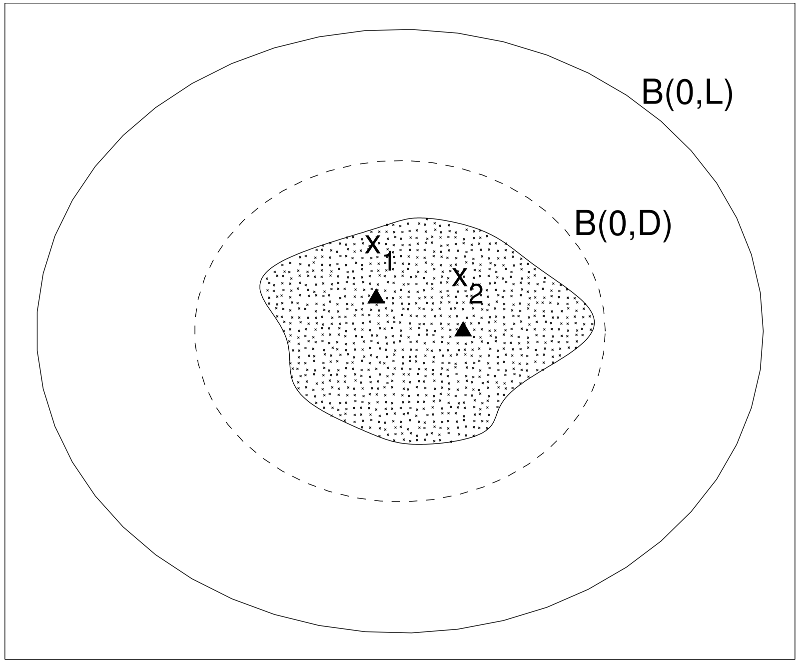
Reciprocity

$$\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{y}) = \hat{G}(\omega, \boldsymbol{y}, \boldsymbol{x})$$

A useful tool: The Helmholtz-Kirchhoff theorem



If the medium is homogeneous (velocity c_o) outside $B(\mathbf{0}, D)$, then $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\mathbf{0}, D)$ we have for $L \gg D$:

$$\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2) - \overline{\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2)} = \frac{2i\omega}{c_o} \int_{\partial B(\mathbf{0}, L)} d\sigma(\boldsymbol{y}) \overline{\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{y})} \hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y})$$

Proof: second Green's identity and Sommerfeld radiation condition.

Useful for: scattering theory, time reversal experiment, and cross correlation.

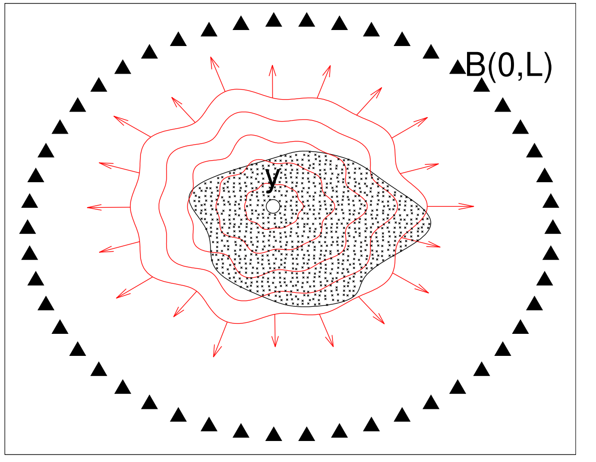
Time-reversal refocusing on a source (1/2)

First part:

A point source at $\boldsymbol{y}$ emits a pulse $f(t)$.

The waves are recorded at the surface $\partial B(\mathbf{0}, L)$:

$\hat{u}(\omega, \boldsymbol{x}) = \hat{G}(\omega, \boldsymbol{x}, \boldsymbol{y})\hat{f}(\omega), \quad \boldsymbol{x} \in \partial B(\mathbf{0}, L)$



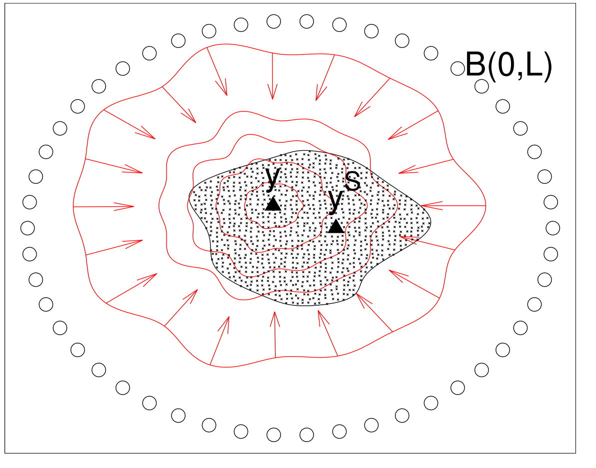
Second part:

The recorded signals are time-reversed

and sent back into the medium.

The signal received at $\boldsymbol{y}^S$ is

$$\hat{u}_{\mathrm{TR}}(\omega, \boldsymbol{y}^S) = \int_{\partial B(\mathbf{0},L)} d\sigma(\boldsymbol{x})\hat{G}(\omega, \boldsymbol{y}^S, \boldsymbol{x})\overline{\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{y})\hat{f}(\omega)}$$



Time-reversal refocusing on a source (2/2)

The signal received at $\boldsymbol{y}^S$ is (using reciprocity: $\hat{G}(\omega,\boldsymbol{x},\boldsymbol{y}) = \hat{G}(\omega,\boldsymbol{y},\boldsymbol{x})$):

$$\hat{u}_{\rm TR}(\omega,\boldsymbol{y}^S) = \int_{\partial B(\boldsymbol{0},L)} d\sigma(\boldsymbol{x}) \overline{\hat{G}(\omega,\boldsymbol{y},\boldsymbol{x})} \hat{G}(\omega,\boldsymbol{y}^S,\boldsymbol{x}) \overline{\hat{f}(\omega)}$$

By Helmholtz-Kirchhoff identity:

$$\hat{G}(\omega,\boldsymbol{y},\boldsymbol{y}^S) - \overline{\hat{G}(\omega,\boldsymbol{y},\boldsymbol{y}^S)} = \frac{2i\omega}{c_o} \int_{\partial B(\boldsymbol{0},L)} d\sigma(\boldsymbol{x}) \overline{\hat{G}(\omega,\boldsymbol{y},\boldsymbol{x})} \hat{G}(\omega,\boldsymbol{y}^S,\boldsymbol{x})$$

we get

$$\hat{u}_{\rm TR}(\omega,\boldsymbol{y}^S) = \frac{\hat{G}(\omega,\boldsymbol{y},\boldsymbol{y}^S) - \overline{\hat{G}(\omega,\boldsymbol{y},\boldsymbol{y}^S)}}{2i\omega/c_o} \overline{\hat{f}(\omega)} = \frac{c_o}{\omega} {\rm Im}\big(\hat{G}(\omega,\boldsymbol{y},\boldsymbol{y}^S)\big) \overline{\hat{f}(\omega)}$$

[Remember: $\boldsymbol{y}$ is the original source location].

In a three-dimensional homogeneous medium:

$$\hat{G}(\omega,\boldsymbol{y},\boldsymbol{y}^S) = \frac{1}{4\pi|\boldsymbol{y}-\boldsymbol{y}^S|} e^{i\frac{\omega|\boldsymbol{y}-\boldsymbol{y}^S|}{c_o}} \quad \Longrightarrow \quad \hat{u}_{\rm TR}(\omega,\boldsymbol{y}^S) = \frac{1}{4\pi} {\rm sinc}\Big(\frac{\omega|\boldsymbol{y}-\boldsymbol{y}^S|}{c_o}\Big) \overline{\hat{f}(\omega)}$$

$\hookrightarrow$ refocusing with a focal spot of diameter $\lambda/2 = \pi c_o/\omega$ (diffraction limit), ${\rm sinc}(s) = \frac{\sin(s)}{s}$.

Least-square inverse problems

Inverse problems

• We look for $\boldsymbol{m} \in \mathcal{X}$, the input parameters of a model, given the observed output $\boldsymbol{y} \in \mathcal{Y}$:

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{m}),$$

where $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$, $\mathcal{X}$ and $\mathcal{Y}$ are abstract spaces (Banach).

• Example: calibration of a model.
We have:
- a parametric family of models $f_{\boldsymbol{m}} : \mathbb{R}^d \to \mathbb{R}$, parameterized by $\boldsymbol{m} \in \mathbb{R}^p$,
- a sample of observations $(\boldsymbol{x}_i, y_i)_{i=1}^n$ with $y_i = f_{\boldsymbol{m}^*}(\boldsymbol{x}_i)(+\text{noise})$, where $\boldsymbol{m}^*$ is unknown.

We want to determine the $\boldsymbol{m}$ that best fits the data.
Here $\mathcal{X} = \mathbb{R}^p$, $\mathcal{Y} = \mathbb{R}^n$, $\boldsymbol{y} = (y_i)_{i=1}^n$ and $\boldsymbol{f}(\boldsymbol{m}) = (f_{\boldsymbol{m}}(\boldsymbol{x}_i))_{i=1}^n$.

- A problem is ill-posed (in Hadamard's sense) if one of three following events occurs:
  - there is no solution,
  - the solution is not unique,
  - the solution is very sensitive to the data $\boldsymbol{y}$.

- In order to solve an inverse problem, the classical approach is to formulate a least-square minimization problem:

$$\underset{\boldsymbol{m}\in\mathcal{X}}{\operatorname{argmin}} \ \|\boldsymbol{y}-\boldsymbol{f}(\boldsymbol{m})\|_{\mathcal{Y}}^{2}.$$

This approach can fail:

- theres is no solution in $\mathcal{X}$,
- there are several minima,
- the solution is very sensitive to the data $\boldsymbol{y}$.

Regularization of ill-posed inverse problems

- A way to regularize the problem is to consider:

$$\underset{\boldsymbol{m}\in\mathcal{X}}{\operatorname{argmin}} \Big(\|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{m})\|_{\mathcal{Y}}^2 + \|\boldsymbol{m} - \boldsymbol{m}_0\|_{\mathcal{X}'}^2 \Big),$$

with a norm $\|\cdot\|_{\mathcal{X}'}$ equal to or stronger than the norm $\|\cdot\|_{\mathcal{X}}$ and $\boldsymbol{m}_0 \in \mathcal{X}'$.

The penalization represents an a priori idea on the structure of the solution.

- Tikhonov regularization: $\|\cdot\|_{\mathcal{X}'} = \alpha\|\cdot\|_{\mathcal{X}}$ for some $\alpha > 0$.

A priori idea: the solution should not have a very large norm $\|\cdot\|_{\mathcal{X}}$.
If α is large enough, then the function to be minimized may become convex.
However: If α is too large, then the regularized problem becomes significantly different from the original problem.

Bayesian approach of inverse problems

• Principle: We do not look for a unique answer, but a probability measure on $\mathcal{X}$ which gives the likelihood of the states $\boldsymbol{m}$ given $\boldsymbol{y}$.

• We assume that:
- $\mathcal{X} = \mathbb{R}^p$ and $\mathcal{Y} = \mathbb{R}^n$.
- We have a priori information on the most likely states $\boldsymbol{m}$ in the form of an a priori distribution on $\mathcal{X}$ with density π_0 (w.r.t. the Lebesgue measure on $\mathcal{X}$).
- The observations are noisy:

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{m}) + \boldsymbol{\eta},$$

where $\boldsymbol{\eta}$ is a random variable taking values in $\mathcal{Y}$ with density ρ (w.r.t. the Lebesgue measure on $\mathcal{Y}$).

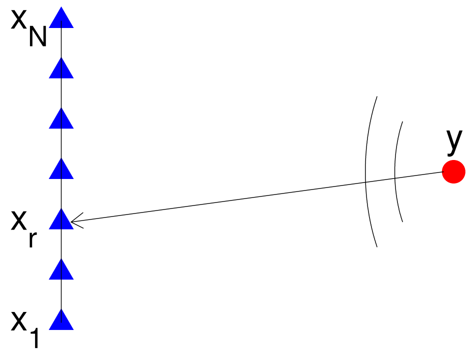
→ The likelihood (distribution of $\boldsymbol{y}$ given $\boldsymbol{m}$) has density $\rho(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{m}))$.

→ Bayes theorem: the a posteriori distribution of $\boldsymbol{m}$ given $\boldsymbol{y}$ has density:

$$\pi_{\boldsymbol{y}}(\boldsymbol{m}) = \frac{\rho(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{m}))\pi_0(\boldsymbol{m})}{\int_{\mathbb{R}^p} \rho(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{m}'))\pi_0(\boldsymbol{m}')d\boldsymbol{m}'}.$$

Source imaging

Source imaging: data acquisition



Passive array $\Longleftrightarrow$ The sensors are receivers.

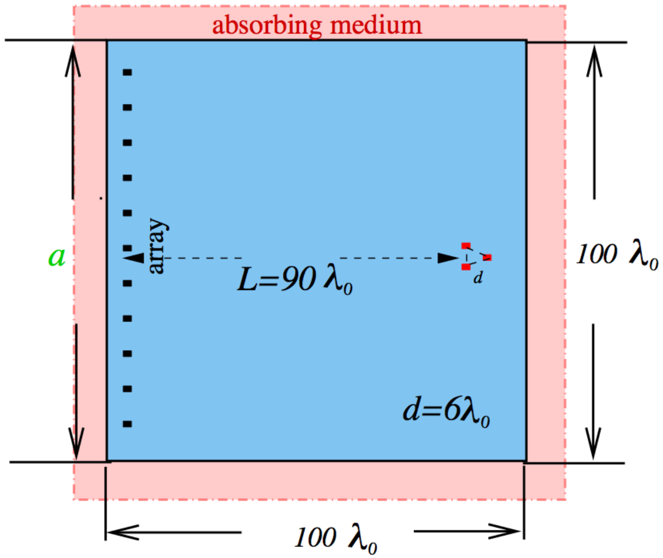
The source $\boldsymbol{y}$ emits a short pulse.

The sensors $(\boldsymbol{x}_r)_{r=1,\ldots,N}$ record the waves.

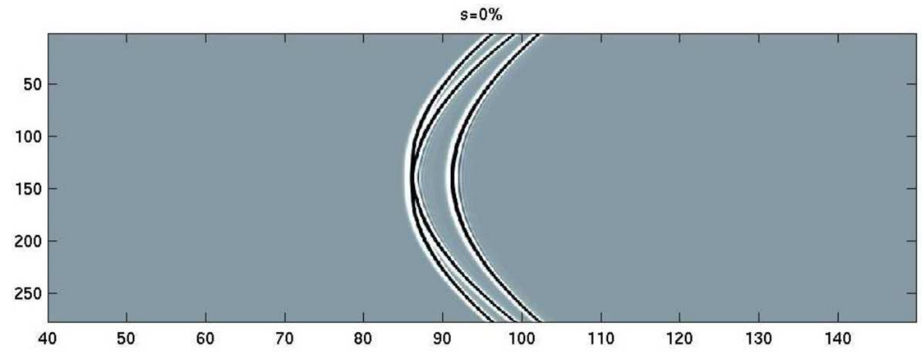
The data set is $\big(u(t,\boldsymbol{x}_r)\big)_{t\in\mathbb{R},r=1,\ldots,N}$.

Goal: find the source position $\boldsymbol{y}$ (more generally, find the *source* region).

Source imaging: simulation

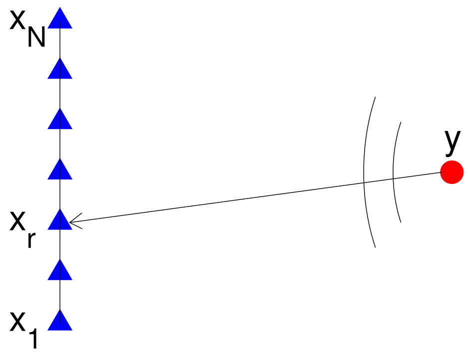


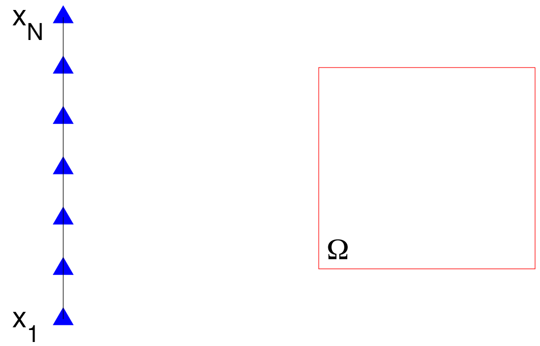
Configuration



Data set $(u(t, \boldsymbol{x}_r))_{80 \le t \le 200, r=1,\ldots,270}$

Source imaging: imaging function





Data acquisition Search region

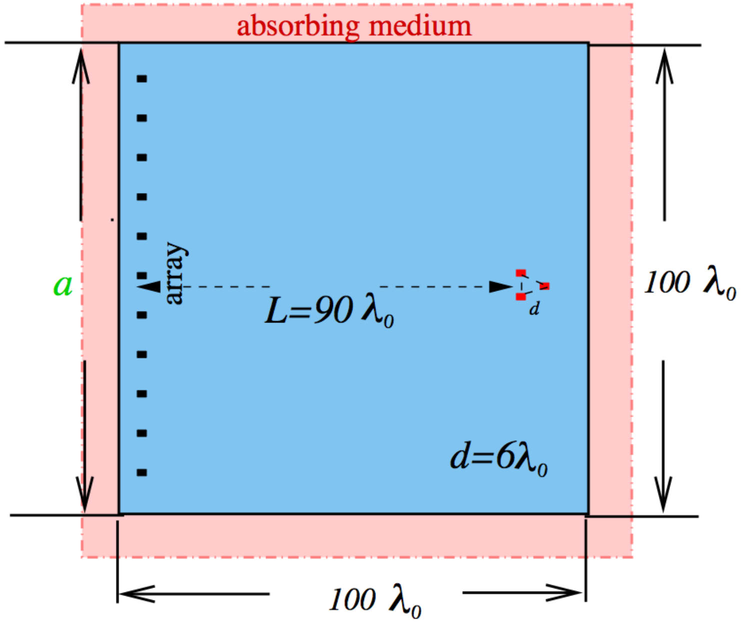
Goal: find the source point $\boldsymbol{y}$ (more generally, find the *source* region).

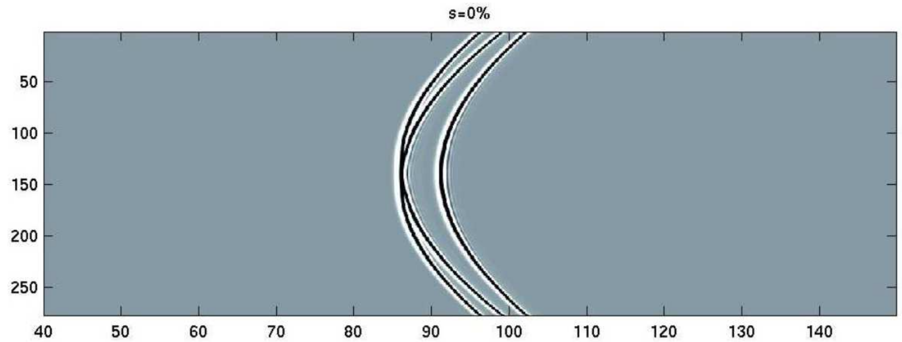
The data set is $\big(u(t,\boldsymbol{x}_r)\big)_{t\in\mathbb{R},r=1,\dots,N}$.

$\hookrightarrow$ Given the data set, build an imaging function in the search region $\Omega \subset \mathbb{R}^3$:

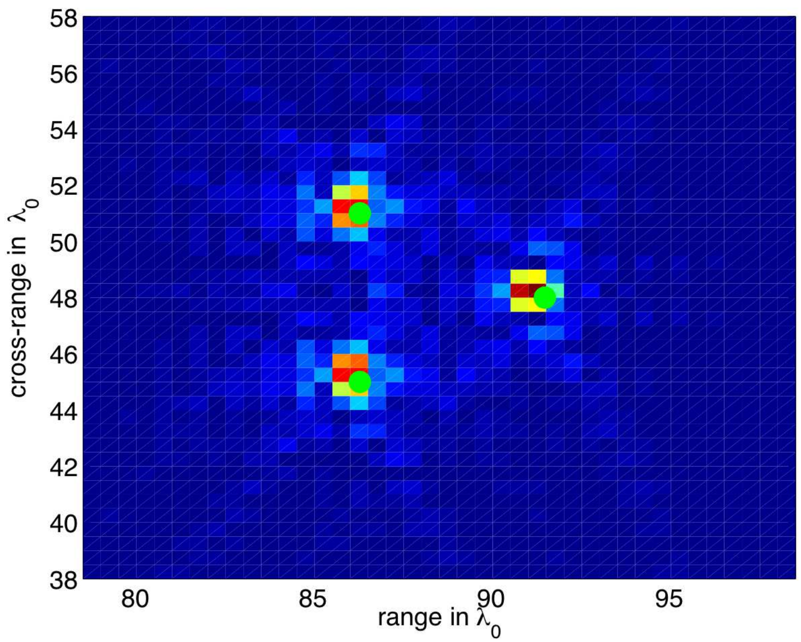
$$\mathcal{I}: \left| \begin{array}{l} \Omega \to \mathbb{R}^+ \\ \boldsymbol{y}^S \mapsto \mathcal{I}(\boldsymbol{y}^S) \end{array} \right.$$ which plots an image of the search region.

Kirchhoff migration for source imaging: simulation





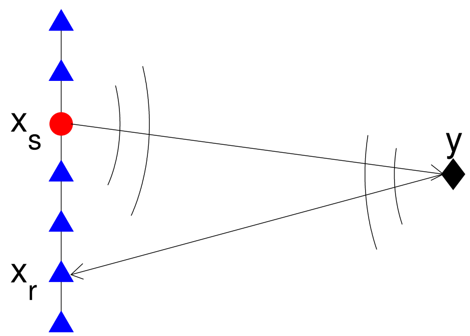
Data $\big(u(t,\boldsymbol{x}_r)\big)_{t\in[40,150],r=1,\ldots,264}$



KM imaging function $\mathcal{I}_{\rm KM}(\boldsymbol{y}^S)$

Reflector imaging

Reflector imaging: data acquisition



Active array $\Longleftrightarrow$ The sensors can be used as sources and/or as receivers.

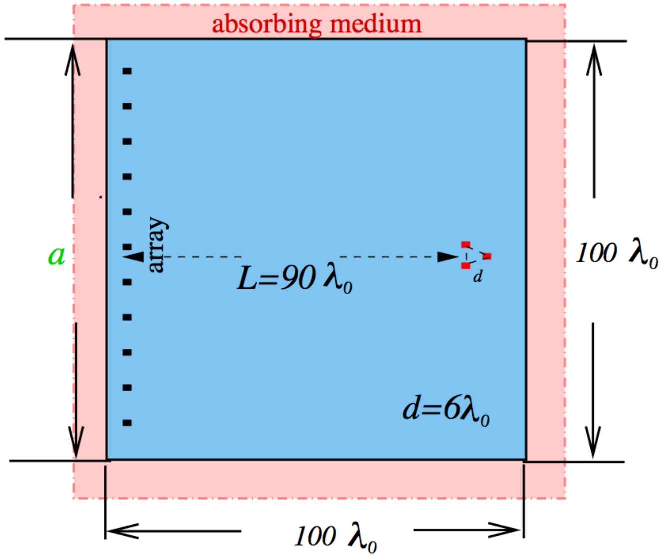
For each $s = 1, \ldots, N$:

- The source $\boldsymbol{x}_s$ emits a short pulse.
- The sensors $(\boldsymbol{x}_r)_{r=1,\ldots,N}$ record the waves.

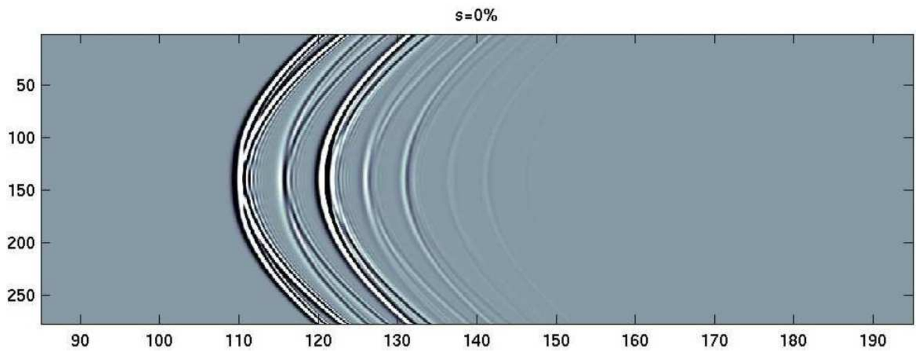
The data set is $(u(t, \boldsymbol{x}_r, \boldsymbol{x}_s))_{t \in \mathbb{R}, r,s=1,\ldots,N}$ (the time-dependent response matrix).

Goal: find the reflector position $\boldsymbol{y}$ (more generally, find the reflectivity function of the *medium*).

Reflector imaging: simulation

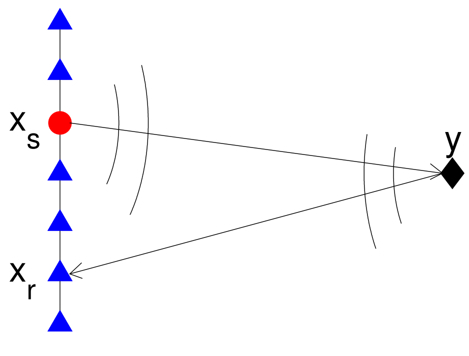


Configuration

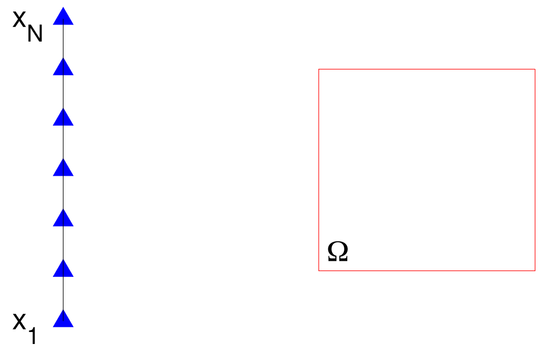


Data set $(u(t, \boldsymbol{x}_r, \boldsymbol{x}_{135}))_{80 \leq t \leq 200, r=1,\ldots,270}$
(traces recorded for a central illumination)

Reflector imaging: imaging function



Data acquisition



Search region

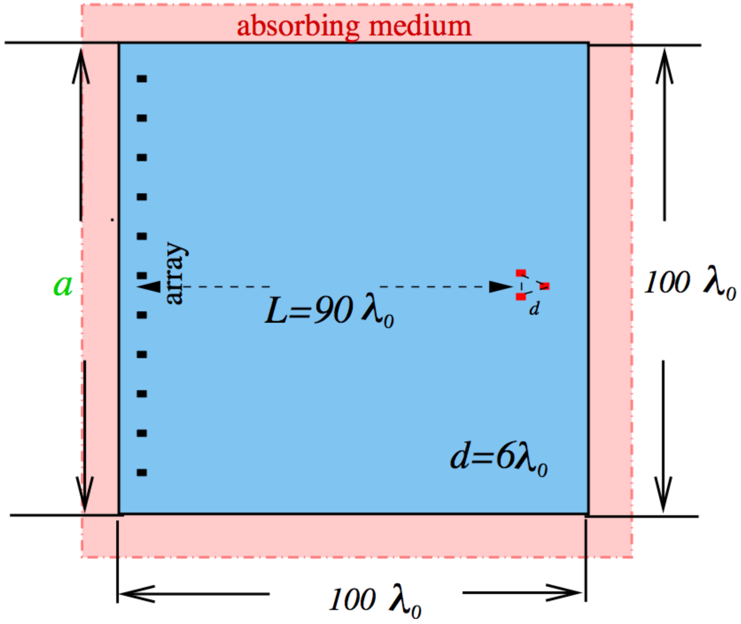
Goal: find the reflector position $\boldsymbol{y}$ (more generally, find the reflectivity function of the *medium*).

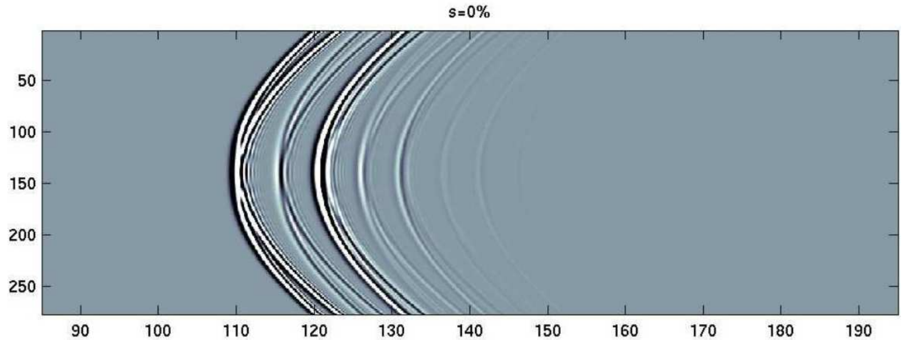
The data set is $(u(t, \boldsymbol{x}_r, \boldsymbol{x}_s))_{t \in \mathbb{R}, r,s=1,\dots,N}$.

$\hookrightarrow$ Given the data set, build an imaging function in the search region $\Omega \subset \mathbb{R}^3$:

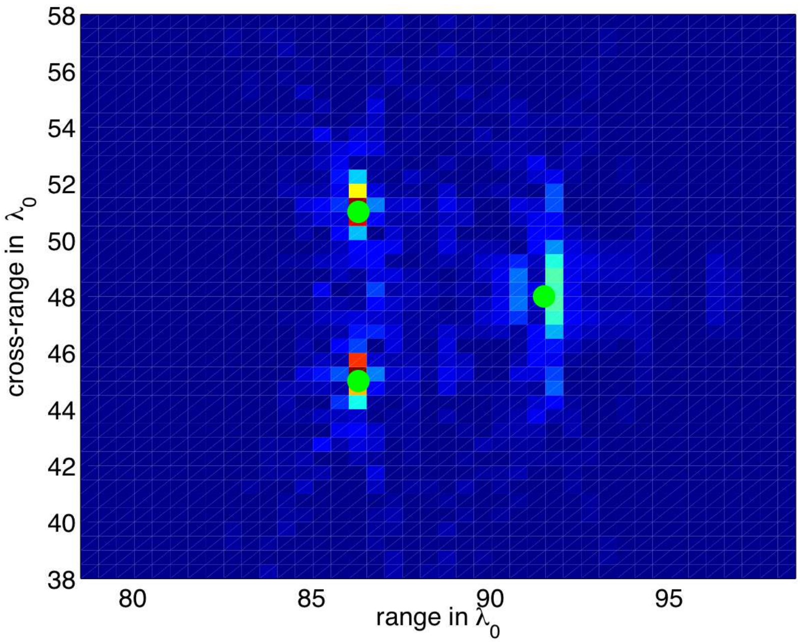
$\mathcal{I} : \left| \begin{array}{l} \Omega \to \mathbb{R}^+ \\ \boldsymbol{y}^S \mapsto \mathcal{I}(\boldsymbol{y}^S) \end{array} \right.$ which plots an image of the search region.

Kirchhoff migration for reflector imaging: simulation



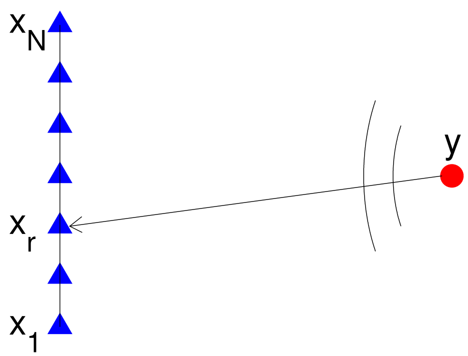


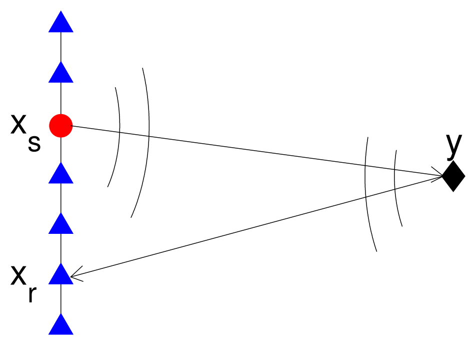
Data $\left(u(t, \boldsymbol{x}_r, \boldsymbol{x}_{128})\right)_{t\in[80,200], r=1,\ldots,264}$



KM imaging function $\mathcal{I}_{\rm KM}(\boldsymbol{y}^S)$

Source and reflector imaging: comparison





Passive array of sensors

The sensors $(\boldsymbol{x}_r)_{r=1,\ldots,N}$ record

$\boldsymbol{y}$ is a source

Data: $\big(u(t, \boldsymbol{x}_r)\big)_{t\in\mathbb{R}, r=1,\ldots,N}$

Active array of sensors/sources

The sensors $(\boldsymbol{x}_r)_{r=1,\ldots,N}$ record

$\boldsymbol{y}$ is a reflector, $\boldsymbol{x}_s$ is a source

Data: $\big(u(t, \boldsymbol{x}_r, \boldsymbol{x}_s)\big)_{t\in\mathbb{R}, r,s=1,\ldots,N}$

Goal: process the data to find $\boldsymbol{y}$.