A quick introduction to Random Matrix Theory

• Different types of results:
- description of the global distribution of the singular values,
- detailed description of the maximal singular value.

• Different types of proofs:
- asymptotic expansions of explicit expressions (special polynomials),
- complex analysis (Stietjes transform of measures).

• The distribution of the singular values depends:
- on the structure of the matrix,
- on the correlation between the random coefficients of the matrix,
- not much on the marginal distribution of the coefficients (in the limit of large matrices).

• Remarkable points: universal results, corresponding to a kind of "central limit theorem", but with unusual scaling and limit distribution.

Realistic models of random matrices

- Gaussian Orthogonal Ensemble (GOE) $\beta = 1$

Real symmetric matrices $\mathbf{W}$ such that:
- the coefficients $(W_{i,j})_{1 \leq i \leq j \leq n}$ are independent.
- $W_{ii} \sim \mathcal{N}(0, 2\sigma^2)$ and $W_{ij} \sim \mathcal{N}(0, \sigma^2)$.

- Gaussian Unitary Ensemble (GUE) $\beta = 2$

Hermitian matrices $\mathbf{W}$ such that:
- the coefficients $(\mathrm{Re}(W_{i,j}))_{1 \leq i \leq j \leq n}$, $(\mathrm{Im}(W_{i,j}))_{1 \leq i \leq j \leq n}$ are independent.
- $W_{ii} \sim \mathcal{N}(0, \sigma^2)$, $\mathrm{Re}(W_{ij}) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$ and $\mathrm{Im}(W_{ij}) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$.

- The joint pdf of the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of $\frac{1}{\sqrt{n}}\mathbf{W}$ is:

$$Q_\beta^{(n)}(\lambda_1, ..., \lambda_n) = \frac{1}{Z_\beta^{(n)}} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta n}{4\sigma^2} \sum_{i=1}^n \lambda_i^2 \right)$$

Obviously: the eigenvalues are not independent; there is level repulsion.

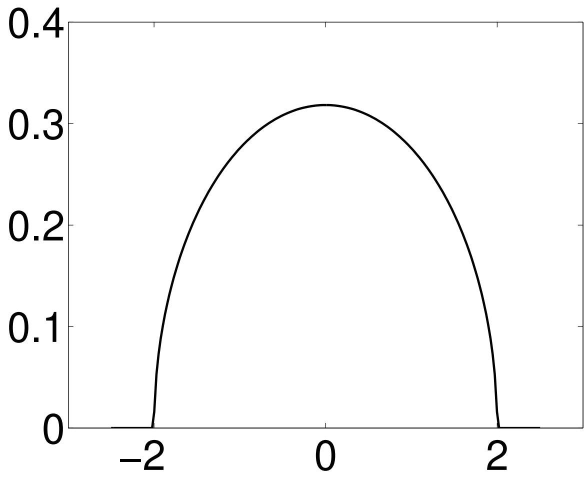
Semi-circle law (for the GOE and the GUE)

The density of states converges almost surely as $n \to \infty$ to a deterministic continuous measure

$$\frac{1}{n}\text{Card}(i = 1, ..., n, \lambda_i \in [a, b]) \stackrel{n\to\infty}{\longrightarrow} \int_a^b \rho^{(\sigma)}(x)dx$$

$$\rho^{(\sigma)}(\lambda) = \frac{1}{\sigma}\rho_{sc}\left(\frac{\lambda}{\sigma}\right), \qquad \rho_{sc}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\mathbf{1}_{[-2,2]}(\lambda)$$

$\rho^{(\sigma)}(\lambda)d\lambda$ gives the proportion of eigenvalues in the interval $[\lambda, \lambda + d\lambda]$.

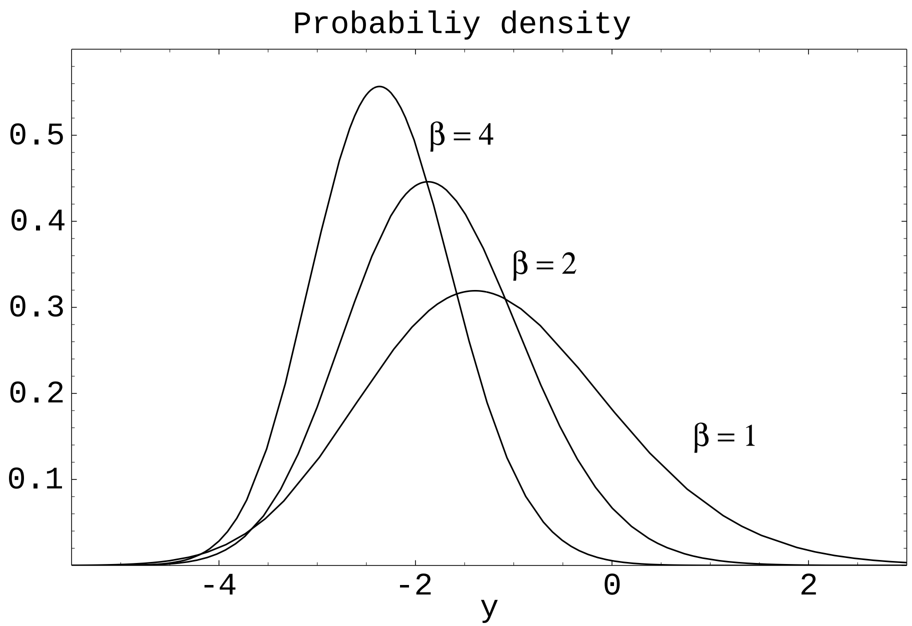


$\rho_{sc}(\lambda)$

The maximal eigenvalue (for the GOE and GUE)

$$n^{2/3}\left(\lambda_{max}^{(n)} - 2\sigma\right) \overset{n\to+\infty}{\longrightarrow} Y$$

in distribution, where Y has a Tracy-Widom distribution of type β.



Tracy-Widom distribution.

Wigner theorem

• A Wigner matrix $\mathbf{W}$ is a $n \times n$ Hermitian random matrix such that:
- the off-diagonal coefficients $(W_{ij})_{1 \leq i < j \leq n}$ are i.i.d. random variables with mean zero and variance σ^2, i.e. $\mathbb{E}[|W_{ij}|^2] = \sigma^2$.
- the diagonal coefficients $(W_{ii})_{1 \leq i \leq n}$ are i.i.d. random variables.

• Theorem: $\mu^{(n)} \overset{n\to\infty}{\longrightarrow} \rho^{(\sigma)}$, i.e. for all a, b,

$$\frac{1}{n}\mathrm{Card}\,(i = 1, ..., n, \lambda_i \in [a, b]) \overset{n\to\infty}{\longrightarrow} \int_a^b \rho^{(\sigma)}(x)dx \qquad \text{a.s.}$$

where - the λ_i are the eigenvalues of $\frac{\mathbf{W}}{\sqrt{n}}$ for $i = 1, ..., n$.
- $\rho^{(\sigma)}$ is the probability measure over $\mathbb{R}$ with density

$$\rho^{(\sigma)}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathbf{1}_{|x| \leq 2\sigma}$$

$\hookrightarrow$ The semi-circle law is universal.

Wishart matrix

- Let $\mathbf{A}$ be a $m \times n$ real random matrix whose coefficients are i.i.d. with mean zero and finite variance.Let

$$\mathbf{W} = \frac{1}{m}\mathbf{A}^T\mathbf{A} = \left(\frac{1}{m}\sum_{k=1}^{m} A_{ki}A_{kj}\right)_{i,j=1,\ldots,n}$$

Assume $m(n)/n \to c > 0$ as $n \to \infty$.

- The density of states of $\mathbf{W}$ converges to the measure

$$\rho(\lambda) = (1-c)^+\delta(\lambda) + \frac{1}{2\pi\sigma^2\lambda}\sqrt{[(\lambda-\lambda_-)(\lambda_+-\lambda)]^+}$$

with $[x]^+ = \max(x,0)$, $\lambda_\pm = (1+c)\sigma^2 \pm 2\sigma^2\sqrt{c}$.

Note: the eigenvalues of $\mathbf{W}$ are the squares of the singular values of $\mathbf{A}$.

- Case $m = n$:

$$\rho(\lambda) = \frac{1}{2\pi\sigma^2\sqrt{\lambda}}\sqrt{4\sigma^2-\lambda}\mathbf{1}_{[0,4\sigma^2]}(\lambda)$$

The density of states of the singular values of $\mathbf{A}$ converges to the quarter-circle law.

Semi-circle law - complex analysis approach

Let $\mathbf{W}^{(n)}$ belongs to the GOE. Denote by $\mu^{(n)}$ the density of states of $\mathbf{W}^{(n)}$.

- Result: $\mu^{(n)}$ converges to μ whose Stietjes transform

$$f(z) = \int_{\mathbb{R}} \frac{\mu(d\lambda)}{\lambda - z}, \qquad \mathrm{Im}(z) > 0$$

satisfies

$$\sigma^2 f(z)^2 + z f(z) + 1 = 0$$

Proof: $f^{(n)}(z) = \int_{\mathbb{R}} \frac{\mu^{(n)}(d\lambda)}{\lambda - z} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_j - z} = \frac{1}{n} \mathrm{Tr}\big[(M - z)^{-1}\big]$ + analysis of the resolvent matrix $(z - M)^{-1}$.

- The measure μ can be recovered by the Perron-Frobenius transform

$$\mu([a,b]) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{[a,b]} \mathrm{Im}(f(\lambda + i\epsilon)) d\lambda$$

We find $\mu(\lambda) = \frac{1}{\pi\sigma^2} \sqrt{4\sigma^2 - \lambda^2} \mathbf{1}_{[-2\sigma, 2\sigma]}(\lambda)$.

Deformed semi-circle law

$$\mathbf{A}^{(n)} = \mathbf{D}^{(n)} + \mathbf{W}^{(n)}$$

where $\mathbf{W}^{(n)}$ belongs to the GOE, $\mathbf{D}^{(n)}$ is a sequence of deterministic matrices whose asymptotic density of states $\mu^{(d)}$ (as $n \to \infty$) in known. Introduce $f^{(d)}$ its Stietjes transform:

$$f^{(d)}(z) = \int_{\mathbb{R}} \frac{\mu^{(d)}(d\lambda)}{\lambda - z}, \qquad \mathrm{Im}(z) > 0$$

Denote by $\mu^{(n)}$ the density of states of $\mathbf{A}^{(n)}$.

- Result: $\mu^{(n)}$ converges to μ whose Stietjes transform

$$f(z) = \int_{\mathbb{R}} \frac{\mu(d\lambda)}{\lambda - z}$$

satisfies

$$f(z) = f^{(d)}\big(z + \sigma^2 f(z)\big)$$

- The measure μ can be recovered by the Perron-Frobenius transform

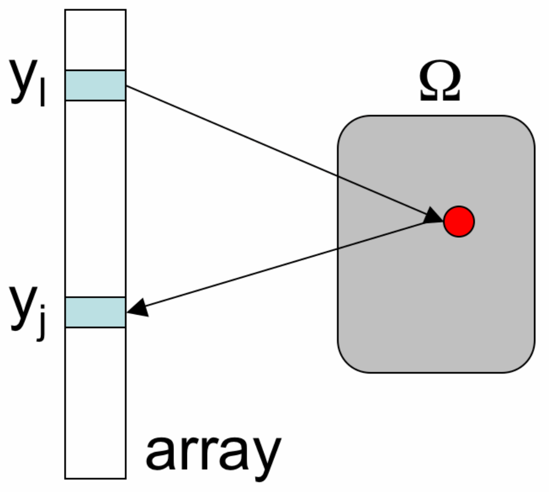
$$\mu([a,b]) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{[a,b]} \mathrm{Im}(f(\lambda + i\epsilon)) d\lambda$$

- Example: if $\mu^{(d)} = \delta_0$, then $f^{(d)}(z) = -1/z$ and $\mu(\lambda) = \frac{1}{\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}\mathbf{1}_{[-2\sigma,2\sigma]}(\lambda)$.

Detection test in the presence of additive noise

The array response matrix (in an ideal world)

- Time-harmonic waves emitted by point sources and recorded by point sensors



- Array of n elements $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$.

- $\hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l)$ = field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits a time-harmonic signal at frequency ω.

- Response matrix $\mathbf{U}^0 = (U^0_{jl})_{j,l=1,\ldots,n}$ in a known environment:

$$U^0_{jl} = \hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l) - \hat{G}_0(\boldsymbol{y}_j, \boldsymbol{y}_l)$$

where
- $\hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l)$ the field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits.
- $\hat{G}_0(\boldsymbol{y}_j, \boldsymbol{y}_l)$ is the incident field (Green's function of the background medium).

A simple model: a point reflector in a constant background (1/2)

Scalar wave equation: $\Delta_{\boldsymbol{x}} \hat{u}(\boldsymbol{x},\boldsymbol{y}) + \frac{\omega^2}{c^2(\boldsymbol{x})} \hat{u}(\boldsymbol{x},\boldsymbol{y}) = -\delta(\boldsymbol{x}-\boldsymbol{y})$.

In the presence of a localized reflector in the search domain Ω:

$$\frac{1}{c^2(\boldsymbol{x})} = \frac{1}{c_0^2}\big(1+\rho_{\rm ref}(\boldsymbol{x})\big)$$

- c_0 is the known background speed (supposed constant),
- the local variation $\rho_{\rm ref}(\boldsymbol{x})$ represents the reflector centered at $\boldsymbol{z}_{\rm ref}$.

• Response matrix $\mathbf{U}^0 = (U^0_{jl})_{j,l=1,\ldots,n}$:

$$U^0_{jl} = \hat{u}(\boldsymbol{y}_j,\boldsymbol{y}_l) - \hat{G}_0(\boldsymbol{y}_j,\boldsymbol{y}_l)$$

- $\hat{u}(\boldsymbol{y}_j,\boldsymbol{y}_l)$ the field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits:

$$\Delta_{\boldsymbol{x}} \hat{u}(\boldsymbol{x},\boldsymbol{y}_l) + \frac{\omega^2}{c_0^2}\big(1+\rho_{\rm ref}(\boldsymbol{x})\big)\hat{u}(\boldsymbol{x},\boldsymbol{y}_l) = -\delta(\boldsymbol{x}-\boldsymbol{y}_l)$$

- $\hat{G}_0(\boldsymbol{y}_j,\boldsymbol{y}_l)$ is the incident field:

$$\hat{G}_0(\boldsymbol{y}_j,\boldsymbol{y}_l) = \frac{e^{i\frac{\omega}{c_0}|\boldsymbol{y}_j-\boldsymbol{y}_l|}}{4\pi|\boldsymbol{y}_j-\boldsymbol{y}_l|}$$

A simple model: a point reflector in a constant background (2/2)

- If the reflector is small, the response matrix has the form (Born approximation):

$$U^0_{jl} = \hat{G}_0(\boldsymbol{x}_j, \boldsymbol{z}_{\rm ref}) \frac{\omega^2}{c_0^2} \Big(\int \rho_{\rm ref}(\boldsymbol{z}) d\boldsymbol{z} \Big) \hat{G}_0(\boldsymbol{z}_{\rm ref}, \boldsymbol{x}_l)$$

or equivalently:

$$\mathbf{U}^0 = \sigma_{\rm ref} \boldsymbol{g}(\boldsymbol{z}_{\rm ref}) \boldsymbol{g}(\boldsymbol{z}_{\rm ref})^T, \text{ with } \quad \sigma_{\rm ref} = \frac{\omega^2}{c_0^2} \Big(\int \rho_{\rm ref}(\boldsymbol{z}) d\boldsymbol{z} \Big) \Big(\sum_{l=1}^n |\hat{G}_0(\boldsymbol{z}_{\rm ref}, \boldsymbol{x}_l)|^2 \Big)$$

- $\sigma_{\rm ref}$ is the scattering amplitude of the reflector,
- $\boldsymbol{g}(\boldsymbol{x})$ is the normalized vector of Green's functions from the array to the point $\boldsymbol{x}$:

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^n |\hat{G}_0(\boldsymbol{x}, \boldsymbol{x}_l)|^2 \big)^{1/2}} \big(\hat{G}_0(\boldsymbol{x}, \boldsymbol{x}_j) \big)_{j=1,\ldots,n}$$

The matrix $\mathbf{U}^0$ has rank 1 and its unique non-zero singular value is $\sigma_{\rm ref}$.

Remark: other possible models: perfectly conducting crack, small inclusion, ...

Question: What is the structure of the (SVD of the) measured matrix $\mathbf{U}$ in the presence of noise ? When do we sound the alarm (presence of a reflector) ?

A point reflector in a noisy environment: the response matrix

• In the presence of a reflector and noise, the response matrix $\mathbf{U}$ has the form

$$\mathbf{U} = \mathbf{U}^0 + \mathbf{W}$$

• The matrix $\mathbf{U}^0$ is the rank-one matrix that corresponds to the reflector.

• The matrix $\mathbf{W}$ models noise.

• In the case of an additive measurement noise, the matrix $\mathbf{W}$ is a complex Gaussian random matrix; after symmetrization, it is a complex symmetric Gaussian random matrix.

• In the case of cluttered noise, in the multiple scattering regime, the matrix $\mathbf{W}$ is a complex symmetric Gaussian random matrix.

• In the case of cluttered noise, in the single scattering regime, the matrix $\mathbf{W}$ is a complex Hankel Gaussian random matrix.

The response matrix without reflector and with measurement noise

$\mathbf{U} = \mathbf{W}$: response matrix with measurement noise (no signal); symmetrize so that $\mathbf{W}$ is a complex symmetric matrix, with $W_{jj} \sim \mathcal{N}\mathcal{C}(0, \delta^2)$ and $W_{jl} \sim \mathcal{N}\mathcal{C}(0, \delta^2/2)$ for $j < l$.

- Denote

- $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \sigma_3^{(n)} \geq \cdots \geq \sigma_n^{(n)}$ the singular values of the response matrix $\mathbf{U}$.
- $\sigma_{\rm c} = \sqrt{n/2}\delta$.

In the regime $n \gg 1$ (number of sensors $\gg 1$):

a) When n is large, the fluctuations of the second moment are Gaussian with a relative amplitude of the order of n^{-1}:

$$\frac{1}{n} \sum_{j=1}^{n} (\sigma_j^{(n)})^2 = \frac{1}{n} \text{Trace}\big(\mathbf{U}^H \mathbf{U}\big) = \frac{1}{n} \sum_{j,l=1}^{n} |U_{jl}|^2 = \sigma_{\rm c}^2 + \frac{\sqrt{2}\sigma_{\rm c}^2}{n} Z_0,$$

where $Z_0 \sim \mathcal{N}(0, 1)$ (remember that $\mathbb{E}[|U_{jl}|^2] = \delta^2/2$).

- Denote

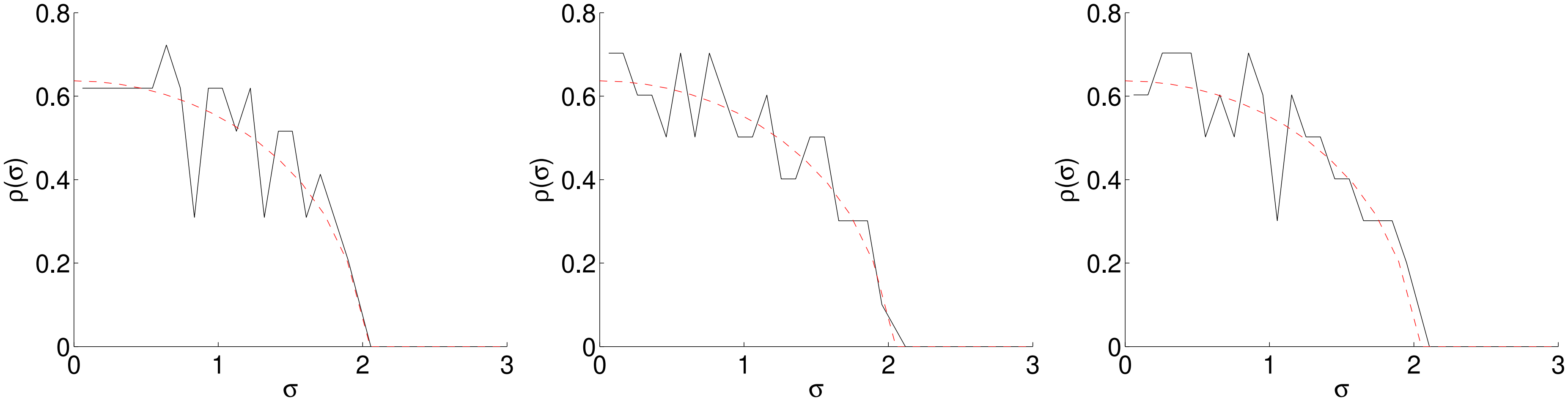
- $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \sigma_3^{(n)} \geq \cdots \geq \sigma_n^{(n)}$ the singular values of the response matrix $\mathbf{U}$.
- $\sigma_c = \sqrt{n/2}\delta$.

In the regime $n \gg 1$ (number of sensors $\gg 1$):

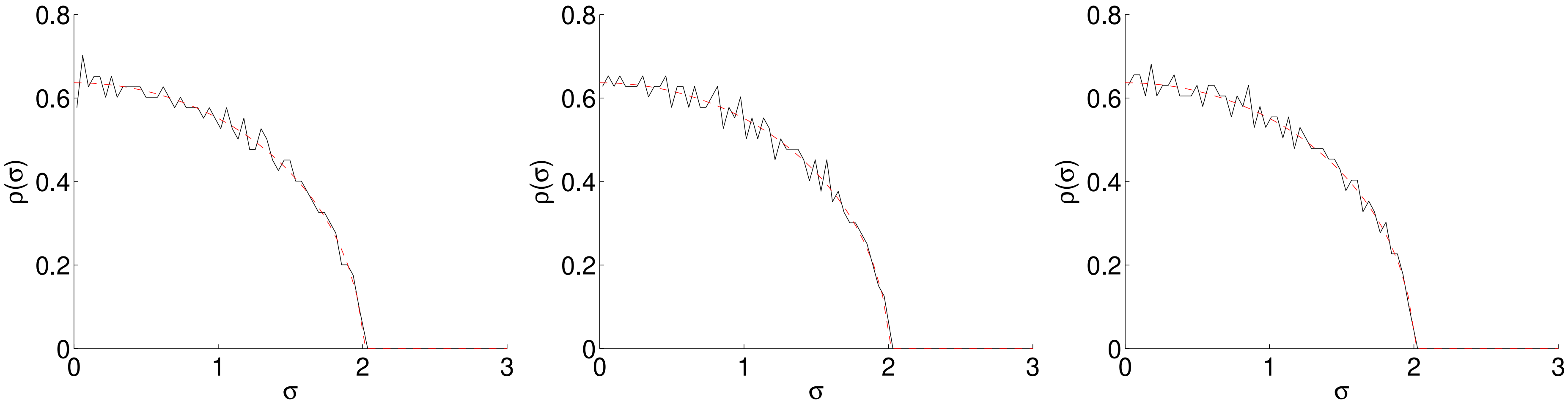
b) The singular values $(\sigma_j^{(n)})_{j=1,\ldots,n}$ follow a quarter-circle distribution:

$$\frac{1}{n}\text{Card}\big(j = 1, \ldots, n\,,\, \sigma_j^{(n)} \in [a, b]\big) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sigma_c}\int_a^b \rho_{\text{qc}}\big(\frac{\sigma}{\sigma_c}\big)d\sigma$$

$$\rho_{\text{qc}}(\sigma) = \begin{cases} \frac{1}{\pi}\sqrt{4-\sigma^2} & \text{if } 0 < \sigma \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$



Histogram of the singular values for three realizations of the random matrix ($n = 100$, $\sigma_c = 1$).



Histogram of the singular values for three realizations of the random matrix ($n = 1000$, $\sigma_c = 1$).

The maximal singular value

- Denote

- $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \sigma_3^{(n)} \geq \cdots \geq \sigma_n^{(n)}$ the singular values of the response matrix $\mathbf{U}$.
- $\sigma_{\rm c} = \sqrt{n/2}\delta$.

In the regime $n \gg 1$ (number of sensors $\gg 1$):

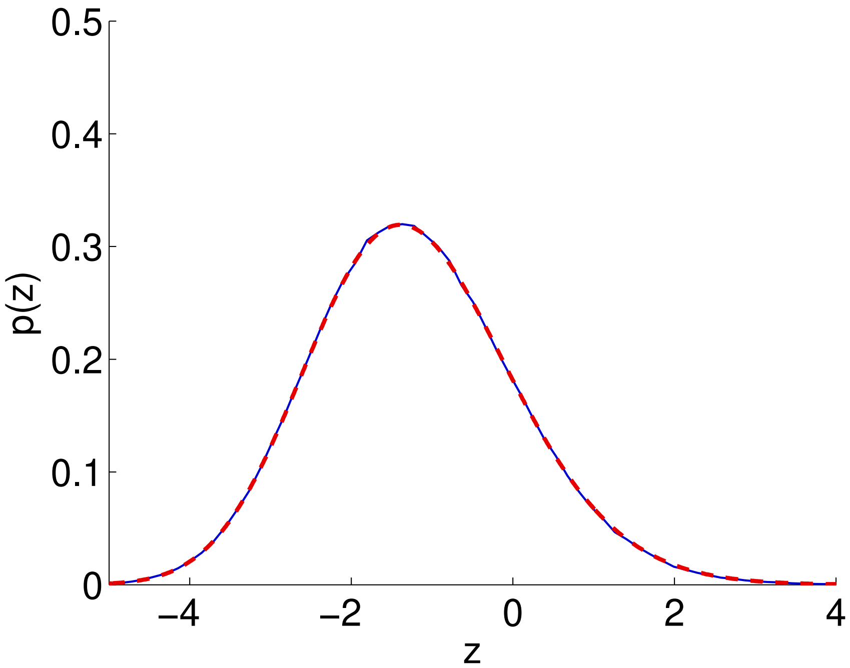
c) The largest singular value is $2\sigma_{\rm c}$ up to a random correction of order $\sigma_{\rm c} n^{-2/3}$:

$$\sigma_1^{(n)} = \max_{j=1,\ldots,n} \{\sigma_j^{(n)}\} = \sigma_{\rm c}\big[2 + 2^{-2/3} n^{-2/3} Z_1 + o(n^{-2/3})\big].$$

The random correction Z_1 is not Gaussian. It is a Tracy-Widom distribution of type $\beta = 1$:

$$\mathbb{P}(Z_1 \leq z) = \int_{-\infty}^{z} p_{\rm TW1}(x)dx = \exp\Big(-\frac{1}{2}\int_z^\infty q(x) + (x-z)q^2(x)dx\Big),$$
$$\mathbb{E}[Z_1] \simeq -1.21, \qquad {\rm Var}(Z_1) \simeq 1.61$$

with $q(x)$ solution of the Painlevé equation $q''(x) = xq(x) + 2q(x)^3$, $q(x) \overset{x\to\infty}{\simeq} {\rm Ai}(x)$.



Pdf of $2^{2/3}n^{2/3}(\max \sigma_j^{(n)}/\sigma_{\rm c} - 2)$ obtained from MC simulations with $n = 50$ (solid) and compared with the theoretical Tracy-Widom distribution of type 1 (dashed).

Therefore, when n is large

$$\frac{\sigma_1^{(n)}}{\left(\frac{1}{n-1}\sum_{j=2}^n \left(\sigma_j^{(n)}\right)^2\right)^{1/2}} = 2 + \frac{1}{2^{2/3}n^{2/3}} Z_1,$$

where Z_1 follows a Tracy-Widom distribution of type 1.

Aparté: The usual extreme value theory

Up to an affine transform, the distribution of the maximum of a large set of random variables is either a Gumbel, a Frechet or a Weibull distribution.

Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. real-valued random variables with the cdf F. Let $M_n = \max_{j=1,\dots,n}(X_j)$. There exists c_n and d_n such that

$$\mathbb{P}\big(c_n^{-1}(M_n - d_n)\big) \stackrel{n\to\infty}{\longrightarrow} \begin{cases} \exp(-\exp(-x)) & \text{Gumbel} \\ \exp(-x^{-\alpha})\mathbf{1}_{(0,\infty)}(x) & \text{Fréchet} \\ \exp(-(-x)^{\alpha})\mathbf{1}_{(-\infty,0)}(x) & \text{Weibull} \end{cases}$$

Independence is crucial.

A point reflector

• Analysis of the response matrix obtained from the backscattering of a strong reflector in the presence of an additive measurement noise or in the case in which the reflector is embedded into a scattering medium.

• The matrix $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{U}^0 + \mathbf{W}.$$

• The matrix $\mathbf{W}$ models noise.

• The rank one matrix $\mathbf{U}_0$ corresponds to the strong reflector:

$$\mathbf{U}^0 = \sigma_{\rm ref} \boldsymbol{g}(\boldsymbol{z}_{\rm ref}) \boldsymbol{g}(\boldsymbol{z}_{\rm ref})^T$$

• In the case of an additive measurement noise, the matrix $\mathbf{W}$ is a symmetric complex Gaussian random matrix (after symmetrization).

The response matrix with a point reflector and with measurement noise

- Denote

- $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \sigma_3^{(n)} \geq \cdots \geq \sigma_n^{(n)}$ the singular values of the response matrix $\mathbf{U}$.
- $\sigma_{\rm c} = \sqrt{n/2}\delta$.

In the regime $n \gg 1$ (number of sensors $\gg 1$):

a)

$$\frac{1}{n-1}\sum_{j=2}^{n}(\sigma_j^{(n)})^2 = \frac{1}{n-1}\Big(\text{Trace}\big(\mathbf{U}^H\mathbf{U}\big) - (\sigma_1^{(n)})^2\Big) = \sigma_{\rm c}^2 + \frac{\sqrt{2}\sigma_{\rm c}^2}{n}Z_0 + \frac{1}{n}z_1 + o(\frac{1}{n})$$

where $Z_0 \sim \mathcal{N}(0,1)$, and $z_1 = \sigma_{\rm ref}^2 - 2\sigma_{\rm c}^2$ if $\sigma_{\rm ref} < \sigma_{\rm c}$ and $z_1 = -\sigma_{\rm c}^4/\sigma_{\rm ref}^2$ if $\sigma_{\rm ref} > \sigma_{\rm c}$.

b) The second singular value satisfies $\sigma_2^{(n)} \simeq 2\sigma_{\rm c}[1 + O(n^{-2/3})]$,
the singular values $(\sigma_j^{(n)})_{j=2,\ldots,n}$ follow a quarter-circle distribution.

c1) If $\sigma_{\rm c} < \sigma_{\rm ref}$, then the largest singular value satisfies

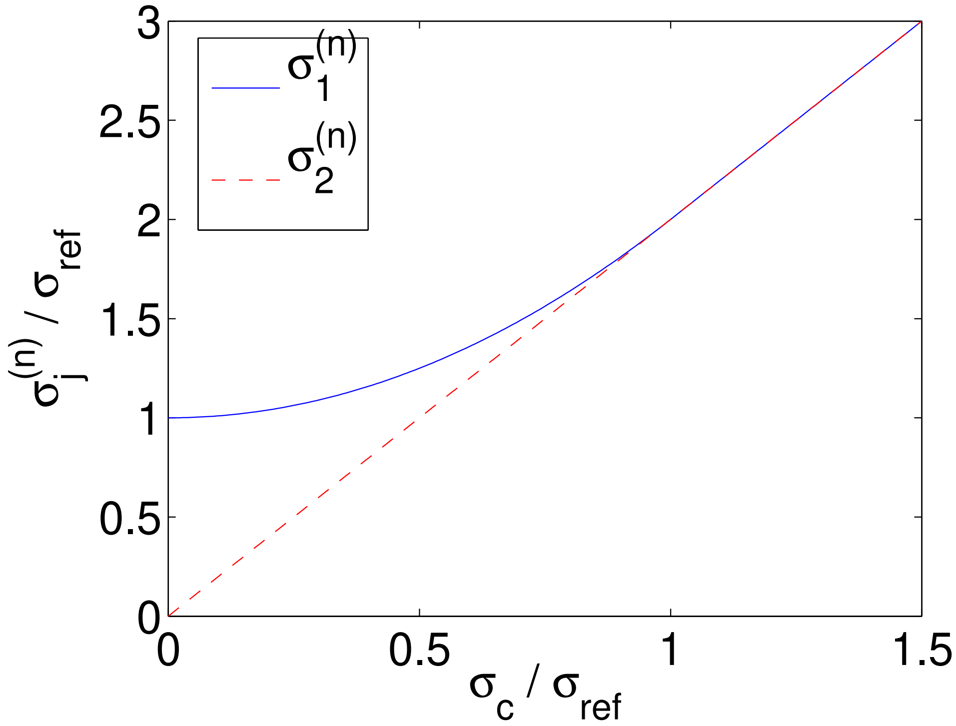
$$\sigma_1^{(n)} = \sigma_{\rm ref}\big[1 + \sigma_{\rm c}^2\sigma_{\rm ref}^{-2} + (2n)^{-1/2}\sigma_{\rm c}\sigma_{\rm ref}^{-1}\big(1 - \sigma_{\rm c}^2\sigma_{\rm ref}^{-2}\big)^{1/2}Z_0 + o(n^{-1/2})\big]$$

where Z_0 follows a standard Gaussian distribution.

c2) If $\sigma_{\rm c} > \sigma_{\rm ref}$, then the largest singular value satisfies

$$\sigma_1^{(n)} = \sigma_{\rm c}\big[2 + 2^{-2/3}n^{-2/3}Z_1 + o(n^{-2/3})\big]$$

where Z_1 follows a type 1 Tracy-Widom distribution.



Mean first and second singular values.

Several interesting features are noticeable:

1) The noise generates many small singular values, whose largest one is $\sigma_2^{(n)}$ which is of the order of $2\sigma_c$.

2) The first singular value, $\sigma_1^{(n)}$, corresponding to the strong reflector, increases as the noise increases. This is a manifestation of the level repulsion: the small singular values (and in particular $\sigma_2^{(n)}$) increase as the noise increases, and the strong singular value is repulsed.

3) The first singular value, corresponding to the strong reflector, and the second singular value, that is the largest singular value generated by the noise, are well separated as long as $\sigma_c < \sigma_{ref}$. In the opposite case $\sigma_c > \sigma_{ref}$ it is not possible to see the reflector.

We can describe the ratio of the maximal singular value over the squared root of the second moment.

a) For $\sigma_{\mathrm{c}} < \sigma_{\mathrm{ref}}$,

$$\frac{\sigma_1^{(n)}}{\left(\frac{1}{n-2}\sum_{j=2}^{n}(\sigma_j^{(n)})^2\right)^{1/2}} = \frac{\sigma_{\mathrm{ref}}}{\sigma_{\mathrm{c}}} + \frac{\sigma_{\mathrm{c}}}{\sigma_{\mathrm{ref}}} + \frac{1}{\sqrt{2n}}\sqrt{1-\sigma_{\mathrm{c}}^2\sigma_{\mathrm{ref}}^{-2}}Z_0,$$

where $Z_0 \sim \mathcal{N}(0,1)$.

b) For $\sigma_{\mathrm{c}} > \sigma_{\mathrm{ref}}$ we have

$$\frac{\sigma_1^{(n)}}{\left(\frac{1}{n-2}\sum_{j=2}^{n}\left(\sigma_j^{(n)}\right)^2\right)^{1/2}} = 2 + \frac{1}{2^{2/3}n^{2/3}}Z_1,$$

where Z_1 follows a Tracy-Widom distribution of type 1.

Complement: Introduction to test theory

- Two hypotheses:
  - *without reflector* : there is no reflector,
  - *with reflector* : there is a reflector.

- We want to test *without reflector* against *with reflector* . Two types of errors can be made:
  - Type I errors correspond to rejecting *without reflector* when it is correct: *false alarm*. The probability of type I error (false alarm rate or FAR) is

$$\text{FAR} := \mathbb{P}\big(\text{accept } \textit{with reflector} \,|\textit{without reflector} \text{ true}\big),$$

  - Type II errors correspond to accepting *without reflector* when it is false: *missed detection*. The probability of type II error (probability of missed detection or PMD) is

$$\text{PDM} := \mathbb{P}\big(\text{accept } \textit{without reflector} \,|\textit{with reflector} \text{ true}\big).$$

The probability of detection (POD) is therefore $1 - \text{PMD}$.

- General result: one cannot maximize the POD and minimize the FAR.

But: amongst all possible tests with a given FAR, it is possible to identify the best test that maximizes the POD (Neyman-Pearson Lemma).

Consider

$$R := \frac{\sigma_1^{(n)}}{\left(\frac{1}{n-2}\sum_{j=2}^{n}(\sigma_j^{(n)})^2\right)^{1/2}}$$

In the regime $n \gg 1$ (number of sensors $\gg 1$)

- In the absence of reflector,

$$R \overset{dist.}{=} 2 + \frac{1}{2^{2/3}n^{2/3}} Z_1$$

where Z_1 follows a type 1 Tracy-Widom distribution.

- In the presence of a reflector, if $\sigma_{\rm ref} > \sigma_{\rm c}$, then

$$R \overset{dist.}{=} \frac{\sigma_{\rm ref}}{\sigma_{\rm c}} + \frac{\sigma_{\rm c}}{\sigma_{\rm ref}} + \frac{1}{\sqrt{2n}}\sqrt{1 - \sigma_{\rm c}^2 \sigma_{\rm ref}^{-2}}\, Z_0$$

where Z_0 follows a standard Gaussian distribution.

If $\sigma_{\rm ref} < \sigma_{\rm c}$ then $R \overset{dist.}{=} 2 + \frac{1}{2^{2/3}n^{2/3}} Z_1$, as in the absence of a reflector !

• *Detection test with level r: If the data gives R, then sound the alarm if $R > r$.*

• The false alarm rate (FAR) is the probability to sound the alarm when there is no reflector:

$$\text{FAR} = \mathbb{P}(R > r | \text{ without reflector })$$

The probability of detection (POD) is the probability to sound the alarm when there is a reflector:

$$\text{POD} = \mathbb{P}(R > r | \text{ with reflector })$$

• Fix $\alpha \in (0,1)$. Choose

$$r_\alpha = 2 + \frac{1}{2^{2/3} n^{2/3}} \Phi_{\text{TW1}}^{-1}(1-\alpha),$$

where Φ_{TW1} is the type 1 Tracy-Widom cumulative distribution function. For instance, $\Phi_{\text{TW1}}^{-1}(0.95) \simeq 0.98$. Main results:

• The FAR of the test $R > r_\alpha$ is α.

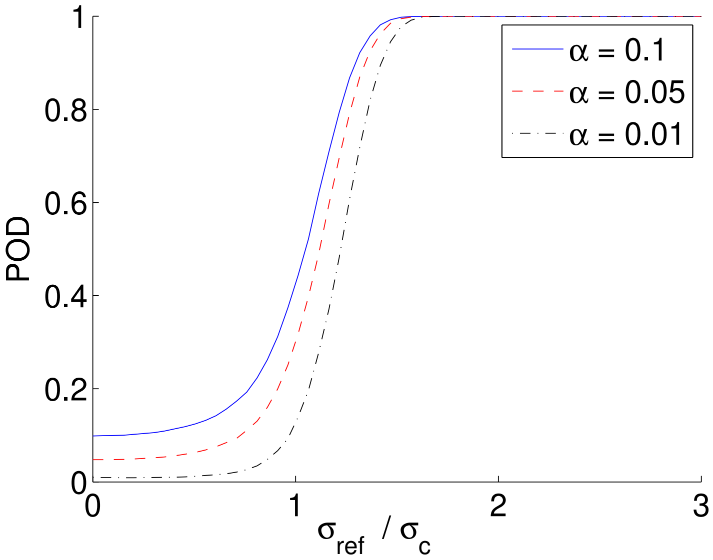
• The POD of the test $R > r_\alpha$ is

$$\text{POD} \simeq \max \left\{ \alpha, \Phi \left(\sqrt{2n} \frac{\frac{\sigma_{\text{ref}}}{\sigma_{\text{c}}} + \frac{\sigma_{\text{c}}}{\sigma_{\text{ref}}} - r_\alpha}{\sqrt{1 - (\sigma_{\text{c}}/\sigma_{\text{ref}})^2}} \right) \right\}$$

where Φ is the standard Gaussian cumulative distribution function.

• By Neyman-Pearson lemma the test $R > r_\alpha$ maximizes the POD for a given FAR α.

The POD increases with the number n of sensors, with $\sigma_{\mathrm{ref}}/\sigma_{\mathrm{c}}$, and with the FAR α.



The SVD-based test becomes powerful when $\sigma_{\mathrm{ref}} > \sigma_{\mathrm{c}}$.

Migration-based detection test

The reverse-time imaging function is defined by

$$\mathcal{I}_{\rm RT}(\boldsymbol{x}) = \overline{\boldsymbol{g}(\boldsymbol{x})}^T \mathbf{U} \overline{\boldsymbol{g}(\boldsymbol{x})}$$

where $\boldsymbol{g}(\boldsymbol{x})$ is the normalized vector of Green's functions:

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^n |\hat{G}_0(\boldsymbol{x},\boldsymbol{x}_l)|^2\big)^{1/2}} \big(\hat{G}_0(\boldsymbol{x},\boldsymbol{x}_j)\big)_{j=1,\ldots,n}$$

• Imaging of a point reflector without noise: $\mathcal{I}_{\rm RT}$ is a peak centered at $\boldsymbol{z}_{\rm ref}$:

$$\mathcal{I}_{\rm RT}(\boldsymbol{x}) = h(\boldsymbol{x}-\boldsymbol{z}_{\rm ref}) = \big(\overline{\boldsymbol{g}(\boldsymbol{x})}^T \boldsymbol{g}(\boldsymbol{z}_{\rm ref})\big)^2$$

Point spread function h maximal at $\mathbf{0}$; $h(\mathbf{0}) = 1$; full aperture: $h(\boldsymbol{x}) = {\rm sinc}^2\big(\frac{\pi|\boldsymbol{x}|}{\lambda_0}\big)$.

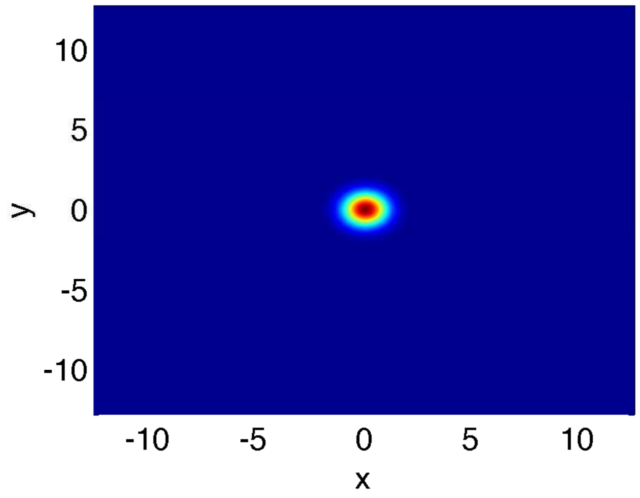
• Imaging of noise without reflector: $\mathcal{I}_{\rm RT}$ is a speckle pattern, i.e. a stationary Gaussian random field with mean zero, variance δ^2, and covariance function:

$$\mathbb{E}\big[\mathcal{I}_{\rm RT}(\boldsymbol{x})\overline{\mathcal{I}_{\rm RT}}(\boldsymbol{y})\big] = \delta^2 h(\boldsymbol{x}-\boldsymbol{y})$$

The hotspot volume is defined by

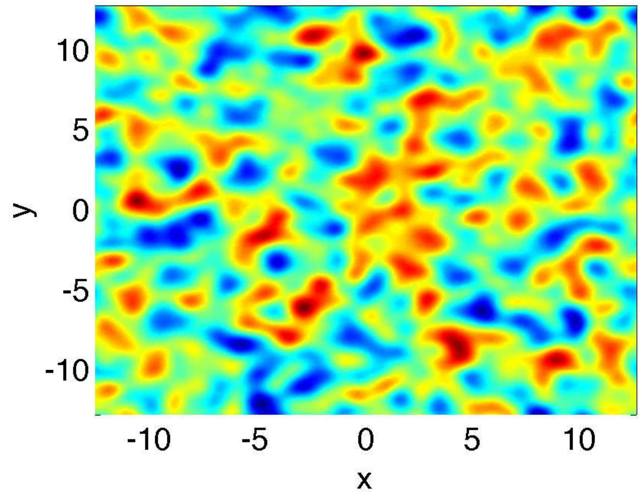
$$V_c = \frac{\pi^{3/2}}{(\det \mathbf{H})^{1/2}}, \qquad \mathbf{H} = \big(-\partial^2_{x_j x_l} h(\mathbf{0})\big)_{j,l=1,\ldots,3}.$$

Full aperture: $V_c = \frac{3^{3/2}}{(2\pi)^{3/2}}\lambda_0^3$.



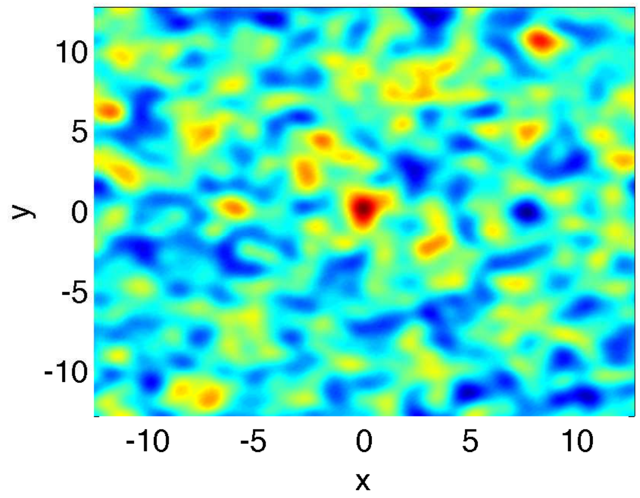
With reflector, without noise

Point spread function



Without reflector, with noise

Speckle pattern



With reflector, with noise

• Consider

$$S := \frac{\|\mathcal{I}_{\mathrm{RT}}\|^2_{L^\infty(\Omega)}|\Omega|}{\|\mathcal{I}_{\mathrm{RT}}\|^2_{L^2(\Omega)}}.$$

In the regime $n \gg 1$ (number of sensors $\gg 1$) and $|\Omega| \gg V_c$ (search volume $\gg$ hotspot volume):

a) In the absence of a reflector, then

$$S \overset{dist.}{=} \ln \frac{|\Omega|}{V_c} + \frac{3}{2} \ln\ln \frac{|\Omega|}{V_c} + Z,$$

where Z follows a Gumbel distribution.

b) In the presence of a reflector, then

$$S \overset{dist.}{=} \max\left\{ \frac{\sigma^2_{\mathrm{ref}}}{\delta^2} + \frac{\sqrt{2}\sigma_{\mathrm{ref}}}{\delta} Z_0 \, , \, \ln \frac{|\Omega|}{V_c} + \frac{3}{2} \ln\ln \frac{|\Omega|}{V_c} + Z \right\},$$

where Z_0 follows a standard Gaussian distribution and is independent of Z.

Proof: Use extreme value theory for random fields.

• *Migration-based detection test: If the data gives S, sound the alarm if $S > s$.*
Fix $\alpha \in (0,1)$. Choose

$$s_\alpha = \ln \frac{|\Omega|}{V_c} + \frac{3}{2} \ln\ln \frac{|\Omega|}{V_c} + \Phi_{\rm G}^{-1}(1-\alpha),$$

where $\Phi_{\rm G}(x) = \exp(-e^{-x})$ is the Gumbel cumulative distribution function. For instance $\Phi_{\rm G}^{-1}(0.95) \simeq 2.97$.

Main results:

- The false alarm rate (FAR) of the test $S > s_\alpha$ is α.
- The probability of detection (POD) of the test $S > s_\alpha$ is

$$\text{POD} = \max\left\{ \Phi\left(\frac{\frac{\sigma_{\rm ref}^2}{\delta^2} - s_\alpha}{\frac{\sqrt{2}\sigma_{\rm ref}}{\delta}} \right), \alpha \right\}$$

where Φ is the standard Gaussian cumulative distribution function.

- By Neyman Pearson lemma, the test $S > s_\alpha$ maximizes the POD for a given FAR α.

The migration-based test becomes powerful when $\sigma_{\rm ref} > \delta \ln^{1/2}(|\Omega|/V_c)$.

Remember: $\sigma_{\rm c} = \sqrt{n/2}\delta$.

Comparison between SVD-based test and migration-based test

• The migration-based test becomes powerful when $\sigma_{\mathrm{ref}} > \delta \ln^{1/2}(|\Omega|/V_c)$.

• The SVD-based test becomes powerful when $\sigma_{\mathrm{ref}} > \delta \sqrt{n/2}$.

• Therefore the migration-based test is more (resp. less) powerful than the SVD-based test when $n >$ (resp. $<$) $2 \ln(|\Omega|/V_c)$.

• In practice, we usually have $n > 2 \ln(|\Omega|/V_c)$, and therefore the migration-based test is more efficient than the SVD-based test.

• The SVD-based test is simpler to implement than the migration-based test.