $m_{\mathcal{Y},0}$ and X_{r_E} become respectively a_T^1, $m_{\mathcal{D}^{(1)},0}$ and X_{r_1} for the first constraint and a_T^2, $m_{\mathcal{D}^{(2)},0}$ and X_{r_2} for the second constraint with:

$$a_T^k(x_d, x_p) = \sqrt{2\log\left(\frac{T}{2\pi}\sqrt{\frac{m_{\mathcal{D}^{(k)},2}\left(x_d, x_p\right)}{m_{\mathcal{D}^{(k)},0}\left(x_d, x_p\right)}}\right)}, \tag{24}$$

for $k = 1,2$. The third constraint of problem (20) is replaced by the approximation given by (18) with

$$\mathcal{F}(x_d, x_p; 0) = \left(\left|\mathcal{D}^{(2)}\left(x_d, x_p; 0\right)\right| - \rho\right)^+, \tag{25}$$

and X_{r_I} becomes X_{r_3}.

2 Constraints defined in terms of a piece-wise stationary Gaussian process

2.1 Definition of the piece-wise stationary process

We consider in this section the optimization problem (3) introduced in section 1 with extreme-based and integral-based constraints except that the process $\mathcal{Y}$ is now piece-wise stationary. The period $[0,T]$ is decomposed into n_T intervals $I_i = [(i-1)\Delta T, i\Delta T]$, $i = 1,\dots,n_T$ and for fixed x_d, x_p the process $\mathcal{Y}$ is defined as:

$$\mathcal{Y}(x_d,x_p;t) = \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d,x_p,s_i;t\right) \mathbb{1}_{I_i}(t) \tag{26}$$

where $s_1,\dots,s_{n_T}$ is a sequence of elements of the set $\{s^1,\dots,s^{n_s}\}$. The processes $\mathcal{Y}_i\left(x_d,x_p,s_1;.\right)$, $i = 1,\dots,n_T$, are independent stationary Gaussian processes with zero mean. The distribution of $\mathcal{Y}_i\left(x_d,x_p,s_i;.\right)$ for $i \in \{1,\dots,n_T\}$ is defined by its spectral density $K_{\mathcal{Y}}(x_d,x_p,s_i;.)$ and we denote $m_{\mathcal{Y},n}\left(x_d,x_p,s_i\right)$ its spectral moment of order n. We denote n^j and p^j, the quantities such that $n^j = \#\{i \in \{1,\dots,n_T\}, s_i = s^j\}$ and $p^j = n^j/n_T$ for $j = 1,\dots,n_s$.

Remark 5 The definitions of the objects introduced in this section are motivated by the industrial case of a floating offshore wind turbine. In this application, each process $\mathcal{Y}_i\left(x_d,x_p,s^j;.\right)$ can be seen as the displacement over time of the floating platform subjected to a sea elevation process characterized by a sea state defined by s^j, for some design and parametric variables x_d and x_p. The constraints then relate to the maximum admissible displacement of the structure or to the accumulated damage of the mooring lines. Furthermore, dividing the time interval $[0,T]$ into n_T intervals is standard in offshore models to represent the different sea states encountered during the period $[0,T]$. To stay close to the engineering terminology, s^j will be called the state of the process $\mathcal{Y}_i\left(x_d,x_p,s^j;.\right)$ in this paper.

The reasoning for extreme-based and integral-based constraints reformulation is the same as for the stationary case: the purpose is to provide good approximations that only rely on spectral properties of the processes $\mathcal{Y}_i\left(x_d,x_p,s_i;.\right)$.

2.2 Approximation of the extreme-based failure probability

To approximate the extreme-based failure probability, we claim that, for fixed x_d,x_p,x_{r_E} and for ΔT large enough:

$$\mathbb{P}_{\mathcal{Y}}\left(\max_{t\in[0,T]} \mathcal{Y}\left(x_d,x_p;t\right) \le x_{r_\mathrm{E}}\right) \simeq \prod_{j=1}^{n_s} \mathbb{P}_{\mathcal{Y}_1}\left(\max_{t\in[0,Tp^j]} \mathcal{Y}_1\left(x_d,x_p,s^j;t\right) \le x_{r_\mathrm{E}}\right) \tag{27}$$

This approximation is justified in section B.1 of Appendix B.

Therefore, if for all x_d,x_p,x_{r_E}, and for all states s^j, the process $\mathcal{Y}_1\left(x_d,x_p,s^j;.\right)$ meets the conditions of theorem 1, it follows from equation (27) and theorem 1 that for ΔT sufficiently large, we can make the following approximation:

$$p_\mathrm{E}(d) \simeq \mathbb{E}_{X_d,X_p,X_{r_\mathrm{E}}}\left[F_\epsilon\left(\sum_{j=1}^{n_s}\exp\left(a_{Tp^j}(X_d,X_p,s^j)^2 - \frac{a_{Tp^j}(X_d,X_p,s^j)X_{r_\mathrm{E}}}{\sqrt{m_{\mathcal{Y},0}\left(X_d,X_p,s^j\right)}}\right)\right)\right] \tag{28}$$

with

$$a_{Tp^j}(x_d,x_p,s^j) = \sqrt{2\log\left(\frac{Tp^j}{2\pi}\sqrt{\frac{m_{\mathcal{Y},2}\left(x_d,x_p,s^j\right)}{m_{\mathcal{Y},0}\left(x_d,x_p,s^j\right)}}\right)}. \tag{29}$$

Bounds on the approximation error of equation (28) are proposed in section B.2 of Appendix B.

2.3 Approximation of the integral-based failure probability

Proposition 1 *We denote by $\mathcal{F}_1\left(x_d,x_p,s^j;.\right)$ the process $f\left(\mathcal{Y}_1\left(x_d,x_p,s^j;.\right)\right)$. If for all x_d, x_p, s^j, the process $\mathcal{F}_1\left(x_d,x_p,s^j;.\right)$ is ergodic, we have for almost every x:*

$$\mathbb{P}_{\mathcal{Y}}\left(\frac{1}{\Delta T}\int_0^T f\left(\mathcal{Y}\left(x_d,x_p;t\right)\right)dt > x\right) \underset{\Delta T\to+\infty}{\longrightarrow} \mathbb{1}_{\sum_{j=1}^{n_s} n^j \mathbb{E}_{\mathcal{F}_1}[\mathcal{F}_1(x_d,x_p,s^j;0)]>x}. \tag{30}$$

The proof of proposition 1 is given in section B.3 of Appendix B.

Using proposition 1, when ΔT is large enough, we can approximate the integral-based failure probability as follows:

$p_{\mathrm{I}}(d)$

$$\simeq \mathbb{E}_{X_d,X_p}\left[F_{r_{\mathrm{I}}}\left(T\sum_{j=1}^{n_s}p^j\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(X_d,X_p,s^j;0\right)\right]\right)\right], \tag{31}$$

with $F_{r_{\mathrm{I}}}$ the cumulative distribution function of $X_{r_{\mathrm{I}}}$. The approximation error made in equation (31) is discussed in section B.4 of Appendix B.

2.4 Optimization problem involving a piece-wise stationary harmonic oscillator

The oscillator problem presented in section 1.4 is slightly modified by considering a piece-wise stationary process $\mathcal{D}$ as defined in section 2.1. For fixed values x_d, x_p and for $i=1,\dots,n_T$, the process $\mathcal{D}_i\left(x_d,x_p,s_i;.\right)$ is solution of the harmonic oscillator equation (19) with an external force $\eta(s_i,.)$. The time-dependent process $\eta(s_i,.)$ is a zero-mean stationary Gaussian process with spectral density:

$$K_\eta\left(s_i;\omega\right)=\frac{s_i}{\sqrt{2\pi}}\exp\left(-\frac{\left(s_i\omega\right)^2}{2}\right), \tag{32}$$

for $s_i>0$. In the piece-wise stationary problem, the processes $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ are defined by the following equation:

$$\mathcal{D}^{(k)}\left(x_d,x_p;t\right)=\sum_{i=1}^{n_T}\mathcal{D}_i^{(k)}\left(x_d,x_p,s_i;t\right)\mathbb{1}_{I_i}(t), \tag{33}$$

for $k=1,2$, with $\mathcal{D}_i^{(k)}\left(x_d,x_p,s_i;.\right)$, the first ($k=1$) and second ($k=2$) time derivatives of $\mathcal{D}_i\left(x_d,x_p,s_i;.\right)$. Hence, the processes $\mathcal{D}_i^{(k)}\left(x_d,x_p,s_i;.\right)$, $k=1,2$, are zero-mean stationary Gaussian processes with respective spectral densities:

$$K_{\mathcal{D}^{(k)}}\left(x_d,x_p,s_i;\omega\right)=\omega^{2k}\left|H_{\mathcal{D}}\left(x_d,x_p;\omega\right)\right|^2K_\eta\left(s_i;\omega\right), \tag{34}$$

for $k=1,2$. We denote $m_{\mathcal{D}^{(k)},n}\left(x_d,x_p,s_i\right)$, $k=1,2$, their spectral moments of order n. Furthermore, the arguments used in the stationary case allow to show that the processes $\mathcal{D}_i^{(k)}\left(x_d,x_p,s_i;.\right)$, $k=1,2$, meet all the conditions to use the reformulation procedure described in section 2.

If ΔT is large enough, it is reasonable to approximate the constraints of problem (20) with a piece-wise stationary process $\mathcal{D}$ by applying the results of sections 2.2 and 2.3. The two first constraints are replaced by the approximation given by equation (28) where a_{Tp^j}, $m_{\mathcal{Y},0}$ and $X_{r_{\mathrm{E}}}$ become respectively $a^1_{Tp^j}$, $m_{\mathcal{D}^{(1)},0}$ and X_{r_1} for the first constraint and $a^2_{Tp^j}$, $m_{\mathcal{D}^{(2)},0}$ and X_{r_2} for the second constraint with:

$$a^k_{Tp^j}\left(x_d,x_p,s^j\right)=\sqrt{2\log\left(\frac{Tp^j}{2\pi}\sqrt{\frac{m_{\mathcal{D}^{(k)},2}\left(x_d,x_p,s^j\right)}{m_{\mathcal{D}^{(k)},0}\left(x_d,x_p,s^j\right)}}\right)}, \tag{35}$$

for $k=1,2$. The third constraint of problem (20) is replaced by the approximation given by equation (31) with

$$\mathcal{F}_1(x_d,x_p,s^j;0)=\left(\left|\mathcal{D}_1^{(2)}\left(x_d,x_p,s^j;0\right)\right|-\rho\right)^+ \tag{36}$$

and $X_{r_{\mathrm{I}}}$ becomes X_{r_3}. We give the relation between $\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(x_d,x_p,s^j;0\right)\right]$ and $m_{\mathcal{D}^{(2)},0}\left(x_d,x_p,s^j\right)$ in appendix B.5.

Remark 6 To evaluate $a^k_{Tp^j}\left(x_d,x_p,s^j\right)$, $k=1,2$ and $\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(x_d,x_p,s^j;0\right)\right]$, the spectral moments of the velocity and acceleration processes need to be numerically computed from the integrals

$$\int_{\mathbb{R}}\omega^n\left|H_{\mathcal{D}}\left(x_d,x_p;\omega\right)\right|^2K_\eta\left(s^j;\omega\right)d\omega, \tag{37}$$

for $n=2,4,6$. Hence, the evaluation of the spectral moments represents the expensive part of the evaluation of the constraints.

3 An active learning Kriging approach for the reformulated optimization problem: AK-ECO

We consider in this section the general problem (3) introduced in section 1 involving a piece-wise stationary process. After the reformulation of the constraints presented in section 2, we end up with the following problem:

$$\begin{aligned}&\min_{d\in\Omega_d}\ cost(d)\quad\text{such that}\\&\mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}}\left[F_\epsilon\left(\sum_{j=1}^{n_s}\exp\left(M_{\mathrm{E}}\left(X_d,X_p,X_{r_{\mathrm{E}}},s^j\right)\right)\right)\right]<p_s\\&\mathbb{E}_{X_d,X_p}\left[F_{r_{\mathrm{I}}}\left(\sum_{j=1}^{n_s}Tp^jM_{\mathrm{I}}\left(X_d,X_p,s^j\right)\right)\right]<p_s\end{aligned} \tag{38}$$

with

$$M_{\mathrm{E}}\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right) = a_{Tp^j}\left(x_d, x_p, s^j\right)^2 - \frac{a_{Tp^j}\left(x_d, x_p, s^j\right) x_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}\left(x_d, x_p, s^j\right)}}, \tag{39}$$

$$M_{\mathrm{I}}\left(x_d, x_p, s^j\right) = \mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(x_d, x_p, s^j; 0\right)\right]. \tag{40}$$

All the notations appearing in problem (38) have been introduced in section 2. The *cost* function is supposed to be fast to evaluate. We remark that the problem is now a time-independent one. Even though the reformulated constraints are easier to evaluate than the initial ones, for each realization x_d, x_p and for each state s^j, the spectral moments of $\mathcal{Y}_1\left(x_d, x_p, s^j; .\right)$ are required to compute the quantities $M_{\mathrm{E}}\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right)$ and $M_{\mathrm{I}}\left(x_d, x_p, s^j\right)$. The evaluation of those spectral moments requires an expensive simulator. Therefore, an estimation of one of the constraints with the Monte Carlo method and a sample of size n_{MC} would impose $n_{MC} \times n_s$ calls to the simulator. This would be too computationally expensive especially when we deal with rare events (i.e. when $p_s \ll 1$). We mentioned in the introduction of the paper existing methods that solve RBDO problems much faster than the brute Monte Carlo. However, we will see the drawbacks of these methods when applied to problem (38). We thus propose a new method, better suited for the reformulated problem and called Adaptive Kriging for Expectation Constraints Optimization (AK-ECO).

Remark 7 Presented as in problem (38), the reformulated constraints depend on the same piece-wise stationary process $\mathcal{Y}$. However the resolution methods that are presented in this section can be applied to constraints that depend on different processes. When several constraints depend on the same process, it can be noticed that the outputs of each simulation can be used in the estimation of the different constraints since they depend on the same quantities (i.e. the spectral moments of the process).

Remark 8 For simplicity, we present a problem with two constraints (one extreme-based and one integral-based) but the resolution methods that we introduce can be applied to several constraints of each type. Furthermore, a problem with extreme and integral based constraints but with a stationary process $\mathcal{Y}$ can be solved with the same approaches since the reformulated problem would be identical to the piece-wise stationary case considering $n_s = 1$.

All the effective approaches in the literature of RBDO methods rely on the assumption that the constraints are expressed as probabilities. Thus, we write the extreme-based and integral-based reformulated constraints of problem (38) as failure probabilities as follows:

$$\mathbb{P}_{X_d, X_p, X_{r_{\mathrm{E}}}, X_\epsilon}\left(X_\epsilon - \sum_{j=1}^{n_s} \exp\left(M_{\mathrm{E}}\left(X_d, X_p, X_{r_{\mathrm{E}}}, s^j\right)\right) < 0\right) < p_s, \tag{41}$$

$$\mathbb{P}_{X_d, X_p, X_{r_{\mathrm{I}}}}\left(X_{r_{\mathrm{I}}} - \sum_{j=1}^{n_s} Tp^j M_{\mathrm{I}}\left(X_d, X_p, s^j\right) < 0\right) < p_s, \tag{42}$$

where X_ϵ is a random variable with an exponential distribution of parameter 1. However, since each evaluation of the functions M_{E} and M_{I} requires a call to an expensive simulator, one evaluation of the performance function of each of these constraints would need n_s simulations. When n_s is large, which is the case for offshore applications (Vorpahl et al. (2013)), the double-loop, single-loop and decoupled loop approaches can be too expensive. This is also the case for the adaptive metamodel approaches since they always replace the whole performance function.

The limitations of the current methods in RBDO have motivated the development of a new approach. For each expectation constraint of problem (38), a metamodel is built to replace the expensive function involved in the reformulated failure probability. We thus obtain as many metamodels as there are constraints. Then, cycles of optimization are carried out. During each cycle, the metamodels are sequentially enriched and the design point is updated. The particularity of our approach lies in the metamodel and active learning strategy which are adapted to the reformulated constraints of the studied problem.

3.1 Metamodel strategy: Kriging

In problem (38), since the expensive functions of the extreme-based and integral-based constraints are M_{E} and M_{I}, we propose to build a metamodel for each of these functions. Unlike other metamodel approaches, the metamodels do not replace the performance functions. Thus, for each $(x_d, x_p, x_{r_{\mathrm{E}}})$, the functions M_{E} and M_{I} need to be evaluated only on the relevant state s^j as we will see below. This could drastically reduce the number of calls to the simulator, especially when n_s is large.

As in Dubourg (2011) and Moustapha and Sudret (2019), for each constraint, we build the metamodel in a so-called augmented space which allows to use and enrich a single model during the whole procedure of

AK-ECO. To do so, the augmented space spans both the design space and the space of uncertainties. The augmented spaces of M_{E} and M_{I} are respectively denoted $\Omega_{\mathrm{E}}^{aug}$ and $\Omega_{\mathrm{I}}^{aug}$. To define precisely those spaces, we need to introduce some notations.

Let $(d_1, \ldots, d_{n_d}) \in \Omega_d = \Omega_{d_1} \times \ldots \times \Omega_{d_{n_d}} \subset \mathbb{R}^{n_d}$ be a design point and $X_{d_1}, \ldots, X_{d_{n_d}}$ the random variables with respective cumulative distribution functions $F_{d_1}, \ldots, F_{d_{n_d}}$ describing the uncertainties at this point. The random vector X_p is composed of n_p random variables $X_{p_i}, (i = 1, \ldots, n_p)$ with cumulative distribution functions $F_{p_i}, (i = 1, \ldots, n_p)$. Finally the cumulative distribution function of $X_{r_{\mathrm{E}}}$ is denoted $F_{r_{\mathrm{E}}}$. The augmented spaces are then defined as follows:

$$\Omega_{\mathrm{E}}^{aug} = \Omega_d^{aug} \times \Omega_p^{aug} \times \Omega_{r_{\mathrm{E}}}^{aug} \times \Omega_S, \tag{43}$$

$$\Omega_{\mathrm{I}}^{aug} = \Omega_d^{aug} \times \Omega_p^{aug} \times \Omega_S, \tag{44}$$

with (denoting by F^{-1} the quantile function associated to a cumulative distribution function F)

$$\Omega_d^{aug} = \prod_{i=1}^{n_d} \Big[\inf_{d_i \in \Omega_{d_i}} F_{d_i}^{-1}(\alpha), \sup_{d_i \in \Omega_{d_i}} F_{d_i}^{-1}(1-\alpha) \Big], \tag{45}$$

$$\Omega_p^{aug} = \prod_{i=1}^{n_p} \left[F_{p_i}^{-1}(\alpha), F_{p_i}^{-1}(1-\alpha) \right], \tag{46}$$

$$\Omega_{r_{\mathrm{E}}}^{aug} = \left[F_{r_{\mathrm{E}}}^{-1}(\alpha), F_{r_{\mathrm{E}}}^{-1}(1-\alpha) \right], \tag{47}$$

$$\Omega_S = \{s^1, \ldots, s^{n_s}\}, \tag{48}$$

and α is a degree of confidence chosen by the user (different values of α could be considered for each set of the Cartesian products Ω_d^{aug}, Ω_p^{aug}, and for $\Omega_{r_{\mathrm{E}}}^{aug}$).

The dimensions of $\Omega_{\mathrm{E}}^{aug}$ and $\Omega_{\mathrm{I}}^{aug}$ are respectively equal to $n_d + n_p + 2$ and $n_d + n_p + 1$. We suppose that $n_d + n_p$ is relatively small (less than 12). Under this assumption, the kriging model is particularly well suited to our approach. When using this technique, an expensive function denoted M is considered as the realization of a Gaussian stochastic process $\widetilde{M}$ whose distribution is characterized by several parameters called hyperparameters. The responses of M over a design of experiments (DoE) are used to calibrate those hyperparameters. The mean μ, the autocorrelation function and the standard deviation σ of the a posteriori distribution $\widetilde{M}$ are then analytically deduced from the hyperparameters and the responses of M (Rasmussen (2004)). Thus, at each point x of the input space of M, the process $\widetilde{M}$ provides a prediction in the form of a Gaussian random variable with distribution $\mathcal{N}\left(\mu(x), \sigma(x)^2\right)$. The mean $\mu(x)$ is used as predictor while the standard deviation $\sigma(x)$ measures the accuracy of the predictor. This latter information makes kriging metamodel particularly well suited to active learning. Therefore, the metamodel used in AK-ECO is the kriging model since it gives good predictions and provides information about the accuracy of its prediction through the variance of the kriging which will be useful for the enrichment of the models.

We can notice that $\Omega_{\mathrm{E}}^{aug} = \Omega_{\mathrm{I}}^{aug} \times \Omega_{r_{\mathrm{E}}}^{aug}$. Thus, to calibrate the metamodels, we only use one DoE of $\Omega_{\mathrm{E}}^{aug}$. Using one DoE is interesting when the two constraints depend on the same process $\mathcal{Y}$ since each simulation can be used to enrich both metamodels. Indeed, in that case, we recall that to evaluate M_{E} and M_{I} at a point $\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right)$, we only need the spectral moments of the process $\mathcal{Y}_1\left(x_d, x_p, s^j; .\right)$.

3.2 Procedure

To solve the reformulated problem (38), AK-ECO begins with the initialization of the design point and the kriging models. Then, the reformulated problem is solved through cycles of optimization. The initialization and the optimization cycle structure are described below.

Initialization: the initial design point d^0 is chosen by the user. An initial DoE, denoted DoE^0, is computed and used to calibrate the initial metamodels $\widetilde{M_{\mathrm{E}}}^0$, $\widetilde{M_{\mathrm{I}}}^0$ of the functions M_{E} and M_{I} (see section 3.3 for more details). At the end of the initialization, the first cycle of optimization ($k = 1$) can begin.

Optimization cycle: we respectively denote d^{k-1}, DoE^{k-1}, $\widetilde{M_{\mathrm{E}}}^{k-1}$, and $\widetilde{M_{\mathrm{I}}}^{k-1}$, the design point, DoE, and kriging models recovered from the initialization if $k = 1$ or from the previous cycle if $k > 1$. Each cycle is numbered k and is decomposed into two steps:

- Step 1. Local enrichment at d^{k-1} of the metamodels $\widetilde{M_{\mathrm{E}}}^{k-1}$ and $\widetilde{M_{\mathrm{I}}}^{k-1}$. For each metamodel:
  - Step 1.a. An accuracy criterion assesses the precision of the metamodel at d^{k-1} (we detail this step in section 3.4).
  - Step 1.b. If the metamodel is inaccurate, one local enrichment is carried out. The local refinement of the metamodel consists in adding to the shared DoE the point x_{enr} selected by the procedure described in section 3.5. The simulator is evaluated at this point and the spectral moments obtained are used to recalibrate all the kriging models.

  Steps 1.a and 1.b are repeated until each kriging model meets the accuracy condition of step 1.a. At the end of step 1, the enriched DoE and kriging models are denoted DoE^k, $\widetilde{M_{\mathrm{E}}}^k$, and $\widetilde{M_{\mathrm{I}}}^k$. For each point of the respective augmented spaces, the predictive means of the kriging models are respectively denoted $\mu_{\mathrm{E}}^{k}\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right)$ and $\mu_{\mathrm{I}}^{k}\left(x_d, x_p, s^j\right)$.

Step 2. The reformulated problem (38) is solved using the optimization algorithm chosen by the user starting from d^{k-1}. At each iteration of the optimization, the constraints are estimated with Monte Carlo and the expensive functions are replaced by their current surrogates. For a design d, those estimations, denoted $p_{\mathrm{E}}^{k}(d)$ and $p_{\mathrm{I}}^{k}(d)$, are given by:

$$p_{\mathrm{E}}^{k}(d) = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} F_{\epsilon}\left(\sum_{j=1}^{n_s} \exp\left(\mu_{\mathrm{E}}^{k}\left(x_d^i, x_p^i, x_{r_{\mathrm{E}}}^i, s^j\right)\right)\right), \tag{49}$$

$$p_{\mathrm{I}}^{k}(d) = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} F_{r_{\mathrm{I}}}\left(\sum_{j=1}^{n_s} Tp^j \mu_{\mathrm{I}}^{k}\left(x_d^i, x_p^i, s^j\right)\right), \tag{50}$$

where $\left\{\left(x_d^i, x_p^i, x_{r_{\mathrm{E}}}^i\right), i = 1, \ldots, n_{MC}\right\} = \Omega_{MC}(d)$ is the Monte Carlo sample of the random vector $(X_d, X_p, X_{r_{\mathrm{E}}})$.

Step 2 does not require any call to the expensive simulator. Once the optimization algorithm has converged, a new design point denoted d^k is obtained.

At the end of each cycle k, the following condition is evaluated:

$$\left\|\overline{d^{k-1}} - \overline{d^k}\right\| < \epsilon_d \ \text{ OR } \ \left|\overline{cost\left(d^{k-1}\right)} - \overline{cost\left(d^k\right)}\right| < \epsilon_{cost} \tag{stopping condition}$$

where $\overline{d}$ and $\overline{cost\,(d)}$ are the normalizations of d and $cost(d)$ in $[0, 1]$. If this condition is met, AK-ECO is stopped and the minimum retained, denoted d^{min}, is d^k, otherwise, $k = k + 1$ and a new cycle begins from step 1. The stopping criterion of AK-ECO does not include a condition on the satisfaction of the constraints since this point is verified at the end of the optimization during step 2. The full procedure of AK-ECO is summarized in Figure 1.

Remark 9 If the random vector X_d depends on d such that $X_d = d + X$ with X a zero-mean random vector, it is possible to use the same Monte Carlo sample Ω_{MC} throughout AK-ECO where $\Omega_{MC} = \left\{\left(x^i, x_p^i, x_{r_{\mathrm{E}}}^i\right), i = 1, \ldots, n_{MC}\right\}$ is a sample of $(X, X_p, X_{r_{\mathrm{E}}})$. It follows that $\Omega_{MC}(d) = \left\{\left(d + x^i, x_p^i, x_{r_{\mathrm{E}}}^i\right), \left(x^i, x_p^i, x_{r_{\mathrm{E}}}^i\right) \in \Omega_{MC}\right\}$.

3.3 Kriging models initialization

The goal of the first kriging models is to provide good predictions of their respective functions over the whole augmented spaces. A space-filling DoE is therefore appropriate. As explained above, we use one DoE for both metamodels: only one sample of $\Omega_{\mathrm{E}}^{aug}$ is needed. Therefore a space-filling DoE of n_{DoE}^0 points of $\Omega_d^{aug} \times \Omega_p^{aug} \times \Omega_{r_{\mathrm{E}}}^{aug}$ is constructed. We then concatenate to this DoE a uniform sample of n_{DoE}^0 points of Ω_S. The resulting DoE is denoted DoE^0. The simulator is evaluated for each point of DoE^0 to calibrate the initial krigings models denoted $\widetilde{M_{\mathrm{E}}}^0$ and $\widetilde{M_{\mathrm{I}}}^0$.

3.4 Accuracy criteria

During step 1 of the k-th cycle of optimization, the current kriging model $\widetilde{M_{\mathrm{E}}}^{k-1}$ at $\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right)$ follows a normal distribution with mean and standard deviation denoted $\mu_{\mathrm{E}}^{k-1}\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right)$ and $\sigma_{\mathrm{E}}^{k-1}\left(x_d, x_p, x_{r_{\mathrm{E}}}, s^j\right)$. To evaluate the precision of the approximation $p_{\mathrm{E}}^{k-1}(d)$ of the true failure probability at d, we adapt the approach proposed by Dubourg (2011) and compute the following quantities:

$$p_{\mathrm{E},\pm}^{k-1}(d) = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} F_{\epsilon}\left(\sum_{j=1}^{n_s} \exp\left(\mu_{\mathrm{E}}^{k-1}\left(x_d^i, x_p^i, x_{r_{\mathrm{E}}}^i, s^j\right) \pm 2\sigma_{\mathrm{E}}^{k-1}\left(x_d^i, x_p^i, x_{r_{\mathrm{E}}}^i, s^j\right)\right)\right). \tag{51}$$

As the exponential function and F_{ϵ} are strictly increasing, we have: $p_{\mathrm{E},-}^{k-1}(d) < p_{\mathrm{E}}^{k-1}(d) < p_{\mathrm{E},+}^{k-1}(d)$. The distance between $p_{\mathrm{E},-}^{k-1}(d)$ and $p_{\mathrm{E},+}^{k-1}(d)$ is an indicator of the uncertainty of the constraint estimation $p_{\mathrm{E}}^{k-1}(d)$. In Dubourg (2011), a criterion based on the ratio between similar optimistic and pessimistic estimations of the failure probability is proposed. However, since it is useless to know precisely the true failure probability if it is far from p_s, we modify this latter criterion. In AK-ECO, the metamodel is considered accurate enough if the following condition is met:

$$\frac{\left|p_{\mathrm{E}}^{k-1}(d^{k-1}) - p_s\right|}{p_{\mathrm{E},+}^{k-1}(d^{k-1}) - p_{\mathrm{E},-}^{k-1}(d^{k-1})} > 1. \tag{Criterion E}$$

This criterion is met if the distance between the low and high estimations of the constraint at d^{k-1} is less than the distance between the estimation of the constraint at d^{k-1} and p_s. In this case, we have reasonable grounds to believe that the kriging model accurately

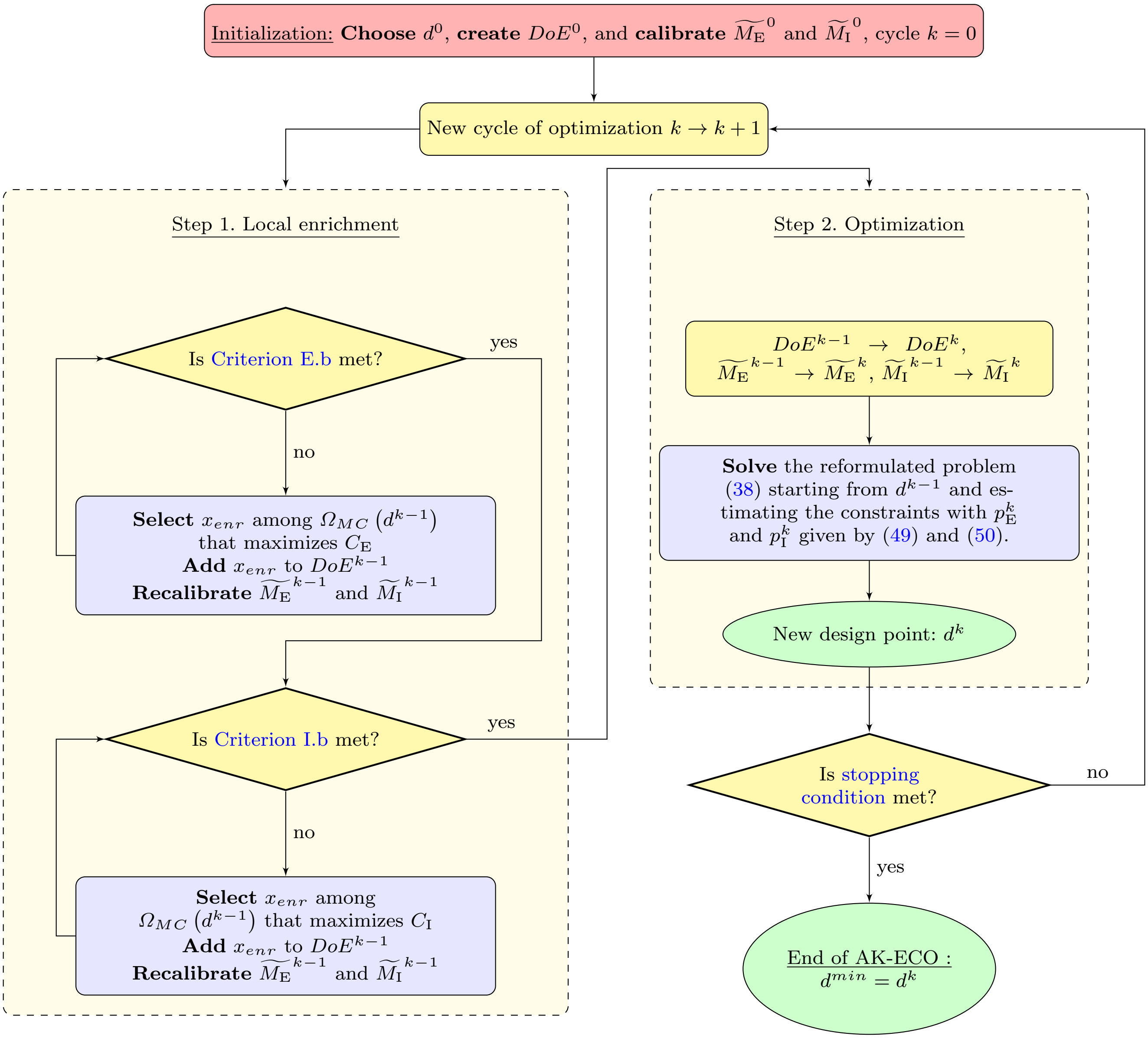


Fig. 1: Flowchart of AK-ECO

predicts whether a point near d^{k-1} belongs or not to the feasible domain.

For the integral-based constraint, a similar criterion is proposed. We consider the kriging model $\widetilde{M_{\mathrm{I}}}^{k-1}$ whose mean and standard deviation at $\left(x_d, x_p, s^j\right)$ are denoted $\mu_{\mathrm{I}}^{k-1}\left(x_d, x_p, s^j\right)$ and $\sigma_{\mathrm{I}}^{k-1}\left(x_d, x_p, s^j\right)$. The function $F_{r_{\mathrm{I}}}$ is also increasing and $p^j > 0$ for $j = 1, \ldots, n_s$. Thus we have, for all d: $p_{\mathrm{I},-}^{k-1}(d) < p_{\mathrm{I}}^{k-1}(d) < p_{\mathrm{I},+}^{k-1}(d)$ with

$$p_{\mathrm{I},\pm}^{k-1}(d) = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} F_{r_{\mathrm{I}}}\left(\sum_{j=1}^{n_s} T p^j \left(\mu_{\mathrm{I}}^{k-1}\left(x_d^i, x_p^i, s^j\right) \pm 2\sigma_{\mathrm{I}}^{k-1}\left(x_d^i, x_p^i, s^j\right)\right)\right). \quad (52)$$

The accuracy criterion for the integral-based constraint at d^{k-1} is:

$$\frac{\left|p_{\mathrm{I}}^{k-1}(d^{k-1}) - p_s\right|}{p_{\mathrm{I},+}^{k-1}(d^{k-1}) - p_{\mathrm{I},-}^{k-1}(d^{k-1})} > 1. \quad \text{(Criterion I)}$$

To avoid a too large number of enrichment steps during the same cycle, a maximal number $n_{\max}$ of enrichment steps is imposed for each metamodel and cycle. Finally, $\widetilde{M_{\mathrm{E}}}^{k-1}$ is considered accurate enough if $p^{k-1}_{\mathrm{E},-}(d^{k-1}) > p_s - \epsilon_p$ and $p^{k-1}_{\mathrm{E},+}(d^{k-1}) < p_s + \epsilon_p$ where ϵ_p is chosen by the user (we proceed similarly for $\widetilde{M_{\mathrm{I}}}^{k-1}$). Hence, $\widetilde{M_{\mathrm{E}}}^{k-1}$, respectively $\widetilde{M_{\mathrm{I}}}^{k-1}$, is enriched if Criterion E.b, respectively Criterion I.b, is not met and those criteria are defined as:

$$\text{Criterion E is met OR } \left(n^k_{\mathrm{E}} \geq n_{\max}\right) \text{ OR } \left(p_s - \epsilon_p < p^{k-1}_{\mathrm{E},-}(d^{k-1}) < p^{k-1}_{\mathrm{E},+}(d^{k-1}) < p_s + \epsilon_p\right) \quad \text{(Criterion E.b)}$$

$$\text{Criterion I is met OR } \left(n^k_{\mathrm{I}} \geq n_{\max}\right) \text{ OR } \left(p_s - \epsilon_p < p^{k-1}_{\mathrm{I},-}(d^{k-1}) < p^{k-1}_{\mathrm{I},+}(d^{k-1}) < p_s + \epsilon_p\right) \quad \text{(Criterion I.b)}$$

where n^k_{E} and n^k_{I} are the numbers of enrichment steps of each metamodel during cycle k. Therefore, during each cycle of AK-ECO, the number of enrichments is at most equal to $n_{\max}$ multiplied by the number of metamodels. When both criteria Criterion E.b and Criterion I.b are met, step 1 ends and step 2 begins.

3.5 Selection of the enrichment point

During the k-th cycle, if Criterion E.b is not met, the model $\widetilde{M_{\mathrm{E}}}^{k-1}$ is not considered sufficiently accurate at d^{k-1}. To improve its precision, a point x_{enr} maximizing a learning function C_{E} is selected among the Monte Carlo sample $\Omega_{MC}\left(d^{k-1}\right)$ used in equation (49) to estimate the extreme-based constraint at d^{k-1}. Hence, x_{enr} is given by:

$$x_{enr} = \underset{\Omega_{MC}\left(d^{k-1}\right)\times\{s^1,\ldots,s^{n_s}\}}{\operatorname{argmax}} C_{\mathrm{E}}(x^i_{d^{k-1}}, x^i_p, x^i_{r_{\mathrm{E}}}, s^j). \quad (53)$$

The goal of criterion C_{E} is to favor points where the uncertainty of prediction of $\widetilde{M_{\mathrm{E}}}^{k-1}$ implies important uncertainties on the constraint estimation at d^{k-1}:

$$\begin{aligned} C_{\mathrm{E}}(x^i_d, x^i_p, x^i_{r_{\mathrm{E}}}, s^j) &= f_{\left(X_d,X_p,X_{r_{\mathrm{E}}}\right)}\left(x^i_d, x^i_p, x^i_{r_{\mathrm{E}}}\right) \\ \times \Bigg[& F_\epsilon\Bigg(e^{\mu^{k-1}_{\mathrm{E}}\left(x^i_d,x^i_p,x^i_{r_{\mathrm{E}}},s^j\right)+2\sigma^{k-1}_{\mathrm{E}}\left(x^i_d,x^i_p,x^i_{r_{\mathrm{E}}},s^j\right)} + \sum_{j'\neq j} e^{\mu^{k-1}_{\mathrm{E}}\left(x^i_d,x^i_p,x^i_{r_{\mathrm{E}}},s^{j'}\right)}\Bigg) \\ & - F_\epsilon\Bigg(e^{\mu^{k-1}_{\mathrm{E}}\left(x^i_d,x^i_p,x^i_{r_{\mathrm{E}}},s^j\right)-2\sigma^{k-1}_{\mathrm{E}}\left(x^i_d,x^i_p,x^i_{r_{\mathrm{E}}},s^j\right)} + \sum_{j'\neq j} e^{\mu^{k-1}_{\mathrm{E}}\left(x^i_d,x^i_p,x^i_{r_{\mathrm{E}}},s^{j'}\right)}\Bigg)\Bigg], \end{aligned} \quad (54)$$

with $f_{\left(X_d,X_p,X_{r_{\mathrm{E}}}\right)}$ the probability density function of the random vector $(X_d, X_p, X_{r_{\mathrm{E}}})$. Once x_{enr} is selected, it is added to the current DoE, DoE^{k-1}, and one call to the simulator is made at this point.

For the integral-based constraint, the idea is the same, if Criterion I.b is not met, a new point x_{enr} is selected as follows:

$$x_{enr} = \underset{\Omega_{MC}\left(d^{k-1}\right)\times\{s^1,\ldots,s^{n_s}\}}{\operatorname{argmax}} C_{\mathrm{I}}(x^i_{d^{k-1}}, x^i_p, s^j), \quad (55)$$

where

$$\begin{aligned} C_{\mathrm{I}}(x^i_d, x^i_p, s^j) &= f_{(X_d,X_p)}\left(x^i_d, x^i_p\right) \\ \times \Bigg[& F_{r_{\mathrm{I}}}\Bigg(Tp^j\left(\mu^{k-1}_{\mathrm{I}}\left(x^i_d, x^i_p, s^j\right) + 2\sigma^{k-1}_{\mathrm{I}}\left(x^i_d, x^i_p, s^j\right)\right) + \sum_{j'\neq j} Tp^{j'}\mu^{k-1}_{\mathrm{I}}\left(x^i_d, x^i_p, s^{j'}\right)\Bigg) \\ & - F_{r_{\mathrm{I}}}\Bigg(Tp^j\left(\mu^{k-1}_{\mathrm{I}}\left(x^i_d, x^i_p, s^j\right) - 2\sigma^{k-1}_{\mathrm{I}}\left(x^i_d, x^i_p, s^j\right)\right) + \sum_{j'\neq j} Tp^{j'}\mu^{k-1}_{\mathrm{I}}\left(x^i_d, x^i_p, s^{j'}\right)\Bigg)\Bigg]. \end{aligned} \quad (56)$$

3.6 Visualization of one cycle of AK-ECO

To illustrate one cycle of AK-ECO, we consider the minimization of a cost function in a two-dimensional design space with extreme-based and integral-based constraints. The level sets of the cost function that we want to minimize are displayed in Figure 2. The infeasible domain is the red hatched area. After initialization of the metamodels, the first cycle of AK-ECO begins by evaluating their accuracy at the initial design point d^0 (step 1.a). Since they are not accurate enough, enrichment candidates of the augmented space are considered (projections in the design space of several candidates are represented with blacks dots). Enrichment points are then selected among these candidates until the precision criteria are met (step 1.b). The projection of 4 of these points are indicated by green dots in the Figure 2. Once the accuracy criteria are met, step 2 begins and the optimization problem is solved with an optimization algorithm using the enriched metamodels. The iterations of this optimization are represented by grey crosses. This resolution provides a new design point d_1 which will be the starting point of the next cycle of AK-ECO.

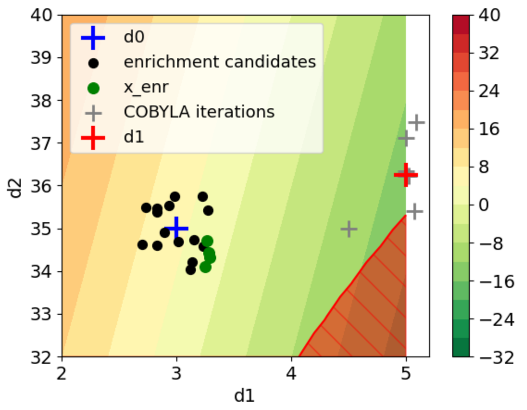


Fig. 2: Visualization of the first cycle of AK-ECO

4 Application to the harmonic oscillator problem

To validate the method proposed in the previous section, we study the resolution of the reformulated problem of the piece-wise stationary harmonic oscillator described in section 2.4. First, a resolution of the problem with a double loop approach using the Monte Carlo method to estimate the constraints is used as a reference. Then, the problem is solved with AK-ECO and the resolution of its probabilistic formulation (see section 3) is carried out with our implementation of RIA, PMA, SORA and the Stieng approach presented in section .

4.1 Cost function, sources of uncertainties and parameters of the problem

Here the cost function is:

$$cost\,(d_1, d_2) = d_2 - 10d_1. \tag{57}$$

The optimization problem is solved on the design space $\Omega_d = [1, 5] \times [20, 50]$ and for the parameters ρ, n_T, ΔT, T, n_s and p_s given in Table 1.

Parameter	ρ	n_T	ΔT	T	n_s	p_s
Value	1	100	216	21600	7	10^{-4}

Table 1: Parameters of the problem

The distributions considered for the random variables X_{d_1}, X_{d_2}, X_p, X_{r_1}, X_{r_2}, and X_{r_3} are given in Table 2 and the couples $\left(s^j, p^j\right)$, $j = 1, \ldots, 7$ in Table 3.

Uncertainty	Distribution
X_{d_1}	$\mathcal{U}\,[d_1 - 0.3, d_1 + 0.3]$
X_{d_2}	$\mathcal{U}\,[d_2 - 1, d_2 + 1]$
X_p	$\mathcal{U}[0.5, 1.5]$
X_{r_1}	$\mathcal{N}\,(1, 0.1^2)$
X_{r_2}	$\mathcal{N}\,(2.5, 0.25^2)$
X_{r_3}	$\mathcal{N}\,(15, 3^2)$

Table 2: Distributions of $X_{d_1}, X_{d_2}, X_p, X_{r_1}, X_{r_2}, X_{r_3}$

The notations $\mathcal{U}[a, b]$ and $\mathcal{N}\left(\mu, \sigma^2\right)$ refer respectively to the uniform distribution on $[a, b]$ and the normal distribution of mean μ and standard deviation σ.

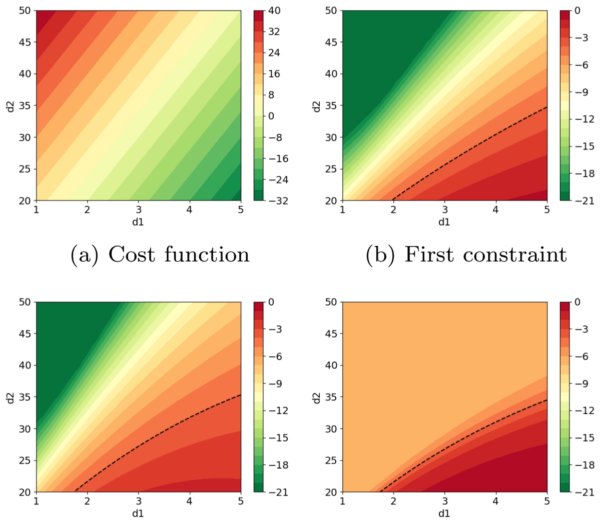
j	1	2	3	4	5	6	7
s^j	1.20	1.16	1.10	1.05	0.99	0.95	0.90
p^j	0.21	0.17	0.18	0.16	0.13	0.09	0.06

Table 3: Couples $\left(s^j, p^j\right)$

Here, the functions involved in the constraints are actually not very expensive and massive Monte Carlo simulations with samples of size 30000 can be carried out. So we are able to check the performances of the different optimization methods. The level sets of the cost function and of the logarithm of the three constraints are displayed in Figure 3. In Figure 3, the black dotted lines correspond to the design points where each failure probability equals p_s.

4.2 Implementations

The reference results are obtained using the COBYLA (Powell (1994)) optimization algorithm and a massive Monte Carlo method to estimate the failure probabilities (this approach is denoted MC). The COBYLA algorithm is also used for the other methods. The FORM method in RIA is performed with the Abdo-Rackwitz algorithm (Abdo and Rackwitz (1991)) available in the python package OpenTURNS (Baudin et al. (2016)). The HMV algorithm (Youn et al. (2003)) is implemented to solve the inverse reliability analysis in PMA. In SORA and Stieng, the SQP (Nocedal and Wright (2006)) algorithm is chosen instead of HMV since it performs better on the studied case. For AK-ECO, the initial space-filling DoE is a Latin Hypercube Sampling (LHS) (McKay et al. (2000)) of size 50. The maximum number of enrichment steps per cycle and per constraint n_{max} is set to 15. The size n_{MC} of the sample used in the MC method is 30000. For Stieng, SORA and AK-ECO,



(a) Cost function (b) First constraint

(c) Second constraint (d) Third constraint

Fig. 3: Level sets of the cost function and of the logarithm of the failure probabilities appearing in each constraint

the cycles of optimization stop if the stopping condition introduced in section 3.2 is met for ϵ_d and ϵ_{cost} equal to 10^{-3}. In Stieng and AK-ECO, the kriging implementation of OpenTURNS is used with a constant trend and a 3/2-Matérn covariance kernel. In AK-ECO, the states s_i are treated as continuous variables by the kriging kernel since, in the oscillator application, the $s^j, j = 1, \ldots, n_s$ are real numbers. In Stieng, the initial DoE is a Sobol sequence (Sobol' (1967)) of size 12. A Sobol sequence of size 40 is used for the second cycle, 160 for the third one and 400 for the next cycles.

4.3 Numerical results

The problem under study is solved with each approach starting from the center $(3, 35)$ of the design space. The results are displayed in Table 4. The first and second rows indicate the design point d^{min} obtained by each approach and the value of the cost function at this point. For $i = 1, 2, 3$, the i-th failure probability at d^{min} is then estimated with a massive Monte Carlo of 30000 points and the result is denoted $p_i^{MC}(d^{min})$. Finally, as explained in remark 6, the expensive part of the constraints is the evaluation of the spectral moments of $\mathcal{D}_1\left(x_d, x_p, s^j; .\right)$. Therefore, during the resolution of the problem, one estimation of the spectral moments for one point $\left(x_d, x_p, s^j\right)$ is considered as one call to the expensive simulator. The number of calls to the simulator by each method is denoted n_{call}. It is important to notice that, unlike usual papers in reliability analysis, the number n_{call} does not refer to the number of calls to the performance functions but to the number of simulations. Hence, for RIA, PMA, SORA and Stieng, the number of calls to the performance functions is equal to $n_{call}/7$ since $n_s = 7$.

We observe that all the methods converge towards the same design point. However, AK-ECO provides the closest design point to the reference point obtained with MC and requires far fewer calls to the expensive simulator than the comparison methods: only 252 calls are required (50 for the initial DoE and 202 for local enrichments of the metamodels during the optimization cycles). This is due to the fact that AK-ECO is well adapted to the reformulated problem: each call to the simulator allows to enrich every kriging models and the simulation is performed only at the relevant states s^j. Furthermore, among the comparison methods and AK-ECO only RIA and AK-ECO provide an estimation of the failure probabilities. At the design point obtained with RIA, the first, second and third failure probabilities are estimated with FORM as 1.1×10^{-4}, 1.3×10^{-4}, 0.8×10^{-4}. The probabilities estimated with AK-ECO at the design point obtained with this algorithm are 0.7×10^{-4}, 1.0×10^{-4}, 0.1×10^{-4}. Hence, with AK-ECO, we observe a good approximation of the failure probabilities since they are close to p_1^{MC}, p_2^{MC}, and p_3^{MC} obtained with Monte Carlo on the real functions.

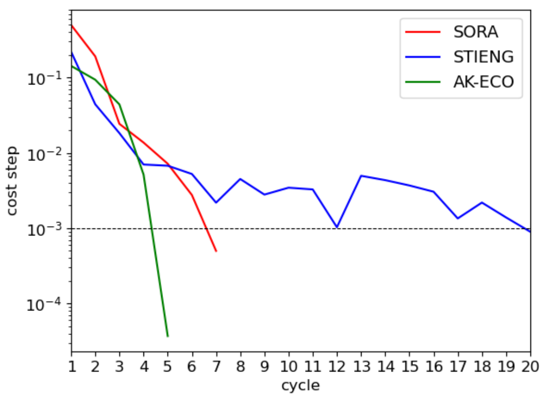
For SORA, the Stieng approach and AK-ECO, at the end of cycle k a design point d^k is obtained. The evolution of $\log\left(\left|\overline{cost}\left(d^{k-1}\right) - \overline{cost}\left(d^k\right)\right|\right)$ for each method and each cycle k is displayed in Figure 4a and the evolution of $\log\left(\left\|\overline{d^k} - \overline{d^{k-1}}\right\|\right)$ in Figure 4b where $\overline{cost}\left(d^{k-1}\right)$ and $\overline{d^k}$ refer respectively to the normalization of $cost\left(d^k\right)$ function and of d^k in $[0, 1]$.

We observe that the resolution of the studied problem takes 5 cycles for AK-ECO to converge while 7 and 20 cycles are necessary for SORA and Stieng to meet the stopping condition. In AK-ECO, the closer the design point is to the true infeasible domain boundary, the more enrichment steps are performed. During the first cycle, 22 points are added to the DoE while 45 enrichment steps are performed during the second, third and last ones.

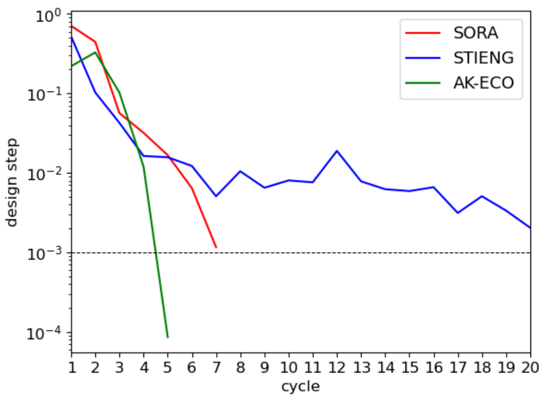
Moreover, the failure probabilities p_1^{MC}, p_2^{MC}, p_3^{MC} have been evaluated with a Monte Carlo of 30000 points at the design point obtained at the end of each cycle of SORA, Stieng and AK-ECO and their evolution is displayed in Figure 5. We can see that with Stieng and AK-ECO, the true constraints are satisfied at the end of each cycle while it takes 4 cycles for SORA (actually, with AK-ECO, p_2^{MC} is slightly above the threshold 10^{-4} at the end of the third cycle).

	MC	RIA	PMA	SORA	Stieng	AK-ECO
d^{min}	(5.0, 35.74)	(5.0, 35.22)	(5.0, 35.04)	(5.0, 36.77)	(5.0, 37.29)	(5.0, 35.73)
$cost(d^{min})$	-14.26	-14.78	-14.96	-13.23	-12.70	-14.27
$p_1^{MC}\left(d^{min}\right)$	0.8×10^{-4}	1.0×10^{-4}	1.0×10^{-4}	0.4×10^{-4}	0.3×10^{-4}	0.8×10^{-4}
$p_2^{MC}\left(d^{min}\right)$	1.0×10^{-4}	1.3×10^{-4}	1.3×10^{-4}	0.6×10^{-4}	0.5×10^{-4}	1.0×10^{-4}
$p_3^{MC}\left(d^{min}\right)$	0.1×10^{-4}	0.8×10^{-4}	0.4×10^{-4}	0	0	0.1×10^{-4}
n_{call}	3.57×10^6	791175	29393	15722	53200	252

Table 4: Results of AK-ECO and the comparison methods for the harmonic oscillator problem



(a) Evolution of $\log\left(\left|\overline{cost\left(d^{k-1}\right)} - \overline{cost\left(d^{k}\right)}\right|\right)$



(b) Evolution of $\log\left(\left\|\overline{d^{k}} - \overline{d^{k-1}}\right\|\right)$

Fig. 4: Evolution of the stopping condition for SORA, Stieng and AK-ECO

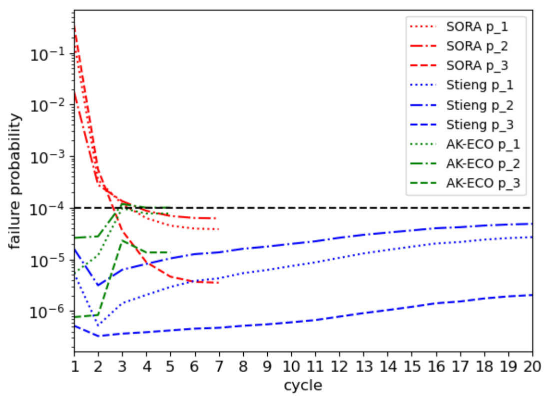


Fig. 5: Evolution of the Monte Carlo estimation of the failure probabilities for SORA, Stieng and AK-ECO

The resolution of the problem has also been repeated with AK-ECO from 20 different starting design points selected with a LHS of the design space. Each time, the initial kriging models are calibrated with a new DoE. The results show that the performance of AK-ECO is not affected by the initial DoE or the initial design point since the algorithm converges towards the good design point each time and with a number of simulations varying from 174 to 416 with a mean number of calls equal to 229.1.

5 Application to the floating offshore wind turbine problem

We now consider a modified version of the floating offshore wind turbine described in Robertson et al. (2014): the wind turbine is equipped with a semi-submersible floater connected to the seabed by three mooring lines. In this section, we aim at minimizing the material cost of the mooring lines while considering reliability constraints. The mooring system must limit the floater movements to ensure the turbine production, avoid compression in the mooring lines and withstand the damage caused by fatigue. The resulting constraints inherit the randomness of the marine conditions, the material properties and the model parameters.

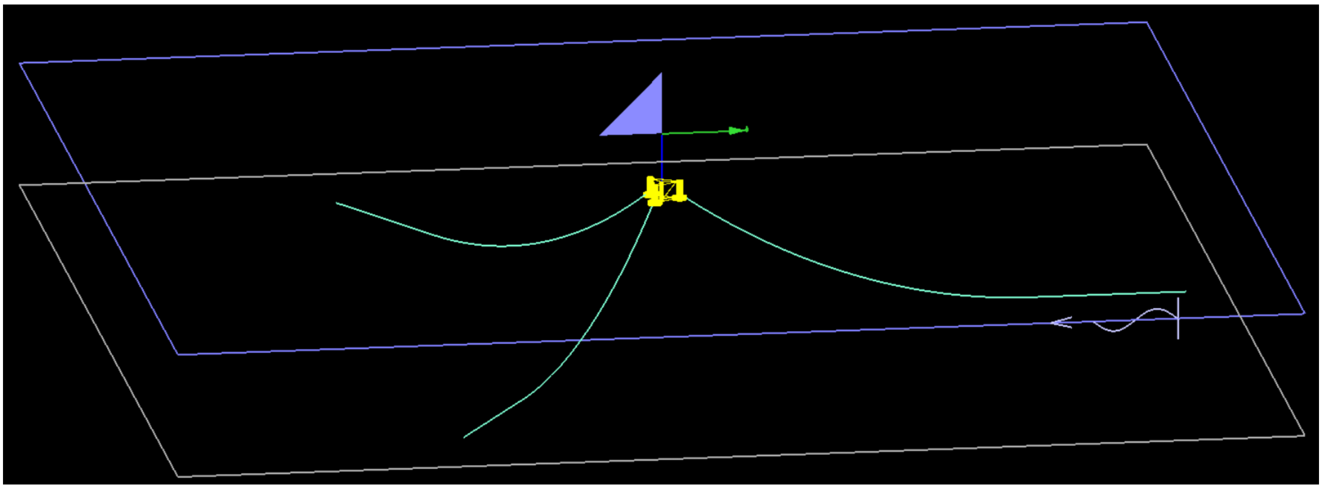


Fig. 6: Perspective view of the floating offshore wind turbine model in Deeplines$^{\text{TM}}$

5.1 Design variables and design space

The material cost of the mooring system and the constraints depend on three design variables: the length d_1 of the mooring line that can be added to, or deducted from, the nominal mooring length of 841.56m (its domain is $[-0.5, 2]$ (in m)), the lineic mass $d_2 \in [70, 180]$ (in kg/m), the position d_3 of the connection of the lines to the side columns of the floater (taking values between 0 and 1 which respectively correspond to the connection at the bottom and at the top of the columns). We denote $d = (d_1, d_2, d_3)$ and the design space $\Omega_d = [-0.5, 2] \times [70, 180] \times [0, 1]$.

5.2 Definition of the time-dependent processes involved in the constraints

The movements of the structure are determined from the environmental loads occurring during the considered period $[0, T]$ (T is equal to one year) and in particular loadings induced by waves.

The swell is modelled as a succession of waves meeting the structure and defined by their height at any time at a given point. To account for all the possible sea states, we discretize the interval of time $[0, T]$ into n_T subintervals I_i, $i = 1, \ldots, n_T$, of length ΔT ($\Delta T = 3$ hours). For each interval I_i, the sea elevation is represented by a zero-mean stationary Gaussian random process defined by its spectral density. In this study, we consider the JONSWAP spectral density (Hasselmann et al. (1980)) which is characterized by three long term parameters (the significant wave height, the peak period, and the mean wind speed) grouped in s_i which will be called the sea state. The sea elevation process on I_i is denoted $\eta_i(s_i; .)$. We consider that $\eta_i(s_i; .)$ and $\eta_j(s_j; .)$ are independent processes for $i \neq j$. We assume that there are only 7 possible sea states $s^1, \ldots, s^7$ (i.e $\forall s_i, i = 1, \ldots, n_T$, $s_i \in \{s^1, \ldots, s^7\}$). Moreover, we consider only constant wind forces (thrust on Rotor-Nacelle-Assembly and turbine torque) applied on the structure for each sea state.

At the end of the optimization problem, the chosen design variables must restrict the platform movements: the horizontal shifting of the structure (called surge) must be less than a conservative threshold $\mathcal{S}_{\max}$ of 5% of the water depth. Tension in the lines must stay positive and the accumulated fatigue damage in the line must remain below a resistance threshold R. We consider only the tension and the fatigue at the top of each line.

For a fixed sea state s_i, the movements of the platform and the lines are solutions of a linearized equation of motion in which the forces come from environmental loads. The surge and tension at the top of the line l, $l \in \{1, 2, 3\}$, can be written as:

$$\mathcal{S}_i(t) = \mu_{\mathcal{S}_i} + \overline{\mathcal{S}_i}(t), \quad \mathcal{T}_i^l(t) = \mu_{\mathcal{T}_i^l} + \overline{\mathcal{T}_i^l}(t), \tag{58}$$

where $\mu_{\mathcal{S}_i}$ and $\mu_{\mathcal{T}_i^l}$ are constants obtained computing the static equilibrium. Moreover, $\overline{\mathcal{S}_i}$ and $\overline{\mathcal{T}_i^l}$ are outputs of linear filters with the sea elevation $\eta_i(s_i; .)$ as input. Taking into account the different sea states, the surge and the tension at the top of the line l, $l \in \{1, 2, 3\}$, on $[0, T]$, denoted respectively $\mathcal{S}$ and $\mathcal{T}^l$, are defined as follows:

$$\mathcal{S}(t) = \sum_{i=1}^{n_T} \mathcal{S}_i(t) \mathbb{1}_{I_i}(t), \quad \mathcal{T}^l(t) = \sum_{i=1}^{n_T} \mathcal{T}_i^l(t) \mathbb{1}_{I_i}(t). \tag{59}$$

We deduce from these definitions and the properties of the sea elevation process that the surge and tension processes are piece-wise stationary Gaussian processes.

We can represent the instantaneous damage occurring at the top of the l-th mooring line, $l \in \{1, 2, 3\}$, as follows:

$$\mathcal{D}^l(t) = \sum_{i=1}^{n_T} \mathcal{D}_i^l(t)\, \mathbb{1}_{I_i}(t)\ ,\forall t \in [0, T], \tag{60}$$

where $\mathcal{D}_i^l(t)$ is the instantaneous damage caused by $\mathcal{T}_i^l(t)$. The accumulated damage over $[0, T]$, also called fatigue, is defined as the integral $\int_0^T \mathcal{D}^l(t)\, dt$. The distributions of the surge process, the tension processes and the fatigue depend on the design variables d and on model parameters denoted x_{p_1}, x_{p_2}, and x_{p_3} (as well as on a parameter x_{d_2} for the fatigue) which are presented below.

5.3 Model and fatigue threshold uncertainties

To account for the lack of knowledge on certain parameters, uncertainties are considered on:

- the wave heading which is represented by a random variable X_{p_1} uniformly distributed between plus and minus $10°$ around the wind turbine axis;
- two quadratic viscous drag coefficients for the surge and the pitch of the floater, denoted X_{p_2} and X_{p_3}, to account for the approximation of the fitting from decay tests (Burmester et al. (2020)). Each of these random variables follows a uniform law respectively on $[10^5, 10^6]$ (in N.s^2.m^{-2}) and $[3 \times 10^{10}, 7 \times 10^{10}]$ (in N.m.s^2.rad^{-2});
- the y-intercept of the fatigue law X_{d_2} accounting for experimental scattering. It follows a log-normal distribution with parameters chosen after the Stiff's fatigue curve for chain, $\sigma_{d_2} = 0.8$ and μ_{d_2} which depends on the lineic mass d_2 via a linear relation of the Breaking Load (see Rossi (2005) for the fatigue law relation);
- the threshold resistance R for approximation of time-independent Palmer Miner damage approach. It is a log-normal distribution of parameters $\mu_R = 1$ and $\sigma_R = 0.3$ (Leira et al. (2005)).

All of these variables are independent and X_p denotes the random vector $X_p = (X_{p_1}, X_{p_2}, X_{p_3})$.

5.4 Optimization problem formulation

Taking into account all the sources of uncertainty, we consider the following optimization problem:

$$\begin{aligned}
&\min_{d \in \Omega_d}\ cost(d) \quad \text{such that} \\
&\mathbb{P}\left(\max_{[0,T]} \mathcal{S}(d, X_p; t) > s_{\max}\right) && < 10^{-4} \\
&\mathbb{P}\left(\min_{[0,T]} \mathcal{T}^l(d, X_p; t) < 0\right) && < 10^{-4}\ ,\ l = 1, 2, 3 \\
&\mathbb{P}\left(\int_0^T \mathcal{D}^l\left(d, X_p, X_{d_2}; t\right) dt > R\right) && < 10^{-4}\ ,\ l = 1, 2, 3.
\end{aligned} \tag{61}$$

The threshold probability of 10^{-4} is recommended by international standards for mooring lines (Det Norske Veritas (2013)) and the cost function is cheap to evaluate. We notice that the surge and tension constraints are extreme-based constraints while the fatigue constraints are integral-based. Besides, since the surge and tension processes are piece-wise stationary Gaussian processes, we can reformulate the problem as described in section 2. We thus obtain a reformulated problem which can be solved with AK-ECO.

5.5 Resolution of the reformulated problem

We now solve the reformulated version of the floating offshore wind turbine problem. For given design variables d, parameters x_p and sea state s^j, the means and the spectral moments of the surge and the tension processes are all outputs of a single simulation and are computed with with the frequency domain solver of the Deeplines$^{\text{TM}}$ software (Le Cunff et al. (2008)) dedicated to offshore engineering applications. This software is based on the finite elements method and the computational time of a one simulation (about 1 minute) is too high to apply naive approaches such as a brut Monte Carlo. The reformulated problem is thus solved with four methods: AK-ECO, SORA, Stieng and a method denoted MC+K1600 which provides the reference results. Since it would be to expensive to evaluate the reformulated constraints with Monte Carlo, in MC+K1600, the expensive functions are replaced by kriging models built from a LHS of 1600 points. The Monte Carlo method is then used with these kriging models to estimate the constraints during the resolution of the optimization problem. For SORA and Stieng, we solve the probabilistic formulation of the problem (see section 3).

The optimization algorithm used for each approach is the COBYLA algorithm. In SORA and Stieng, the

SQP algorithm is chosen to solve the inverse reliability analyses. For AK-ECO, the initial DoE is a LHS of size 60. The maximum number of enrichment steps per cycle and per constraint n_{max} is set to 15. For Stieng, SORA and AK-ECO, the cycles of optimization stop if the stopping condition introduced in section 3.2 is met for ϵ_d and ϵ_{cost} equal to 10^{-3}. For AK-ECO, MC+K1600, and Stieng, a kriging with constant trend and 3/2-Matérn covariance kernel is used for every metamodel. A Sobol sequence of 30 points is used to calibrate the kriging models of the first cycle of Stieng. For the second cycle, the size of the sequence is 90 and 300 for the next ones. The same Monte Carlo sample of X_p of size 30000 is used in MC+K1600 and AK-ECO to estimate the reformulated failure probabilities.

5.6 Numerical results

The results obtained by each approach, considering an initial design point at the center of the design space $(0.75, 125, 0.5)$ are given in Table 5. The design point obtained by each method is denoted d^{min} and $\overline{cost(d^{min})}$ is the cost function evaluated at this point normalized between $[0, 1]$. The surge, tension, and fatigue failure probabilities at d^{min} obtained by each method have been evaluated with Monte Carlo and the kriging models of the MC+K1600 method. The results are denoted $p_{\mathcal{S}}^{K1600}\left(d^{min}\right)$, $p_{\mathcal{T}^l}^{K1600}\left(d^{min}\right)$ and $p_{\mathcal{D}^l}^{K1600}\left(d^{min}\right)$ $(l = 1, 2, 3)$. Finally, the number of simulations performed during the resolution of the studied problem with each method is denoted n_{call}.

We observe that each method provides a reliable optimum. However, the design proposed by AK-ECO is much closer to the reference result obtained with MC+K1600 than the ones proposed by SORA and Stieng. This difference is due to the inverse reliability analyses performed during these methods which underestimate the reliability associated with the last constraint. This leads to a sub-optimal design point provided by these approaches. Furthermore, AK-ECO requires much less evaluations of the simulator since only 305 calls were needed (60 for the initial DoE and 245 for the enrichment procedure during the optimization cycles).

6 Conclusion

We have considered in this paper a time-dependent RBDO problem with two types of constraints which involve the maximum and the integral function of a time-dependent stationary or piece-wise stationary Gaussian random process. This kind of problems is inspired by applications that arise in offshore wind turbine design where the movements of a structure are time-dependent stochastic processes whose properties can be derived from input processes representing the marine conditions (i.e. the wind speed and the sea elevation).

To solve this problem efficiently, we have proposed a two-step procedure. First, we use limit theorems to reformulate the constraints into easier to evaluate constraints that are time-independent and depend only on uncertainties represented by random vectors. The extreme value theory enables us to approximate the extreme-based failure probability while the ergodicity of the process appearing in the integral-based constraint is used to reformulate it. Therefore, we obtain a RBDO problem with constraints involving expectations instead of failure probabilities.

A sum of n_s terms appears in each expectation constraint (where n_s is the number of different stationary distributions involved in the piece-wise stationary process) and one call to an expensive simulator is required to compute each of these terms. We thus propose a new method based on adaptive kriging and called AK-ECO that solves the reformulated problem much faster than the approaches in the literature. In the usual approaches, for each constraint, a metamodel is built to replace the performance function and one evaluation of the performance function leads to n_s simulations. In AK-ECO, the metamodel replaces the expensive function involved in the sum. Thus, one enrichment of the metamodel only requires one simulation. The number of calls to the expensive simulator is dramatically reduced especially when n_s is large as in the offshore wind turbine application.

To illustrate the procedure, a problem involving a harmonic oscillator is introduced. We apply the reformulation and then, we solve the obtained problem with AK-ECO and state-of-the-art algorithms in RBDO. The results show that AK-ECO is better suited than the presented competing approaches for the type of problems considered in our paper. The two-step procedure is then applied to minimize the material cost of a mooring system of a floating offshore wind turbine under extreme and integral based constraints.

The AK-ECO efficiency relies on the assumptions that the cost function is fast to evaluate and that the dimension of the augmented space is small (say, less than 12). In high dimension, kriging models perform poorly and additional work would be necessary to adapt AK-ECO. In addition, when n_s becomes large, the estimation of the failure probabilities with Monte Carlo as well as the selection of enrichment points can become cumbersome. Hence, AK-ECO would benefit from being coupled with a variance reduction technique. Furthermore, it is assumed that the inital DoE quality is

	MC+K1600	SORA	Stieng	AK-ECO
d^{min}	$(0.862, 108.86, 0)$	$(2, 141.22, 0)$	$(2, 140.58, 0)$	$(0.947, 109.52, 0)$
$\overline{cost(d^{min})}$	0.2818	0.5883	0.5819	0.2867
$p_{\mathcal{S}}^{K1600}\left(d^{min}\right)$	1.0×10^{-4}	0	0	0.9×10^{-4}
$p_{\mathcal{T}^l}^{K1600}\left(d^{min}\right)\ (l=1,2,3)$	0	0	0	0
$p_{\mathcal{D}^l}^{K1600}\left(d^{min}\right)\ (l=1,2)$	0	0	0	0
$p_{\mathcal{D}^3}^{K1600}\left(d^{min}\right)$	1.0×10^{-4}	0.2×10^{-4}	0.2×10^{-4}	0.9×10^{-4}
n_{call}	1600	16394	5754	305

Table 5: Results of AK-ECO and the comparison methods for the wind turbine problem

good enough to capture the shape of the feasible domain since the metamodels are only enriched locally during AK-ECO. To improve the quality of the initial DoE, it would be interesting to implement a procedure before starting AK-ECO, that enriches the metamodels and thus the failure probabilities over the whole design space, to ensure a good estimation of the feasible domain.

Finally, AK-ECO has been implemented to be coupled with a local optimization algorithm and thus provides a local optimum. To obtain a global optimum, it is possible to perform multistart optimization with AK-ECO from several initial design points.

A Appendix of section 1

A.1 Extreme-based failure probability reformulation error

Bounds can be obtained on the approximation (15) using the theorem 2.2 of Kratz and Rootzén (1997). We present here the theorem for a process ξ satisfying the conditions of theorem 1 and we show how it can be applied to control the reformulation error.

A.1.1 Rate of convergence of theorem 1

Theorem 2 (Theorem 2.2 of Kratz and Rootzén (1997)) *Let ξ be the process introduced in theorem 1 satisfying condition (9) and the following conditions:*

$$\mathbb{E}\left[(\xi'(t)-\xi'(0))\right]^2 = 2\left(k_\xi''(\tau)-k_\xi''(0)\right) \leq c\tau^2, \quad \tau \geq 0, \quad (62)$$

$$|k_\xi(\tau)| \leq Ct^{-\alpha}, \quad |k_\xi(\tau)| + k_\xi'(\tau)^2 \leq C_0 t^{-\alpha}, \quad \tau \geq 0, \quad (63)$$

for some $\alpha > 2$ and constants c, C, C_0. Then there is a constant K which depends on k_ξ but not on u or T such that, for $T \geq T_0 > 1$,

$$\left|\mathbb{P}\left(\max_{t\in[0,T]} \frac{\xi(t)}{\sqrt{m_{\xi,0}}} \leq u\right) - \exp\left(-\sqrt{\frac{m_{\xi,2}}{m_{\xi,0}}}T\mu(u)\right)\right| \leq K\frac{\log\left(\sqrt{\frac{m_{\xi,2}}{m_{\xi,0}}}T\right)^{1+1/\alpha}}{\left(\sqrt{\frac{m_{\xi,2}}{m_{\xi,0}}}T\right)^{\delta}}, \quad (64)$$

with $\mu(u) = \frac{1}{2\pi}e^{-\frac{u^2}{2}}$, $\delta = \min\{1/2, \inf_{\tau\geq 0}\rho(\tau)\}$ and $\rho(\tau) = \frac{\left(1-\frac{k_\xi(\tau)}{m_{\xi,0}}\right)^2}{1-\frac{1}{m_{\xi,0}^2}k_\xi(\tau)^2+\frac{1}{m_{\xi,0}m_{\xi,2}}k_\xi'(\tau)\left|k_\xi'(\tau)\right|}$.

When the process ξ is known through its spectral density K_ξ, conditions (62) and (63) are met if we have:

$$m_{\xi,4} < \infty, \quad (65)$$

$$K_\xi \in C^3, K_\xi^{(i)} \text{ and } \omega K_\xi^{(j)} \text{ are integrable for } i=0,1,2,3 \text{ and } j=0,1,2, \quad (66)$$

with $K_\xi^{(i)}$ the i-th derivative of K_ξ.

Proposition 2 *If a process ξ meets all the conditions of theorem 2, it follows that for all $x_r \in \mathbb{R}$:*

$$\left|\mathbb{P}\left(\max_{t\in[0,T]}\xi(t) \leq x_r\right) - \exp\left(-e^{a_T^2 - \frac{a_T x_r}{\sqrt{m_{\xi,0}}}}\right)\right| \leq K\frac{\log\left(2\pi\frac{T}{T_c}\right)^{1+1/\alpha}}{\left(2\pi\frac{T}{T_c}\right)^{\delta}} + \exp\left(-\frac{T}{T_c}\exp^{-\frac{x_r^2}{2m_{\xi,0}}}\right). \quad (67)$$

with $a_T = \sqrt{2\log\left(\frac{T}{T_c}\right)}$ and $T_c = 2\pi\sqrt{\frac{m_{\xi,0}}{m_{\xi,2}}}$.

Proof (Proof of proposition 2) We have, $\forall x_r \in \mathbb{R}$:

$$a_T^2 - a_T\frac{x_r}{\sqrt{m_{\xi,0}}} + \frac{x_r^2}{2m_{\xi,0}} \geq \min_{x_r'\in\mathbb{R}} a_T^2 - a_T\frac{x_r'}{\sqrt{m_{\xi,0}}} + \frac{{x_r'}^2}{2m_{\xi,0}} = \frac{a_T^2}{2}. \quad (68)$$

Taking the exponential of right- and left-hand sides gives:

$$\exp\left(-e^{a_T^2-\frac{a_T x_r}{\sqrt{m_{\xi,0}}}}\right)\leq\exp\left(-\frac{T}{T_c}\exp\left(-\frac{x_r^2}{2m_{\xi,0}}\right)\right),\forall x_r\in\mathbb{R}. \tag{69}$$

It follows from this result and theorem 2 with $u=\frac{x_r}{\sqrt{m_{\xi,0}}}$ that:

$$\begin{aligned}&\left|\mathbb{P}\left(\max_{t\in[0,T]}\xi(t)\leq x_r\right)-\exp\left(-e^{a_T^2-\frac{a_T x_r}{\sqrt{m_{\xi,0}}}}\right)\right|\\ \leq&\left|\mathbb{P}\left(\max_{t\in[0,T]}\xi(t)\leq x_r\right)-\exp\left(-\frac{T}{T_c}\exp\left(-\frac{x_r^2}{2m_{\xi,0}}\right)\right)\right|\\ &+\left|\exp\left(-e^{a_T^2-\frac{a_T x_r}{\sqrt{m_{\xi,0}}}}\right)-\exp\left(-\frac{T}{T_c}\exp\left(-\frac{x_r^2}{2m_{\xi,0}}\right)\right)\right|\\ \leq&K\frac{\log\left(2\pi\frac{T}{T_c}\right)^{1+1/\alpha}}{\left(2\pi\frac{T}{T_c}\right)^{\delta}}+\exp\left(-\frac{T}{T_c}\exp\left(-\frac{x_r^2}{2m_{\xi,0}}\right)\right)\end{aligned} \tag{70}$$

□

A.1.2 Application of proposition 2 to extreme-based failure probability approximation

Proposition 3 *We denote by $f_{r_{\mathrm{E}}}$ the probability density function of $X_{r_{\mathrm{E}}}$ and $T_c(x_d,x_p)=2\pi\sqrt{\frac{m_{\mathcal{Y},0}(x_d,x_p)}{m_{\mathcal{Y},2}(x_d,x_p)}}$. We assume the following conditions:*

(1) $\exists K,\alpha,\delta$ such that $\forall x_d,x_p$, the process $\mathcal{Y}(x_d,x_p;.)$ satisfies the conditions of theorem 2,
(2) $\exists T_1>0,\ T_2>0,\ m_1>0$ such that $\forall x_d,\ x_p,\ T_1\leq T_c(x_d,x_p)\leq T_2$ and $m_1\leq m_{\mathcal{Y},0}(x_d,x_p)$,
(3) $\exists c_1>0,\ c_r>0$ such that $\forall x,\ |x|\geq c_1,\ f_{r_{\mathrm{E}}}(x)\leq c_r\exp\left(-\frac{x^2}{m_1}\right)$.

Then, if $\sqrt{m_1\log(T)}>c_1$, the error made in the approximation (15) can be bounded as follows:

$$\left|p_{\mathrm{E}}(d)-\mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}}\left[F_\epsilon\left(e^{a_T(X_d,X_p)^2-\frac{a_T(X_d,X_p)X_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(X_d,X_p)}}}\right)\right]\right|\leq K\frac{\log\left(2\pi\frac{T}{T_1}\right)^{1+1/\alpha}}{\left(2\pi\frac{T}{T_2}\right)^{\delta}}+\frac{T_2+c_r\sqrt{\pi m_1}}{\sqrt{T}}. \tag{71}$$

Proof (Proof of proposition 3)

$$\begin{aligned}&\left|p_{\mathrm{E}}(d)-\mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}}\left[F_\epsilon\left(e^{a_T(X_d,X_p)^2-\frac{a_T(X_d,X_p)X_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(X_d,X_p)}}}\right)\right]\right|\\ \leq&\mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}}\left[\left|\mathbb{P}_{\mathcal{Y}}\left(\max_{t\in[0,T]}\mathcal{Y}(X_d,X_p;t)\leq X_{r_{\mathrm{E}}}\right)-\exp\left(-e^{a_T(X_d,X_p)^2-\frac{a_T(X_d,X_p)X_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(X_d,X_p)}}}\right)\right|\right]\\ \leq&\mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}}\left[\left|K\frac{\log\left(2\pi\frac{T}{T_c(X_d,X_p)}\right)^{1+1/\alpha}}{\left(2\pi\frac{T}{T_c(X_d,X_p)}\right)^{\delta}}+\exp\left(-\frac{T}{T_c(X_d,X_p)}\exp\left(-\frac{X_{r_{\mathrm{E}}}^2}{2m_{\mathcal{Y},0}(X_d,X_p)}\right)\right)\right|\right]\\ \leq&K\frac{\log\left(2\pi\frac{T}{T_1}\right)^{1+1/\alpha}}{\left(2\pi\frac{T}{T_2}\right)^{\delta}}+\mathbb{E}_{X_{r_{\mathrm{E}}}}\left[\exp\left(-\frac{T}{T_2}\exp\left(-\frac{X_{r_{\mathrm{E}}}^2}{2m_1}\right)\right)\right].\end{aligned}$$

The two last equations are obtained using successively proposition 2 and assumption (2). Denoting $\alpha_T=\sqrt{m_1\log(T)}$, it follows from assumption (3):

$$\begin{aligned}&\mathbb{E}_{X_{r_{\mathrm{E}}}}\left[\exp\left(-\frac{T}{T_2}\exp\left(-\frac{X_{r_{\mathrm{E}}}^2}{2m_1}\right)\right)\right]\\ =&\int_{-\alpha_T}^{\alpha_T}\exp\left(-\frac{T}{T_2}\exp\left(-\frac{x^2}{2m_1}\right)\right)f_{r_{\mathrm{E}}}(x)dx\\ &+\int_{\mathbb{R}/[-\alpha_T,\alpha_T]}\exp\left(-\frac{T}{T_2}\exp\left(-\frac{x^2}{2m_1}\right)\right)f_{r_{\mathrm{E}}}(x)dx \qquad (72)\\ \leq&\exp\left(-\frac{T}{T_2}\exp\left(-\frac{\alpha_T^2}{2m_1}\right)\right)+\int_{\mathbb{R}/[-\alpha_T,\alpha_T]}f_{r_{\mathrm{E}}}(x)dx\\ \leq&\exp\left(-\frac{\sqrt{T}}{T_2}\right)+c_r\int_{\mathbb{R}/[-\alpha_T,\alpha_T]}\exp\left(-\frac{x^2}{m_1}\right)dx\\ \leq&\frac{T_2}{\sqrt{T}}+c_r\left(\sqrt{\pi m_1}-\sqrt{\pi m_1}\sqrt{1-\exp\left(-\frac{\alpha_T^2}{m_1}\right)}\right) \qquad (73)\\ \leq&\frac{T_2}{\sqrt{T}}+c_r\sqrt{\pi m_1}\left(1-\sqrt{1-\frac{1}{T}}\right)\\ \leq&\frac{T_2}{\sqrt{T}}+c_r\sqrt{\frac{\pi m_1}{T}}. \qquad (74)\end{aligned}$$

Equation (73) is obtained considering $I=\int_0^{\alpha_T}\exp\left(-\frac{x^2}{m_1}\right)dx$, then I^2 can be bounded working in polar coordinates. □

A.2 Integral-based failure probability reformulation error

Proposition 4 *We denote by $f_{r_{\mathrm{I}}}$ the probability density function of $X_{r_{\mathrm{I}}}$ and for fixed values x_d, x_p, $Z_T(x_d,x_p)$ the random variable $\frac{1}{T}\int_0^T\mathcal{F}(x_d,x_p;t)\,dt$. We assume the following conditions:*

(1) $\exists c_1>0,c_r>0$ such that $\forall x\geq c_1,\ f_{r_{\mathrm{I}}}(x)\leq\frac{c_r}{x}$,
(2) $\exists T_0,c_2>0$ such that $\forall x_d,x_p$ and $\forall T>T_0,\ Z_T(x_d,x_p)\geq c_2$ almost surely,

(3) $\exists c_{\mathcal{F}} > 0$ *such that for all* x_d, x_p:

$$\int_{\mathbb{R}} |k_{\mathcal{F}}(x_d, x_p; \tau)| \, d\tau < c_{\mathcal{F}}. \tag{75}$$

Then, if $T > T_0$ *and* $T > \frac{c_1}{c_2}$, *we have:*

$$\left| p_{\mathrm{I}}(d) - \mathbb{E}_{X_d, X_p}\left[F_{r_{\mathrm{I}}}\left(T\mathbb{E}_{\mathcal{F}}\left[\mathcal{F}\left(X_d, X_p; 0\right)\right]\right)\right]\right| \leq \frac{c_r}{c_2}\sqrt{\frac{2c_{\mathcal{F}}}{T}}. \tag{76}$$

Proof (Proof of proposition 4)

$$\begin{aligned}
&\Big| p_{\mathrm{I}}(d) - \mathbb{E}_{X_d, X_p}\left[F_{r_{\mathrm{I}}}\left(T\mathbb{E}_{\mathcal{F}}\left[\mathcal{F}\left(X_d, X_p; 0\right)\right]\right)\right] \Big| \\
\leq& \mathbb{E}_{X_d, X_p, \mathcal{Y}}\left[\Big| F_{r_{\mathrm{I}}}\left(TZ_T(X_d, X_p)\right) - F_{r_{\mathrm{I}}}\left(T\mathbb{E}_{\mathcal{F}}\left[Z_T\left(X_d, X_p\right)\right]\right) \Big|\right] \\
\leq& \mathbb{E}_{X_d, X_p, \mathcal{Y}}\Bigg[\Big| TZ_T(X_d, X_p) - T\mathbb{E}_{\mathcal{F}}\left[Z_T\left(X_d, X_p\right)\right] \Big| \\
& \qquad \max\left(\frac{c_r}{\left(TZ_T\left(X_d, X_p\right)\right)}, \frac{c_r}{\left(T\mathbb{E}_{\mathcal{F}}\left[Z_T\left(X_d, X_p\right)\right]\right)}\right)\Bigg] \qquad (77) \\
\leq& \mathbb{E}_{X_d, X_p, \mathcal{Y}}\left[T\Big| Z_T(X_d, X_p) - \mathbb{E}_{\mathcal{F}}\left[Z_T\left(X_d, X_p\right)\right]\Big| \frac{c_r}{c_2 T}\right] \\
\leq& \frac{c_r}{c_2}\, \mathbb{E}_{X_d, X_p}\left[\sqrt{\mathrm{Var}\left(Z_T\left(X_d, X_p\right)\right)}\right] \qquad (78) \\
\leq& \frac{c_r}{c_2}\sqrt{\frac{2c_{\mathcal{F}}}{T}}. \qquad (79)
\end{aligned}$$

Conditions (1), (2) and $T \geq \frac{c_1}{c_2}$ imply equation (77). In (78), we apply the Cauchy–Schwarz inequality and equation (79) follows from:

$$\begin{aligned}
\mathrm{Var}(Z_T\left(x_d, x_p\right)) &= \frac{1}{T^2}\int_0^T \int_0^T k_{\mathcal{F}}\left(x_d, x_p; |t - t'|\right) dt dt' \\
&\leq \frac{2}{T}\int_0^T \left|k_{\mathcal{F}}\left(x_d, x_p; \tau\right)\right| d\tau. \qquad (80)
\end{aligned}$$

□

A.3 Proof of the sufficient conditions to reformulate the stationary harmonic oscillator problem

Proof To apply the approximation (15) to the first and second constraints of the oscillator problem we need to show that for all x_d, x_p and for $k = 1, 2$, the following conditions are satisfied:

$$m_{\mathcal{D}^{(k)},0}(x_d, x_p) < \infty, m_{\mathcal{D}^{(k)},2}(x_d, x_p) < \infty \tag{81}$$

$$K_{\mathcal{D}^{(k)}}(x_d, x_p, .) \in C^1, K_{\mathcal{D}^{(k)}}(x_d, x_p, .) \text{ and } K'_{\mathcal{D}^{(k)}}(x_d, x_p, .) \text{ are integrable.} \tag{82}$$

As we have the relation $K_{\mathcal{D}^{(k)}}\left(x_d, x_p; \omega\right) = \omega^{2k} K_{\mathcal{D}}\left(x_d, x_p; \omega\right)$ $\forall \omega$, to show (81) and (82), it is sufficient to prove that $\omega^i K_{\mathcal{D}}^{(j)}$ is integrable for the appropriate values of i and j where $K_{\mathcal{D}}^{(j)}$ is the j-th derivative of $K_{\mathcal{D}}$. In fact, in our case it is true for all i and j. Indeed, it follows from relation (22) that $K_{\mathcal{D}}\left(x_d, x_p; .\right)$ is the product of a rational function (with no real pole) and a Gaussian function. Hence, we can demonstrate that $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}$, $\omega^i K_{\mathcal{D}}^{(j)}\left(x_d, x_p; .\right)$ is integrable.

□

B Appendix of section 2

B.1 Extreme-based failure probability reformulation

For $j = 1, \ldots, n_s$, let us denote $I^j = \bigcup_{i, s_i = s^j} I_i$. Hence, I^j is the union of n^j intervals of length ΔT. For fixed $x_d, x_p, x_{r_{\mathrm{E}}}$, we have:

$$\begin{aligned}
&\mathbb{P}_{\mathcal{Y}}\left(\max_{t \in [0,T]} \mathcal{Y}(x_d, x_p; t) \leq x_{r_{\mathrm{E}}}\right) \\
&= \mathbb{P}_{\mathcal{Y}}\left(\max_{t \in [0,T]} \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, s_i; t\right) \mathbb{1}_{I_i}(t) \leq x_{r_{\mathrm{E}}}\right) \\
&= \mathbb{P}_{\mathcal{Y}}\Bigg(\max_{t \in I^1} \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, s^1; t\right) \mathbb{1}_{I_i}(t) \leq x_{r_{\mathrm{E}}}, \ldots, \\
&\qquad \max_{t \in I^{n_s}} \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, s^{n_s}; t\right) \mathbb{1}_{I_i}(t) \leq x_{r_{\mathrm{E}}}\Bigg) \\
&= \prod_{j=1}^{n_s} \mathbb{P}_{\mathcal{Y}_1, \ldots, \mathcal{Y}_{n_T}}\left(\max_{t \in I^j} \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, s^j; t\right) \mathbb{1}_{I_i}(t) \leq x_{r_{\mathrm{E}}}\right) \qquad (83) \\
&= \prod_{j=1}^{n_s} \mathbb{P}_{\mathcal{Y}_1, \ldots, \mathcal{Y}_{n^j}}\left(\max_{t \in [0, n^j \Delta T]} \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, s^j; t\right) \mathbb{1}_{I_i}(t) \leq x_{r_{\mathrm{E}}}\right). \qquad (84)
\end{aligned}$$

The independence of $\mathcal{Y}_i\left(x_d, x, s_i; .\right)$ and $\mathcal{Y}_{i'}\left(x_d, x_p, s_{i'}; .\right)$ for all $i \neq i'$ is used to obtain equation (83). The last equation results from the fact that $\mathcal{Y}_i\left(x_d, x_p, s^j; .\right)$ and $\mathcal{Y}_{i'}\left(x_d, x_p, s^j; .\right)$ for $i \neq i'$ are i.i.d processes.

Finally, when ΔT is large, we consider that each term of the product appearing in equation (84) can be approached by $\mathbb{P}_{\mathcal{Y}_1}\left(\max_{t \in [0, n^j \Delta T]} \mathcal{Y}_1\left(x_d, x_p, s^j; t\right) \leq x_{r_{\mathrm{E}}}\right)$ which leads to approximation (27) since $n^j \Delta T = T p^j$.

B.2 Extreme-based failure probability reformulation error

Proposition 5 *We denote* $f_{r_{\mathrm{E}}}$ *the probability density function of* $X_{r_{\mathrm{E}}}$ *and* $T_c\left(x_d, x_p, s^j\right) = 2\pi\sqrt{\frac{m_{\mathcal{Y},0}\left(x_d, x_p, s^j\right)}{m_{\mathcal{Y},2}\left(x_d, x_p, s^j\right)}}$. *We assume the following conditions:*

(1) $\exists K, \alpha, \delta$ *such that* $\forall x_d, x_p, s^j$, *the process* $\mathcal{Y}_1\left(x_d, x_p, s^j; .\right)$ *satisfies the conditions of theorem 2,*

(2) $\exists T_1 > 0$, $T_2 > 0$, $m_1 > 0$ *such that* $\forall x_d$, x_p, s^j, $T_1 \leq T_c\left(x_d, x_p, s^j\right) \leq T_2$ *and* $m_1 \leq m_{\mathcal{Y},0}\left(x_d, x_p, s^j\right)$,

(3) $\exists c_1 > 0$, $c_r > 0$ *such that* $\forall x$, $|x| \geq c_1$, $f_{r_{\mathrm{E}}}(x) \leq c_r \exp\left(-\frac{x^2}{m_1}\right)$.

Then, if $\sqrt{m_1 \log(\Delta T)} > c_1$ the error made in equation (28) can be bounded as follows:

$$\Big| p_{\mathrm{E}}(d) - \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[F_\epsilon \left(\sum_{j=1}^{n_s} e^{a_{Tp^j}(X_d,X_p,s^j)^2 - \frac{a_{Tp^j}(X_d,X_p,s^j) X_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(X_d,X_p,s^j)}}} \right) \right] \Big| \leq n_s \left(n_T K \frac{\log\left(2\pi \frac{\Delta T}{T_1}\right)^{1+1/\alpha}}{\left(2\pi \frac{\Delta T}{T_2}\right)^{\delta}} + (n_T+1) \frac{T_2 + c_r\sqrt{\pi m_1}}{\sqrt{\Delta T}} \right). \tag{85}$$

Proof (Proof of proposition 5) We introduce the following notations:

$$P_j(T) = \mathbb{P}_{\mathcal{Y}_1}\left(\max_{t\in[0,T]} \mathcal{Y}_1\left(x_d, x_p, s^j; t\right) \leq x_{r_{\mathrm{E}}} \right), \tag{86}$$

$$E_j(T) = \exp\left(-e^{a_T(x_d,x_p,s^j)^2 - \frac{a_T(x_d,x_p,s^j) x_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(x_d,x_p,s^j)}}} \right), \tag{87}$$

$$E_-(T) = \min_{j=1,\dots,n_s} E_j(T)\ ,\ E_+(T) = \max_{j=1,\dots,n_s} E_j(T), \tag{88}$$

$$p^- = \min_{j=1,\dots,n_s} p^j\ ,\ p^+ = \max_{j=1,\dots,n_s} p^j. \tag{89}$$

For fixed values of $x_d, x_p, x_{r_{\mathrm{E}}}$, we have $\forall j \in \{1,\dots,n_s\}$:

$$\left| P_j(\Delta T)^{n^j} - E_j\left(Tp^j\right) \right| \leq \left| P_j(\Delta T)^{n^j} - E_j(\Delta T)^{n^j} \right| + \left| E_j(\Delta T)^{n^j} - E_j\left(Tp^j\right) \right|. \tag{90}$$

Besides, using $P_j(\Delta T) \in [0,1]$, $E_j(\Delta T) \in [0,1]$, proposition 2 and assumption (2), we obtain:

$$\left| P_j(\Delta T)^{n^j} - E_j(\Delta T)^{n^j} \right| \leq n^j \left| P_j(\Delta T) - E_j(\Delta T) \right| \leq n_T \left(K \frac{\log\left(2\pi \frac{\Delta T}{T_1}\right)^{1+1/\alpha}}{\left(2\pi \frac{\Delta T}{T_2}\right)^{\delta}} + \exp\left(-\frac{\Delta T}{T_2} \exp\left(-\frac{x_{r_{\mathrm{E}}}^2}{2m_1} \right) \right) \right). \tag{91}$$

Since $E_j(T)$ is a decreasing function with respect to T and that its images are in $[0,1]$, we have for $n^j > 0$:

$$-E_+\left(Tp^-\right) \leq E_j(\Delta T)^{n^j} - E_j\left(Tp^j\right) \leq E_+(\Delta T). \tag{92}$$

It follows, with $Tp^- > 1$ (i.e. if $n^j > 0, \forall j$) and applying the preliminary result of proof A.1.1:

$$\left| E_j(\Delta T)^{n^j} - E_j\left(Tp^j\right) \right| \leq E_+(\Delta T) \leq \exp\left(-\frac{\Delta T}{T_2} \exp\left(-\frac{x_{r_{\mathrm{E}}}^2}{2m_1} \right) \right). \tag{93}$$

We therefore can use equations (91) and (93) to bound the approximation error:

$$\begin{aligned}
&\Big| p_{\mathrm{E}}(d) - \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \Big[F_\epsilon \left(\sum_{j=1}^{n_s} e^{a_{Tp^j}(X_d,X_p,s^j)^2 - \frac{a_{Tp^j}(X_d,X_p,s^j) X_{r_{\mathrm{E}}}}{\sqrt{m_{\mathcal{Y},0}(X_d,X_p,s^j)}}} \right) \Big] \Big| \\
=& \Big| \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[1 - \prod_{j=1}^{n_s} P_j(\Delta T)^{n^j} \right] - \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[1 - \prod_{j=1}^{n_s} E_j\left(Tp^j\right) \right] \Big| \quad (94)\\
\leq& \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[\left| \prod_{j=1}^{n_s} P_j(\Delta T)^{n^j} - \prod_{j=1}^{n_s} E_j\left(Tp^j\right) \right| \right] \quad (95)\\
\leq& n_s \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[n_T K \frac{\log\left(2\pi \frac{\Delta T}{T_1}\right)^{1+1/\alpha}}{\left(2\pi \frac{\Delta T}{T_2}\right)^{\delta}} + (n_T+1) \exp\left(-\frac{\Delta T}{T_2} \exp\left(-\frac{X_{r_{\mathrm{E}}}^2}{2m_1} \right) \right) \right] \quad (96)\\
\leq& n_s \left(n_T K \frac{\log\left(2\pi \frac{\Delta T}{T_1}\right)^{1+1/\alpha}}{\left(2\pi \frac{\Delta T}{T_2}\right)^{\delta}} + (n_T+1) \frac{T_2 + c_r\sqrt{\pi m_1}}{\sqrt{\Delta T}} \right) \quad (97)
\end{aligned}$$

In equation (94), we use the following equalities:

$$\begin{aligned}
p_{\mathrm{E}}(d) &= \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[1 - \prod_{j=1}^{n_s} \mathbb{P}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{n^j}} \left(\max_{t\in[0,n^j\Delta T]} \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, s^j; t\right) \mathbb{1}_{I_i}(t) \leq x_{r_{\mathrm{E}}} \right) \right] \\
&= \mathbb{E}_{X_d,X_p,X_{r_{\mathrm{E}}}} \left[1 - \prod_{j=1}^{n_s} \mathbb{P}_{\mathcal{Y}_1} \left(\max_{t\in[0,\Delta T]} \mathcal{Y}_1\left(x_d, x_p, s^j; t\right) \leq x_{r_{\mathrm{E}}} \right)^{n^j} \right]. \quad (98)
\end{aligned}$$

Equation (95) holds since, for $a_i, b_i \in [0,1], n \in \mathbb{N}^*$, if $|a_i - b_i| < c, i = 1,\dots,n$ then $\left|\prod_i^n a_i - \prod_i^n b_i\right| < nc$. Finally, in equation (97), $\mathbb{E}_{X_{r_{\mathrm{E}}}}\left[\exp\left(-\frac{\Delta T}{T_2} \exp\left(-\frac{X_{r_{\mathrm{E}}}^2}{2m_1} \right) \right) \right]$ is bounded applying the reasoning of proof A.1.2. □

B.3 Proof proposition 1

Proof (Proof of proposition 1) For fixed values x_d, x_p and x, we have:

$$\mathbb{P}_{\mathcal{Y}}\left(\frac{1}{\Delta T}\int_0^T f\left(\sum_{i=1}^{n_T}\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\mathbb{1}_{I_i}(t)\right)dt>x\right)$$
$$=\mathbb{P}_{\mathcal{Y}}\left(\frac{1}{\Delta T}\sum_{i=1}^{n_T}\int_{I_i} f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right)dt>x\right)$$
$$=\mathbb{P}_{\mathcal{Y}}\left(\sum_{i=1}^{n_T}\frac{1}{\Delta T}\int_0^{\Delta T} f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right)dt>x\right). \quad (99)$$

The last equality is obtained using the stationarity of the processes $f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;.\right)\right)$ and the independence between $f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;.\right)\right)$ and $f\left(\mathcal{Y}_j\left(x_d,x_p,s_{i'};.\right)\right)$ for $i\neq i'$. Let $U_{i,\Delta T}$ be the random variable $\frac{1}{\Delta T}\int_0^{\Delta T} f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right)dt$. It follows from the assumption that $f\left(\mathcal{Y}_1\left(x_d,x_p,s^j;.\right)\right)$ is ergodic for all s^j that:

$$U_{i,\Delta T}\xrightarrow[\Delta\to+\infty]{\mathbb{P}} u_i, \quad (100)$$

with $u_i=\mathbb{E}_{\mathcal{Y}_i}\left[f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;0\right)\right)\right]$. Using the independence of $U_{i,\Delta T}$ and $U_{i',\Delta T}$ for $i\neq i'$, we deduce that:

$$\sum_{i=1}^{n_T}U_{i,\Delta T}\xrightarrow[\Delta T\to+\infty]{\mathbb{P}}\sum_{i=1}^{n_T}u_i. \quad (101)$$

Therefore, we have the convergence in distribution:

$$\mathbb{P}_{\mathcal{Y}}\left(\sum_{i=1}^{n_T}\frac{1}{\Delta T}\int_0^{\Delta T} f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right)dt>x\right)\xrightarrow[\Delta\to+\infty]{}\mathbb{1}_{\sum_{i=1}^{n_T}\mathbb{E}_{\mathcal{Y}_i}\left[f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;0\right)\right)\right]>x}, \quad (102)$$

for all $x\neq\sum_{i=1}^{n_T}\mathbb{E}_{\mathcal{Y}_i}\left[f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;0\right)\right)\right]$, with

$$\sum_{i=1}^{n_T}\mathbb{E}_{\mathcal{Y}_i}\left[f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;0\right)\right)\right]=\sum_{j=1}^{n_s}n^j\mathbb{E}_{\mathcal{Y}_1}\left[f\left(\mathcal{Y}_1\left(x_d,x_p,s^j;0\right)\right)\right]. \quad (103)$$

□

B.4 Integral-based failure probability reformulation error

Proposition 6 *We denote f_{r_1} the probability density function of X_{r_1} and for fixed values x_d, x_p:*

$$Z_{\Delta T}(x_d,x_p)=\frac{1}{\Delta T}\int_0^T f\left(\mathcal{Y}\left(x_d,x_p;t\right)\right)dt \quad (104)$$
$$=\frac{1}{\Delta T}\sum_{i=1}^{n_T}\int_{I_i} f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right)dt, \quad (105)$$

with $k_{\mathcal{F}_1}\left(x_d,x_p,s_i;.\right)$ the autocovariance function of the process $\mathcal{F}_1\left(x_d,x_p,s_i;.\right)$. We assume that the following assumptions are valid:

(1) $\exists c_1>0, c_r>0$ such that $\forall x\geq c_1$, $f_{r_1}(x)\leq\frac{c_r}{x}$,
(2) $\exists\Delta T_0$, $\exists c_2>0$ such that $\forall x_d$, x_p and $\Delta T>\Delta T_0$, $Z_{\Delta T}\left(x_d,x_p\right)\geq c_2$ almost surely,
(3) for each state s^j, $\exists c_{\mathcal{F},s^j}>0$ such that for all x_d, x_p:

$$\int_{\mathbb{R}}\left|k_{\mathcal{F}_1}\left(x_d,x_p,s^j;\tau\right)\right|d\tau<c_{\mathcal{F},s^j}. \quad (106)$$

Then, if $\Delta T>T_0$ and $\Delta T\geq\frac{c_1}{c_2}$, the approximation error made in equation (31) can be bounded as follows:

$$\left|p_{\mathrm{I}}(d)-\mathbb{E}_{X_d,X_p}\left[F_{r_1}\left(T\sum_{j=1}^{n_s}p^j\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(X_d,X_p,s^j;0\right)\right]\right)\right]\right|\leq\frac{c_r}{c_2}\sqrt{\frac{2n_T}{\Delta T}\sum_{j_1}^{n_s}p^jc_{\mathcal{F},s^j}}. \quad (107)$$

Proof [Proof of proposition 6]
We denote in this proof, for fixed values x_d,x_p:

$$A_i=\int_{I_i} f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right)dt, \quad (108)$$
$$\mathcal{F}_i\left(s_i;t\right)=f\left(\mathcal{Y}_i\left(x_d,x_p,s_i;t\right)\right), \quad (109)$$

and

$$k_{\mathcal{F}_i,\mathcal{F}_j}(s_i,s_j;t,t')=\mathbb{E}_{\mathcal{F}_i,\mathcal{F}_j}\left[\mathcal{F}_i\left(s_i;t\right)\mathcal{F}_j\left(s_j;t'\right)\right]-\mathbb{E}_{\mathcal{F}_i}\left[\mathcal{F}_i\left(s_i;t\right)\right]\mathbb{E}_{\mathcal{F}_j}\left[\mathcal{F}_j\left(s_j;t'\right)\right]. \quad (110)$$

We calculate $\mathbb{E}_{\mathcal{Y}}\left[A_i\right],\mathbb{E}_{\mathcal{Y}}\left[A_iA_j\right]$, and $\mathrm{Var}_{\mathcal{Y}}\left(Z_{\Delta T}\left(x_d,x_p\right)\right)$ which are used further down in the proof. For the first quantity, we use Fubini's theorem to obtain:

$$\mathbb{E}_{\mathcal{Y}}\left[A_i\right]=\Delta T\mathbb{E}_{\mathcal{F}_i}\left[\mathcal{F}_i\left(s_i;0\right)\right]. \quad (111)$$

Besides,

$$\mathbb{E}_{\mathcal{Y}}\left[A_iA_j\right]=\mathbb{E}_{\mathcal{Y}}\left[\int_{I_i}\int_{I_j}\mathcal{F}_i\left(s_i;t\right)\mathcal{F}_j\left(s_j;t'\right)dtdt'\right] \quad (112)$$
$$=\int_{I_i}\int_{I_j}\left(k_{\mathcal{F}_i,\mathcal{F}_j}(s_i,s_j,t,t')+\mathbb{E}_{\mathcal{F}_i}\left[\mathcal{F}_i\left(s_i;t\right)\right]\mathbb{E}_{\mathcal{F}_j}\left[\mathcal{F}_j\left(s_j;t'\right)\right]\right)dtdt'. \quad (113)$$

- if $i\neq j$: by independence of $\mathcal{F}_i\left(s_i;.\right)$ and $\mathcal{F}_j\left(s_j;.\right)$ we have $k_{\mathcal{F}_i,\mathcal{F}_j}\left(s_i,s_j;t,t'\right)=0$ and it follows that $\mathbb{E}_{\mathcal{Y}}\left[A_iA_j\right]=\mathbb{E}_{\mathcal{Y}}\left[A_i\right]\mathbb{E}_{\mathcal{Y}}\left[A_j\right]$,
- if $i=j$: we use the stationarity of $\mathcal{F}_i\left(s_i,.\right)$ to obtain the following equalities:

$$\mathbb{E}_{\mathcal{Y}}\left[A_i^2\right]=\int_{I_i}\int_{I_i}\left(k_{\mathcal{F}_i,\mathcal{F}_i}(s_i,s_i,t,t')+\mathbb{E}_{\mathcal{F}_i}\left[\mathcal{F}_i\left(s_i;t\right)\right]^2\right)dtdt' \quad (114)$$
$$=\int_{I_i}\int_{I_i}k_{\mathcal{F}_i}(s_i,|t-t'|)dtdt'+\Delta T^2\mathbb{E}_{\mathcal{F}_i}\left[\mathcal{F}_i\left(s_i;0\right)\right]^2 \quad (115)$$

Let us calculate $\text{Var}_{\mathcal{Y}}\left(Z_{\Delta T}\left(x_d, x_p\right)\right)$:

$$\begin{aligned}
&\text{Var}_{\mathcal{Y}}\left(Z_{\Delta T}\left(x_d, x_p\right)\right)\\
&= \mathbb{E}_{\mathcal{Y}}\left[Z_{\Delta T}\left(x_d, x_p\right)^2\right] - \mathbb{E}_{\mathcal{Y}}\left[Z_{\Delta T}\left(x_d, x_p\right)\right]^2 && (116)\\
&= \frac{1}{\Delta T^2}\left(\sum_{i=1}^{n_T}\mathbb{E}_{\mathcal{Y}}\left[A_i^2\right] + 2\sum_{i<j}\mathbb{E}_{\mathcal{Y}}\left[A_iA_j\right]\right.\\
&\qquad\left. - \sum_{i=1}^{n_T}\mathbb{E}_{\mathcal{Y}}\left[A_i\right]^2 - 2\sum_{i<j}\mathbb{E}_{\mathcal{Y}}\left[A_i\right]\mathbb{E}_{\mathcal{Y}}\left[A_j\right]\right) && (117)\\
&= \frac{1}{\Delta T^2}\left(\sum_{i=1}^{n_T}\int_{I_i}\int_{I_i}k_{\mathcal{F}_i}(s_i, |t-t'|)dtdt'\right)\\
&\leq \frac{2}{\Delta T}\sum_{i=1}^{n_T}\int_0^{\Delta T}\left|k_{\mathcal{F}_i}\left(s_i;\tau\right)\right|d\tau. && (118)
\end{aligned}$$

To bound the approximation error made in equation (31), the reasoning used in proof A.2 in the stationary case is applied here: assumptions (1), (2) and $\Delta T > \frac{c_1}{c_2}$ imply the second inequality while the Cauchy-Schwarz inequality is used to obtain the third inequality:

$$\begin{aligned}
&\left|p_{\mathrm{I}}(d) - \mathbb{E}_{X_d,X_p}\left[F_{r_{\mathrm{I}}}\left(T\sum_{j=1}^{n_s}p^j\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(X_d, X_p, s^j;0\right)\right]\right)\right]\right|\\
\leq &\mathbb{E}_{X_d,X_p,\mathcal{Y}}\left[\left|F_{X_{r_{\mathrm{I}}}}\left(\Delta T Z_{\Delta T}(X_d, X_p)\right)\right.\right.\\
&\qquad\left.\left. - F_{X_{r_{\mathrm{I}}}}\left(\Delta T\mathbb{E}_{\mathcal{Y}}\left[Z_{\Delta T}\left(X_d, X_p\right)\right]\right)\right|\right]\\
\leq &\frac{c_r}{c_2}\mathbb{E}_{X_d,X_p,\mathcal{Y}}\left[\left|Z_{\Delta T}(X_d,X_p) - \mathbb{E}_{\mathcal{Y}}\left[Z_{\Delta T}\left(x_d,x_p\right)\right]\right|\right]\\
\leq &\frac{c_r}{c_2}\mathbb{E}_{X_d,X_p}\left[\sqrt{\text{Var}_{\mathcal{Y}}\left(Z_{\Delta T}\left(X_d,X_p\right)\right)}\right]\\
\leq &\frac{c_r}{c_2}\sqrt{\frac{2n_T}{\Delta T}\sum_{j=1}^{n_s}p^j c_{\mathcal{F},s^j}}.
\end{aligned}$$

□

B.5 Computation of the quantity involved in the integral-based constraint of the oscillator problem

We recall that the random variable $\mathcal{D}_1^{(2)}\left(x_d, x_p, s^j;0\right)$ follows a normal distribution with zero-mean and standard deviation $\sigma_{\mathcal{D}^{(2)}}\left(x_d, x_p, s^j\right)$. We denote the latter σ in the following calculations. Thus, we have for the oscillator problem:

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(x_d, x_p, s^j;0\right)\right] = \mathbb{E}\left[\left(\left|\mathcal{D}_1^{(2)}\left(x_d,x_p,s^j;0\right)\right| - \rho\right)^+\right]\\
&= \int_{\mathbb{R}}\left(|y|-\rho\right)^+\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy. && (119)
\end{aligned}$$

The integral in the last equation is decomposed into integrals over $(-\infty, -\rho]$ and $[\rho, +\infty)$ and the calculations lead to:

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_1}\left[\mathcal{F}_1\left(x_d, x_p, s^j;0\right)\right] &= \sqrt{\frac{2}{\pi}}\sigma\exp\left(-\frac{\rho^2}{2\sigma^2}\right)\\
&\quad + 2\rho\left(\Phi\left(\frac{\rho}{\sigma}\right) - 1\right), && (120)
\end{aligned}$$

with Φ the cumulative distribution function of the standard normal distribution.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Replication of results The procedure and the flowchart of AK-ECO are provided in section 3 and the implementation is detailed in section 4.

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