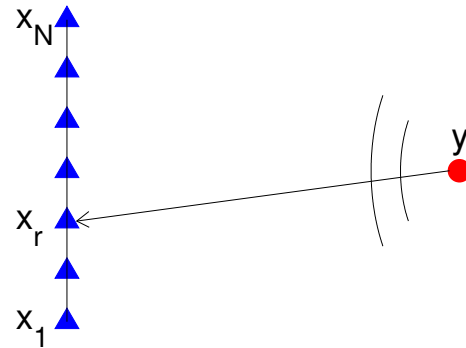


Source imaging

Source imaging: data acquisition



Passive array \iff The sensors are receivers.

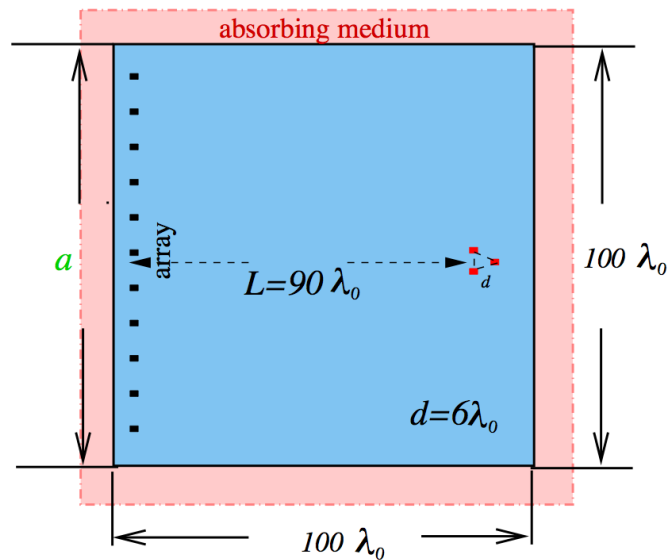
The source \mathbf{y} emits a short pulse.

The sensors $(\mathbf{x}_r)_{r=1,\dots,N}$ record the waves.

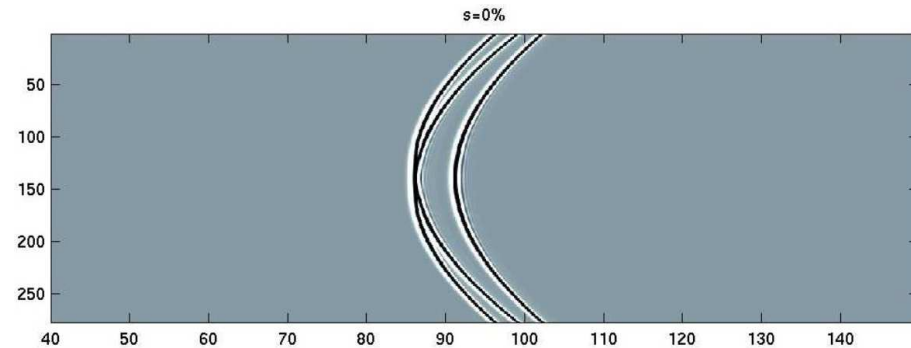
The data set is $(u(t, \mathbf{x}_r))_{t \in \mathbb{R}, r=1,\dots,N}$.

Goal: find the source position \mathbf{y} (more generally, find the *source* region).

Source imaging: simulation

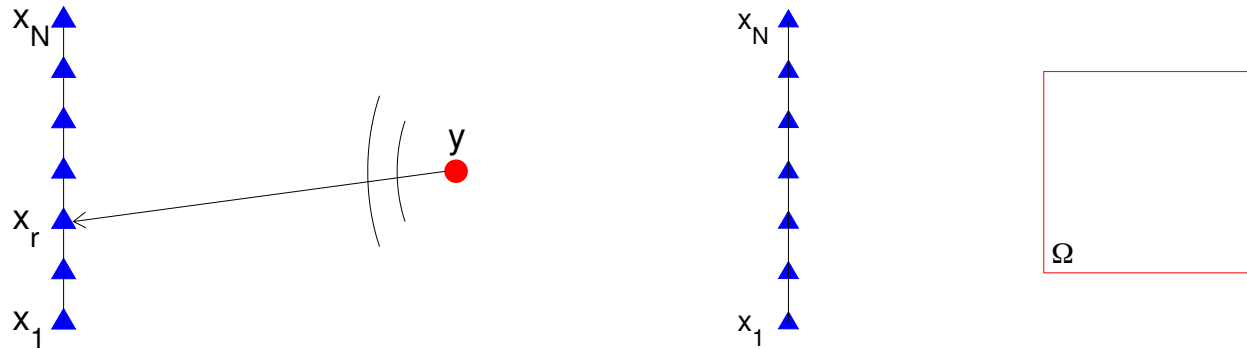


Configuration



Data set $(u(t, \mathbf{x}_r))_{80 \leq t \leq 200, r=1, \dots, 270}$

Source imaging: imaging function



Data acquisition

Search region

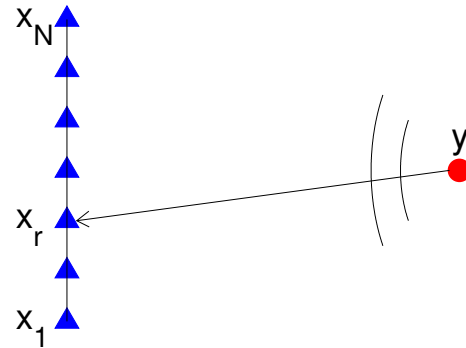
Goal: find the source point \mathbf{y} (more generally, find the *source* region).

The data set is $(u(t, \mathbf{x}_r))_{t \in \mathbb{R}, r=1, \dots, N}$.

\Leftrightarrow Given the data set, build an imaging function in the search region $\Omega \subset \mathbb{R}^3$:

$$\mathcal{I} : \begin{cases} \Omega \rightarrow \mathbb{R}^+ \\ \mathbf{y}^S \mapsto \mathcal{I}(\mathbf{y}^S) \end{cases} \quad \text{which plots an image of the search region.}$$

Source imaging - the linear forward operator



The source term is of the form $n(t, \mathbf{y}) = \rho_{\text{real}}(\mathbf{y})\delta(t)$.

Goal: find the source function ρ_{real} .

Here: the Green's function is known.

The data set is $\hat{\mathbf{u}} = (\hat{u}(\omega, \mathbf{x}_r))_{\omega \in \mathcal{B}, r=1, \dots, N}$ with $\hat{u}(\omega, \mathbf{x}_r) = \int_{\Omega} \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \rho_{\text{real}}(\mathbf{y}) d\mathbf{y}$.

We define

$$[\hat{\mathbf{A}}\rho](\omega, \mathbf{x}_r) = \int_{\Omega} \hat{G}(\omega, \mathbf{x}_r, \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y}.$$

$\hat{\mathbf{A}}$ is the linear operator that maps the source function to the array data $\hat{\mathbf{u}}$:

$$\hat{\mathbf{u}} = \hat{\mathbf{A}}\rho_{\text{real}}$$

Source imaging - the adjoint problem

The forward operator $\hat{\mathbf{A}}$ is the linear operator from $L^2(\Omega)$ to $L^2(\mathcal{B} \times \{1, \dots, N\})$:

$$[\hat{\mathbf{A}}\rho](\omega, \mathbf{x}_r) = \int_{\Omega} \hat{G}(\omega, \mathbf{x}_r, \mathbf{y})\rho(\mathbf{y})d\mathbf{y}.$$

The adjoint operator is such that, for all test functions $\hat{u}(\omega, \mathbf{x}_r)$ and $\rho(\mathbf{y})$:

$$\sum_{r=1}^N \int_{\mathcal{B}} \overline{\hat{u}(\omega, \mathbf{x}_r)} [\hat{\mathbf{A}}\rho](\omega, \mathbf{x}_r) d\omega = \int_{\Omega} \overline{[\hat{\mathbf{A}}^* \hat{u}](\mathbf{y})} \rho(\mathbf{y}) d\mathbf{y}$$

(Existence and uniqueness guaranteed by Riesz representation theorem)

The adjoint operator is

$$[\hat{\mathbf{A}}^* \hat{u}](\mathbf{y}) = \sum_{r=1}^N \int_{\mathcal{B}} \overline{\hat{G}(\omega, \mathbf{y}, \mathbf{x}_r)} \hat{u}(\omega, \mathbf{x}_r) d\omega$$

The normal operator $\hat{\mathbf{A}}^* \hat{\mathbf{A}}$ is the operator with kernel

$$[\hat{\mathbf{A}}^* \hat{\mathbf{A}}\rho](\mathbf{y}) = \int_{\Omega} a(\mathbf{y}, \mathbf{y}') \rho(\mathbf{y}') d\mathbf{y}'$$
$$a(\mathbf{y}, \mathbf{y}') = \sum_{r=1}^N \int_{\mathcal{B}} \overline{\hat{G}(\omega, \mathbf{y}, \mathbf{x}_r)} \hat{G}(\omega, \mathbf{y}', \mathbf{x}_r) d\omega$$

Source imaging - the inverse problem

The least squares inverse problem is to minimize $J[\rho]$ where

$$J[\rho] = \int d\omega \sum_r |\hat{u}(\omega, \mathbf{x}_r) - [\hat{\mathbf{A}}(\omega)\rho](\mathbf{x}_r)|^2$$

Solution:

$$\rho_{\text{LS}} = (\hat{\mathbf{A}}^* \hat{\mathbf{A}})^{-1} (\hat{\mathbf{A}}^* \hat{\mathbf{u}})$$

(Mercer's theorem gives an explicit representation of the kernel of the normal operator $\hat{\mathbf{A}}^* \hat{\mathbf{A}}$, which then gives an expression of the -regularized- inverse).

The adjoint operator is

$$[\hat{\mathbf{A}}^* \hat{\mathbf{u}}](\mathbf{y}) = \sum_r \int \overline{\hat{G}(\omega, \mathbf{y}, \mathbf{x}_r)} \hat{u}(\omega, \mathbf{x}_r) d\omega$$

Note: complex conjugation = time reversal.

Adjoint operator = backpropagation to the test point \mathbf{y} .

Source imaging - the reverse-time imaging function

- Least-squares imaging function:

$$\mathcal{I}_{\text{LS}}(\mathbf{y}^S) = \left[(\hat{\mathbf{A}}^* \hat{\mathbf{A}})^{-1} (\hat{\mathbf{A}}^* \hat{\mathbf{u}}) \right] (\mathbf{y}^S)$$

with

$$[\hat{\mathbf{A}}^* \hat{\mathbf{u}}](\mathbf{y}) = \sum_r \int \overline{\hat{G}(\omega, \mathbf{y}, \mathbf{x}_r)} \hat{u}(\omega, \mathbf{x}_r) d\omega$$

- The operator $\hat{\mathbf{A}}^* \hat{\mathbf{A}}$ is approximately proportional to the identity operator (related to time-reversal refocusing property) \rightarrow drop the normalizing factor in the LS function.

\Leftrightarrow Reverse Time imaging function for the search point \mathbf{y}^S :

$$\mathcal{I}_{\text{RT}}(\mathbf{y}^S) = \frac{1}{2\pi} [\hat{\mathbf{A}}^* \hat{\mathbf{u}}](\mathbf{y}^S) = \frac{1}{2\pi} \int d\omega \sum_r \overline{\hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r)} \hat{u}(\omega, \mathbf{x}_r)$$

Kirchhoff Migration (or travel-time migration) for source imaging

- Reverse Time imaging function for the search point \mathbf{y}^S :

$$\mathcal{I}_{\text{RT}}(\mathbf{y}^S) = \frac{1}{2\pi} \int d\omega \sum_r \overline{\hat{G}(\omega, \mathbf{y}^S, \mathbf{x}_r)} \hat{u}(\omega, \mathbf{x}_r)$$

- If we take $\hat{G}(\omega, \mathbf{x}, \mathbf{y}) \simeq \exp[i\omega\mathcal{T}(\mathbf{x}, \mathbf{y})]$, where $\mathcal{T}(\mathbf{x}, \mathbf{y})$ is the travel time from \mathbf{x} to \mathbf{y} , then we get the Kirchhoff Migration imaging function:

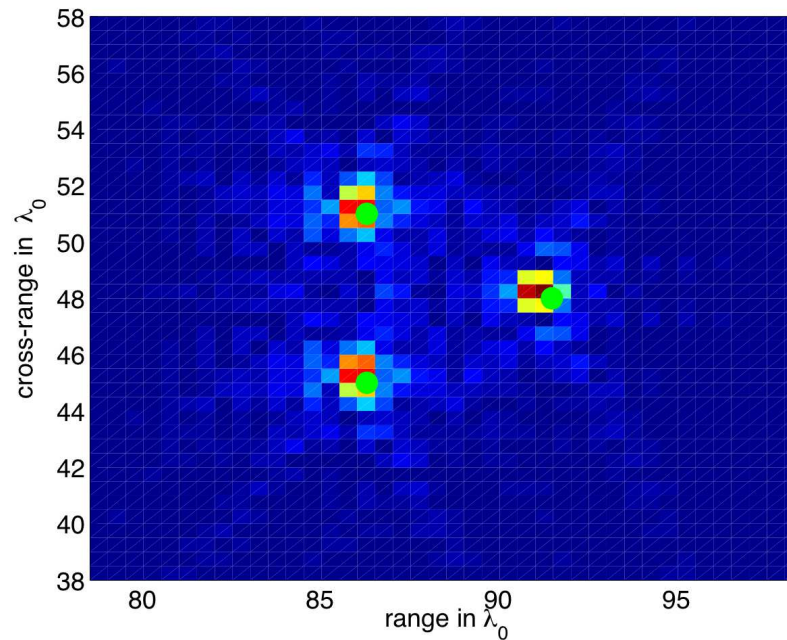
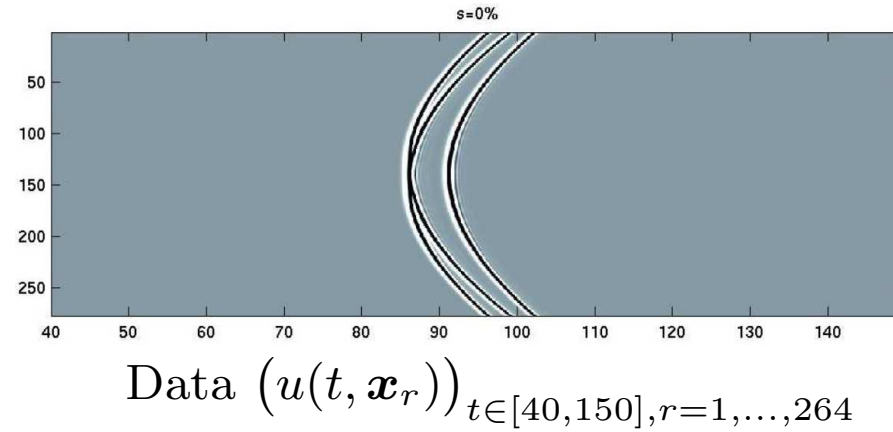
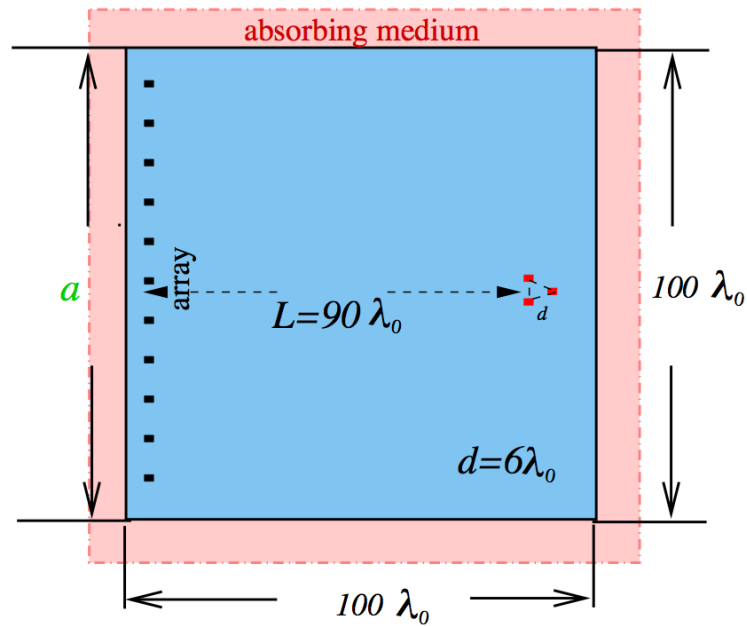
$$\begin{aligned} \mathcal{I}_{\text{KM}}(\mathbf{y}^S) &= \frac{1}{2\pi} \int d\omega \sum_r \exp[-i\omega\mathcal{T}(\mathbf{x}_r, \mathbf{y}^S)] \hat{u}(\omega, \mathbf{x}_r) \\ &= \sum_r u(\mathcal{T}(\mathbf{x}_r, \mathbf{y}^S), \mathbf{x}_r) \end{aligned}$$

If the background medium is homogeneous, then $\mathcal{T}(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|/c_o$.

Kirchhoff Migration (or travel time migration) has been analyzed in detail and is used extensively in practice.

Cf: N. Bleistein, J. K. Cohen, and J. W. Stockwell Jr, Mathematics of multidimensional seismic imaging, migration, and inversion, Springer, 2001.

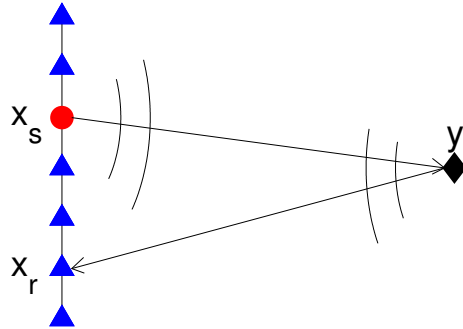
Kirchhoff migration for source imaging: simulation



KM imaging function
 $\mathcal{I}_{\text{KM}}(\mathbf{y}^S)$

Reflector imaging

Reflector imaging: data acquisition



Active array \iff The sensors can be used as sources and/or as receivers.

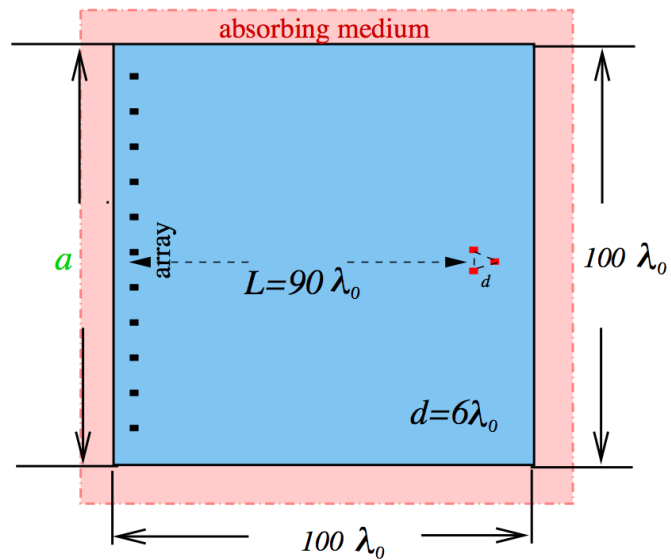
For each $s = 1, \dots, N$:

- The source \mathbf{x}_s emits a short pulse.
- The sensors $(\mathbf{x}_r)_{r=1, \dots, N}$ record the waves.

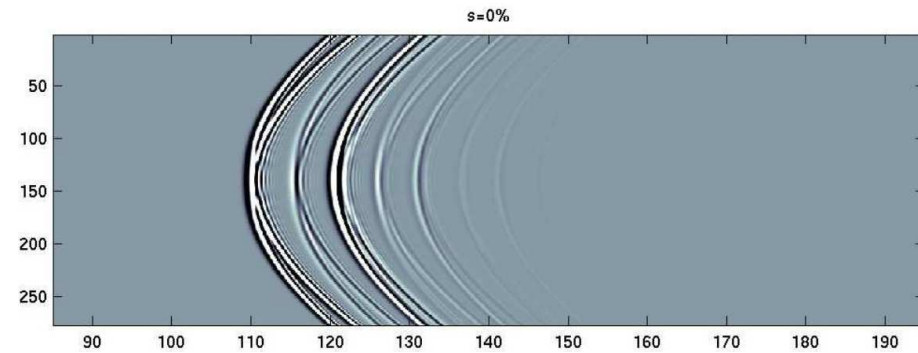
The data set is $(u(t, \mathbf{x}_r, \mathbf{x}_s))_{t \in \mathbb{R}, r, s=1, \dots, N}$ (the time-dependent response matrix).

Goal: find the reflector position \mathbf{y} (more generally, find the reflectivity function of the *medium*).

Reflector imaging: simulation

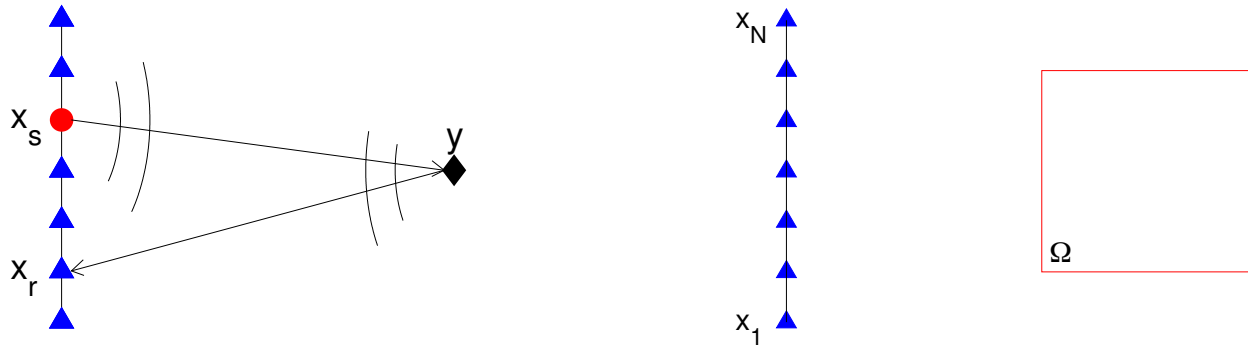


Configuration



Data set $(u(t, \mathbf{x}_r, \mathbf{x}_{135}))_{80 \leq t \leq 200, r=1, \dots, 270}$
(traces recorded for a central illumination)

Reflector imaging: imaging function



Data acquisition

Search region

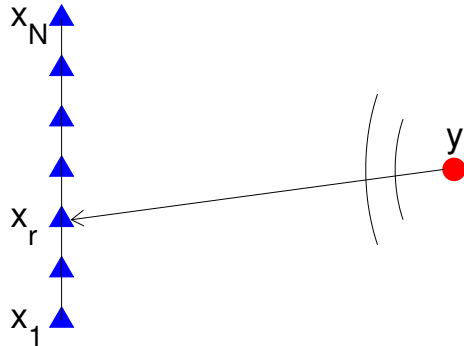
Goal: find the reflector position \mathbf{y} (more generally, find the reflectivity function of the *medium*).

The data set is $(u(t, \mathbf{x}_r, \mathbf{x}_s))_{t \in \mathbb{R}, r, s=1, \dots, N}$.

\Leftrightarrow Given the data set, build an imaging function in the search region $\Omega \subset \mathbb{R}^3$:

$$\mathcal{I} : \begin{cases} \Omega \rightarrow \mathbb{R}^+ \\ \mathbf{y}^S \mapsto \mathcal{I}(\mathbf{y}^S) \end{cases} \quad \text{which plots an image of the search region.}$$

Source and reflector imaging: comparison

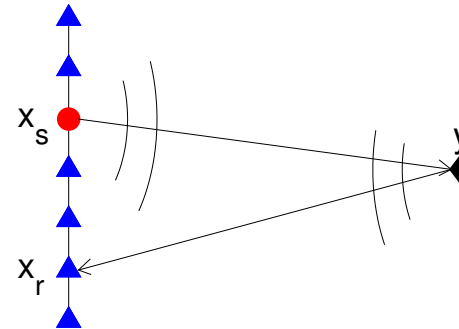


Passive array of sensors

The sensors $(\mathbf{x}_r)_{r=1,\dots,N}$ record

\mathbf{y} is a source

Data: $(u(t, \mathbf{x}_r))_{t \in \mathbb{R}, r=1,\dots,N}$



Active array of sensors/sources

The sensors $(\mathbf{x}_r)_{r=1,\dots,N}$ record

\mathbf{y} is a reflector, \mathbf{x}_s is a source

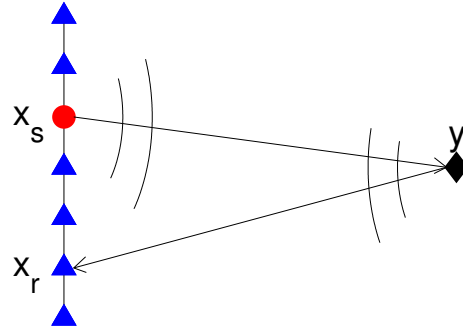
Data: $(u(t, \mathbf{x}_r, \mathbf{x}_s))_{t \in \mathbb{R}, r,s=1,\dots,N}$

Goal: process the data to find \mathbf{y} .

Passive and active array imaging: important features

- Different data acquisition geometries: Dense arrays or distributed networks, synthetic aperture arrays (zero-offset), full aperture imaging (medical).
- Point targets or extended targets.
- Narrowband (microwaves), broadband (ultrasound), and noise probing signals.
- Coherent (radar, sonar, seismic ...) and incoherent (infrared, optical tomography, X-ray tomography) imaging.
- Amount of data.
- Real time processing.
- Signal-to-Noise ratio (SNR) issues. Stability with respect to additive noise (electronic noise) and to partial knowledge of the medium (clutter noise).

Reflector imaging - modeling



Goal: find the propagation speed $c_{\text{real}}(\mathbf{x})$.

The time Fourier transform of the data $\hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s)$ is

$$\hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s) = \hat{G}(\omega, \mathbf{x}_r, \mathbf{x}_s; c_{\text{real}})$$

$\hat{G}(\omega, \mathbf{x}, \mathbf{y}; c)$ is the Green's function that solves the Helmholtz equation

$$\Delta \hat{G} + \frac{\omega^2}{c^2(\mathbf{x})} \hat{G} = -\delta(\mathbf{x} - \mathbf{y}),$$

with the Sommerfeld radiation condition. It depends on the velocity $c(\mathbf{x})$.

Reflector imaging - nonlinear inversion

Data:

$$\hat{\mathbf{u}} = (\hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s))_{\omega \in \mathcal{B}, r, s=1, \dots, N} = (\hat{G}(\omega, \mathbf{x}_r, \mathbf{x}_s, c_{\text{real}}))_{\omega \in \mathcal{B}, r, s=1, \dots, N}$$

Goal: find the propagation speed $c_{\text{real}}(\mathbf{x})$.

The inverse problem, the array least squares problem, is:

Minimize

$$J[c] \left(+ \alpha \|c\|_{\text{REG}}^2 \right),$$

where

$$J[c] = \int d\omega \sum_{r,s} |\hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s) - \hat{G}(\omega, \mathbf{x}_r, \mathbf{x}_s; c)|^2$$

(and α is a strength of regularization parameter).

→ this is a nonlinear problem for the unknown propagation speed $c(\mathbf{x})$.

Reflector imaging - linearization

$$\frac{1}{c^2(\mathbf{x})} = \frac{1}{c_o^2} (n_0^2(\mathbf{x}) + \rho(\mathbf{x}))$$

where

- c_o is a reference speed,
- $n_0(\mathbf{x})$ is a smooth background index of refraction (known, typically constant),
- $\rho(\mathbf{x})$ is the target reflectivity (unknown but small).

The Green's function satisfies:

$$\Delta \hat{G} + \frac{\omega^2}{c_o^2} (n_0^2(\mathbf{x}) + \rho(\mathbf{x})) \hat{G} = -\delta(\mathbf{x} - \mathbf{y})$$

The background Green's function is defined by:

$$\Delta \hat{G}_0 + \frac{\omega^2}{c_o^2} n_0^2(\mathbf{x}) \hat{G}_0 = -\delta(\mathbf{x} - \mathbf{y})$$

Born approximation:

$$\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \hat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \frac{\omega^2}{c_o^2} \int \hat{G}_0(\omega, \mathbf{x}, \mathbf{z}) \rho(\mathbf{z}) \hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) d\mathbf{z}$$

First term: direct waves.

Second term: single-scattered waves $\mathbf{y} \rightarrow \mathbf{z} \rightarrow \mathbf{x}$.

Proof of the Born approximation (1/2)

Consider

$$\begin{aligned}\Delta_{\mathbf{z}}\hat{G}(\omega, \mathbf{z}, \mathbf{x}) + \frac{\omega^2}{c_0^2}n_0^2(\mathbf{z})\hat{G}(\omega, \mathbf{z}, \mathbf{x}) &= -\frac{\omega^2}{c_0^2}\rho(\mathbf{z})\hat{G}(\omega, \mathbf{z}, \mathbf{x}) - \delta(\mathbf{z} - \mathbf{x}) \\ \Delta_{\mathbf{z}}\hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) + \frac{\omega^2}{c_0^2}n_0^2(\mathbf{z})\hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) &= -\delta(\mathbf{z} - \mathbf{y})\end{aligned}$$

We multiply the first equation by $\hat{G}_0(\omega, \mathbf{x}, \mathbf{y})$ and subtract the second equation multiplied by $\hat{G}(\omega, \mathbf{x}, \mathbf{z})$:

$$\begin{aligned}&\nabla_{\mathbf{z}} \cdot \left[\hat{G}_0(\omega, \mathbf{z}, \mathbf{y})\nabla_{\mathbf{z}}\hat{G}(\omega, \mathbf{z}, \mathbf{x}) - \hat{G}(\omega, \mathbf{z}, \mathbf{y})\nabla_{\mathbf{z}}\hat{G}_0(\omega, \mathbf{z}, \mathbf{x}) \right] \\ &= -\frac{\omega^2}{c_0^2}\rho(\mathbf{z})\hat{G}(\omega, \mathbf{z}, \mathbf{x})\hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) - \hat{G}_0(\omega, \mathbf{z}, \mathbf{y})\delta(\mathbf{z} - \mathbf{x}) + \hat{G}(\omega, \mathbf{z}, \mathbf{x})\delta(\mathbf{z} - \mathbf{y}) \\ &= -\frac{\omega^2}{c_0^2}\rho(\mathbf{z})\hat{G}(\omega, \mathbf{z}, \mathbf{x})\hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) - \hat{G}_0(\omega, \mathbf{x}, \mathbf{y})\delta(\mathbf{z} - \mathbf{x}) + \hat{G}(\omega, \mathbf{y}, \mathbf{x})\delta(\mathbf{z} - \mathbf{y}) \\ &\stackrel{\text{reciprocity}}{=} -\frac{\omega^2}{c_0^2}\rho(\mathbf{z})\hat{G}(\omega, \mathbf{x}, \mathbf{z})\hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) - \hat{G}_0(\omega, \mathbf{x}, \mathbf{y})\delta(\mathbf{z} - \mathbf{x}) + \hat{G}(\omega, \mathbf{x}, \mathbf{y})\delta(\mathbf{z} - \mathbf{y})\end{aligned}$$

We integrate over $B(\mathbf{0}, L)$ (with L large enough so that it encloses the support of ρ):

$$0 = -\frac{\omega^2}{c_0^2} \int \hat{G}(\omega, \mathbf{x}, \mathbf{z})\rho(\mathbf{z})\hat{G}_0(\omega, \mathbf{z}, \mathbf{y})d\mathbf{z} - \hat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \hat{G}(\omega, \mathbf{x}, \mathbf{y})$$

Proof of the Born approximation (2/2)

- Lippmann-Schwinger equation (exact):

$$\hat{G}(\omega, \mathbf{x}, \mathbf{y}) = \hat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \frac{\omega^2}{c_0^2} \int \hat{G}(\omega, \mathbf{x}, \mathbf{z}) \rho(\mathbf{z}) \hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) d\mathbf{z}$$

- Born approximation (approximate):

$$\hat{G}(\omega, \mathbf{x}, \mathbf{y}) \simeq \hat{G}_0(\omega, \mathbf{x}, \mathbf{y}) + \frac{\omega^2}{c_0^2} \int \hat{G}_0(\omega, \mathbf{x}, \mathbf{z}) \rho(\mathbf{z}) \hat{G}_0(\omega, \mathbf{z}, \mathbf{y}) d\mathbf{z}$$

Reflector imaging - the linearized forward operator

The data set is modeled by:

$$\hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s) = \frac{\omega^2}{c_o^2} \int \hat{G}_0(\omega, \mathbf{x}_r, \mathbf{z}) \rho(\mathbf{z}) \hat{G}_0(\omega, \mathbf{z}, \mathbf{x}_s) d\mathbf{z}$$

(remove $\hat{G}_0(\omega, \mathbf{x}_r, \mathbf{x}_s)$).

We define

$$[\hat{\mathbf{A}}\rho](\omega, \mathbf{x}_r, \mathbf{x}_s) = \int \hat{G}_0(\omega, \mathbf{x}_r, \mathbf{z}) \rho(\mathbf{z}) \hat{G}_0(\omega, \mathbf{z}, \mathbf{x}_s) d\mathbf{z}$$

It is the linear operator that maps the reflectivity function to the array data.

Reflector imaging - inversion

The least squares linearized inverse problem is to minimize $J_{\text{LS}}[\rho]$ where

$$J_{\text{LS}}[\rho] = \int d\omega \sum_{r,s} |\hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s) - [\hat{\mathbf{A}}\rho](\omega, \mathbf{x}_r, \mathbf{x}_s)|^2$$

Solution:

$$\rho_{\text{LS}} = (\hat{\mathbf{A}}^* \hat{\mathbf{A}})^{-1} (\hat{\mathbf{A}}^* \hat{\mathbf{u}})$$

The adjoint operator is

$$[\hat{\mathbf{A}}^* \hat{\mathbf{u}}](\mathbf{y}) = \sum_{r,s} \int \overline{\hat{G}_0(\omega, \mathbf{y}, \mathbf{x}_r) \hat{G}_0(\omega, \mathbf{x}_s, \mathbf{y})} \hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s) d\omega$$

Remember: complex conjugation = time reversal.

Adjoint operator = Backpropagation both from \mathbf{x}_r and from \mathbf{x}_s to the test point \mathbf{y} .

Kirchhoff migration (or travel-time migration) for reflector imaging

- Reverse-time migration (using the full Green's function for migration).

Imaging function for the search point \mathbf{y}^S :

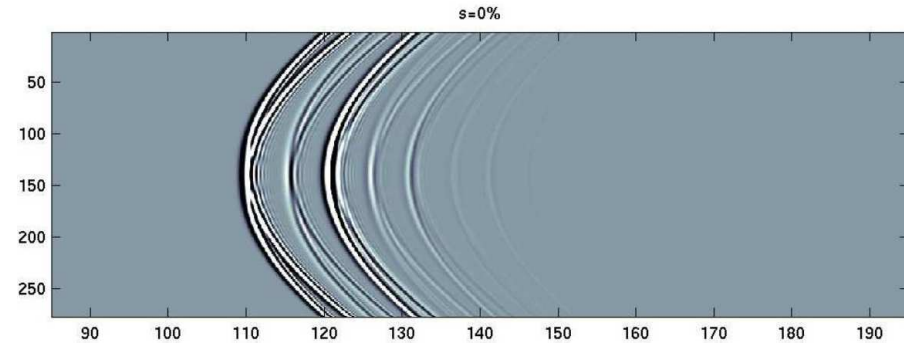
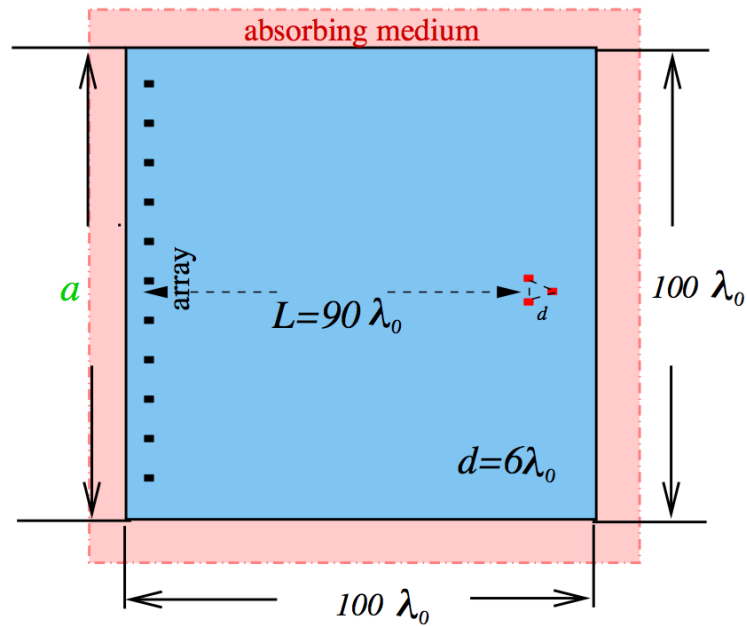
$$\mathcal{I}_{\text{RT}}(\mathbf{y}^S) = \frac{1}{2\pi} \int d\omega \sum_{r,s} \overline{\hat{G}_0(\omega, \mathbf{y}^S, \mathbf{x}_r)} \hat{G}_0(\omega, \mathbf{x}_s, \mathbf{y}^S) \hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s)$$

- Kirchhoff migration (or travel time migration).

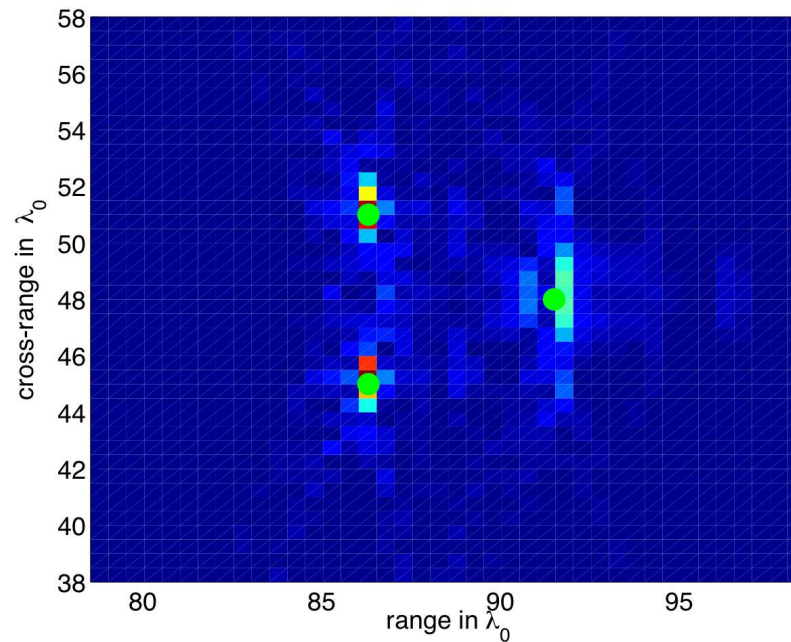
If we take $\hat{G}(\omega, \mathbf{x}, \mathbf{y}) \simeq \exp[i\omega\mathcal{T}(\mathbf{x}, \mathbf{y})]$, where $\mathcal{T}(\mathbf{x}, \mathbf{y})$ is the travel time from \mathbf{x} to \mathbf{y} , then we get the KM imaging function:

$$\begin{aligned} \mathcal{I}_{\text{KM}}(\mathbf{y}^S) &= \frac{1}{2\pi} \int d\omega \sum_{r,s} \exp[-i\omega(\mathcal{T}(\mathbf{x}_r, \mathbf{y}^S) + \mathcal{T}(\mathbf{x}_s, \mathbf{y}^S))] \hat{u}(\omega, \mathbf{x}_r, \mathbf{x}_s) \\ &= \sum_{r,s} u(\mathcal{T}(\mathbf{x}_r, \mathbf{y}^S) + \mathcal{T}(\mathbf{x}_s, \mathbf{y}^S), \mathbf{x}_r, \mathbf{x}_s) \end{aligned}$$

Kirchhoff migration for reflector imaging: simulation



Data $(u(t, \mathbf{x}_r, \mathbf{x}_{128}))_{t \in [80, 200], r=1, \dots, 264}$



KM imaging function
 $\mathcal{I}_{\text{KM}}(\mathbf{y}^S)$