Array imaging with additive noise

Active array imaging

- In the presence of a point reflector at $\boldsymbol{z}_{\rm ref}$ and in the Born approximation, the response matrix is

$$\big(u_0(\omega)\big)_{rs} = \hat{G}(\omega, \boldsymbol{x}_r, \boldsymbol{z}_{\rm ref}) \frac{\omega^2}{c_0^2} \sigma_{\rm ref} l_{\rm ref}^3 \hat{G}(\omega, \boldsymbol{z}_{\rm ref}, \boldsymbol{x}_s), \qquad r,s = 1,\dots,N$$

or equivalently:

$$\mathbf{u}_0(\omega) = \tau_{\rm ref} \boldsymbol{g}(\omega, \boldsymbol{z}_{\rm ref}) \boldsymbol{g}(\omega, \boldsymbol{z}_{\rm ref})^T, \ \text{ with } \quad \tau_{\rm ref} = \frac{\omega^2}{c_0^2} \sigma_{\rm ref} l_{\rm ref}^3 \Big(\sum_{l=1}^N |\hat{G}(\omega, \boldsymbol{z}_{\rm ref}, \boldsymbol{x}_l)|^2 \Big)$$

- $\tau_{\rm ref}$ is the scattering amplitude of the reflector,
- $l_{\rm ref}^3$ is the volume of the reflector,
- $\boldsymbol{g}(\omega, \boldsymbol{x})$ is the normalized vector of Green's functions from the array to the point $\boldsymbol{x}$:

$$\boldsymbol{g}(\omega, \boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^N |\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_l)|^2 \big)^{1/2}} \big(\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_j) \big)_{j=1,\dots,N}$$

The matrix $\mathbf{u}_0$ is symmetric and it has rank 1.

- With an additive Gaussian white noise (with variance δ^2), the data set is

$$\mathbf{u}(\omega) = \mathbf{u}_0(\omega) + \mathbf{W}(\omega)$$

- We want to estimate $\boldsymbol{z}_{\rm ref}$ from the symmetrized matrix $\mathbf{u}^s(\omega) = \frac{1}{2}\big(\mathbf{u}(\omega) + \mathbf{u}(\omega)^T\big)$.
- RT imaging functional:

$$\mathcal{I}_{\rm RT}(\boldsymbol{z}^S) = \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}^T \mathbf{u}^s(\omega) \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}$$

where $\boldsymbol{g}$ is the normalized vector of Green's functions.

- KM imaging functional:

$$\mathcal{I}_{\rm KM}(\boldsymbol{z}^S) = \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}^T \mathbf{u}^s(\omega) \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}$$

where

$$\boldsymbol{d}(\omega, \boldsymbol{z}) = \frac{1}{\sqrt{N}} \Big(\exp\big(i\omega\mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z})\big)\Big)_{j=1,\dots,N}$$

- MUSIC imaging functional:

$$\mathcal{I}_{\rm MU}(\boldsymbol{z}^S) = \frac{1}{\Big|\boldsymbol{g}(\omega, \boldsymbol{z}^S) - \big\langle \boldsymbol{v}_1(\omega), \boldsymbol{g}(\omega, \boldsymbol{z}^S)\big\rangle \boldsymbol{v}_1(\omega)\Big|^2}$$

where $\boldsymbol{v}_1(\omega)$ is the first vector of the symmetric SVD (Takagi factorization) of $\mathbf{u}^s(\omega) = \mathbf{V}(\omega)\boldsymbol{\Sigma}(\omega)\mathbf{V}(\omega)^T$.

Data acquisition for array imaging

• We consider that there are M sources and N receivers.
We want to measure the $N \times M$ response matrix $\mathbf{u}_0(\omega)$.
We can do M experiments.
Each source can emit a unit-amplitude time-harmonic signal.
The measures are noisy: the signal measured by a receiver is corrupted by an additive noise (a complex Gaussian random variable with mean zero and variance δ^2).

• Standard Acquisition
The response matrix is measured during a sequence of M experiments:
- In the mth experience, $m = 1, \ldots, M$, the mth source generates a time-harmonic signal with unit amplitude.
- the N receivers record the backscattered waves:

$$u_{nm} = u_{0,nm} + W_{nm}, \quad n = 1, \ldots, N$$

$\hookrightarrow$ we obtain

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{W},$$

where $\mathbf{u}_0$ is the unperturbed response matrix and W_{nm} are independent complex Gaussian random variables with mean zero and variance δ^2.

• Optimal Acquisition: Hadamard Technique

Noise reduction technique in the presence of additive noise.

Definition: A real Hadamard matrix $\mathbf{H}$ *of order* M *is a* $M \times M$ *matrix whose elements are* -1 *or* $+1$ *and such that* $\mathbf{H}^T\mathbf{H} = M\mathbf{I}$.

Real Hadamard matrices do not exist for all M. A necessary condition for the existence is that $M = 1, 2$ or a multiple of 4. A sufficient condition is that M is a power of two. Explicit examples are known for all M multiple of 4 up to $M = 664$. Hadamard conjecture: a Hadamard matrix of order $4k$ exists for every integer k.

Definition: A complex Hadamard matrix $\mathbf{H}$ *of order* M *is a* $M \times M$ *matrix whose elements are of modulus one and such that* $\mathbf{H}^\dagger\mathbf{H} = M\mathbf{I}$.

Complex Hadamard matrices exist for all M. For instance the Fourier matrix

$$H_{nm} = \exp\Big[i2\pi\frac{(n-1)(m-1)}{M}\Big], \qquad m, n = 1, \ldots, M,$$

is a complex Hadamard matrix.

Proposition: A Hadamard matrix has maximal determinant among matrices with complex entries in the closed unit disk.

More exactly the determinant of any complex $M \times M$ matrix $\mathbf{H}$ with $|H_{mn}| \leq 1$ satisfies $|\det \mathbf{H}| \leq M^{M/2}$, with equality attained by a complex Hadamard matrix.

Let $\mathbf{H}$ be a complex inversible $M \times M$ matrix with $|H_{mn}| \leq 1$.

Multi-source acquisition scheme:

- In the mth experience, $m = 1, \ldots, M$, all sources generate time-harmonic signals, the m' source generating $H_{m'm}$ (the amplitude is bounded by one).
- The N receivers record the backscattered waves:

$$\widetilde{u}_{nm} = \sum_{m'=1}^{M} H_{m'm} u_{0,nm'} + W_{nm} = (\mathbf{u}_0 \mathbf{H})_{nm} + W_{nm}, \qquad n = 1, \ldots, N,$$

where W_{nm} are independent complex Gaussian random variables with mean zero and variance δ^2.

- The measured response matrix $\mathbf{u}$ is obtained by right multiplying by $\mathbf{H}^{-1}$:

$$\mathbf{u} := \widetilde{\mathbf{u}} \mathbf{H}^{-1} = \mathbf{u}_0 \mathbf{H} \mathbf{H}^{-1} + \mathbf{W} \mathbf{H}^{-1}$$

$\hookrightarrow$ we get the unperturbed matrix $\mathbf{u}_0$ up to a new noise

$$\mathbf{u} = \mathbf{u}_0 + \widetilde{\mathbf{W}}, \qquad \widetilde{\mathbf{W}} = \mathbf{W} \mathbf{H}^{-1}$$

The choice of the matrix $\mathbf{H}$ should fulfill the property that the new noise matrix $\widetilde{\mathbf{W}} = \mathbf{W}\mathbf{H}^{-1}$ has independent complex entries with mean zero and minimal variance. We have

$$\begin{aligned}
\mathbb{E}\Big[\overline{\widetilde{W}_{nm}}\widetilde{W}_{n'm'}\Big] &= \sum_{q,q'=1}^{M} \overline{(\mathbf{H}^{-1})_{qm}}(\mathbf{H}^{-1})_{q'm'}\mathbb{E}\big[\overline{W_{nq}}W_{n'q'}\big] \\
&= \delta^2 \sum_{q,q'=1}^{M} \overline{(\mathbf{H}^{-1})_{qm}}(\mathbf{H}^{-1})_{q'm'}\mathbf{1}_n(n')\mathbf{1}_q(q') \\
&= \delta^2 \sum_{q=1}^{M} ((\mathbf{H}^{-1})^\dagger)_{mq}(\mathbf{H}^{-1})_{qm'}\mathbf{1}_n(n') \\
&= \delta^2 ((\mathbf{H}^{-1})^\dagger \mathbf{H}^{-1})_{mm'}\mathbf{1}_n(n'),
\end{aligned}$$

This shows that we look for a complex matrix $\mathbf{H}$ with entries in the unit disk such that $(\mathbf{H}^{-1})^\dagger\mathbf{H}^{-1} = c\mathbf{I}$ with a minimal c. This is equivalent to require that $\mathbf{H}$ is unitary and that $|\det \mathbf{H}|$ is maximal.
$\hookrightarrow$ The optimal matrix $\mathbf{H}$ that minimizes the noise variance should be a Hadamard matrix. For instance, take the Fourier matrix (in the case of a linear array, this corresponds to an illumination in the form of plane waves with regularly sampled angles).

When the multi-source acquisition scheme is used with a Hadamard technique, we measure

$$\mathbf{u} = \mathbf{u}_0 + \widetilde{\mathbf{W}},$$

where the new noise matrix $\widetilde{\mathbf{W}}$ has independent complex entries with Gaussian statistics, mean zero, and variance δ^2/M:

$$\mathbb{E}\Big[\overline{\widetilde{W}_{nm}}\widetilde{W}_{n'm'}\Big] = \frac{\delta^2}{M}\mathbf{1}_m(m')\mathbf{1}_n(n').$$

This gain of a factor $\sqrt{M}$ in the signal-to-noise ratio is called the Hadamard advantage.

A quick introduction to Random Matrix Theory

• Different types of results:
- description of the global distribution of the singular values,
- detailed description of the maximal singular value.

• Different types of proofs:
- asymptotic expansions of explicit expressions (special polynomials),
- complex analysis (Stietjes transform of measures).

• The distribution of the singular values depends:
- on the structure of the matrix,
- on the correlation between the random coefficients of the matrix,
- not much on the marginal distribution of the coefficients (in the limit of large matrices).

• Remarkable points: universal results, corresponding to a kind of "central limit theorem", but with unusual scaling and limit distribution.

Quantities of interest

Consider a $n \times n$ diagonalizable random matrix $\mathbf{A}$ with eigenvalues $(\lambda_1, ..., \lambda_n)$.

Quantities of interest:

1) (integrated) density of states

$$\mu^{(n)}([a,b]) = \frac{1}{n}\mathrm{Card}\,(i = 1, ..., n, \text{ such that } \lambda_i \in [a,b])$$

2) spectral gap probability:

$$T^{(n)}(a,b) = \mathbb{P}\,(\text{ no eigenvalue in } [a,b]\ ) = \mathbb{P}(\mu^{(n)}([a,b]) = 0)$$

2bis) distance between two successive eigenvalues.

3) the maximal eigenvalue

$$\lambda_{max}^{(n)} = \max_{i=1,\dots,n} (\lambda_i)$$

A toy model

The simplest random matrix is the $n \times n$ matrix:

$$\mathbf{A} = \mathrm{Diag}(X_1, ..., X_n)$$

where X_i are independent and identically distributed (i.i.d.) random variables (r.v.) with probability density function (pdf) f:

$$\mathbb{P}(X_1 \in [a,b]) = \int_a^b f(x)dx$$

1) density of states. $\mu^{(n)} \stackrel{n\to\infty}{\longrightarrow}$ continuous measure over $\mathbb{R}$ with density f.

$$\begin{aligned} \mu^{(n)}([a,b]) &= \frac{1}{n}\mathrm{Card}(i = 1, ..., n, \lambda_i \in [a,b]) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[a,b]}(\lambda_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[a,b]}(X_i) \stackrel{n\to\infty}{\longrightarrow} \mathbb{E}[\mathbf{1}_{[a,b]}(X_1)] \\ &\stackrel{n\to\infty}{\longrightarrow} \mathbb{P}(X_1 \in [a,b]) = \int_a^b f(x)dx \end{aligned}$$

Conclusion: the density of eigenvalues around a is $\sim nf(a)$.

2) spectral gap probability.

$$\begin{aligned} T^{(n)}(a,b) &= \mathbb{P}(\text{ 0 eigenvalue in } [a,b]) = \mathbb{P}(X_1 \notin [a,b], ..., X_n \notin [a,b]) \\ &= \mathbb{P}(X_1 \notin [a,b])^n = \left(1 - \int_a^b f(x)dx\right)^n \end{aligned}$$

goes to 0 as soon as $\int_a^b f(x)dx > 0$.

Let us look more carefully. We zoom in so that the average density of eigenvalues is equal to 1:

$$b = a + \frac{s}{nf(a)}$$

with a such that $f(a) > 0$. Then

$$T^{(n)}\left(a, a + \frac{s}{nf(a)}\right) \simeq \left(1 - \frac{s}{n}\right)^n \overset{n\to\infty}{\longrightarrow} e^{-s}$$

It can be shown that

$$\mathbb{P}\Big(k \text{ eigenvalues in } \big[a, a + \frac{s}{nf(a)}\big]\Big) \overset{n\to\infty}{\longrightarrow} \frac{s^k}{k!}e^{-s}$$

→ the number of eigenvalues in a given interval follows a Poisson distribution.

2 bis) gap between two successive eigenvalues.

Assume that there is an eigenvalue in λ_0. Denote by $\lambda_0 + \Delta\lambda$ the following eigenvalue. We have

$$\mathbb{P}(\Delta\lambda > s) = T^{(n-1)}(\lambda_0, \lambda_0 + s)$$

Zoom in so that so that the average density of eigenvalues is equal to 1:

$$\Delta\lambda = \frac{Z_n}{nf(\lambda_0)}$$

with a such that $f(a) > 0$. We have

$$\mathbb{P}(Z_n > s) = T^{(n-1)}\Big(\lambda_0, \lambda_0 + \frac{Z_n}{nf(\lambda_0)}\Big) \simeq \Big(1 - \frac{s}{n}\Big)^n \stackrel{n\to\infty}{\longrightarrow} e^{-s}$$

Therefore $Z_n \stackrel{n\to\infty}{\longrightarrow} Z$ where the pdf of Z is $\exp(-z)\mathbf{1}_{[0,\infty)}(z)$ (independent de f).

Conclusion: the distance between two successive eigenvalues follows an exponential distribution.

3) the maximal eigenvalue

$$\mathbb{P}(\lambda_{max}^{(n)} \leq x) = \mathbb{P}(X_1 \leq x, ..., X_n \leq x) = \mathbb{P}(X \leq x)^n = \left(1 - \int_x^{\infty} f(s)ds\right)^n$$

Assume: we can find a_n and b_n such that $\mathbb{P}(\lambda_{max}^{(n)} \leq a_n + b_n x)$ converges to a cumulative distribution function $F(x)$.

Theorem (Fisher-Tippett-Gnedenko). $F(x)$ belongs to one of the three following types:
- $F_1(x) = e^{-e^{-x}}$ with support $\mathbb{R}$ (Weibull).
- $F_{2,\alpha}(x) = e^{-1/x^\alpha}$ with support $\mathbb{R}^+$, $\alpha > 0$ (Gumbel).
- $F_{3,\alpha}(x) = e^{-(-x)^\alpha}$ with support $\mathbb{R}^-$, $\alpha > 0$ (Fréchet).

Conclusion:

$$\lambda_{max}^{(n)} = a_n + b_n Y$$

where Y follows one of the three "extreme" distributions.

Example 1: $X_i \sim \mathcal{U}([0,1])$, i.e. $f(x) = \mathbf{1}_{[0,1]}(x)$.
Then $a_n = 1$, $b_n = 1/n$, and $F = F_{3,1}$.

Example 2: $X_i \sim \mathcal{E}(1)$, i.e. $f(x) = e^{-x}\mathbf{1}_{[0,\infty)}(x)$.
Then $a_n = \ln(n)$, $b_n = 1$, and $F = F_1$.

Realistic models of random matrices

• Gaussian Orthogonal Ensemble (GOE) $\beta = 1$

Real symmetric matrices $\mathbf{W}$ such that:

- the coefficients $(W_{i,j})_{1 \le i \le j \le n}$ are independent.
- $W_{ii} \sim \mathcal{N}(0, 2\sigma^2)$ and $W_{ij} \sim \mathcal{N}(0, \sigma^2)$.

• Gaussian Unitary Ensemble (GUE) $\beta = 2$

Hermitian matrices $\mathbf{W}$ such that:

- the coefficients $(\mathrm{Re}(W_{i,j}))_{1 \le i \le j \le n}$, $(\mathrm{Im}(W_{i,j}))_{1 \le i \le j \le n}$ are independent.
- $W_{ii} \sim \mathcal{N}(0, \sigma^2)$, $\mathrm{Re}(W_{ij}) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$ and $\mathrm{Im}(W_{ij}) \sim \mathcal{N}(0, \frac{\sigma^2}{2})$.

• The joint pdf of the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $\frac{1}{\sqrt{n}}\mathbf{W}$ is:

$$Q_\beta^{(n)}(\lambda_1, ..., \lambda_n) = \frac{1}{Z_\beta^{(n)}} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta n}{4\sigma^2} \sum_{i=1}^n \lambda_i^2 \right)$$

Obviously: the eigenvalues are not independent; there is level repulsion.