Random processes

Random process

• Random variable Z = random number = application from a probability space to $\mathbb{R}$.
A realization of the random variable = a real number.
Distribution of a continuous random variable Z is characterized by its pdf $p_Z(z)$.
Example: $Z \sim \mathcal{N}(0,1) \mapsto p_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

• Random process $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ = random function = application from a probability space to $\mathbb{R}^{\mathbb{R}^d}$.
A realization of the process = a function from $\mathbb{R}^d$ to $\mathbb{R}$.
Distribution of $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ characterized by the finite-dimensional distributions $(Z(\boldsymbol{x}_1),\ldots,Z(\boldsymbol{x}_n))$, for any n, $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \in \mathbb{R}^d$.

Gaussian process

- We say that a random process $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ is Gaussian if any linear combination $Z_\lambda = \sum_{i=1}^n \lambda_i Z(\boldsymbol{x}_i)$ has Gaussian distribution.

$\to$ $(Z(\boldsymbol{x}_1),\dots,Z(\boldsymbol{x}_n))$ is a Gaussian vector with

$$\mathbb{E}\big[e^{i\boldsymbol{\lambda}^T\boldsymbol{Z}}\big] = \exp\Big(i\boldsymbol{\lambda}^T\boldsymbol{\mu} - \frac{\boldsymbol{\lambda}^T\mathbf{C}\boldsymbol{\lambda}}{2}\Big)$$

with $\boldsymbol{\mu} = (\mathbb{E}[\mu(\boldsymbol{x}_j)])_{j=1}^n$ and $\mathbf{C} = (\mathbb{E}[Z(\boldsymbol{x}_i)Z(\boldsymbol{x}_j)] - \mathbb{E}[Z(\boldsymbol{x}_i)]\mathbb{E}[Z(\boldsymbol{x}_j)])_{i,j=1}^n$.

$\to$ Z_λ has Gaussian distribution with pdf

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma_\lambda}\exp\big(-\frac{(z-\mu_\lambda)^2}{2\sigma_\lambda^2}\big)$$

where $\mu_\lambda = \sum_{i=1}^n \lambda_i\mathbb{E}[Z(\boldsymbol{x}_i)]$ and $\sigma_\lambda^2 = \sum_{i,j=1}^n \lambda_i\lambda_j\mathbb{E}[Z(\boldsymbol{x}_i)Z(\boldsymbol{x}_j)] - \mu_\lambda^2$, provided $\sigma_\lambda > 0$.

$\to$ The first two moments $\mu(\boldsymbol{x}_1) = \mathbb{E}[Z(\boldsymbol{x}_1)]$ and $R(\boldsymbol{x}_1,\boldsymbol{x}_2) = \mathbb{E}[Z(\boldsymbol{x}_1)Z(\boldsymbol{x}_2)]$ characterize the finite-dimensional distribution of the process $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$.

$\to$ The distribution of a Gaussian process is characterized by its first two moments $\mu(\boldsymbol{x}_1) = \mathbb{E}[Z(\boldsymbol{x}_1)]$ and $R(\boldsymbol{x}_1,\boldsymbol{x}_2) = \mathbb{E}[Z(\boldsymbol{x}_1)Z(\boldsymbol{x}_2)]$.

$\to$ Any linear transform of a Gaussian process is a Gaussian process.

• Simulation: in order to simulate $(Z(\boldsymbol{x}_1), \ldots, Z(\boldsymbol{x}_n))$:
- compute the mean vector $M_i = \mathbb{E}[Z(\boldsymbol{x}_i)]$ and the covariance matrix $C_{ij} = \mathbb{E}[Z(\boldsymbol{x}_i)Z(\boldsymbol{x}_j)] - \mathbb{E}[Z(\boldsymbol{x}_i)]\mathbb{E}[Z(\boldsymbol{x}_j)]$.
- generate a random vector $\boldsymbol{G} = (G_1, \ldots, G_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $\boldsymbol{Z} = \boldsymbol{M} + \mathbf{C}^{1/2}\boldsymbol{G}$. The vector $\boldsymbol{Z}$ has the distribution of $(Z(\boldsymbol{x}_1), \ldots, Z(\boldsymbol{x}_n))$.

Note: the computation of the square root is expensive (use Cholesky method).

Proof. $\boldsymbol{Z}$ is a Gaussian vector.
The mean of $\boldsymbol{Z}$ is the mean of the vector $(Z(\boldsymbol{x}_1), \ldots, Z(\boldsymbol{x}_n))$. The covariance matrix of $\boldsymbol{Z}$ is the covariance matrix of the vector $(Z(\boldsymbol{x}_1), \ldots, Z(\boldsymbol{x}_n))$.

Brownian motion

• Brownian motion $(W_t)_{t\geq 0}$= Gaussian process with mean 0 and covariance function

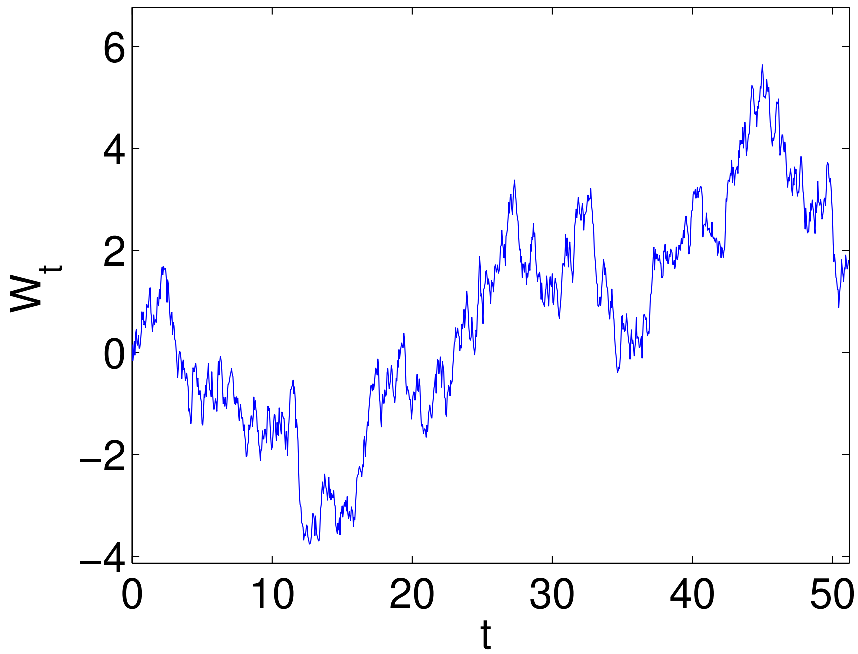
$$\mathbb{E}[W_t W_{t'}] = t \wedge t'$$

The increments of the Brownian motion are independent:
if $t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq t_0 = 0$, then $(W_{t_n} - W_{t_{n-1}}, \ldots, W_{t_2} - W_{t_1}, W_{t_1})$ are independent Gaussian random variables with mean 0 and variance

$$\mathbb{E}\big[(W_{t_j} - W_{t_{j-1}})^2\big] = t_j - t_{j-1}$$

• Simulation: in order to simulate $(W_h, W_{2h}, ..., W_{nh})$, one can use the Cholesky method, but the following method is more rapid:
- generate a random vector $X = (X_1, ..., X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $Y_j = \sqrt{h} \sum_{i=1}^{j} X_i$. The vector $\boldsymbol{Y}$ has the distribution of $(W_h, W_{2h}, ..., W_{nh})$.

• Simulation: in order to simulate $(W_h, W_{2h}, ..., W_{nh})$:
- generate a random vector $\boldsymbol{G} = (G_1, ..., G_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $Y_j = \sqrt{h} \sum_{i=1}^{j} G_i$. The vector $\boldsymbol{Y}$ has the distribution of $(W_h, W_{2h}, ..., W_{nh})$.



Stationary random process

• $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ is stationary if $(Z(\boldsymbol{x}+\boldsymbol{x}_0))_{\boldsymbol{x}\in\mathbb{R}^d}$ has the same distribution as $(Z(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^d}$ for any $\boldsymbol{x}_0 \in \mathbb{R}^d$.
Sufficient and necessary condition:

$$\mathbb{E}\big[\phi(Z(\boldsymbol{x}_1),\dots,Z(\boldsymbol{x}_n))\big] = \mathbb{E}\big[\phi(Z(\boldsymbol{x}_0+\boldsymbol{x}_1),\dots,Z(\boldsymbol{x}_0+\boldsymbol{x}_n))\big]$$

for any n, $\boldsymbol{x}_0,\dots,\boldsymbol{x}_n \in \mathbb{R}^d$, $\phi \in \mathcal{C}_b(\mathbb{R}^n,\mathbb{R})$.

• Let $Z(\boldsymbol{x})$ be a stationary process (with finite second-order moments). Then its mean is constant and its covariance function $\mathrm{Cov}(Z(\boldsymbol{x}), Z(\boldsymbol{x}'))$ depends on $\boldsymbol{x}-\boldsymbol{x}'$ only.

• A Gaussian process $Z(\boldsymbol{x})$ is stationary iff its mean is constant and its covariance function $\mathrm{Cov}(Z(\boldsymbol{x}), Z(\boldsymbol{x}'))$ is of the form $c(\boldsymbol{x}-\boldsymbol{x}')$.
This function c is even, maximal at $\mathbf{0}$, and its Fourier transform is nonnegative.
• Bochner's theorem: a function $c(\boldsymbol{x})$ is a covariance function of a stationary process iff its Fourier transform is nonnegative.

Spectral representation of a real-valued stationary Gaussian process

$$Z(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \sqrt{\hat{c}(\boldsymbol{k})} \hat{n}_{\boldsymbol{k}} d\boldsymbol{k}$$

with $\hat{n}_{\boldsymbol{k}}$ complex white noise, i.e.:
$\hat{n}_{\boldsymbol{k}}$ complex-valued, Gaussian, $\hat{n}_{-\boldsymbol{k}} = \overline{\hat{n}_{\boldsymbol{k}}}$, $\mathbb{E}\left[\hat{n}_{\boldsymbol{k}}\right] = 0$, and $\mathbb{E}\left[\hat{n}_{\boldsymbol{k}}\overline{\hat{n}_{\boldsymbol{k}'}}\right] = (2\pi)^d \delta(\boldsymbol{k} - \boldsymbol{k}')$.
(the representation is formal, one should use stochastic integrals $d\hat{W}_{\boldsymbol{k}} = \hat{n}_{\boldsymbol{k}} d\boldsymbol{k}$).

We have $\hat{n}_{\boldsymbol{k}} = \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} n(\boldsymbol{x}) d\boldsymbol{x}$ where $n(\boldsymbol{x})$ is a real white noise, i.e.:
$n(\boldsymbol{x})$ real-valued, Gaussian, $\mathbb{E}\left[n(\boldsymbol{x})\right] = 0$, and $\mathbb{E}\left[n(\boldsymbol{x})n(\boldsymbol{x}')\right] = \delta(\boldsymbol{x} - \boldsymbol{x}')$.
(in 1D, formally, $n(x) = dW_x/dx$).

- Simulation ($d = 1$): in order to simulate $(Z(x_1), \dots, Z(x_n))$, $x_j = (j-1)h$:
- compute the covariance vector $\boldsymbol{C} = (c(x_1), \dots, c(x_n))$.
- generate a random vector $\boldsymbol{G} = (G_1, \dots, G_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- filter with the square root of the Fourier transform of $\boldsymbol{C}$:

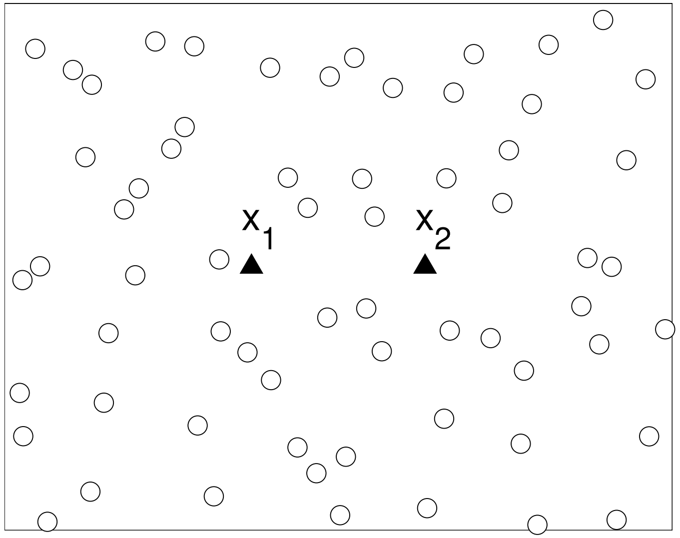
$$\boldsymbol{Z} = \mathrm{IFT}\big(\sqrt{\mathrm{DFT}(\boldsymbol{C})} \times \mathrm{DFT}(\boldsymbol{G})\big)$$

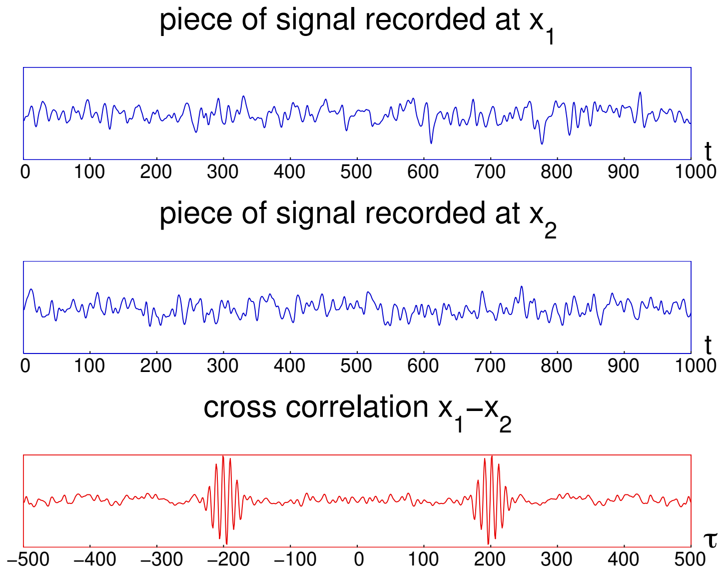
$\hookrightarrow$ $\boldsymbol{Z}$ is a realization of $(Z(x_1), \dots, Z(x_n))$ (in practice, use FFT and IFFT).

Passive imaging with noise sources

Green's function estimation by cross correlation of ambient noise signals

- Ambient noise sources ($\circ$) emit stationary random signals.
- The waves propagate in the (inhomogeneous) medium.
- The signals $u(t, \boldsymbol{x}_1)$ and $u(t, \boldsymbol{x}_2)$ are recorded at two sensors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.





- Compute the empirical cross correlation:

$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_1) u(t + \tau, \boldsymbol{x}_2) dt$$

- $C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$ is related to the Green's function from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$!

Wave equation

$$\frac{1}{c^2(\boldsymbol{x})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\boldsymbol{x}} u = n(t,\boldsymbol{x})$$

• Three-dimensional inhomogeneous medium (background velocity $c(\boldsymbol{x})$).

• Sources $n(t,\boldsymbol{x})$: Gaussian process, with mean zero, with covariance

$$\langle n(t_1,\boldsymbol{y}_1)n(t_2,\boldsymbol{y}_2)\rangle = F(t_2-t_1)\Gamma(\boldsymbol{y}_1,\boldsymbol{y}_2)$$

• Stationary in time: $\big(n(t,\boldsymbol{y})\big)_{t,\boldsymbol{y}}$ and $\big(n(t+h,\boldsymbol{y})\big)_{t,\boldsymbol{y}}$ have the same statistical distribution for any $h \Longrightarrow$ the time correlation function F depends only on t_2-t_1.

• The spatial distribution of the sources is characterized by $\Gamma(\boldsymbol{y}_1,\boldsymbol{y}_2)$. To simplify:

$$\Gamma(\boldsymbol{y}_1,\boldsymbol{y}_2) = K(\boldsymbol{y}_1)\delta(\boldsymbol{y}_1-\boldsymbol{y}_2)$$

The function K characterizes the spatial support of the sources.
[For more general Γ, use multiscale or microlocal analysis, see below.]

Remember: the Green's function

Solution u of the wave equation:

$$u(t,\boldsymbol{x}) = \int\int G(s,\boldsymbol{x},\boldsymbol{y}) n(t-s,\boldsymbol{y}) ds d\boldsymbol{y}$$

Green's function:

$$\frac{1}{c^2(\boldsymbol{x})}\frac{\partial^2 G}{\partial t^2} - \Delta_{\boldsymbol{x}} G = \delta(t)\delta(\boldsymbol{x}-\boldsymbol{y})$$

starting from $G(0,\boldsymbol{x},\boldsymbol{y}) = \partial_t G(0,\boldsymbol{x},\boldsymbol{y}) = 0 \quad (G(t,\boldsymbol{x},\boldsymbol{y}) = 0$ for $t<0)$.

Time-harmonic Green's function:

$$\hat{G}(\omega,\boldsymbol{x},\boldsymbol{y}) = \int G(t,\boldsymbol{x},\boldsymbol{y}) e^{i\omega t} dt$$

solution of

$$\frac{\omega^2}{c^2(\boldsymbol{x})}\hat{G} + \Delta_{\boldsymbol{x}}\hat{G} = -\delta(\boldsymbol{x}-\boldsymbol{y})$$

+ Sommerfeld radiation condition.

Example: If $c(\boldsymbol{x}) \equiv c_0$ (and in dimension 3)

$$G(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta\Big(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0} - t\Big)$$

$$\hat{G}(\omega,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} e^{i\omega\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}}$$

Empirical cross correlation and statistical cross correlation

Empirical cross correlation:

$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T} \int_0^T u(t, \boldsymbol{x}_1) u(t+\tau, \boldsymbol{x}_2) dt$$

with $u(t, \boldsymbol{x}) = \iint G(s, \boldsymbol{x}, \boldsymbol{y}) n(t-s, \boldsymbol{y}) ds d\boldsymbol{y}$ and $G =$ causal time-dependent Green's function.

1. The expectation of C_T (with respect to the distribution of the sources) is independent of the integration time T:

$$\langle C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \rangle = C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$$

where the statistical cross correlation $C^{(1)}$ is given by

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi} \int d\boldsymbol{y} \int d\omega \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y}) \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}) K(\boldsymbol{y}) \hat{F}(\omega) e^{-i\omega\tau}$$

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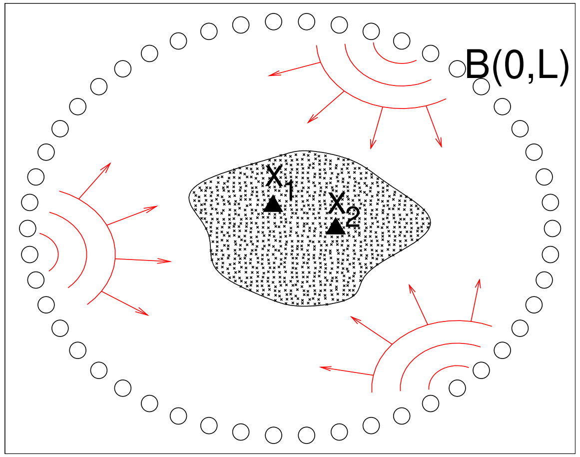
2. The empirical cross correlation is a self-averaging quantity:

$$C_T(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \xrightarrow{T \to \infty} C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)$$

in probability.
More precisely, the fluctuations of C_T around its expectation $C^{(1)}$ are of order $T^{-1/2}$.

Emergence of the Green's function for an extended distribution of sources in an inhomogeneous open medium



Cross correlation with noise sources distributed on a <u>closed surface</u> $\partial B(\mathbf{0}, L)$:

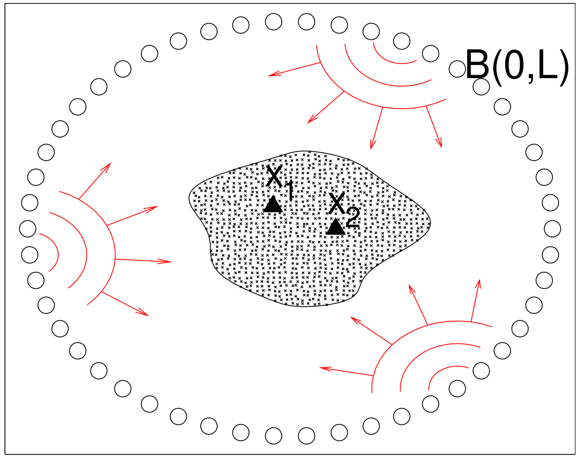
$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi} \int d\omega \int_{\partial B(\mathbf{0},L)} d\sigma(\boldsymbol{y}) \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y}) \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}) \hat{F}(\omega) e^{-i\omega\tau}$$

By Helmholtz-Kirchhoff identity,

$$\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2) - \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{2i\omega}{c_0} \int_{\partial B(\mathbf{0},L)} d\sigma(\boldsymbol{y}) \overline{\hat{G}}(\omega, \boldsymbol{x}_1, \boldsymbol{y}) \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y})$$

we have

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{c_0}{2\pi} \int \frac{\hat{F}(\omega)}{\omega} \mathrm{Im}\big(\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2)\big) e^{-i\omega\tau} d\omega$$

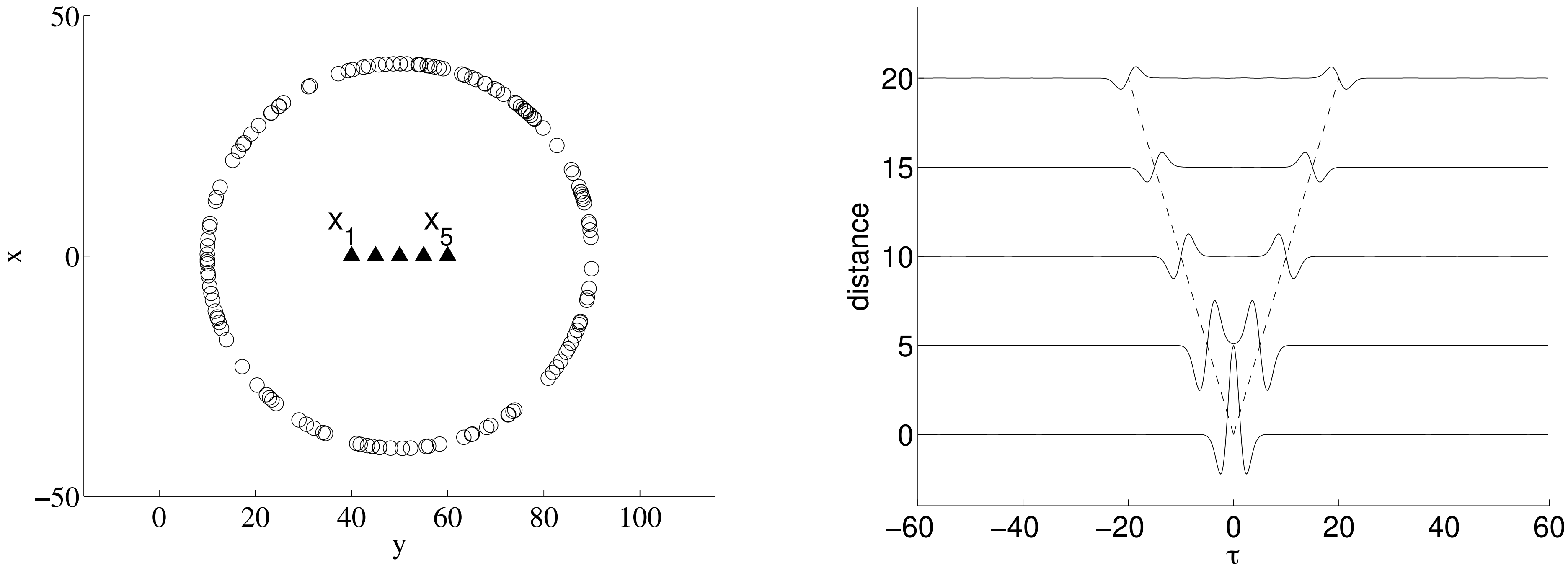


$$\partial_\tau C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = -\frac{ic_0}{2\pi}\int \hat{F}(\omega)\mathrm{Im}\big(\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2)\big)e^{-i\omega\tau}d\omega$$
$$= -\frac{c_0}{2}\Big(F *_\tau G(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) - F *_\tau G(-\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)\Big)$$

• The cross correlation of noise signals recorded by two passive sensors is related to the Green's function between the sensors.
$\hookrightarrow$ this is the classical proof that passive sensors can be transformed into virtual sources (known in seismology).

• This proof requires the noise sources to surround the region of interest.
The previous proofs also establish exact relations in ideal conditions.
Other proofs can justify approximate relations (enough for travel time estimation) in realistic conditions.

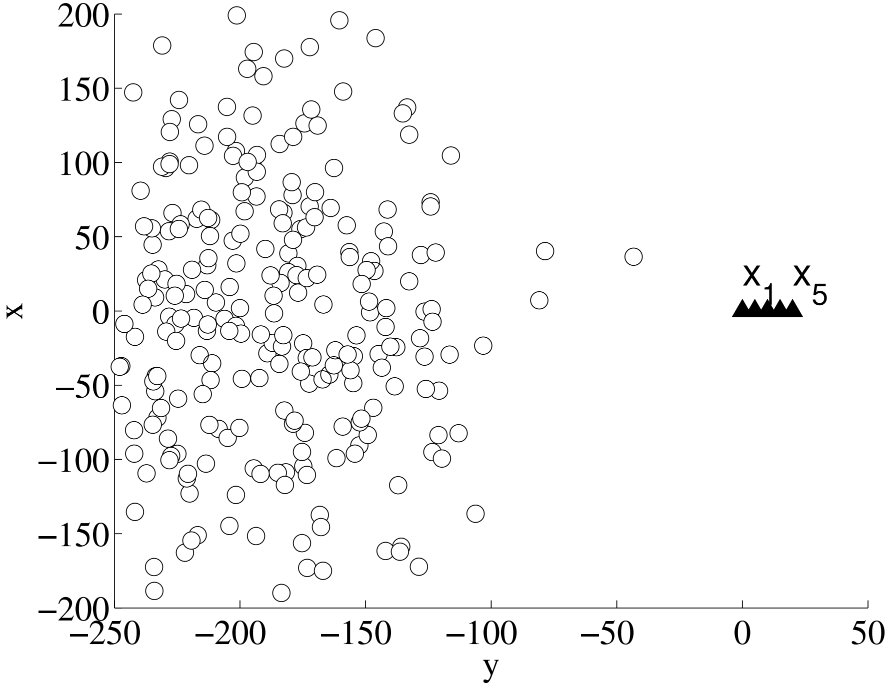
Ideal illumination:

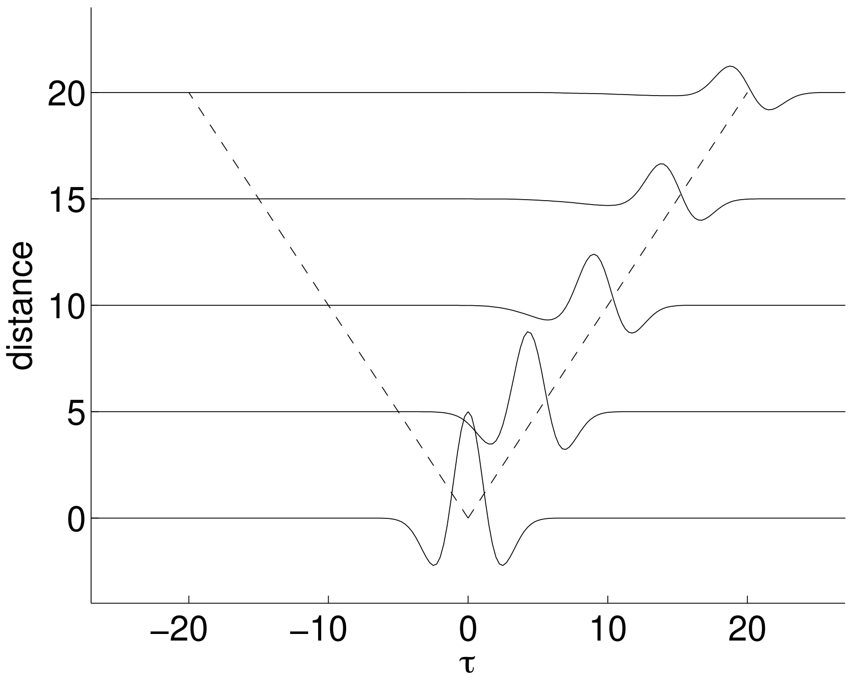


Left: The circles are the noise sources and the triangles are the sensors (the distance between two successive sensors is <u>5</u>). Here $\hat{F}(\omega) = \omega^2 \hat{f}(\omega)$, $\hat{f}(\omega) = \exp(-\omega^2)$, $c_0 = 1$.
Right: Cross correlation $\tau \to C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ between the pairs of sensors $(\boldsymbol{x}_1, \boldsymbol{x}_j)$, $j = 1, \ldots, 5$, versus the distance $|\boldsymbol{x}_j - \boldsymbol{x}_1|$. In theory:

$$
\begin{aligned}
C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j) &= \frac{c_0}{8\pi |\boldsymbol{x}_1 - \boldsymbol{x}_j|} \Big[f'\Big(\tau - \frac{|\boldsymbol{x}_j - \boldsymbol{x}_1|}{c_0}\Big) - f'\Big(\tau + \frac{|\boldsymbol{x}_j - \boldsymbol{x}_1|}{c_0}\Big)\Big], \quad j \geq 2 \\
C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_1) &= -\frac{1}{4\pi} f''(\tau)
\end{aligned}
$$

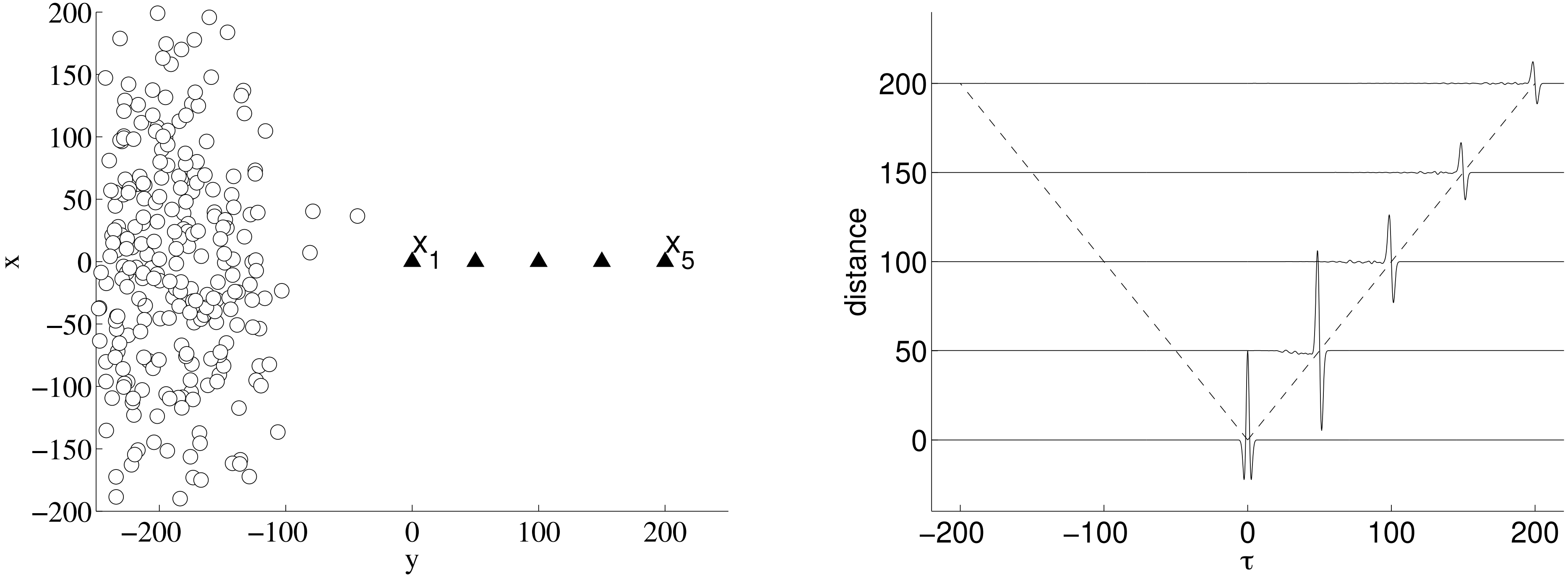
Realistic illumination:





Left: The circles are the noise sources and the triangles are the sensors (the distance between two successive sensors is 5).
Right: Cross correlation $\tau \to C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ between the pairs of sensors $(\boldsymbol{x}_1, \boldsymbol{x}_j)$, $j = 1, \ldots, 5$, versus the distance $|\boldsymbol{x}_j - \boldsymbol{x}_1|$.

Realistic illumination, high-frequency regime:



Left: The circles are the noise sources and the triangles are the sensors (the distance between two successive sensors is 50).

Right: Cross correlation $\tau \to C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_j)$ between the pairs of sensors $(\boldsymbol{x}_1, \boldsymbol{x}_j)$, $j = 1, \ldots, 5$, versus the distance $|\boldsymbol{x}_j - \boldsymbol{x}_1|$.