Array imaging with additive noise

Active array imaging

• In the presence of a point reflector at $\boldsymbol{z}_{\rm ref}$ and in the Born approximation, the response matrix is

$$\big(u_0(\omega)\big)_{rs} = \hat{G}(\omega, \boldsymbol{x}_r, \boldsymbol{z}_{\rm ref}) \frac{\omega^2}{c_0^2} \sigma_{\rm ref} l_{\rm ref}^3 \hat{G}(\omega, \boldsymbol{z}_{\rm ref}, \boldsymbol{x}_s), \qquad r, s = 1, \dots, N$$

or equivalently:

$$\mathbf{u}_0(\omega) = \tau_{\rm ref} \boldsymbol{g}(\omega, \boldsymbol{z}_{\rm ref}) \boldsymbol{g}(\omega, \boldsymbol{z}_{\rm ref})^T, \text{ with } \quad \tau_{\rm ref} = \frac{\omega^2}{c_0^2} \sigma_{\rm ref} l_{\rm ref}^3 \Big(\sum_{l=1}^N |\hat{G}(\omega, \boldsymbol{z}_{\rm ref}, \boldsymbol{x}_l)|^2 \Big)$$

- $\tau_{\rm ref}$ is the scattering amplitude of the reflector,
- $l_{\rm ref}^3$ is the volume of the reflector,
- $\boldsymbol{g}(\omega, \boldsymbol{x})$ is the normalized vector of Green's functions from the array to the point $\boldsymbol{x}$:

$$\boldsymbol{g}(\omega, \boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^N |\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_l)|^2\big)^{1/2}} \big(\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_j)\big)_{j=1,\dots,N}$$

The matrix $\mathbf{u}_0$ is symmetric and it has rank 1.

• With an additive Gaussian white noise (with variance δ^2), the data set is

$$\mathbf{u}(\omega) = \mathbf{u}_0(\omega) + \mathbf{W}(\omega)$$

- We want to estimate $\boldsymbol{z}_{\mathrm{ref}}$ from the symmetrized matrix $\mathbf{u}^s(\omega) = \frac{1}{2}\big(\mathbf{u}(\omega) + \mathbf{u}(\omega)^T\big)$.
- RT imaging functional:

$$\mathcal{I}_{\mathrm{RT}}(\boldsymbol{z}^S) = \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}^T \mathbf{u}^s(\omega) \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}$$

where $\boldsymbol{g}$ is the normalized vector of Green's functions.

- KM imaging functional:

$$\mathcal{I}_{\mathrm{KM}}(\boldsymbol{z}^S) = \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}^T \mathbf{u}^s(\omega) \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}$$

where

$$\boldsymbol{d}(\omega, \boldsymbol{z}) = \frac{1}{\sqrt{N}} \Big(\exp\big(i\omega \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z})\big)\Big)_{j=1,\dots,N}$$

- MUSIC imaging functional:

$$\mathcal{I}_{\mathrm{MU}}(\boldsymbol{z}^S) = \frac{1}{\Big|\boldsymbol{g}(\omega, \boldsymbol{z}^S) - \big\langle \boldsymbol{v}_1(\omega), \boldsymbol{g}(\omega, \boldsymbol{z}^S)\big\rangle \boldsymbol{v}_1(\omega)\Big|^2}$$

where $\boldsymbol{v}_1(\omega)$ is the first vector of the symmetric SVD (Takagi factorization) of $\mathbf{u}^s(\omega) = \mathbf{V}(\omega)\boldsymbol{\Sigma}(\omega)\mathbf{V}(\omega)^T$.

Data acquisition for array imaging

• We consider that there are M sources and N receivers.
We want to measure the $N \times M$ response matrix $\mathbf{u}_0(\omega)$.
We can do M experiments.
Each source can emit a unit-amplitude time-harmonic signal.
The measures are noisy: the signal measured by a receiver is corrupted by an additive noise (a complex Gaussian random variable with mean zero and variance δ^2).

• Standard Acquisition
The response matrix is measured during a sequence of M experiments:
- In the mth experience, $m = 1, \ldots, M$, the mth source generates a time-harmonic signal with unit amplitude.
- the N receivers record the backscattered waves:

$$u_{nm} = u_{0,nm} + W_{nm}, \quad n = 1, \ldots, N$$

$\hookrightarrow$ we obtain

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{W},$$

where $\mathbf{u}_0$ is the unperturbed response matrix and W_{nm} are independent complex Gaussian random variables with mean zero and variance δ^2.

A quick introduction to Random Matrix Theory

• Different types of results:
- description of the global distribution of the singular values,
- detailed description of the maximal singular value.

• Different types of proofs:
- asymptotic expansions of explicit expressions (special polynomials),
- complex analysis (Stietjes transform of measures).

• The distribution of the singular values depends:
- on the structure of the matrix,
- on the correlation between the random coefficients of the matrix,
- not much on the marginal distribution of the coefficients (in the limit of large matrices).

• Remarkable points: universal results, corresponding to a kind of "central limit theorem", but with unusual scaling and limit distribution.

Realistic models of random matrices

• Gaussian Orthogonal Ensemble (GOE) $\beta = 1$

Real symmetric matrices $\mathbf{W}$ such that:

- the coefficients $(W_{i,j})_{1 \le i \le j \le n}$ are independent.
- $W_{ii} \sim \mathcal{N}(0, 2\sigma^2)$ and $W_{ij} \sim \mathcal{N}(0, \sigma^2)$.

• Gaussian Unitary Ensemble (GUE) $\beta = 2$

Hermitian matrices $\mathbf{W}$ such that:

- the coefficients $(\mathrm{Re}(W_{i,j}))_{1 \le i \le j \le n}$, $(\mathrm{Im}(W_{i,j}))_{1 \le i \le j \le n}$ are independent.
- $W_{ii} \sim \mathcal{N}(0, \sigma^2)$, $\mathrm{Re}(W_{ij}) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$ and $\mathrm{Im}(W_{ij}) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$.

• The joint pdf of the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of $\frac{1}{\sqrt{n}}\mathbf{W}$ is:

$$Q_\beta^{(n)}(\lambda_1, ..., \lambda_n) = \frac{1}{Z_\beta^{(n)}} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta n}{4\sigma^2} \sum_{i=1}^n \lambda_i^2 \right)$$

Obviously: the eigenvalues are not independent; there is level repulsion.