Extrema of Gaussian processes

Global maximum of a Gaussian process

Let us consider a stationary Gaussian process $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ with mean zero and covariance function $c(\boldsymbol{x})$, such that $c(\mathbf{0}) = 1$.

When $|\Omega| \gg l_c^d$, the number N_I^Ω of local maxima of $Z(\boldsymbol{x})^2$ in a domain Ω that have peak values larger than I is

$$N_I^\Omega \simeq \frac{2|\Omega|}{V_c} I^{(d-1)/2} e^{-I/2}$$

Intuitively, the global maximum $I_\Omega = \max_{\boldsymbol{x}\in\Omega} Z(\boldsymbol{x})^2$ should be such that $N_{I_\Omega}^\Omega \sim 1$.

Cf. R. Adler's book:

$$I_\Omega = 2\ln\Big(\frac{2|\Omega|}{V_c}\Big) + (d-1)\ln\Big[2\ln\Big(\frac{2|\Omega|}{V_c}\Big)\Big] + 2Z_g$$

where Z_g follows a Gumbel distribution $\mathbb{P}(Z_g \leq z) = \exp(-e^{-z})$.

Array imaging with additive noise

Source imaging

- In the presence of a point source at $\boldsymbol{z}_{\rm so}$ the response vector is

$$\big(\hat{u}_0(\omega)\big)_r = \hat{G}(\omega, \boldsymbol{x}_r, \boldsymbol{z}_{\rm so})\sigma_{\rm so}(\omega), \qquad r = 1, \dots, N$$

or equivalently:

$$\hat{\boldsymbol{u}}_0(\omega) = \tau_{\rm so}(\omega)\boldsymbol{g}(\omega, \boldsymbol{z}_{\rm so}), \text{ with } \quad \tau_{\rm so}(\omega) = \sigma_{\rm so}(\omega)\Big(\sum_{l=1}^{N} |\hat{G}(\omega, \boldsymbol{z}_{\rm so}, \boldsymbol{x}_l)|^2\Big)^{1/2}$$

- $\tau_{\rm so}(\omega)$ is proportional to the emission power,
- $\boldsymbol{g}(\omega, \boldsymbol{x})$ is the normalized vector of Green's functions from the array to the point $\boldsymbol{x}$:

$$\boldsymbol{g}(\omega, \boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^{N} |\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_l)|^2\big)^{1/2}} \big(\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_j)\big)_{j=1,\dots,N}$$

- With an additive Gaussian white noise (with variance δ^2), the data set is

$$\boldsymbol{u}(\omega) = \boldsymbol{u}_0(\omega) + \boldsymbol{W}(\omega)$$

where $(W_r(\omega))_{r=1,\dots,N}$ are complex-valued, independent and identically distributed Gaussian $\mathcal{N}(0, \delta^2)$ ($W_r = W_{r1} + iW_{r2}$ with W_{r1} and W_{r2} real-valued, independent and identically distributed Gaussian $\mathcal{N}(0, \delta^2/2)$).

- We want to estimate $\boldsymbol{z}_{\rm so}$ from the measured vector $\boldsymbol{u}(\omega)$:

$$\boldsymbol{u}(\omega) = \boldsymbol{u}_0(\omega) + \boldsymbol{W}(\omega)$$

- RT imaging functional:

$$\mathcal{I}_{\rm RT}(\boldsymbol{z}^S) = \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}^T \boldsymbol{u}(\omega)$$

where $\boldsymbol{g}$ is the normalized vector of Green's functions:

$$\boldsymbol{g}(\omega, \boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^N |\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_l)|^2\big)^{1/2}} \big(\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_j)\big)_{j=1,\dots,N}$$

- KM imaging functional:

$$\mathcal{I}_{\rm KM}(\boldsymbol{z}^S) = \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}^T \boldsymbol{u}(\omega)$$

where

$$\boldsymbol{d}(\omega, \boldsymbol{z}) = \frac{1}{\sqrt{N}} \Big(\exp\big(i\omega \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z})\big)\Big)_{j=1,\dots,N}$$

Active array imaging

• In the presence of a point reflector at $\boldsymbol{z}_{\rm ref}$ and in the Born approximation, the response matrix is

$$\big(u_0(\omega)\big)_{rs} = \hat{G}(\omega,\boldsymbol{x}_r,\boldsymbol{z}_{\rm ref})\frac{\omega^2}{c_0^2}\sigma_{\rm ref} l_{\rm ref}^3 \hat{G}(\omega,\boldsymbol{z}_{\rm ref},\boldsymbol{x}_s), \qquad r,s=1,\ldots,N$$

or equivalently:

$$\mathbf{u}_0(\omega) = \tau_{\rm ref}\boldsymbol{g}(\omega,\boldsymbol{z}_{\rm ref})\boldsymbol{g}(\omega,\boldsymbol{z}_{\rm ref})^T, \text{ with } \quad \tau_{\rm ref} = \frac{\omega^2}{c_0^2}\sigma_{\rm ref} l_{\rm ref}^3 \Big(\sum_{l=1}^N |\hat{G}(\omega,\boldsymbol{z}_{\rm ref},\boldsymbol{x}_l)|^2\Big)$$

- $\tau_{\rm ref}$ is the scattering amplitude of the reflector,
- $l_{\rm ref}^3$ is the volume of the reflector,
- $\boldsymbol{g}(\omega,\boldsymbol{x})$ is the normalized vector of Green's functions from the array to the point $\boldsymbol{x}$:

$$\boldsymbol{g}(\omega,\boldsymbol{x}) = \frac{1}{\big(\sum_{l=1}^N |\hat{G}(\omega,\boldsymbol{x},\boldsymbol{x}_l)|^2\big)^{1/2}}\big(\hat{G}(\omega,\boldsymbol{x},\boldsymbol{x}_j)\big)_{j=1,\ldots,N}$$

The matrix $\mathbf{u}_0$ is symmetric and it has rank 1.

• With an additive Gaussian white noise (with variance δ^2), the data set is

$$\mathbf{u}(\omega) = \mathbf{u}_0(\omega) + \mathbf{W}(\omega)$$

- We want to estimate $\boldsymbol{z}_{\rm ref}$ from the symmetrized matrix $\mathbf{u}^s(\omega) = \frac{1}{2}\big(\mathbf{u}(\omega) + \mathbf{u}(\omega)^T\big)$.
- RT imaging functional:

$$\mathcal{I}_{\rm RT}(\boldsymbol{z}^S) = \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}^T \mathbf{u}^s(\omega) \overline{\boldsymbol{g}(\omega, \boldsymbol{z}^S)}$$

where $\boldsymbol{g}$ is the normalized vector of Green's functions.

- KM imaging functional:

$$\mathcal{I}_{\rm KM}(\boldsymbol{z}^S) = \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}^T \mathbf{u}^s(\omega) \overline{\boldsymbol{d}(\omega, \boldsymbol{z}^S)}$$

where

$$\boldsymbol{d}(\omega, \boldsymbol{z}) = \frac{1}{\sqrt{N}} \Big(\exp\big(i\omega \mathcal{T}(\boldsymbol{x}_j, \boldsymbol{z})\big)\Big)_{j=1,\dots,N}$$

- MUSIC imaging functional:

$$\mathcal{I}_{\rm MU}(\boldsymbol{z}^S) = \frac{1}{\Big|\boldsymbol{g}(\omega, \boldsymbol{z}^S) - \big\langle \boldsymbol{v}_1(\omega), \boldsymbol{g}(\omega, \boldsymbol{z}^S)\big\rangle \boldsymbol{v}_1(\omega)\Big|^2}$$

where $\boldsymbol{v}_1(\omega)$ is the first vector of the symmetric SVD (Takagi factorization) of $\mathbf{u}^s(\omega) = \mathbf{V}(\omega)\boldsymbol{\Sigma}(\omega)\mathbf{V}(\omega)^T$.