WAVE PROPAGATION IN PERIODIC AND RANDOM TIME-DEPENDENT MEDIA

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Abstract. We study wave propagation in materials in which the refractive index varies in time. We compare the situations in which the variations are periodic and random. If the modulation period (in the periodic case) or the coherence time (in the random case) of the refractive index is much smaller than the central wave period, then the two situations are analogous and effective dispersion is the main effect. If the modulation period or coherence time is of the same order as the central wave period, then wavenumber-dependent amplification and dispersion are the main effects, but they have different characteristics: the amplification is very narrowband in the periodic case and broadband in the random case. In the random case, we get an analog of the O'Doherty-Anstey theory in randomly spatially varying media: the wave front experiences a random shift and a deterministic deformation, but contrarily to the spatially varying case, the deterministic deformation induced by random time variations involves amplification and dispersion; moreover, the incoherent wave fluctuations grow faster than the coherent wave front, so that the former eventually dominate the latter.

Key words. Electromagnetic waves, random media, time-dependent media, pulse stabilization.

AMS subject classifications. 78A48, 35R60, 35Q61, 60F05.

1. Introduction.

The propagation and transformation of waves through spatially homogeneous and time-dependent media has been studied for a long time [22, 27], and it is still the topic of active research in order to get reflection and transmission coefficients in the case of a sudden change [5, 28] or in the case of a smooth transition occurring during a finite period [11, 12]. Related fundamental questions on refraction, splitting, and guiding have been considered [14, 13, 21, 20, 24, 25]. Amongst the possible applications of sudden changes, time-reversal experiments based on a new concept of instantaneous time-reversal mirrors have attracted attention [1, 2, 7]. Amongst the possible applications of smooth variations, the concept of photonic crystals (with variations that are periodic in time) has emerged [15, 19] and wave propagation in randomly time-dependent and randomly layered media in a particular scaling regime has been studied in [4]. In both cases amplification of the propagating wave has been reported.

In this paper we want to clarify the effects of time-dependent media on wave propagation. We address a situation that can be reduced to a one-dimensional wave equation with small-amplitude time-dependent medium variations. We apply tools from averaging and diffusion-approximation theories [8]. Our approach is valid when the amplitudes of the fluctuations of the medium parameters are small and the propagation time is large, so that the effects of the time-dependent medium are of order one. In the random case we give a full statistical description of the transmitted wave and we underline that the mean-field approach gives a misleading prediction, because it averages out a random shift that obeys Gaussian statistics, which induces an apparent, non-physical diffusion for the mean field which compensates for the true physical amplification.

We compare the situations in which the time variations are periodic and random. When the modulation period (in the periodic case) or the coherence time (in the random case) is smaller than the central wave period, then the two situations are

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analogous and effective dispersion is the main effect. When the modulation period or coherence time is of the same order as the central wave period, then wavenumber-dependent amplification and dispersion are the main effects but they have different characteristics: amplification is very narrowband and stronger than dispersion in the periodic case (see section 3), while it is broadband and of the same order as dispersion in the random case (see section 4). In the random case, we get an analog of the O'Doherty-Anstey theory that was established in randomly spatially varying media [6, 8, 9, 10, 23] and in a random space- and time-dependent medium in [4]: the wave front experiences a random shift and a deterministic deformation, but contrarily to the spatially varying case, the deformation induced by random time variations involves amplification and dispersion. Moreover, the incoherent wave fluctuations grow faster than the coherent wave front, so that the former eventually dominate the latter. We characterize the Wigner transform (the local power spectral density) of the incoherent wave fluctuations in this regime. Finally, in section 5 we extend the results to three-dimensional time-varying media and radially symmetric fields in a straightforward manner.

2. Electromagnetic wave equation. The electromagnetic system that we analyze is a spatially homogeneous material with permittivity which is modulated in time. The field is polarized in the x-direction and propagates in the z-direction. This model was studied in [3, 19, 29, 30] when the modulation is periodic, which showed that this system yields a photonic time crystal.

The magnetic field $B(z,t)$ (along the y-direction) and the electric displacement field $D(z,t)$ (along the x-direction) satisfy (from the Maxwell-Ampère equation $\nabla \times \boldsymbol{B} = \mu_o \partial_t \boldsymbol{D}$):

$$\partial_z B = -\mu_o \partial_t D. \tag{2.1}$$

In general, $D(z,t) = \int_0^\infty R(t,\tau) E(z,t-\tau)$, where $E(z,t)$ is the electric field and $R(t,\tau)$ is the response function of the material [18], which in our case is spatially homogeneous and changes in time. We use an approximation of instantaneous response, meaning that $R(t,\tau) = \epsilon(t)\delta(\tau)$ as in [3, 19]:

$$D(z,t) = \epsilon(t) E(z,t). \tag{2.2}$$

The electric displacement field $D(z,t)$ and the magnetic field $B(z,t)$ also satisfy (from the Maxwell-Faraday equation $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$):

$$\partial_z D = -\epsilon(t) \partial_t B. \tag{2.3}$$

The displacement electric field $D(z,t)$ therefore satisfies the time-dependent wave equation

$$\partial_z^2 D - \epsilon(t)\mu_o \partial_t^2 D = 0. \tag{2.4}$$

We assume that the time modulation of the electric permittivity is of the form

$$\epsilon(t) = \frac{\epsilon_o}{1 + \delta \nu(t) \mathbf{1}_{[0,\infty)}(t)}, \tag{2.5}$$

where $\nu(t)$ is a bounded function and δ is a (small) parameter that characterizes the relative amplitude of the modulation. For negative times the electric permittivity is constant and equal to ϵ_o, which corresponds to the propagation speed $c_o = 1/\sqrt{\epsilon_o \mu_o}$.

We assume that the wavefield is a forward-going wave for $t < 0$:

$$D(z,t) = D_o(z - c_o t). \tag{2.6}$$

We assume that the Fourier transform $\hat{D}_o$ of D_o is compactly supported.

As the medium is homogeneous we take a Fourier transform in space:

$$\hat{D}(k,t) = \int_{\mathbb{R}} D(z,t) \exp(-ikz) dz. \tag{2.7}$$

It satisfies the equation of a randomly modulated harmonic oscillator:

$$c_o^2 k^2 \big[1 + \delta\nu(t)\big] \hat{D} + \partial_t^2 \hat{D} = 0. \tag{2.8}$$

We introduce the forward- and backward-going mode amplitude:

$$a(k,t) = \frac{1}{2}\big(\hat{D} + \frac{i}{c_o k}\partial_t \hat{D}\big) \exp(ic_o kt), \tag{2.9}$$

$$b(k,t) = \frac{1}{2}\big(\hat{D} - \frac{i}{c_o k}\partial_t \hat{D}\big) \exp(-ic_o kt). \tag{2.10}$$

They satisfy the coupled first-order system:

$$\partial_t a = -\frac{ic_o k\delta}{2}\nu(t)\big(a + b\exp(2ic_o kt)\big), \tag{2.11}$$

$$\partial_t b = \frac{ic_o k\delta}{2}\nu(t)\big(a\exp(-2ic_o kt) + b\big), \tag{2.12}$$

starting from the initial conditions $a(k, t = 0) = \hat{D}_o(k)$ and $b(k, t = 0) = 0$. By linearity we have

$$a(k,t) = \hat{D}_o(k)\alpha(k,t), \qquad b(k,t) = \hat{D}_o(k)\beta(k,t), \tag{2.13}$$

where $(\alpha(k,t), \beta(k,t))$ satisfy

$$\partial_t \alpha = -\frac{ic_o k\delta}{2}\nu(t)\big(\alpha + \beta\exp(2ic_o kt)\big), \tag{2.14}$$

$$\partial_t \beta = \frac{ic_o k\delta}{2}\nu(t)\big(\alpha\exp(-2ic_o kt) + \beta\big), \tag{2.15}$$

starting from the initial conditions $\alpha(k, t = 0) = 1$ and $\beta(k, t = 0) = 0$.

For propagation times of the order of δ^{-p}, $p > 0$, the electric displacement field has the form

$$D\big(z, \frac{t}{\delta^p}\big) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(ikz)\hat{D}_o(k)\big[\alpha\big(k, \frac{t}{\delta^p}\big)\exp\big(-ic_o k\frac{t}{\delta^p}\big) + \beta\big(k, \frac{t}{\delta^p}\big)\exp\big(ic_o k\frac{t}{\delta^p}\big)\big]dk. \tag{2.16}$$

By taking into account that $\hat{D}(-k,t) = \overline{\hat{D}(k,t)}$ (and similarly for α, β), we also have

$$D\big(z, \frac{t}{\delta^p}\big) = \frac{1}{2\pi}\int_0^\infty \exp(ikz)\hat{D}_o(k)\big[\alpha\big(k, \frac{t}{\delta^p}\big)\exp\big(-ic_o k\frac{t}{\delta^p}\big) + \beta\big(k, \frac{t}{\delta^p}\big)\exp\big(ic_o k\frac{t}{\delta^p}\big)\big]dk + c.c., \tag{2.17}$$

where *c.c.* stands for complex conjugate. Similarly, the magnetic field has the form

$$B\big(z,\frac{t}{\delta^p}\big)=\frac{\mu_o c_o}{2\pi}\int_{\mathbb{R}}\exp(ikz)\hat{D}_o(k)\big[\alpha\big(k,\frac{t}{\delta^p}\big)\exp\big(-ic_ok\frac{t}{\delta^p}\big)$$
$$-\beta\big(k,\frac{t}{\delta^2}\big)\exp\big(ic_ok\frac{t}{\delta^p}\big)\big]dk. \tag{2.18}$$

3. Wave propagation with periodic variations. In this section we address the case where ν is a periodic function:

$$\nu(t)=\sigma\cos(\omega_c t). \tag{3.1}$$

We denote

$$k_c=\frac{\omega_c}{2c_o}. \tag{3.2}$$

This section shows that the transmitted wave profile experiences a wavenumber-dependent amplification that affects strongly (for propagation times of the order of δ^{-1}) the Fourier components with wavenumbers close to k_c and a dispersion that affects weakly (for propagation times of the order of δ^{-2}) all Fourier components.

3.1. Narrowband amplification. By standard periodic averaging theory [26] we first prove the following lemma.

LEMMA 3.1. *1. For any $k\neq k_c$, we have $a(k,t/\delta)\to\hat{D}_o(k)$ and $b(k,t/\delta)\to 0$ when $\delta\to 0$.*

2. For wavenumbers close to k_c, of the form $k_c+\delta k$, we have

$$\alpha\big(k_c+\delta k,\frac{t}{\delta}\big)\overset{\delta\to 0}{\longrightarrow}\mathcal{A}(k,t)\exp(ic_okt), \tag{3.3}$$
$$\beta\big(k_c+\delta k,\frac{t}{\delta}\big)\overset{\delta\to 0}{\longrightarrow}\mathcal{B}(k,t)\exp(-ic_okt), \tag{3.4}$$

where

$$\mathcal{A}(k,t)=\cosh\Big(\sqrt{\frac{\omega_c^2\sigma^2}{64}-c_o^2k^2}t\Big)-\frac{ic_ok}{\sqrt{\frac{\omega_c^2\sigma^2}{64}-c_o^2k^2}}\sinh\Big(\sqrt{\frac{\omega_c^2\sigma^2}{64}-c_o^2k^2}t\Big), \tag{3.5}$$
$$\mathcal{B}(k,t)=-\frac{i\omega_c\sigma}{8\sqrt{\frac{\omega_c^2\sigma^2}{64}-c_o^2k^2}}\sinh\Big(\sqrt{\frac{\omega_c^2\sigma^2}{64}-c_o^2k^2}t\Big). \tag{3.6}$$

This lemma shows that the Fourier components with wavenumbers in the interval $(k_c(1-\delta\sigma/4),k_c(1+\delta\sigma/4))$ are amplified.

Proof. By (2.14-2.15) the rescaled functions $\alpha^\delta(k,t)=\alpha(k,t/\delta)$ and $\beta^\delta(k,t)=\beta(k,t/\delta)$ satisfy

$$\partial_t\alpha^\delta=-\frac{ic_ok\sigma}{2}\big(\cos(\omega_c\frac{t}{\delta})\alpha^\delta+\cos(\omega_c\frac{t}{\delta})\exp\big(2ic_ok\frac{t}{\delta}\big)\beta^\delta\big),$$
$$\partial_t\beta^\delta=\frac{ic_ok\sigma}{2}\big(\cos(\omega_c\frac{t}{\delta})\exp\big(-2ic_ok\frac{t}{\delta}\big)\alpha^\delta+\cos(\omega_c\frac{t}{\delta})\beta^\delta\big).$$

If $k\neq k_c$, then periodic averaging theory [26] shows that $(\alpha^\delta,\beta^\delta)$ converges to the solution of $\partial_t\alpha=0$, $\partial_t\beta=0$, which gives the first item of the lemma.

By (2.14-2.15) the rescaled functions $\alpha^\delta(k,t) = \alpha(k_c + \delta k, t/\delta)\exp(-ic_o kt)$ and $\beta^\delta(k,t) = \beta(k_c + \delta k, t/\delta)\exp(ic_o kt)$ satisfy

$$\partial_t \alpha^\delta = -\frac{i\omega_c \sigma}{4}\big(\cos(\omega_c \frac{t}{\delta})\alpha^\delta + \cos(\omega_c \frac{t}{\delta})\exp(i\omega_c \frac{t}{\delta})\beta^\delta\big) - ic_o k\alpha^\delta + O(\delta),$$
$$\partial_t \beta^\delta = \frac{i\omega_c \sigma}{4}\big(\cos(\omega_c \frac{t}{\delta})\exp(-i\omega_c \frac{t}{\delta})\alpha^\delta + \cos(\omega_c \frac{t}{\delta})\beta^\delta\big) + ic_o k\beta^\delta + O(\delta).$$

Periodic averaging theory shows that $(\alpha^\delta, \beta^\delta)$ converges to the solution of

$$\partial_t \alpha = -\frac{i\omega_c\sigma}{8}\beta - ic_o k\alpha, \qquad \partial_t \beta = \frac{i\omega_c\sigma}{8}\alpha + ic_o k\beta.$$

This gives the second item of the lemma. □

The following proposition gives the form of the electric displacement field for propagation times of the order of δ^{-1} when the spectrum of the initial field is narrowband and concentrated around k_c.

PROPOSITION 3.2. *If the spectrum is of the form*

$$\hat{D}_o(k) = \frac{1}{\delta}D_{oo}\Big(\frac{k-k_c}{\delta}\Big) + \frac{1}{\delta}\overline{D_{oo}}\Big(-\frac{k+k_c}{\delta}\Big), \tag{3.7}$$

where D_{oo} is compactly supported, then

$$D\big(\frac{z}{\delta},\frac{t}{\delta}\big) \overset{\delta\to 0}{\longrightarrow} \exp\Big(i\frac{\omega_c}{2\delta}\big(\frac{z}{c_o}-t\big)\Big)\frac{1}{2\pi}\int_{\mathbb{R}} e^{ikz}D_{oo}(k)\mathcal{A}(k,t)dk$$
$$+\exp\Big(i\frac{\omega_c}{2\delta}\big(\frac{z}{c_o}+t\big)\Big)\frac{1}{2\pi}\int_{\mathbb{R}} e^{ikz}D_{oo}(k)\mathcal{B}(k,t)dk + cc. \tag{3.8}$$

Proof. By (2.14-2.15) and Gronwall's lemma the functions $\alpha(k,t/\delta)$ and $\beta(k,t/\delta)$ are uniformly bounded in δ and k in the support of D_{oo}. The proposition is proved by applying Lebesgue's dominated convergence theorem to (2.16) and by using Lemma 3.1. □

The amplification of the wavefield is characterized by a exponential growth of the Fourier components with wavenumbers $k_c + \delta k$, $k \in (-k_c\sigma/4, k_c\sigma/4)$ of the form $\exp\Big(\sqrt{\frac{\omega_c^2\sigma^2}{64} - c_o^2k^2}t\Big)$. Note that the overall picture is that the wave does not propagate anymore: it oscillates and is amplified locally. In Figure 3.1 we present some numerical simulations to illustrate the amplifying effect of time-dependent media. Here we solve (2.4-2.5) by using a finite-difference time-domain (FDTD) method with $\delta = 1$, $c_o = 1$, $\nu(t) = \sigma\cos(\omega_c t)$, $\sigma = 0.5$, and $\omega_c = 5$. The initial condition at time $t = 0$ is a narrowband Gaussian profile $D_o(z) = \exp(-z^2/r_o^2)\cos(k_c z)$, $\hat{D}_o(k) = (r_o/\sqrt{\pi})\exp(-(|k|-k_c)^2 r_o^2/4)$, with $k_c = \omega_c/(2c_o) = 2.5$ and $r_o = 20$ [Note that $\hat{D}_o$ is not compactly supported, but it decays fast enough and the theoretical results derived in the paper can certainly be extended beyond the compactly supported case]. The agreement between the theoretical predictions and the numerical simulations is perfect.

3.2. Broadband dispersion. The following lemma shows that all Fourier components with wavenumbers outside the small amplification band around k_c experience a phase modulation for long propagation times of the order of δ^{-2}.

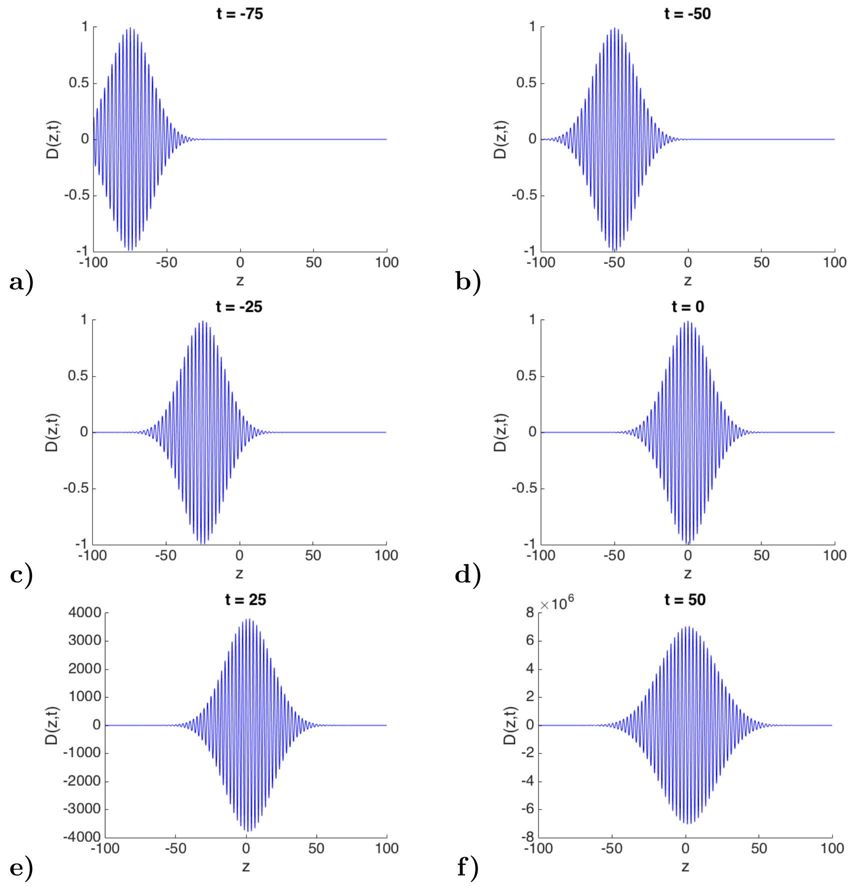

FIG. 3.1. *Propagation of a narrowband Gaussian profile* $D_o(z) = \exp(-z^2/r_o^2)\cos(k_c z)$. *The time variations of the medium* $\nu(t) = \sigma\cos(\omega_c t)$ *start at* $t = 0$. *Before* $t = 0$ *the wave is going to the right with speed of propagation* c_o. *After* $t = 0$ *the wave is amplified as described in Proposition 3.2. Here* $\delta = 1$, $c_o = 1$, $\sigma = 0.5$, $\omega_c = 5$ *(*$k_c = 2.5$*),* $r_o = 20$.

LEMMA 3.3. *For any* $|k| \neq k_c$, *we have*

$$\alpha\Big(k, \frac{t}{\delta^2}\Big) \stackrel{\delta\to 0}{\longrightarrow} \exp\Big(-\frac{i}{4}\frac{c_o^3 k^3 \sigma^2}{\omega_c^2 - 4c_o^2 k^2} t\Big), \qquad \beta\Big(k, \frac{t}{\delta^2}\Big) \stackrel{\delta\to 0}{\longrightarrow} 0. \tag{3.9}$$

Proof. Let $|k| \neq k_c$. The four rescaled functions $\alpha_r^\delta = \mathrm{Re}(\alpha(k, t/\delta^2))$, $\alpha_i^\delta = \mathrm{Im}(\alpha(k, t/\delta^2))$, $\beta_r^\delta = \mathrm{Re}(\beta(k, t/\delta^2))$, and $\beta_i^\delta = \mathrm{Im}(\beta(k, t/\delta^2))$ satisfy

$$
\begin{aligned}
\partial_t \begin{pmatrix} \alpha_r^\delta \\ \alpha_i^\delta \\ \beta_r^\delta \\ \beta_i^\delta \end{pmatrix} = & \frac{c_o k \sigma}{2\delta} \cos\big(\frac{\omega_c}{\delta^2} t\big) \begin{pmatrix} \alpha_i^\delta \\ -\alpha_r^\delta \\ -\beta_i^\delta \\ \beta_r^\delta \end{pmatrix} \\
& + \frac{c_o k \sigma}{4\delta} \cos\big(\frac{\omega_c + 2c_o k}{\delta^2} t\big) \begin{pmatrix} \beta_i^\delta \\ -\beta_r^\delta \\ -\alpha_i^\delta \\ \alpha_r^\delta \end{pmatrix} + \frac{c_o k \sigma}{4\delta} \sin\big(\frac{\omega_c + 2c_o k}{\delta^2} t\big) \begin{pmatrix} \beta_r^\delta \\ \beta_i^\delta \\ \alpha_r^\delta \\ \alpha_i^\delta \end{pmatrix} \\
& + \frac{c_o k \sigma}{4\delta} \cos\big(\frac{\omega_c - 2c_o k}{\delta^2} t\big) \begin{pmatrix} \beta_i^\delta \\ -\beta_r^\delta \\ -\alpha_i^\delta \\ \alpha_r^\delta \end{pmatrix} + \frac{c_o k \sigma}{4\delta} \sin\big(\frac{\omega_c - 2c_o k}{\delta^2} t\big) \begin{pmatrix} -\beta_r^\delta \\ -\beta_i^\delta \\ -\alpha_r^\delta \\ -\alpha_i^\delta \end{pmatrix} .
\end{aligned}
$$

By applying Proposition A.1 (a special averaging theorem) we get that the function $(\alpha_r^\delta, \alpha_i^\delta, \beta_r^\delta, \beta_i^\delta)$ converges to $(\alpha_r, \alpha_i, \beta_r, \beta_i)$ solution of

$$
\partial_t \begin{pmatrix} \alpha_r \\ \alpha_i \\ \beta_r \\ \beta_i \end{pmatrix} = \frac{c_o^3 k^3 \sigma^2}{4(\omega_c^2 - 4c_o^2 k^2)} \begin{pmatrix} \alpha_i \\ -\alpha_r \\ -\beta_i \\ \beta_r \end{pmatrix},
$$

or equivalently

$$
\partial_t \alpha = -i \frac{c_o^3 k^3 \sigma^2}{4(\omega_c^2 - 4c_o^2 k^2)} \alpha, \qquad \partial_t \beta = i \frac{c_o^3 k^3 \sigma^2}{4(\omega_c^2 - 4c_o^2 k^2)} \beta.
$$

This proves the desired result. □

The following proposition gives the form of a broadband electric displacement field for propagation times of the order of δ^{-2}.

PROPOSITION 3.4. *Let us assume that the support of $\hat{D}_o(k)$ does not contain k_c. When $\delta \to 0$, the transmitted field at time t/δ^2 around the expected arrival point $c_o t/\delta^2$ converges to a dispersive profile:*

$$
D\big(c_o \frac{t}{\delta^2} + z, \frac{t}{\delta^2}\big) \stackrel{\delta \to 0}{\longrightarrow} \mathcal{D}(z, t), \tag{3.10}
$$

$$
\mathcal{D}(z, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{D}_o(k) \exp\Big(ikz - \frac{i}{4} \frac{c_o^3 k^3 \sigma^2}{\omega_c^2 - 4c_o^2 k^2} t\Big) dk. \tag{3.11}
$$

Proof. By [16] the functions $\alpha(k, t/\delta^2)$ and $\beta(k, t/\delta^2)$ can be bounded uniformly with respect to δ and k in the support of $\hat{D}_0$. The proposition is proved by applying Lebesgue's dominated convergence theorem to (2.16) and by using Lemma 3.3. □

If, for instance, k_c is much larger than the spatial bandwidth of the initial field, then this proposition shows that the wavefield experiences a third-order dispersion:

$$
\mathcal{D}(z, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{D}_o(k) \exp\big(ikz - i\eta_c k^3 t\big) dk, \tag{3.12}
$$

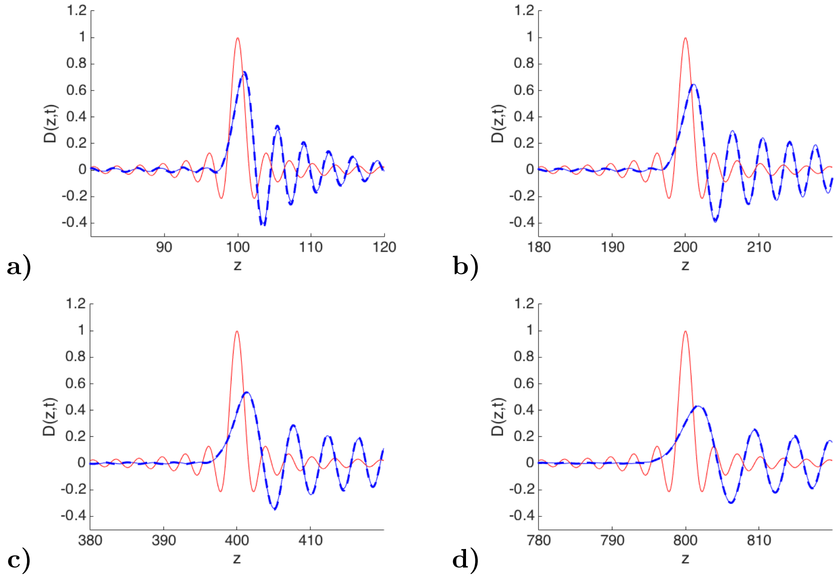

FIG. 3.2. *Profile of an input sinc pulse propagating in a periodic time-dependent medium at time $t = 100$ (a), $t = 200$ (b), $t = 400$ (c), and $t = 800$ (d). The red line is the output profile in a homogeneous stationary medium (i.e., the shifted input profile), the solid blue line is the numerical output profile, the dashed blue line is the theoretical profile (3.11).*

with $\eta_c = \frac{c_o^3\sigma^2}{4\omega_c^2}$. Under such circumstances, $\mathcal{D}(z,t)$ satisfies the effective equation $\partial_t \mathcal{D} = \eta_c \partial_z^3 \mathcal{D}$.

In Figure 3.2 we present some numerical simulations to illustrate the strong dispersive effect of time-dependent media with high modulation frequency. Here we solve (2.4-2.5) by a FDTD method with $\delta = 1$, $c_o = 1$, $\nu(t) = \sigma\cos(\omega_c t)$, $\sigma = 0.5$, and $\omega_c = 5$. The initial condition is a sinc profile: $D_o(z) = \mathrm{sinc}(2z) = \sin(2z)/(2z)$ ($\hat{D}_o(k) = (\pi/2)\mathbf{1}_{[-2,2]}(k)$). The agreement between the theoretical predictions and the numerical simulations is excellent.

4. Wave propagation with random variations. In this section we assume that the perturbation ν in (2.5) is a zero-mean, stationary random process with good ergodic properties, such as the ones described in [8, Chapter 6] (the process is bounded, Markov, and ergodic) or in [17, 4-6-2] (the process is ϕ-mixing with $\phi \in L^{1/2}$).

4.1. Coherent wave propagation. We introduce

$$\gamma^{\mathrm{c}}(\omega) = 2\int_0^\infty \mathbb{E}[\nu(0)\nu(t)]\cos(2\omega t)dt, \tag{4.1}$$

$$\gamma^{\mathrm{s}}(\omega) = 2\int_0^\infty \mathbb{E}[\nu(0)\nu(t)]\sin(2\omega t)dt. \tag{4.2}$$

Note that $\gamma^{\mathrm{c}}(\omega)$ is nonnegative by Bochner's theorem.

PROPOSITION 4.1. *When $\delta \to 0$ the transmitted field at time t/δ^2 around the expected arrival point $c_o t/\delta^2$ converges in distribution to a profile with a random shift and a deterministic profile:*

$$\Big(D\big(\frac{c_o t}{\delta^2} + z, \frac{t}{\delta^2}\big)\Big)_{z\in\mathbb{R}} \overset{\delta\to 0}{\longrightarrow} \Big(\mathcal{D}_t(z - Z_t)\Big)_{z\in\mathbb{R}}. \tag{4.3}$$

The deterministic profile has the form

$$\mathcal{D}_t(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(ikz) \mathcal{K}_{\mathrm{coh},t}(k) \hat{D}_o(k) dk, \tag{4.4}$$

with the kernel $\mathcal{K}_{\mathrm{coh},t}$ *given by*

$$\mathcal{K}_{\mathrm{coh},t}(k) = \exp\Big(\frac{\gamma^{\mathrm{c}}(c_o k) c_o^2 k^2 t}{8} + i \frac{\gamma^{\mathrm{s}}(c_o k) c_o^2 k^2 t}{8} \Big). \tag{4.5}$$

The random shift has the form

$$Z_t = \frac{\sqrt{\gamma^{\mathrm{c}}(0)} c_o}{2} W_0(t), \tag{4.6}$$

where W_0 *is a standard Brownian motion.*
The convergence holds in the space of continuous functions (with the topology associated to the supremum norm over compact intervals).

We also have the following convergence result for the magnetic field:

$$\Big(B\big(\frac{c_o t}{\delta^2} + z, \frac{t}{\delta^2}\big) \Big)_{z \in \mathbb{R}} \stackrel{\delta \to 0}{\longrightarrow} \Big(\mu_o c_o \mathcal{D}_t(z - Z_t) \Big)_{z \in \mathbb{R}}. \tag{4.7}$$

Moreover, for any $t > 0$ and for any position $z_0 \neq c_o t$, we have

$$\Big(D\big(\frac{z_0}{\delta^2} + z, \frac{t}{\delta^2}\big) \Big)_{z \in \mathbb{R}} \stackrel{\delta \to 0}{\longrightarrow} 0, \qquad \Big(B\big(\frac{z_0}{\delta^2} + z, \frac{t}{\delta^2}\big) \Big)_{z \in \mathbb{R}} \stackrel{\delta \to 0}{\longrightarrow} 0, \tag{4.8}$$

which expresses the fact that there is only one coherent wave front and it is located around the expected arrival point $c_o t / \delta^2$.

Before giving the proof we can make a few remarks.

1. Proposition 4.1 shows that the transmitted wave profile experiences a deterministic wavenumber-dependent amplification (because γ^{c} is nonnegative valued) and a deterministic dispersion. The fact that the wave is amplified was obtained by [4] by a time-space approach based on the method described in [8, Section 8.1]. Note that amplification and dispersion are of the same order and affect all Fourier components contrarily to the periodic case addressed in the previous section.

2. The mean-field approach gives a misleading prediction, in the sense that a typical realization of the transmitted wave front does not look like the mean transmitted wave front. Indeed we find from Proposition 4.1 that the mean transmitted wave front is

$$\Big(\mathbb{E}\big[D\big(\frac{c_o t}{\delta^2} + z, \frac{t}{\delta^2}\big) \big] \Big)_{z \in \mathbb{R}} \stackrel{\delta \to 0}{\longrightarrow} \Big(\mathcal{D}_{\mathrm{mean},t}(z) \Big)_{z \in \mathbb{R}}, \tag{4.9}$$

where

$$\mathcal{D}_{\mathrm{mean},t}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(ikz) \mathcal{K}_{\mathrm{mean},t}(k) \hat{D}_o(k) dk, \tag{4.10}$$

$$\mathcal{K}_{\mathrm{mean},t}(k) = \exp\Big(\frac{[\gamma^{\mathrm{c}}(c_o k) - \gamma^{\mathrm{c}}(0)] c_o^2 k^2 t}{8} + i \frac{\gamma^{\mathrm{s}}(c_o k) c_o^2 k^2 t}{8} \Big). \tag{4.11}$$

The mean-field approach averages out the random shift that obeys Gaussian statistics: $\mathbb{E}[\exp(ikZ_t)] = \exp(-\gamma^{\mathrm{c}}(0) c_o^2 k^2 t / 8)$. The amplification exhibited by Proposition 4.1 is hidden in the mean-field formulation because $\gamma^{\mathrm{c}}(c_o k) \leq \gamma^{\mathrm{c}}(0)$.

3. In the special case where the random perturbation derives from a rapid and smooth process $\nu(t) = \sigma\mu'(\omega_c t)$, with μ a stationary process with smooth autocorrelation function $\mathcal{R}$, then the autocorrelation function of ν is of the form $\mathbb{E}[\nu(0)\nu(t)] = -\sigma^2\mathcal{R}''(\omega_c t)$. If $\omega_c \gg \omega$, then we have $\gamma^{\rm c}(\omega) \simeq 8\omega^2\sigma^2\int_0^\infty \mathcal{R}(s)ds/\omega_c^3$ and $\gamma^{\rm s}(\omega) \simeq -4\omega\sigma^2\mathcal{R}(0)/\omega_c^2$. The kernel is then a third-order dispersion kernel to leading order:

$$\mathcal{K}_{{\rm coh},t}(k) \simeq \exp\Big(-i\frac{\sigma^2\mathcal{R}(0)c_o^3k^3t}{2\omega_c^2}\Big), \tag{4.12}$$

which is very similar to the result obtained in the periodic case. Indeed, we recover Eq. (3.12) if we interpret $\mu(\cdot) = \sin(\cdot)$ as a stationary process with variance $\mathcal{R}(0) = 1/2$ (because the "variance" of sin is the average of $\sin^2$ equal to 1/2).

Proof of Proposition 4.1. The proof is based on a time-wavenumber approach and it follows the lines of the one in [8, Section 8.2]. We consider moments of the transmitted field of the form

$$\begin{aligned}
\mathbb{E}\Big[\prod_{l=1}^m D\big(\frac{c_ot}{\delta^2}+z_l,\frac{t}{\delta^2}\big)^{n_l}\Big] &= \mathbb{E}\Big[\prod_{l=1}^m \Big(\frac{1}{2\pi}\int_{\mathbb{R}} \hat{D}_o(k)\big[\alpha(k,\frac{t}{\delta^2})\exp(ikz_l) \\
&\quad + \beta(k,\frac{t}{\delta^2})\exp(-ikz_l-2ikc_o\frac{t}{\delta^2})\big]dk\Big)^{n_l}\Big] \\
&= \frac{1}{(2\pi)^{\sum_{l=1}^m n_l}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\prod_{l=1}^m\prod_{j_l=1}^{n_l} dk_{j_l}\prod_{l=1}^m\prod_{j_l=1}^{n_l}\hat{D}_o(k_{j_l}) \\
\times\,\mathbb{E}\Big[\prod_{l=1}^m\prod_{j_l=1}^{n_l}&\big[\alpha(k_{j_l},\frac{t}{\delta^2})\exp(ik_{j_l}z_l)+\beta(k_{j_l},\frac{t}{\delta^2})\exp(-ik_{j_l}z_l-2ik_{j_l}c_o\frac{t}{\delta^2})\big]\Big],
\end{aligned}$$

where $m\in\mathbb{N}$, $z_l\in\mathbb{R}$, $n_l\in\mathbb{N}$, $l=1,\dots,m$. The main point consists in studying the limits of specific moments of the form $\mathbb{E}[\prod_{j=1}^n\alpha(k_j,t/\delta^2)]$ for distinct wavenumbers k_j, $j=1,\dots,n$. We consider the vector $(\alpha(k_j,t/\delta^2),\beta(k_j,t/\delta^2))_{j=1}^n$ solution of [by (2.14-2.15)]:

$$\begin{aligned}
\partial_t\alpha(k_j,\frac{t}{\delta^2}) &= -\frac{ic_ok_j}{2\delta}\nu(\frac{t}{\delta^2})\big(\alpha(k_j,\frac{t}{\delta^2})+\beta(k_j,\frac{t}{\delta^2})\exp(2ic_ok_j\frac{t}{\delta^2})\big), \\
\partial_t\beta(k_j,\frac{t}{\delta^2}) &= \frac{ic_ok_j}{2\delta}\nu(\frac{t}{\delta^2})\big(\alpha(k_j,\frac{t}{\delta^2})\exp(-2ic_ok_j\frac{t}{\delta^2})+\beta(k_j,\frac{t}{\delta^2})\big),
\end{aligned}$$

starting from the initial conditions $\alpha(k_j,t=0)=1$ and $\beta(k_j,t=0)=0$. By applying the diffusion-approximation theorem [8, Theorem 6.5] we find that

$$\mathbb{E}\Big[\prod_{j=1}^n\alpha\big(k_j,\frac{t}{\delta^2}\big)\Big]\stackrel{\delta\to0}{\longrightarrow}\Big[\prod_{j=1}^n\mathcal{K}_{{\rm coh},t}(k_j)\Big]\exp\Big(-\big(\sum_{j=1}^n k_j\big)^2\frac{\gamma^{\rm c}(0)c_o^2}{8}t\Big),$$

and the limit can be identified as

$$\mathbb{E}\Big[\prod_{j=1}^n\mathcal{K}_{{\rm coh},t}(k_j)\exp(-ik_jZ_t)\Big],$$

where Z_t is given by (4.6). Similarly, we find

$$\mathbb{E}\Big[\prod_{j=1}^n\beta\big(k_j,\frac{t}{\delta^2}\big)\prod_{j'=1}^{n'}\alpha\big(k_{j'},\frac{t}{\delta^2}\big)\Big]\stackrel{\delta\to0}{\longrightarrow}0$$

as soon as $n \geq 1$. We can use the perturbed test function method as in [8, section 6.3.5] to prove that the moments of $\alpha(k, t/\delta^2)$ and $\beta(k, t/\delta^2)$ are bounded uniformly with respect to δ and k in the support of $\hat{D}_o$. Thus, we obtain the convergence of the finite-dimensional moments: for any z_l, n_l, $l = 1, \dots, m$:

$$\mathbb{E}\Big[\prod_{l=1}^m D\big(\frac{c_o t}{\delta^2} + z_l, \frac{t}{\delta^2}\big)^{n_l}\Big] \stackrel{\delta\to 0}{\longrightarrow} \mathbb{E}\Big[\prod_{l=1}^m \Big(\frac{1}{2\pi}\int_{\mathbb{R}} \hat{D}_o(k)\mathcal{K}_{\mathrm{coh},t}(k)\exp(ik(z_l - Z_t))dk\Big)^{n_l}\Big].$$

The tightness in the space of continuous functions can be obtained by Kolmogorov continuity criterium. We have

$$\begin{aligned}
&\mathbb{E}\Big[\big|D\big(\frac{c_o t}{\delta^2} + z, \frac{t}{\delta^2}\big) - D\big(\frac{c_o t}{\delta^2} + z', \frac{t}{\delta^2}\big)\big|^2\Big] \\
&\leq 2\mathbb{E}\Big[\Big|\frac{1}{2\pi}\int \hat{D}_o(k)\alpha\big(k, \frac{t}{\delta^2}\big)\big(e^{ikz} - e^{ikz'}\big)dk\Big|^2\Big] \\
&\quad + 2\mathbb{E}\Big[\Big|\frac{1}{2\pi}\int \hat{D}_o(k)\beta\big(k, \frac{t}{\delta^2}\big)\big(e^{-ikz} - e^{-ikz'}\big)e^{-2ikc_o\frac{t}{\delta^2}}dk\Big|^2\Big] \\
&\leq C|z - z'|^2 \int_{\mathbb{R}} k^2|\hat{D}_o(k)|^2\Big\{\mathbb{E}\big[\big|\alpha\big(k, \frac{t}{\delta^2}\big)\big|^2\big] + \mathbb{E}\big[\big|\beta\big(k, \frac{t}{\delta^2}\big)\big|^2\big]\Big\}dk,
\end{aligned}$$

because the support of $\hat{D}_o(k)$ is bounded. The integral in the right-hand side has a finite limit as $\delta \to 0$, because the proof of the next proposition shows that the expectations have finite limits for all k, and they are bounded uniformly with respect to δ and k in the support of $\hat{D}_o$. The tightness and the convergence of the finite-dimensional moments gives the desired result. □

We have carried out a few numerical simulations to illustrate the effective deterministic amplification and dispersion and the random shift of the wave front propagating in random time-dependent media. In the simulations the process ν is an Ornstein-Uhlenbeck process with autorrelation function

$$\mathbb{E}[\nu(0)\nu(t)] = \sigma^2 \exp(-\omega_c|t|). \tag{4.13}$$

By Proposition 4.1 the theoretical prediction is that, at time t, the wave front experiences a random shift that has Gaussian distribution, mean zero, and variance $\mathbb{E}[Z_t^2] = \sigma^2 c_o^2 t/(2\omega_c)$ and a deterministic deformation determined by the kernel

$$\mathcal{K}_{\mathrm{coh},t}(k) = \exp\Big(\frac{\sigma^2 c_o^2 k^2}{4\omega_c(1 + 4c_o^2k^2/\omega_c^2)}\big(1 + 2i\frac{c_o k}{\omega_c}\big)t\Big). \tag{4.14}$$

In Figures 4.1-4.2 we solve (2.4-2.5) by a FDTD method with $\delta = 1$, $c_o = 1$, $\sigma = 0.1$, and $\omega_c = 1$. The initial condition is a sinc profile: $D_o(z) = \mathrm{sinc}(2z)$. We can see in Figure 4.1 that the result (4.3) predicts well the transmitted profiles, which are indeed deterministic up to random shifts. We can see in Figure 4.2 that the theoretical formulas (4.4-4.6) predict well the random shifts. We can also observe in Figure 4.1 that, for larger and larger propagation distances, the fluctuations of the waves (outside the main transmitted wave front) become more and more important. This is described in the next section.

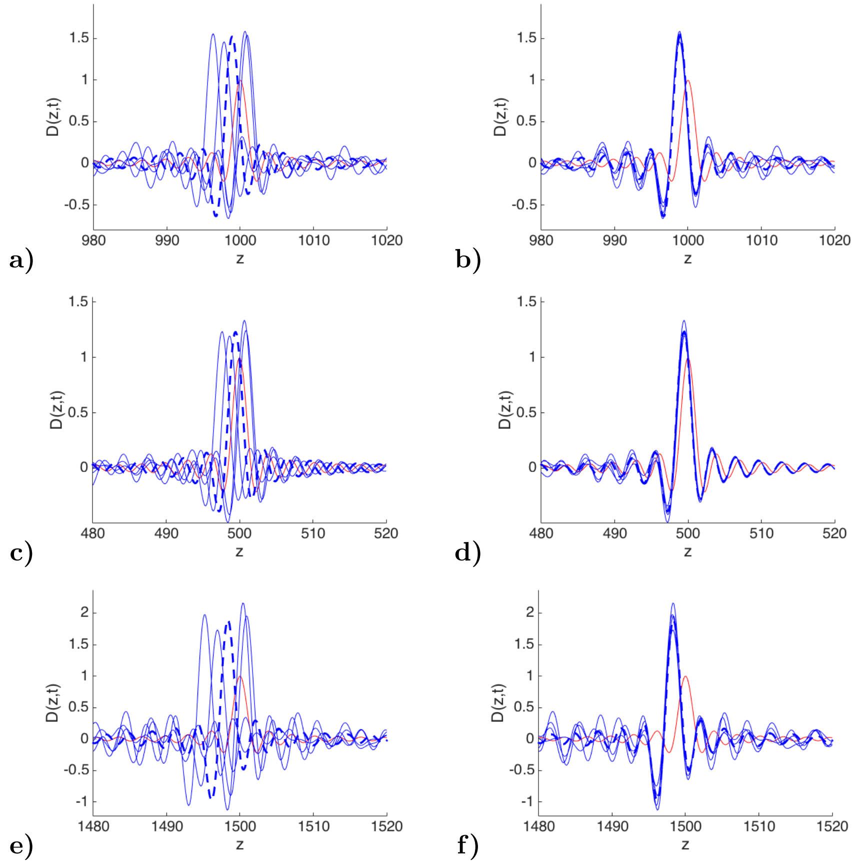

FIG. 4.1. *Profile of an input sinc pulse propagating in a random time-dependent medium at time $t = 500$ (a,b), $t = 1000$ (c,d), and $t = 1500$ (e,f). The red line is the theoretical output profile in a homogeneous stationary medium (i.e., the shifted input profile), the solid blue lines are the numerical output profiles for four independent realizations of the process ν, the dashed blue line is the theoretical deterministic profile (4.4). In figures (a), (c), and (e), we plot the true numerical output profiles, which shows a deterministic form but a random shift (that depends on the realization of the random process ν). In figures (b), (d), and (f), we remove the shifts by aligning the maxima and we plot the resulting numerical output profiles, which shows that the centered theoretical profile (4.4) predicts the transmitted profile with accuracy.*

4.2. Self-averaging amplification of the energy. We consider the energy density

$$e(z,t) = \frac{1}{2\epsilon_o}|D(z,t)|^2 + \frac{1}{2\mu_o}|B(z,t)|^2. \tag{4.15}$$

The energy can be expressed in terms of the mode amplitudes:

$$\mathcal{E}(t) = \int_{\mathbb{R}} e(z,t)dz = \frac{1}{2\pi\epsilon_o}\int_{\mathbb{R}} |\hat{D}_o(k)|^2 \big(|\alpha(k,t)|^2 + |\beta(k,t)|^2\big)dk. \tag{4.16}$$

PROPOSITION 4.2. *When $\delta \to 0$ the energy at time t/δ^2 converges in probability to a deterministic quantity:*

$$\mathcal{E}\big(\frac{t}{\delta^2}\big) \xrightarrow{\delta\to 0} \mathcal{E}_t, \tag{4.17}$$

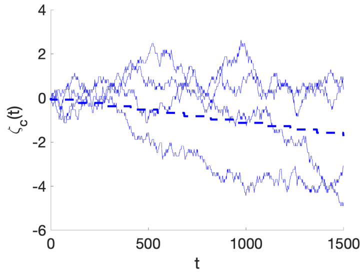

FIG. 4.2. *Evolution of the spatial center $\zeta_c(t)$ of an input sinc pulse propagating in a random time-dependent medium up to time $t = 1500$ (same setup as in Figure 4.1). The deterministic component $c_o t$ is removed from the plotted center. The solid blue lines are the numerical centers for four independent realizations of the process ν, the dashed blue line is the center of the deterministic theoretical profile (4.4). The spatial center of the pulse has two contributions that come from the random medium fluctuations: 1) a deterministic contribution given by the deterministic deformation of the pulse profile $\mathcal{D}_t(z)$ given by (4.4) which results in a shift of its center; 2) a random contribution Z_t proportional to a Brownian motion as described by (4.6).*

with

$$\mathcal{E}_t = \frac{1}{2\pi\epsilon_o}\int_{\mathbb{R}} \exp\Big(\frac{\gamma^{\rm c}(c_o k)c_o^2 k^2 t}{2}\Big)|\hat{D}_o(k)|^2 dk. \tag{4.18}$$

Note that, by Proposition 4.1 and (4.7), the energy of the coherent field

$$\mathcal{E}_{{\rm coh},t} = \int_{\mathbb{R}} e_{{\rm coh},t}(z)dz, \qquad e_{{\rm coh},t}(z) = \frac{1}{2\epsilon_o}|\mathcal{D}_t(z-Z_t)|^2 + \frac{1}{2\mu_o}|\mu_o c_o \mathcal{D}_t(z-Z_t)|^2, \tag{4.19}$$

is deterministic and given by

$$\mathcal{E}_{{\rm coh},t} = \frac{1}{2\pi\epsilon_o}\int_{\mathbb{R}} \exp\Big(\frac{\gamma^{\rm c}(c_o k)c_o^2 k^2 t}{4}\Big)|\hat{D}_o(k)|^2 dk. \tag{4.20}$$

The comparison of (4.18) and (4.20) shows that most of the energy is carried by incoherent wave fluctuations rather than the coherent wave front for large propagation times (times t such that the typical values of $\gamma^{\rm c}(c_o k)c_o^2 k^2 t$ for k in the bandwidth of $\hat{D}_o$ become larger than one).

Proof. Here the proof follows the lines of the one in [8, Chapter 7]. We consider the moments:

$$\mathbb{E}\big[\mathcal{E}(\frac{t}{\delta^2})^n\big] = \mathbb{E}\Big[\prod_{j=1}^n \frac{1}{2\pi\epsilon_o}\int_{\mathbb{R}} |\hat{D}_o(k_j)|^2\big(|\alpha(k_j,t)|^2 + |\beta(k_j,t)|^2\big)dk_j\Big],$$

where $n \in \mathbb{N}$. In fact, $n = 1$ and 2 are sufficient. We study the limits of specific moments of the form $\mathbb{E}[\prod_{j=1}^n |\alpha(k_j, t/\delta^2)|^2]$ for distinct wavenumbers k_j. The result for $n = 1$ gives the convergence for $\mathbb{E}[\mathcal{E}(t/\delta^2)]$:

$$\mathbb{E}\Big[\big|\alpha\big(k,\frac{t}{\delta^2}\big)\big|^2 + \big|\beta\big(k,\frac{t}{\delta^2}\big)\big|^2\Big] \stackrel{\delta\to 0}{\longrightarrow} \exp\Big(\frac{\gamma^{\rm c}(c_o k)c_o^2 k^2 t}{2}\Big).$$

The result for $n = 2$ gives

$$\mathbb{E}\Big[\Big(\big|\alpha\big(k,\frac{t}{\delta^2}\big)\big|^2 + \big|\beta\big(k,\frac{t}{\delta^2}\big)\big|^2\Big)^2\Big] \stackrel{\delta\to 0}{\longrightarrow} \exp\Big(2\gamma^{\rm c}(c_o k)c_o^2 k^2 t\Big),$$

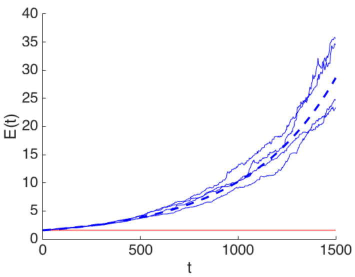

FIG. 4.3. *Evolution of the energy of an input sinc pulse propagating in a random time-dependent medium up to time* $t = 1500$ *(same setup as in Figure 4.1). The red line is the theoretical (constant) energy in a homogeneous stationary medium, the solid blue lines are the numerical energies for four independent realizations of the process* ν*, the dashed blue line is the theoretical energy (4.21), which shows that the output energies have weak relative variances and expectations that are well predicted by the theoretical formula (4.21).*

and the decorrelation of the square amplitude of α in wavenumber:

$$\mathbb{E}\Big[\big|\alpha\big(k,\frac{t}{\delta^2}\big)\big|^2\big|\alpha\big(k',\frac{t}{\delta^2}\big)\big|^2\Big] - \mathbb{E}\Big[\big|\alpha\big(k,\frac{t}{\delta^2}\big)\big|^2\Big]\mathbb{E}\Big[\big|\alpha\big(k',\frac{t}{\delta^2}\big)\big|^2\Big] \stackrel{\delta\to 0}{\longrightarrow} 0,$$

for $k \neq k'$, and similarly for β, which in turn gives the self-averaging property of the energy. $\square$

In Figure 4.3 we consider the numerical simulations carried out in the previous subsection (with the Ornstein-Uhlenbeck processes and an initial sinc profile). The theoretical energy (4.18) then reads

$$\mathcal{E}_t = \frac{1}{2\pi\epsilon_o}\int_{\mathbb{R}} \exp\Big(\frac{\sigma^2 c_o^2 k^2}{\omega_c(1+4c_o^2k^2/\omega_c^2)}t\Big)|\hat{D}_o(k)|^2 dk, \qquad \hat{D}_o(k) = \frac{\pi}{2}\mathbf{1}_{[-2,2]}(k). \quad (4.21)$$

We can see that the theoretical formula (4.21) predicts well the transmitted energy, which is indeed deterministic up to small relative fluctuations.

4.3. Incoherent wave fluctuations. In this section we study the incoherent wave fluctuations. Indeed the previous subsection shows that most of the energy is not in the transmitted coherent wave front but in incoherent wave fluctuations when the propagation time becomes large. These incoherent wave fluctuations have mean zero and we here study their two-point covariance function

$$\begin{aligned} C^\delta(z,\zeta,t) =& \frac{1}{2\epsilon_o}\mathbb{E}\big[D\big(\frac{z}{\delta^2}+\frac{\zeta}{2},\frac{t}{\delta^2}\big)D\big(\frac{z}{\delta^2}-\frac{\zeta}{2},\frac{t}{\delta^2}\big)\big] \\ &+ \frac{1}{2\mu_o}\mathbb{E}\big[B\big(\frac{z}{\delta^2}+\frac{\zeta}{2},\frac{t}{\delta^2}\big)B\big(\frac{z}{\delta^2}-\frac{\zeta}{2},\frac{t}{\delta^2}\big)\big], \end{aligned} \quad (4.22)$$

where z/δ^2 is the mid-point and ζ is the offset. It can be expressed as

$$\begin{aligned} C^\delta(z,\zeta,t) =& \frac{\delta^2}{(2\pi)^2\epsilon_o}\iint_{\mathbb{R}^2} \hat{D}_o\big(k+\frac{h}{2}\delta^2\big)\overline{\hat{D}_o}\big(k-\frac{h}{2}\delta^2\big) \\ &\times \big[W^\delta_{\alpha\alpha}(h,k,t) + W^\delta_{\beta\beta}(h,k,t)\big]e^{ihz+ik\zeta}dkdh, \end{aligned} \quad (4.23)$$

where we have introduced

$$W^\delta_{\alpha\alpha}(h,k,t) = \mathbb{E}\big[\alpha\big(k+\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\overline{\alpha}\big(k-\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\big]e^{-ihc_ot}, \tag{4.24}$$

$$W^\delta_{\beta\beta}(h,k,t) = \mathbb{E}\big[\beta\big(k+\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\overline{\beta}\big(k-\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\big]e^{ihc_ot}. \tag{4.25}$$

Note that $C^\delta(z,0,t) = \mathbb{E}[e(z/\delta^2,t/\delta^2)]$ is the mean energy density and the Fourier transform in ζ of $C^\delta(z,\zeta,t)$ is the Wigner transform of the field.

Proposition 4.3. *The rescaled covariance function $\delta^{-2}C^\delta(z,\zeta,t)$ converges to*

$$\mathcal{C}(z,\zeta,t) = \frac{1}{(2\pi)^2\epsilon_o}\iint_{\mathbb{R}^2}|\hat{D}_o(k)|^2\mathcal{W}(h,k,t)e^{ihz+ik\zeta}dkdh, \tag{4.26}$$

where

$$\mathcal{W}(h,k,t) = \exp\Big(\frac{\gamma^c(c_ok)c_o^2k^2}{4}t\Big)\Big\{\cosh\Big(\sqrt{\frac{\gamma^c(c_ok)^2c_o^4k^4}{16}-h^2c_o^2}t\Big) + \frac{\frac{\gamma^c(c_ok)c_o^2k^2}{4}-ihc_o}{\sqrt{\frac{\gamma^c(c_ok)^2c_o^4k^4}{16}-h^2c_o^2}}\sinh\Big(\sqrt{\frac{\gamma^c(c_ok)^2c_o^4k^4}{16}-h^2c_o^2}t\Big)\Big\}. \tag{4.27}$$

The convergence is in the weak sense in z, that is to say, for any ζ and for any test function φ (such that $\hat{\varphi}\in L^1$), $\int_{\mathbb{R}}\delta^{-2}C^\delta(z,\zeta,t)\varphi(z)dz \to \int_{\mathbb{R}}C(z,\zeta,t)\varphi(z)dz$. We can check, however, that $\mathcal{E}_t = \int_{\mathbb{R}}\mathcal{C}(z,0,t)dz$.

This proposition shows that the incoherent wave fluctuations at time t/δ^2 is of typical amplitude δ and of spatial support of the order of δ^{-2}, but they grow faster than the coherent wave front. The Wigner transform of the field at time t/δ^2 and at position z/δ^2 in the regime $\delta\to 0$ is

$$\int C^\delta(z,\zeta,t)e^{-ik\zeta}d\zeta \simeq \frac{\delta^2}{2\pi\epsilon_o}|\hat{D}_o(k)|^2\int_{\mathbb{R}}\mathcal{W}(h,k,t)e^{ihz}dh. \tag{4.28}$$

For large propagation times (times t such that the typical values of $\gamma^c(c_ok)c_o^2k^2t$ for k in the bandwidth of $\hat{D}_o$ become larger than one) it is of the form

$$\int C^\delta(z,\zeta,t)e^{-ik\zeta}dk \simeq \frac{\delta^2}{2\pi\epsilon_o}|\hat{D}_o(k)|^2\sqrt{\frac{\pi\gamma^c(c_ok)k^2}{2t}}\exp\Big(\frac{\gamma^c(c_ok)k^2}{2}\big(c_o^2t-\frac{z^2}{4t}\big)\Big), \tag{4.29}$$

which is maximal at $z=0$. This means that, at large times, the most important incoherent wave fluctuations are around the position of the wave field at the onset time of the random forcing. This is somewhat similar to the observation at the end of section 3.1 for periodic media and this can be observed in the results of the numerical simulations carried out in the previous subsections (with the Ornstein-Uhlenbeck processes and an initial sinc profile) and shown in Figure 4.4.

Proof. We introduce

$$w^\delta_{\alpha\alpha}(h,k,t) = \alpha\big(k+\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\overline{\alpha}\big(k-\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)e^{-ihc_ot},$$

$$w^\delta_{\alpha\beta}(h,k,t) = \alpha\big(k+\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\overline{\beta}\big(k-\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big),$$

$$w^\delta_{\beta\alpha}(h,k,t) = \beta\big(k+\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\overline{\alpha}\big(k-\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big),$$

$$w^\delta_{\beta\beta}(h,k,t) = \beta\big(k+\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)\overline{\beta}\big(k-\frac{h}{2}\delta^2,\frac{t}{\delta^2}\big)e^{ihc_ot}.$$

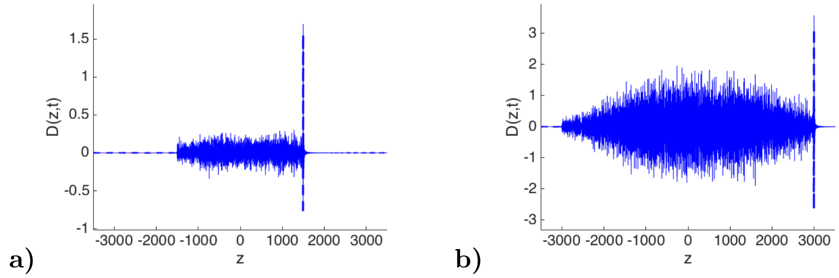

FIG. 4.4. *Wave field generated by an input sinc pulse propagating in a random time-dependent medium at time $t = 1500$ (a) and $t = 3000$ (b) (same setup as in Figure 4.1). One can observe the coherent wave front around $z = 1500$ (a) and $z = 3000$ (b), and the incoherent wave fluctuations supported in $z \in (-1500, 1500)$ (a) and $z \in (-3000, 3000)$ (b), with a maximum around $z = 0$ (b) (when $t = 3000$, more generally, when t is large enough).*

The process $(w^\delta_{\alpha\alpha}, w^\delta_{\alpha\beta}, w^\delta_{\beta\alpha}, w^\delta_{\beta\beta})$ satisfies

$$\begin{aligned}
\partial_t w^\delta_{\alpha\alpha} &= \frac{ic_o k}{2\delta}\nu\big(\frac{t}{\delta^2}\big)\big[-w^\delta_{\beta\alpha}e^{2ic_o k\frac{t}{\delta^2}} + w^\delta_{\alpha\beta}e^{-2ic_o k\frac{t}{\delta^2}}\big] - ihc_o w^\delta_{\alpha\alpha},\\
\partial_t w^\delta_{\alpha\beta} &= \frac{ic_o k}{2\delta}\nu\big(\frac{t}{\delta^2}\big)\big[-2w^\delta_{\alpha\beta} - w^\delta_{\alpha\alpha}e^{2ic_o k\frac{t}{\delta^2}} - w^\delta_{\beta\beta}e^{2ic_o k\frac{t}{\delta^2}}\big],\\
\partial_t w^\delta_{\beta\alpha} &= \frac{ic_o k}{2\delta}\nu\big(\frac{t}{\delta^2}\big)\big[2w^\delta_{\beta\alpha} + w^\delta_{\alpha\alpha}e^{-2ic_o k\frac{t}{\delta^2}} + w^\delta_{\beta\beta}e^{-2ic_o k\frac{t}{\delta^2}}\big],\\
\partial_t w^\delta_{\beta\beta} &= \frac{ic_o k}{2\delta}\nu\big(\frac{t}{\delta^2}\big)\big[w^\delta_{\alpha\beta}e^{-2ic_o k\frac{t}{\delta^2}} - w^\delta_{\beta\alpha}e^{2ic_o k\frac{t}{\delta^2}}\big] + ihc_o w^\delta_{\beta\beta}.
\end{aligned}$$

Using the diffusion-approximation theorem [8, Theorem 6.5], we find that, as $\delta \to 0$, $W^\delta_{\alpha\alpha} = \mathbb{E}[w^\delta_{\alpha\alpha}] \to \mathcal{W}_{\alpha\alpha}$, $W^\delta_{\beta\beta} = \mathbb{E}[w^\delta_{\beta\beta}] \to \mathcal{W}_{\beta\beta}$, where $(\mathcal{W}_{\alpha\alpha}, \mathcal{W}_{\beta\beta})$ satisfies

$$\begin{aligned}
\partial_t \mathcal{W}_{\alpha\alpha} &= \frac{\gamma^c(c_o k)c_o^2 k^2}{4}\big(\mathcal{W}_{\alpha\alpha} + \mathcal{W}_{\beta\beta}\big) - ihc_o\mathcal{W}_{\alpha\alpha},\\
\partial_t \mathcal{W}_{\beta\beta} &= \frac{\gamma^c(c_o k)c_o^2 k^2}{4}\big(\mathcal{W}_{\alpha\alpha} + \mathcal{W}_{\beta\beta}\big) + ihc_o\mathcal{W}_{\beta\beta}.
\end{aligned}$$

The resolution of this system of ordinary differential equations gives the convergence of $W^\delta_{\alpha\alpha}(h,k,t) + W^\delta_{\beta\beta}(h,k,t)$ towards $|\hat{D}_o(k)|^2\mathcal{W}(h,k,t)$. The uniform bound with respect to k in the support of $\hat{D}_o$ finally gives the desired result. $\square$

5. Three-dimensional wave propagation. In this section we consider the three-dimensional scalar wave equation with time-dependent and spatially homogeneous speed of propagation:

$$c(t) = \frac{c_o}{\sqrt{1 + \delta\nu(t)\mathbf{1}_{[0,\infty)}(t)}}. \tag{5.1}$$

The scalar field $E(\boldsymbol{x}, t)$ satisfies

$$\Delta E - c^{-2}(t)\partial_t^2 E = -c_o^{-2}\delta(t - t_o)\psi(|\boldsymbol{x}|), \tag{5.2}$$

with $E(\boldsymbol{x}, t) = 0$ $\forall t < t_o$, where $t_o \in (-\infty, 0)$. This means that the wave is generated at time t_o by a radially symmetric source centered at $\mathbf{0}$. We will see that the situation is very similar to the one-dimensional case addressed in the previous sections, because the wave field conserves its spatial radially symmetric form and the problem can be reduced to a one-dimensional wave propagation problem.

In this section, we assume that the function ψ is smooth and decays fast, we introduce the radially symmetric three-dimensional Fourier transform:

$$\check{\psi}(k) = 4\pi \int_0^\infty \mathrm{sinc}(kx)\psi(x)x^2 dx, \tag{5.3}$$

whose inverse is

$$\psi(x) = \frac{1}{2\pi^2} \int_0^\infty \mathrm{sinc}(kx)\check{\psi}(k)k^2 dk. \tag{5.4}$$

5.1. Evolution of the radially symmetric field. The Fourier transform of the scalar field:

$$\hat{E}(\boldsymbol{k}, t) = \int_{\mathbb{R}^3} E(\boldsymbol{x}, t) \exp(-i\boldsymbol{k} \cdot \boldsymbol{x}) d\boldsymbol{x} \tag{5.5}$$

satisfies for $t \in (t_o, 0)$:

$$c_o^2 |\boldsymbol{k}|^2 \hat{E} + \partial_t^2 \hat{E} = 0, \tag{5.6}$$

starting from $\hat{E}(\boldsymbol{k}, t = t_o) = 0$ and $\partial_t \hat{E}(\boldsymbol{k}, t = t_o) = \check{\psi}(|\boldsymbol{k}|)$, and for $t > 0$:

$$c_o^2 |\boldsymbol{k}|^2 \big[1 + \delta\nu(t)\big] \hat{E} + \partial_t^2 \hat{E} = 0, \tag{5.7}$$

with the continuity of $\hat{E}$ and $\partial_t \hat{E}$ at $t = 0$.

We first look at the form of the wave for $t \in (t_o, 0)$, that is to say, before the onset of the time-variations of the medium.

PROPOSITION 5.1. *For any* $t \in (t_o, 0)$*:*

$$E(\boldsymbol{x}, t) = \frac{1}{4\pi^2 c_o |\boldsymbol{x}|} \int_0^\infty \check{\psi}(k) \big[\cos\big(k(c_o(t - t_o) - |\boldsymbol{x}|)\big) - \cos\big(k(c_o(t - t_o) + |\boldsymbol{x}|)\big)\big] dk. \tag{5.8}$$

As soon as $c_o(t - t_o)$ is larger than the initial support of the wave, the first term is dominant and we get that the field is a divergent (outgoing) spherical wave:

$$E(\boldsymbol{x}, t) = \frac{1}{4\pi^2 c_o |\boldsymbol{x}|} \int_0^\infty \check{\psi}(k) \cos(k(|\boldsymbol{x}| - c_o(t - t_o))) dk, \tag{5.9}$$

that is supported in the neighborhood of the surface of the sphere with center at $\mathbf{0}$ and radius $c_o(t - t_o)$. As an example, if $\psi(x) = \exp(-x^2/r_o^2)$, then

$$\begin{aligned} E(\boldsymbol{x}, t) &= \frac{r_o^2}{4c_o|\boldsymbol{x}|} \Big[\exp\Big(-\frac{(|\boldsymbol{x}| - c_o(t - t_o))^2}{r_o^2}\Big) - \exp\Big(-\frac{(|\boldsymbol{x}| + c_o(t - t_o))^2}{r_o^2}\Big)\Big] \\ &\simeq \frac{r_o^2}{4c_o|\boldsymbol{x}|} \exp\Big(-\frac{(|\boldsymbol{x}| - c_o(t - t_o))^2}{r_o^2}\Big) \text{ if } c_o(t - t_o) \gg r_o. \end{aligned} \tag{5.10}$$

Proof. We introduce

$$a(\boldsymbol{k}, t) = \frac{1}{2}\Big(\hat{E}(\boldsymbol{k}, t) + \frac{i}{c_o|\boldsymbol{k}|}\partial_t \hat{E}(\boldsymbol{k}, t)\Big) \exp\big(ic_o|\boldsymbol{k}|t\big), \tag{5.11}$$

$$b(\boldsymbol{k}, t) = \frac{1}{2}\Big(\hat{E}(\boldsymbol{k}, t) - \frac{i}{c_o|\boldsymbol{k}|}\partial_t \hat{E}(\boldsymbol{k}, t)\Big) \exp\big(-ic_o|\boldsymbol{k}|t\big). \tag{5.12}$$

We have, for any $t \in (t_o, 0)$, $\partial_t a = \partial_t b = 0$ and

$$a(\boldsymbol{k}, t = t_o) = \frac{i}{2c_o|\boldsymbol{k}|}\check{\psi}(|\boldsymbol{k}|)\exp\big(ic_o|\boldsymbol{k}|t_o\big), \tag{5.13}$$

$$b(\boldsymbol{k}, t = t_o) = -\frac{i}{2c_o|\boldsymbol{k}|}\check{\psi}(|\boldsymbol{k}|)\exp\big(-ic_o|\boldsymbol{k}|t_o\big). \tag{5.14}$$

Thus,

$$\begin{aligned}\hat{E}(\boldsymbol{k}, t) &= a(\boldsymbol{k}, t)\exp\big(-ic_o|\boldsymbol{k}|t\big) + b(\boldsymbol{k}, t)\exp\big(ic_o|\boldsymbol{k}|t\big)\\ &= \frac{i}{2c_o|\boldsymbol{k}|}\check{\psi}(|\boldsymbol{k}|)\big[\exp(-ic_o|\boldsymbol{k}|(t-t_o)) - \exp(ic_o|\boldsymbol{k}|(t-t_o))\big].\end{aligned}$$

Using the inverse Fourier transform

$$E(\boldsymbol{x}, t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\hat{E}(\boldsymbol{k}, t)d\boldsymbol{k},$$

the identity $\int_{\partial B(\boldsymbol{0},1)}\exp(ik\hat{\boldsymbol{k}}\cdot\boldsymbol{x})d\hat{\boldsymbol{k}} = 4\pi\mathrm{sinc}(k|\boldsymbol{x}|) = -2\pi i(e^{ik|\boldsymbol{x}|} - e^{-ik|\boldsymbol{x}|})/(k|\boldsymbol{x}|)$, and the fact that $\check{\psi}$ is real-valued, this gives

$$\begin{aligned}E(\boldsymbol{x}, t) &= \frac{1}{8\pi^2 c_o|\boldsymbol{x}|}\int_0^\infty \check{\psi}(k)\big[e^{ik(c_o(t_o-t)+|\boldsymbol{x}|)} - e^{ik(c_o(t-t_o)+|\boldsymbol{x}|)}\big]dk + c.c.\\ &= \frac{1}{2\pi^2 c_o|\boldsymbol{x}|}\int_0^\infty \check{\psi}(k)\sin(k(c_o(t-t_o))\sin(k|\boldsymbol{x}|)dk\\ &= \frac{1}{4\pi^2 c_o|\boldsymbol{x}|}\int_0^\infty \check{\psi}(k)\big[\cos\big(k(c_o(t-t_o) - |\boldsymbol{x}|)\big) - \cos\big(k(c_o(t-t_o)+|\boldsymbol{x}|)\big)\big]dk,\end{aligned}$$

which is the desired result. □

The following proposition shows that the three-dimensional radially symmetric problem has the same structure as the one-dimensional problem.

PROPOSITION 5.2. *For any* $t \geq 0$*:*

$$\begin{aligned}E(\boldsymbol{x}, t) = \frac{1}{8\pi^2 c_o|\boldsymbol{x}|}\int_0^\infty \check{\psi}(k)\big[&\alpha(k,t)e^{ik(c_o(t_o-t)+|\boldsymbol{x}|)} - \overline{\alpha(k,t)}e^{ik(c_o(t-t_o)+|\boldsymbol{x}|)}\\ &+\beta(k,t)e^{ik(c_o(t_o+t)+|\boldsymbol{x}|)} - \overline{\beta(k,t)}e^{ik(c_o(-t-t_o)+|\boldsymbol{x}|)}\big]dk + c.c.,\end{aligned} \tag{5.15}$$

where (α, β) *is solution of (2.14-2.15) with the initial conditions* $\alpha(k, t = 0) = 1$ *and* $\beta(k, t = 0) = 0$.

Proof. We introduce $(a(\boldsymbol{k}, t), b(\boldsymbol{k}, t))$ as in (5.11-5.12). They satisfy for $t > 0$:

$$\partial_t a = -\frac{ic_o|\boldsymbol{k}|\delta}{2}\nu(t)\big(a + b\exp(2ic_o|\boldsymbol{k}|t)\big), \tag{5.16}$$

$$\partial_t b = \frac{ic_o|\boldsymbol{k}|\delta}{2}\nu(t)\big(a\exp(-2ic_o|\boldsymbol{k}|t) + b\big), \tag{5.17}$$

i.e., the same equation as (2.11-2.12), with the initial conditions (5.13-5.14) at $t = 0$. Using the linearity and the symmetry of the equations (if (α, β) is a solution of (2.14-2.15), then $(\overline{\beta}, \overline{\alpha})$ is also a solution), we have

$$a(\boldsymbol{k}, t) = \frac{i}{2c_o|\boldsymbol{k}|}\check{\psi}(|\boldsymbol{k}|)\big[\alpha(|\boldsymbol{k}|, t)\exp(ic_o|\boldsymbol{k}|t_o) - \overline{\beta(|\boldsymbol{k}|, t)}\exp(-ic_o|\boldsymbol{k}|t_o)\big], \tag{5.18}$$

$$b(\boldsymbol{k}, t) = \frac{i}{2c_o|\boldsymbol{k}|}\check{\psi}(|\boldsymbol{k}|)\big[\beta(|\boldsymbol{k}|, t)\exp(ic_o|\boldsymbol{k}|t_o) - \overline{\alpha(|\boldsymbol{k}|, t)}\exp(-ic_o|\boldsymbol{k}|t_o)\big], \tag{5.19}$$

where $(\alpha(k,t),\beta(k,t))$ is solution of (2.14-2.15). For any $t\geq 0$, we have

$$E(\boldsymbol{x},t)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\big[a(\boldsymbol{k},t)\exp(-ic_o|\boldsymbol{k}|t)+b(\boldsymbol{k},t)\exp(ic_o|\boldsymbol{k}|t)\big]d\boldsymbol{k},$$

which gives the desired result. □

5.2. Periodic time variations with narrowband source. Here we address small-amplitude time-periodic variations of the medium of the form (3.1) and long propagation times. We consider that
1) the source emission time is t_o/δ instead of t_o (for $t_o<0$) and we look for times of the form t/δ (for $t>0$),
2) the source is narrowband and of the form

$$\check{\psi}(k)=\frac{1}{\delta}\check{\psi}_o\Big(\frac{k-k_c}{\delta}\Big), \tag{5.20}$$

where $\check{\psi}_o$ is even. For example, we may think at $\psi_o(x)=\exp(-x^2/r_o^2)$ and $\check{\psi}_o(k)=r_o\pi^{1/2}\exp(-k^2r_o^2/4)$. We then have (to leading order in δ):

$$\psi(x)=\frac{k_c^2}{\pi}\operatorname{sinc}(k_cx)\psi_o(\delta x),\qquad \psi_o(x)=\frac{1}{\pi}\int_0^\infty\cos(kx)\check{\psi}_o(k)dk, \tag{5.21}$$

in other words, ψ is essentially a sinc function.

By Proposition 5.2, the field for $t>0$ is

$$\begin{aligned}E\big(\boldsymbol{x},\frac{t}{\delta}\big)=&\frac{1}{8\pi^2c_o|\boldsymbol{x}|}\int_{\mathbb{R}}\check{\psi}_o(k)\big[\alpha^\delta(k,t)e^{i(k_c+\delta k)(c_o\frac{(t_o-t)}{\delta}+|\boldsymbol{x}|)}\\&-\overline{\alpha^\delta(k,t)}e^{i(k_c+\delta k)(c_o\frac{(t-t_o)}{\delta}+|\boldsymbol{x}|)}+\beta^\delta(k,t)e^{i(k_c+\delta k)(c_o\frac{(t_o+t)}{\delta}+|\boldsymbol{x}|)}\\&-\overline{\beta^\delta(k,t)}e^{i(k_c+\delta k)(c_o\frac{(-t_o-t)}{\delta}+|\boldsymbol{x}|)}\big]dk+c.c.,\end{aligned} \tag{5.22}$$

where $\alpha^\delta(k,t)=\alpha(k_c+\delta k,t/\delta)$ and $\beta^\delta(k,t)=\beta(k_c+\delta k,t/\delta)$.

Using Lemma 3.1, we get that the field is, for $t>0$,

$$\begin{aligned}E\big(\frac{\boldsymbol{x}}{\delta},\frac{t}{\delta}\big)=&\frac{\delta}{8\pi^2c_o|\boldsymbol{x}|}e^{i\frac{k_c}{\delta}(c_o(t_o-t)+|\boldsymbol{x}|)}\int_{\mathbb{R}}\check{\psi}_o(k)\mathcal{A}(k,t)e^{ik(c_ot_o+|\boldsymbol{x}|)}dk\\&-\frac{\delta}{8\pi^2c_o|\boldsymbol{x}|}e^{i\frac{k_c}{\delta}(c_o(t-t_o)+|\boldsymbol{x}|)}\int_{\mathbb{R}}\check{\psi}_o(k)\mathcal{A}(k,t)e^{ik(-c_ot_o+|\boldsymbol{x}|)}dk\\&+\frac{\delta}{8\pi^2c_o|\boldsymbol{x}|}e^{i\frac{k_c}{\delta}(c_o(t_o+t)+|\boldsymbol{x}|)}\int_{\mathbb{R}}\check{\psi}_o(k)\mathcal{B}(k,t)e^{ik(c_ot_o+|\boldsymbol{x}|)}dk\\&-\frac{\delta}{8\pi^2c_o|\boldsymbol{x}|}e^{i\frac{k_c}{\delta}(c_o(-t_o-t)+|\boldsymbol{x}|)}\int_{\mathbb{R}}\check{\psi}_o(k)\mathcal{B}(k,t)e^{ik(-c_ot_o+|\boldsymbol{x}|)}dk+c.c.,\end{aligned} \tag{5.23}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined by (3.5) and (3.6). If c_ot_o is larger than the source radius, then this can be simplfied:

$$\begin{aligned}E\big(\frac{\boldsymbol{x}}{\delta},\frac{t}{\delta}\big)=&\frac{\delta}{8\pi^2c_o|\boldsymbol{x}|}e^{i\frac{k_c}{\delta}(c_o(t_o-t)+|\boldsymbol{x}|)}\int_{\mathbb{R}}\check{\psi}_o(k)\mathcal{A}(k,t)e^{ik(c_ot_o+|\boldsymbol{x}|)}dk\\&+\frac{\delta}{8\pi^2c_o|\boldsymbol{x}|}e^{i\frac{k_c}{\delta}(c_o(t_o+t)+|\boldsymbol{x}|)}\int_{\mathbb{R}}\check{\psi}_o(k)\mathcal{B}(k,t)e^{ik(c_ot_o+|\boldsymbol{x}|)}dk+c.c..\end{aligned} \tag{5.24}$$

We can observe an oscillatory, amplifying, but non-propagating, spherical wave whose support is localized around the sphere with center at $\mathbf{0}$ and radius $-c_o t_o$ (the position of the field at the time when the periodic variations start). This is a straightforward extension of the one-dimensional result described in Section 3.1.

Finally, we remark that we can revisit Section 3.2 for rapid periodic time variations with broadband source and Section 4 for random time variations similarly.

6. Conclusions. We can now summarize our main findings.

When the medium has periodic time-dependent variations, the wave field experiences wavenumber-dependent amplification in a narrow (spatial) bandwidth. If the spatial spectrum of the initial field does not intersect this bandwidth, then the wave field experiences dispersion for long propagation times.

When the medium has random stationary time-dependent variations, the transmitted wave front, i.e. the transmitted field around its expected arrival point, is randomly shifted and experiences a deterministic deformation that contains both amplification and dispersion. We have explained that the mean transmitted wave front may be misleading in the sense that a typical realization of the transmitted wave front does not at all look like the mean transmitted wave front, because of the averaging of the random shift with Gaussian distribution that produces diffusion. We have also shown that, for large propagation times, the incoherent wave fluctuations beyond the transmitted wave front become dominant: they are small at small propagation times but their amplification growth rates are larger than the growth rate of the coherent wave front.

To be complete, we mention that the analysis of wave propagation in spatially heterogeneous and time independent media (carried out in [8]) leads to results that are fundamentally different from the ones obtained in this paper when the medium is spatially homogeneous and time dependent. Indeed, although one can formally exchange the time and space variables in the one-dimensional wave equation to go from one situation to the other one, the initial and boundary conditions are different and they play a fundamental role in the analysis. As an illustration of the fundamental difference, there cannot be any amplification when the medium is spatially heterogeneous and time independent, because of energy conservation.

Appendix A. An averaging theorem for highly oscillatory differential equations. The following proposition is obtained from [16].

PROPOSITION A.1. *Let N, Q be positive integers and $t_0 < t_1$. Let us consider the solution $\boldsymbol{X}^\delta = (X_j^\delta)_{j=1,\dots,N}$ in $\mathcal{C}([t_0,t_1],\mathbb{R}^N)$ of the initial value problem*

$$\frac{d\boldsymbol{X}^\delta}{dt} = \frac{1}{\delta}\sum_{q=1}^{Q}\left(\boldsymbol{F}^{(q)}(\boldsymbol{X}^\delta)\cos\Big(\frac{\omega^{(q)}t}{\delta^2}\Big) + \boldsymbol{G}^{(q)}(\boldsymbol{X}^\delta)\sin\Big(\frac{\omega^{(q)}t}{\delta^2}\Big)\right), \qquad \text{(A.1)}$$

starting from $\boldsymbol{X}^\delta(t_0) = \boldsymbol{X}_{\rm ini}$. Here $\boldsymbol{F}^{(q)} : \mathbb{R}^N \to \mathbb{R}^N$ and $\boldsymbol{G}^{(q)} : \mathbb{R}^N \to \mathbb{R}^N$ are smooth functions with bounded derivatives and the $\omega^{(q)} \in (0,\infty)$ are distinct. Then $\boldsymbol{X}^\delta$ converges as $\delta \to 0$ to $\boldsymbol{X}$ the solution of the effective equation

$$\frac{d\boldsymbol{X}}{dt} = -\sum_{q=1}^{Q}\frac{1}{2\omega^{(q)}}\big[\boldsymbol{F}^{(q)}(\boldsymbol{X}),\boldsymbol{G}^{(q)}(\boldsymbol{X})\big], \qquad \boldsymbol{X}(t_0) = \boldsymbol{X}_{\rm ini}, \qquad \text{(A.2)}$$

where $[\cdot,\cdot]$ *stands for the Lie brackets defined by*

$$\big[\boldsymbol{F}(\boldsymbol{X}),\boldsymbol{G}(\boldsymbol{X})\big]=\sum_{j=1}^{N}\Big(\frac{\partial\boldsymbol{F}(\boldsymbol{X})}{\partial X_j}G_j(\mathbf{X})-\frac{\partial\boldsymbol{G}(\boldsymbol{X})}{\partial X_j}F_j(\boldsymbol{X})\Big)\,. \tag{A.3}$$

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