

INTENSITY FLUCTUATIONS IN RANDOM WAVEGUIDES

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Abstract. An asymptotic analysis of wave propagation in randomly perturbed waveguides is carried out in order to identify the effective Markovian dynamics of the guided mode powers. The main result consists in a quantification of the fluctuations of the mode powers and wave intensities that increase exponentially with the propagation distance. The exponential growth rate is studied in detail so as to determine its dependence with respect to the waveguide geometry, the statistics of the random perturbations, and the operating wavelength.

Keywords. Waveguide; wave propagation in random media; diffusion approximation.

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1. Introduction We consider wave propagation in randomly perturbed waveguides. The random perturbations may affect the index of refraction within the core of the waveguide or the geometry of the core boundary. An asymptotic analysis based on a separation of scales technique can be applied when the amplitude of the random perturbation is small, its correlation length is of the same order as the operating wavelength, and the propagation distance is large so that the net effect of the perturbations is of order one. The overall result is that the scalar wavefield can be expanded on the complete basis of the modes of the unperturbed waveguide, that contains guided modes, radiating modes and evanescent modes, and the complex mode amplitudes of this decomposition follow an effective Markovian dynamics. In particular the guided mode powers form a Markovian process with a generator that describes random exchange of powers between the guided modes and power leakage (towards the radiating modes) that can be expressed as a deterministic mode-dependent dissipation. These results can be found in different forms in the physics literature [5, 7, 18] and in the mathematics literature [10, 13, 15]. In this paper we present a unified framework, we clarify the relationships between the mode-dependent dissipation coefficients and the statistics of the random perturbations, and we give a precise characterization of the mode power fluctuations, which is the main result of the paper and which can be summarized as follows.

The effective Markovian description of the guided mode powers makes it possible to analyze their first- and second-order moments (that are second- and fourth-order moments of the mode amplitudes), which in turn gives a statistical description of the intensity distribution of the wavefield. We find that the relative fluctuations of the intensity are, in general, characterized by an exponential growth with the propagation distance, whose rate can be defined as the difference of the first eigenvalues of two self-adjoint operators. When the effective dissipation is negligible, we recover the well-known equipartition result [8, 10]: The exponential growth rate is zero and the power becomes equipartitioned amongst the guided modes. When there is effective dissipation, the exponential growth rate can be positive, which means that power fluctuations may become very large, as first noticed in the physics literature by Creamer [6]. In fact we show that the exponential growth rate is positive as soon as two effective mode-dependent dissipation coefficients are different. The growth rate increases when the effective mode-dependent dissipation coefficients become more different, and it de-

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creases when the number of guided modes increases. We analyse a special regime, the continuum approximation, in which the operating frequency is large so that the number of guided modes becomes large. Under such circumstances, we find that the exponential growth rate vanishes. The exponential growth of the intensity fluctuations can, therefore, only be observed when there is a limited number of guided modes, and we recover the standard result that, in open random medium, the wavefield behaves like a Gaussian-distributed complex field for large propagation distances and the scintillation index that measures the relative intensity fluctuations becomes equal to one.

The paper is organized as follows. In Section 2 we formulate the problem and present the waveguide geometry. In Section 3 we review the spectral analysis of the ideal waveguide, when the medium inside the core is homogeneous and the boundaries are straight. In Section 4 we explain that the wavefield in the random waveguide can be expanded on the set of eigenmodes of the ideal waveguide and we identify the set of coupled equations satisfied by the mode amplitudes. In Section 5 we present the effective Markovian dynamics for the mode amplitudes and in Section 6 we remark that the mode powers also satisfy Markovian dynamics. The long-range behavior of the mean mode powers is described in Section 7, and the fluctuation analysis in Section 8 reveals that the normalized variance of the intensity grows exponentially with the propagation distance.

2. Wave propagation in waveguides Our model consists of a two-dimensional waveguide with range axis denoted by $z \in \mathbb{R}$ and transverse coordinate denoted by $x \in \mathbb{R}$ (see Figure 2.1). This may model a dielectric slab waveguide for instance. A point-like source at a fixed position $(x, z) = (x_s, 0)$ transmits a time-harmonic signal. The wavefield $p(x, z)$ satisfies the Helmholtz equation:

$$\left[(\partial_x^2 + \partial_z^2) + k^2 n^2(x, z) \right] p(x, z) = \delta(z) \delta(x - x_s), \quad (2.1)$$

for $(x, z) \in \mathbb{R}^2$, where k is the homogeneous wavenumber and $n(x, z)$ is the index of refraction at position (x, z) .

In the case of ideal (unperturbed) waveguides, the index of refraction is range-independent and equal to

$$n^{(0)}(x)^2 = \begin{cases} n^2 & \text{if } x \in (-d/2, d/2), \\ 1 & \text{otherwise,} \end{cases} \quad (2.2)$$

where $n > 1$ is the relative index of the core and $d > 0$ is its diameter.

We are interested in randomly perturbed waveguides. In this paper we address two types of random waveguides.

Type I perturbation: in the first type, the index of refraction within the core region $x \in (-d/2, d/2)$ is randomly perturbed [2, 4, 5, 13, 15]:

$$n^{(\varepsilon)}(x, z)^2 = \begin{cases} n^2 + \varepsilon \nu(x, z) & \text{if } x \in (-d/2, d/2) \text{ and } z \in (0, L^{(\varepsilon)}), \\ 1 & \text{otherwise.} \end{cases} \quad (2.3)$$

The fluctuations are modeled by the zero-mean, bounded, stationary in z random process $\nu(x, z)$ with smooth covariance function

$$\mathcal{R}_1(x, x', z') = \mathbb{E}[\nu(x, z) \nu(x', z + z')]. \quad (2.4)$$

It satisfies strong mixing conditions in z as defined for example in [19, section 2]. The typical amplitude of the fluctuations of index of refraction is assumed to be much smaller than 1 and it is modeled by the small and positive dimensionless parameter ε .

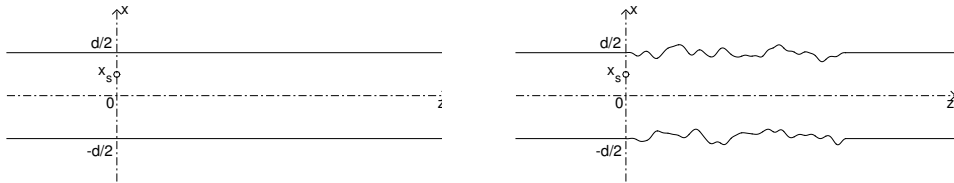


FIG. 2.1. *Left: An ideal two-dimensional waveguide. Right: A two-dimensional waveguide with cross-section perturbed by random fluctuations of the top and bottom boundaries. The point source is in the plane $z=0$.*

Type II perturbation: in the second type (see Figure 2.1), the boundaries of the core are randomly perturbed [1, 3, 14, 17, 18]:

$$n^{(\varepsilon)}(x, z)^2 = \begin{cases} n^2 & \text{if } x \in (\mathcal{D}_-^{(\varepsilon)}(z), \mathcal{D}_+^{(\varepsilon)}(z)) \text{ and } z \in (0, L^{(\varepsilon)}), \\ 1 & \text{otherwise,} \end{cases} \quad (2.5)$$

where

$$\mathcal{D}_-^{(\varepsilon)}(z) = -d/2 + \varepsilon d\nu_1(z), \quad (2.6)$$

$$\mathcal{D}_+^{(\varepsilon)}(z) = d/2 + \varepsilon d\nu_2(z). \quad (2.7)$$

The fluctuations are modeled by the zero-mean, bounded, independent and identically distributed stationary random processes ν_1 and ν_2 with smooth covariance function

$$\mathcal{R}_{\text{II}}(z') = \mathbb{E}[\nu_q(z)\nu_q(z+z')], \quad q=1,2. \quad (2.8)$$

They satisfy strong mixing conditions. The typical amplitude of the fluctuations of the boundaries is assumed to be much smaller than the core diameter d and it is modeled in (2.6-2.7) by the small and positive dimensionless parameter ε .

We study the wavefield at $z > 0$, satisfying

$$p(x, z) \in \mathcal{C}^0((0, +\infty), H^2(\mathbb{R})) \cap \mathcal{C}^2((0, +\infty), L^2(\mathbb{R})), \quad (2.9)$$

and to set radiation conditions, we have assumed that the random fluctuations are supported in the range interval $(0, L^{(\varepsilon)})$. We will see that net scattering effect of these fluctuations becomes of order one at range distances of order ε^{-2} , so we consider the interesting case $L^{(\varepsilon)} = L/\varepsilon^2$.

3. Homogeneous waveguide In this section, we consider an index of refraction of the form (2.2), which is stepwise constant. There is no fluctuation of the medium along the z -axis. The analysis of the perfect waveguide is classical [16, 21], we only give the main results. The Helmholtz operator has a spectrum of the form

$$(-\infty, k^2) \cup \{\beta_{N-1}^2, \dots, \beta_0^2\}, \quad (3.1)$$

where the N modal wavenumbers β_j are positive and $k^2 < \beta_{N-1}^2 < \dots < \beta_0^2 < n^2 k^2$. The generalized eigenfunctions $\phi_{t,\gamma}$, $t \in \{e, o\}$, associated to the spectral parameter γ in the continuous spectrum $(-\infty, k^2)$ and the eigenfunctions ϕ_j , $j=0, \dots, N-1$, associated to the discrete spectrum, are given in Appendix A. The generalized eigenfunctions $\phi_{e,\gamma}$ are even and $\phi_{o,\gamma}$ are odd. The eigenfunctions ϕ_j are even for even j and odd for odd j . Any

function can be expanded on the complete set of the eigenfunctions of the Helmholtz operator. In particular, any solution of the Helmholtz equation in homogeneous medium can be expanded as

$$p(x, z) = \sum_{j=0}^{N-1} p_j(z) \phi_j(x) + \sum_{t \in \{e, o\}} \int_{-\infty}^{k^2} p_{t, \gamma}(z) \phi_{t, \gamma}(x) d\gamma. \quad (3.2)$$

The modes for $j=0, \dots, N-1$ are guided, the modes for $\gamma \in (0, k^2)$ are radiating, the modes for $\gamma \in (-\infty, 0)$ are evanescent. Indeed, the complex mode amplitudes satisfy

$$\partial_z^2 p_j + \beta_j^2 p_j = 0, \quad j=0, \dots, N-1, \quad (3.3)$$

$$\partial_z^2 p_{t, \gamma} + \gamma p_{t, \gamma} = 0, \quad \gamma \in (-\infty, k^2), \quad (3.4)$$

for any $z \neq 0$. Therefore, if the source is of the form (2.1), we have for $z > 0$:

$$\begin{aligned} p(x, z) &= \sum_{j=0}^{N-1} \frac{a_{j, s}}{\sqrt{\beta_j}} e^{i\beta_j z} \phi_j(x) + \sum_{t \in \{e, o\}} \int_0^{k^2} \frac{a_{t, \gamma, s}}{\gamma^{1/4}} e^{i\sqrt{\gamma} z} \phi_{t, \gamma}(x) d\gamma \\ &+ \sum_{t \in \{e, o\}} \int_{-\infty}^0 \frac{a_{t, \gamma, s}}{|\gamma|^{1/4}} e^{-\sqrt{|\gamma|} z} \phi_{t, \gamma}(x) d\gamma, \end{aligned} \quad (3.5)$$

where the mode amplitudes are constant and determined by the source:

$$a_{j, s} = \frac{\sqrt{\beta_j}}{2} \phi_j(x_s), \quad j=0, \dots, N-1, \quad (3.6)$$

$$a_{t, \gamma, s} = \frac{|\gamma|^{1/4}}{2} \phi_{t, \gamma}(x_s), \quad \gamma \in (-\infty, k^2), \quad t \in \{e, o\}. \quad (3.7)$$

4. Random waveguide We consider the two types of random perturbations described in Section 2. In both cases we can write

$$n^2(x, z) = n^{(0)}(x)^2 + V^{(\varepsilon)}(x, z) \mathbf{1}_{(0, L^{(\varepsilon)})}(z),$$

where the perturbation is of the form

$$V^{(\varepsilon)}(x, z) = \varepsilon \nu(x, z) \quad (4.1)$$

for type I perturbations, and

$$\begin{aligned} V^{(\varepsilon)}(x, z) &= (n^2 - 1) \left[-\mathbf{1}_{(-d/2, -d/2 + \varepsilon d\nu_1(z))}(x) \mathbf{1}_{(0, +\infty)}(\nu_1(z)) \right. \\ &\quad \left. + \mathbf{1}_{(-d/2 + \varepsilon d\nu_1(z), -d/2)}(x) \mathbf{1}_{(-\infty, 0)}(\nu_1(z)) \right] \\ &+ (n^2 - 1) \left[\mathbf{1}_{(d/2, d/2 + \varepsilon d\nu_2(z))}(x) \mathbf{1}_{(0, +\infty)}(\nu_2(z)) \right. \\ &\quad \left. - \mathbf{1}_{(d/2 + \varepsilon d\nu_2(z), d/2)}(x) \mathbf{1}_{(-\infty, 0)}(\nu_2(z)) \right] \end{aligned} \quad (4.2)$$

for type II perturbations.

The solution of the perturbed Helmholtz equation (2.1) can be expanded as (3.2) and the complex mode amplitudes satisfy the coupled equations for $z \in (0, L^{(\varepsilon)})$:

$$\partial_z^2 p_j + \beta_j^2 p_j = -k^2 \sum_{l=0}^{N-1} C_{j, l}^{(\varepsilon)}(z) p_l - k^2 \sum_{t' \in \{e, o\}} \int_{-\infty}^{k^2} C_{j, t', \gamma'}^{(\varepsilon)}(z) p_{t', \gamma'} d\gamma', \quad (4.3)$$

for $j=0, \dots, N-1$,

$$\partial_z^2 p_{t,\gamma} + \gamma p_{t,\gamma} = -k^2 \sum_{l=0}^{N-1} C_{t,\gamma,l}^{(\varepsilon)}(z) p_l - k^2 \sum_{t' \in \{e,o\}} \int_{-\infty}^{k^2} C_{t,\gamma,t',\gamma'}^{(\varepsilon)}(z) p_{t',\gamma'} d\gamma', \quad (4.4)$$

for $\gamma \in (-\infty, k^2)$ and $t \in \{e,o\}$, with

$$C_{j,l}^{(\varepsilon)}(z) = \left(\phi_j, \phi_l V^{(\varepsilon)}(\cdot, z) \right)_{L^2}, \quad (4.5)$$

$$C_{j,t',\gamma'}^{(\varepsilon)}(z) = \left(\phi_j, \phi_{t',\gamma'} V^{(\varepsilon)}(\cdot, z) \right)_{L^2}, \quad (4.6)$$

$$C_{t,\gamma,l}^{(\varepsilon)}(z) = \left(\phi_{t,\gamma}, \phi_l V^{(\varepsilon)}(\cdot, z) \right)_{L^2}, \quad (4.7)$$

$$C_{t,\gamma,t',\gamma'}^{(\varepsilon)}(z) = \left(\phi_{t,\gamma}, \phi_{t',\gamma'} V^{(\varepsilon)}(\cdot, z) \right)_{L^2}, \quad (4.8)$$

and $(\cdot, \cdot)_{L^2}$ stands for the standard scalar product in $L^2(\mathbb{R})$ (see (A.2)). These equations are obtained by substituting the ansatz (3.2) into (2.1) and by projecting onto the eigenmodes.

From the definitions (4.1) or (4.2) of $V^{(\varepsilon)}(x, z)$ and the Taylor expansions of the eigenfunctions $\phi_j(x)$ and $\phi_{t,\gamma}(x)$ around $x = \pm d/2$, we obtain power series (in ε) expressions of the coefficients $C_{j,l}^{(\varepsilon)}$:

$$C_{j,l}^{(\varepsilon)}(z) = \varepsilon C_{j,l}(z) + \varepsilon^2 c_{j,l}(z) + o(\varepsilon^2), \quad (4.9)$$

$$C_{j,l}(z) = \begin{cases} (\phi_j, \phi_l \nu(\cdot, z))_{L^2} & \text{type I} \\ (n^2 - 1)d \left\{ -\nu_1(z) [\phi_j \phi_l] \left(-\frac{d}{2}\right) + \nu_2(z) [\phi_j \phi_l] \left(\frac{d}{2}\right) \right\} & \text{type II} \end{cases}, \quad (4.10)$$

$$c_{j,l}(z) = \begin{cases} 0 & \text{type I} \\ \frac{(n^2 - 1)d^2}{2} \left\{ -\nu_1^2(z) \partial_x [\phi_j \phi_l] \left(-\frac{d}{2}\right) + \nu_2^2(z) \partial_x [\phi_j \phi_l] \left(\frac{d}{2}\right) \right\} & \text{type II} \end{cases}, \quad (4.11)$$

and similarly for $C_{j,t,\gamma}^{(\varepsilon)}$, $C_{t,\gamma,l}^{(\varepsilon)}$, and $C_{t,\gamma,t',\gamma'}^{(\varepsilon)}$.

We finally introduce the generalized forward-going and backward-going mode amplitudes:

$$\{a_j(z), b_j(z), j=0, \dots, N-1\} \quad \text{and} \quad \{a_{t,\gamma}(z), b_{t,\gamma}(z), \gamma \in (0, k^2)\}, \quad (4.12)$$

for $t \in \{e,o\}$, which are defined such that

$$\begin{aligned} p_j(z) &= \frac{1}{\sqrt{\beta_j}} \left(a_j(z) e^{i\beta_j z} + b_j(z) e^{-i\beta_j z} \right), \\ \partial_z p_j(z) &= i\sqrt{\beta_j} \left(a_j(z) e^{i\beta_j z} - b_j(z) e^{-i\beta_j z} \right), \quad j=0, \dots, N-1, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} p_{t,\gamma}(z) &= \frac{1}{\gamma^{1/4}} \left(a_{t,\gamma}(z) e^{i\sqrt{\gamma}z} + b_{t,\gamma}(z) e^{-i\sqrt{\gamma}z} \right), \\ \partial_z p_{t,\gamma}(z) &= i\gamma^{1/4} \left(a_{t,\gamma}(z) e^{i\sqrt{\gamma}z} - b_{t,\gamma}(z) e^{-i\sqrt{\gamma}z} \right), \quad \gamma \in (0, k^2), \quad t \in \{e,o\}. \end{aligned} \quad (4.14)$$

We can substitute (4.13–4.14) into (4.3–4.4) in order to obtain the first-order system of coupled random differential equations satisfied by the mode amplitudes (4.12):

$$\begin{aligned} \partial_z a_j(z) &= \frac{ik^2}{2} \sum_{l'=0}^{N-1} \frac{C_{j,l'}^{(\varepsilon)}(z)}{\sqrt{\beta_{l'}\beta_j}} \left[a_{l'}(z)e^{i(\beta_{l'}-\beta_j)z} + b_{l'}(z)e^{i(-\beta_{l'}-\beta_j)z} \right] \\ &+ \frac{ik^2}{2} \sum_{t' \in \{e,o\}} \int_0^{k^2} \frac{C_{j,t',\gamma'}^{(\varepsilon)}(z)}{\sqrt[4]{\gamma'}\sqrt{\beta_j}} \left[a_{t',\gamma'}(z)e^{i(\sqrt{\gamma'}-\beta_j)z} + b_{t',\gamma'}(z)e^{i(-\sqrt{\gamma'}-\beta_j)z} \right] d\gamma' \\ &+ \frac{ik^2}{2} \sum_{t' \in \{e,o\}} \int_{-\infty}^0 \frac{C_{j,t',\gamma'}^{(\varepsilon)}(z)}{\sqrt{\beta_j}} p_{t',\gamma'}(z)e^{-i\beta_j z} d\gamma', \end{aligned} \quad (4.15)$$

$$\begin{aligned} \partial_z a_{t,\gamma}(z) &= \frac{ik^2}{2} \sum_{l'=0}^{N-1} \frac{C_{t,\gamma,l'}^{(\varepsilon)}(z)}{\sqrt[4]{\gamma}\sqrt{\beta_{l'}}} \left[a_{l'}(z)e^{i(\beta_{l'}-\sqrt{\gamma})z} + b_{l'}(z)e^{i(-\beta_{l'}-\sqrt{\gamma})z} \right] \\ &+ \frac{ik^2}{2} \sum_{t' \in \{e,o\}} \int_0^{k^2} \frac{C_{t,\gamma,t',\gamma'}^{(\varepsilon)}(z)}{\sqrt[4]{\gamma'}\gamma} \left[a_{t',\gamma'}(z)e^{i(\sqrt{\gamma'}-\sqrt{\gamma})z} + b_{t',\gamma'}(z)e^{i(-\sqrt{\gamma'}-\sqrt{\gamma})z} \right] d\gamma' \\ &+ \frac{ik^2}{2} \sum_{t' \in \{e,o\}} \int_{-\infty}^0 \frac{C_{t,\gamma,t',\gamma'}^{(\varepsilon)}(z)}{\sqrt[4]{\gamma}} p_{t',\gamma'}(z)e^{-i\sqrt{\gamma}z} d\gamma', \end{aligned} \quad (4.16)$$

with similar equations for b_j and $b_{t,\gamma}$. This system is complemented with the boundary conditions at $z=0$ and $z=L^{(\varepsilon)}$:

$$a_j(0) = a_{j,s}, \quad b_j(L^{(\varepsilon)}) = 0, \quad a_{t,\gamma}(0) = a_{t,\gamma,s}, \quad b_{t,\gamma}(L^{(\varepsilon)}) = 0,$$

where $a_{j,s}$ and $a_{t,\gamma,s}$ are defined by (3.6–3.7). The evanescent mode amplitudes $p_{t,\gamma}$, $t \in \{e,o\}$, $\gamma \in (-\infty, 0)$, satisfy (4.4).

5. The effective Markovian dynamics for the mode amplitudes We re-name the complex mode amplitudes in the long-range scaling as

$$a_j^\varepsilon(z) = a_j\left(\frac{z}{\varepsilon^2}\right), \quad b_j^\varepsilon(z) = b_j\left(\frac{z}{\varepsilon^2}\right), \quad j = 0, \dots, N-1, \quad (5.1)$$

$$a_{t,\gamma}^\varepsilon(z) = a_{t,\gamma}\left(\frac{z}{\varepsilon^2}\right), \quad b_{t,\gamma}^\varepsilon(z) = b_{t,\gamma}\left(\frac{z}{\varepsilon^2}\right), \quad \gamma \in (0, k^2), \quad t \in \{e,o\}. \quad (5.2)$$

We can follow the lines of [13] to get the following results.

1) In the regime $\varepsilon \ll 1$ the evanescent mode amplitudes, that satisfy (4.4), can be expressed to leading order in closed forms as functions of the guided and radiating mode amplitudes (5.1–5.2). Indeed it is possible to invert the operator $\partial_z^2 + \gamma$ in (4.4) for $\gamma < 0$ by using the Green's function that satisfies the radiation condition and to obtain:

$$\begin{aligned} p_{t,\gamma}\left(\frac{z}{\varepsilon^2}\right) &= \frac{\varepsilon k^2}{2\sqrt{|\gamma|}} \int_0^{L/\varepsilon^2} \sum_{l'=0}^{N-1} \left\{ \frac{C_{t,\gamma,l'}(z')}{\sqrt{\beta_{l'}}} \left[a_{l'}^\varepsilon(z')e^{i\beta_{l'}z'} + b_{l'}^\varepsilon(z')e^{-i\beta_{l'}z'} \right] \right. \\ &+ \left. \int_0^{k^2} \frac{C_{t,\gamma,t',\gamma'}(z')}{\sqrt[4]{\gamma'}} \left[a_{t',\gamma'}^\varepsilon(z')e^{i\sqrt{\gamma'}z'} + b_{t',\gamma'}^\varepsilon(z')e^{-i\sqrt{\gamma'}z'} \right] d\gamma' \right\} e^{-\sqrt{|\gamma|}\left|\frac{z}{\varepsilon^2}-z'\right|} dz' \\ &+ O(\varepsilon^2), \end{aligned} \quad (5.3)$$

for $z > 0, \gamma < 0$ and $t \in \{e, o\}$. Here we recognize $G_\gamma(z, z') = \frac{1}{2\sqrt{|\gamma|}} e^{-\sqrt{|\gamma|}|z-z'|}$ that is the Green's function of the equation $\partial_z^2 G_\gamma(z, z') + \gamma G_\gamma(z, z') = -\delta(z - z')$ for $\gamma < 0$.

2) Under the assumption that the power spectral density $\widehat{\mathcal{R}}_{\text{I}}(\kappa, x, x')$ for type-I perturbations (or $\widehat{\mathcal{R}}_{\text{II}}(\kappa)$ for type-II perturbations) has compact support or fast decay, the forward-scattering approximation can be proved, i.e. the coupling between forward-going and backward-going mode amplitudes is negligible, so that we have

$$b_j^\varepsilon(z) \approx 0, \quad j = 0, \dots, N-1, \quad b_{t,\gamma}^\varepsilon(z) \approx 0, \quad \gamma \in (0, k^2), \quad t \in \{e, o\}.$$

3) The forward-going guided mode amplitudes $(a_j^\varepsilon)_{j=0}^{N-1}$ and radiating mode amplitudes $(a_{t,\gamma}^\varepsilon)_{\gamma \in (0, k^2), t \in \{e, o\}}$ then satisfy a closed linear system of the form

$$\frac{d\mathbf{a}^\varepsilon}{dz} = \frac{1}{\varepsilon} \mathbf{F}\left(\frac{z}{\varepsilon^2}\right) \mathbf{a}^\varepsilon + \mathbf{G}\left(\frac{z}{\varepsilon^2}\right) \mathbf{a}^\varepsilon + o(1),$$

with initial conditions for \mathbf{a}^ε at $z = 0$. Here \mathbf{F} , resp. \mathbf{G} , is an operator with zero mean, resp. non-zero mean, and ergodic properties inherited from those of the processes ν .

We can finally apply a diffusion approximation theorem to establish the following result (see [13] for the full statement or [15] for a first version in which the contributions of the evanescent modes is neglected, which means that the operator \mathcal{L}^3 is missing in the expression of the generator \mathcal{L}).

PROPOSITION 5.1. *The random process*

$$\left((a_j^\varepsilon(z))_{j=0}^{N-1}, (a_{t,\gamma}^\varepsilon(z))_{\gamma \in (0, k^2), t \in \{e, o\}} \right)$$

converges in distribution in $\mathcal{C}^0([0, L], \mathbb{C}^N \times L^2((0, k^2))^2)$, the space of continuous functions from $[0, L]$ to $\mathbb{C}^N \times L^2((0, k^2))^2$, to the Markov process

$$\left((\mathbf{a}_j(z))_{j=0}^{N-1}, (\mathbf{a}_{t,\gamma}(z))_{\gamma \in (0, k^2), t \in \{e, o\}} \right)$$

with infinitesimal generator \mathcal{L} . Here $\mathbb{C}^N \times L^2((0, k^2))^2$ is equipped with the weak topology and the infinitesimal generator has the form $\mathcal{L} = \mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3$, where \mathcal{L}^j , $1 \leq j \leq 3$, are the differential operators:

$$\begin{aligned} \mathcal{L}^1 &= \frac{1}{2} \sum_{j,l=0}^{N-1} \Gamma_{jl} (\mathbf{a}_j \bar{\mathbf{a}}_l \partial_{\mathbf{a}_l} \partial_{\bar{\mathbf{a}}_l} + \mathbf{a}_l \bar{\mathbf{a}}_j \partial_{\mathbf{a}_j} \partial_{\bar{\mathbf{a}}_j} - \mathbf{a}_j \mathbf{a}_l \partial_{\mathbf{a}_j} \partial_{\mathbf{a}_l} - \bar{\mathbf{a}}_j \bar{\mathbf{a}}_l \partial_{\bar{\mathbf{a}}_j} \partial_{\bar{\mathbf{a}}_l}) \mathbf{1}_{j \neq l} \\ &+ \frac{1}{2} \sum_{j,l=0}^{N-1} \Gamma_{jl}^1 (\mathbf{a}_j \bar{\mathbf{a}}_l \partial_{\mathbf{a}_j} \partial_{\bar{\mathbf{a}}_l} + \bar{\mathbf{a}}_j \mathbf{a}_l \partial_{\bar{\mathbf{a}}_j} \partial_{\mathbf{a}_l} - \mathbf{a}_j \mathbf{a}_l \partial_{\mathbf{a}_j} \partial_{\mathbf{a}_l} - \bar{\mathbf{a}}_j \bar{\mathbf{a}}_l \partial_{\bar{\mathbf{a}}_j} \partial_{\bar{\mathbf{a}}_l}) \\ &+ \frac{1}{2} \sum_{j=0}^{N-1} (\Gamma_{jj} - \Gamma_{jj}^1) (\mathbf{a}_j \partial_{\mathbf{a}_j} + \bar{\mathbf{a}}_j \partial_{\bar{\mathbf{a}}_j}) + \frac{i}{2} \sum_{j=0}^{N-1} \Gamma_{jj}^s (\mathbf{a}_j \partial_{\mathbf{a}_j} - \bar{\mathbf{a}}_j \partial_{\bar{\mathbf{a}}_j}), \end{aligned} \quad (5.4)$$

$$\mathcal{L}^2 = -\frac{1}{2} \sum_{j=0}^{N-1} (\Lambda_j + i\Lambda_j^s) \mathbf{a}_j \partial_{\mathbf{a}_j} + (\Lambda_j - i\Lambda_j^s) \bar{\mathbf{a}}_j \partial_{\bar{\mathbf{a}}_j}, \quad (5.5)$$

$$\mathcal{L}^3 = i \sum_{j=0}^{N-1} \kappa_j (\mathbf{a}_j \partial_{\mathbf{a}_j} - \bar{\mathbf{a}}_j \partial_{\bar{\mathbf{a}}_j}). \quad (5.6)$$

In these definitions we use the classical complex derivative: if $\zeta = \zeta_r + i\zeta_i$, then $\partial_\zeta = (1/2)(\partial_{\zeta_r} - i\partial_{\zeta_i})$ and $\partial_{\bar{\zeta}} = (1/2)(\partial_{\zeta_r} + i\partial_{\zeta_i})$, and the coefficients of the operators (5.4-5.6) are defined for $j, l = 0, \dots, N-1$, as follows:

- For all $j \neq l$, Γ_{jl} and Γ_{jl}^s are given by

$$\Gamma_{jl} = \frac{k^4}{2\beta_j\beta_l} \int_0^\infty \mathcal{R}_{jl}(z) \cos((\beta_l - \beta_j)z) dz, \quad (5.7)$$

$$\Gamma_{jl}^s = \frac{k^4}{2\beta_j\beta_l} \int_0^\infty \mathcal{R}_{jl}(z) \sin((\beta_l - \beta_j)z) dz, \quad (5.8)$$

with $\mathcal{R}_{jl}(z)$ defined by

$$\mathcal{R}_{jl}(z) := \mathbb{E}[C_{j,l}(0)C_{j,l}(z)], \quad (5.9)$$

$$\mathbb{E}[C_{j,l}(0)C_{j',l'}(z)] = \begin{cases} \int_{-d/2}^{d/2} \int_{-d/2}^{d/2} \phi_j \phi_l(x) \mathcal{R}_I(x, x', z) \phi_{j'} \phi_{l'}(x') dx dx' & \text{type I} \\ (n^2 - 1)^2 d^2 \left[\phi_j \phi_l \phi_{j'} \phi_{l'} \left(-\frac{d}{2}\right) \right. \\ \quad \left. + \phi_j \phi_l \phi_{j'} \phi_{l'} \left(\frac{d}{2}\right) \right] \mathcal{R}_{II}(z) & \text{type II} \end{cases} \quad (5.10)$$

- For all j, l :

$$\Gamma_{jl}^1 = \frac{k^4}{4\beta_j\beta_l} \int_0^\infty \mathbb{E}[C_{j,j}(0)C_{l,l}(z)] + \mathbb{E}[C_{l,l}(0)C_{j,j}(z)] dz.$$

- For all j , Λ_j is defined by

$$\Lambda_j = \int_0^{k^2} \frac{k^4}{2\sqrt{\gamma}\beta_j} \sum_{t \in \{e, o\}} \int_0^\infty \mathcal{R}_{j,t,\gamma}(z) \cos((\sqrt{\gamma} - \beta_j)z) dz d\gamma \quad (5.11)$$

and

$$\begin{aligned} \Gamma_{jj} &= - \sum_{l=0, l \neq j}^{N-1} \Gamma_{jl}, & \Gamma_{jj}^s &= - \sum_{l=0, l \neq j}^{N-1} \Gamma_{jl}^s, \\ \Lambda_j^s &= \sum_{t \in \{e, o\}} \int_0^{k^2} \frac{k^4}{2\sqrt{\gamma}\beta_j} \int_0^\infty \mathcal{R}_{j,t,\gamma}(z) \sin[(\sqrt{\gamma} - \beta_j)z] dz d\gamma, \\ \kappa_j &= \sum_{t \in \{e, o\}} \int_{-\infty}^0 \frac{k^4}{2\sqrt{|\gamma|}\beta_j} \int_0^\infty \mathcal{R}_{j,t,\gamma}(z) \cos(\beta_j z) e^{-\sqrt{|\gamma|}z} dz d\gamma + \frac{k^2}{2\beta_j} \mathbb{E}[c_{j,j}(0)], \end{aligned}$$

where $\mathcal{R}_{j,t,\gamma}(z) = \mathbb{E}[C_{j,t,\gamma}(0)C_{j,t,\gamma}(z)]$ is defined as in (5.9) upon substitution (t, γ) for l and

$$\mathbb{E}[c_{j,j}(0)] = \begin{cases} 0 & \text{type I} \\ (n^2 - 1)d^2 \mathcal{R}_{II}(0) \partial_x [\phi_j^2] \left(\frac{d}{2}\right) & \text{type II} \end{cases}$$

We give some remarks before focusing our attention on the mode powers.

1) The convergence result holds in the weak topology. This means that we can only compute quantities of the form $\mathbb{E}[F(\mathbf{a}_0, \dots, \mathbf{a}_{N-1}, \int_0^{k^2} \alpha_{e,\gamma} \mathbf{a}_{e,\gamma} d\gamma, \int_0^{k^2} \alpha_{o,\gamma} \mathbf{a}_{o,\gamma} d\gamma)]$ for any test functions $\alpha_e, \alpha_o \in L^2((0, k^2))$ and $F: \mathbb{R}^{N+2} \rightarrow \mathbb{R}$. These quantities are the limits of $\mathbb{E}[F(a_0^\varepsilon, \dots, a_{N-1}^\varepsilon, \int_0^{k^2} \alpha_{e,\gamma} a_{e,\gamma}^\varepsilon d\gamma, \int_0^{k^2} \alpha_{o,\gamma} a_{o,\gamma}^\varepsilon d\gamma)]$ as $\varepsilon \rightarrow 0$.

2) The generator \mathcal{L} does not involve $\partial_{\mathbf{a}_{t,\gamma}}$ or $\partial_{\bar{\mathbf{a}}_{t,\gamma}}$. Therefore $(a_j^\varepsilon(z))_{j=0}^{N-1}$ converges in distribution in $\mathcal{C}^0([0, L], \mathbb{C}^N)$ to the Markov process $(\mathbf{a}_j(z))_{j=0}^{N-1}$ with generator \mathcal{L} .

The weak and strong topologies are the same in \mathbb{C}^N , so we can compute any moment of the form $\mathbb{E}[F(\mathbf{a}_0, \dots, \mathbf{a}_{N-1})]$, which are the limits of $\mathbb{E}[F(a_0^\varepsilon, \dots, a_{N-1}^\varepsilon)]$.

3) \mathcal{L}_1 is the contribution of the coupling between guided modes, which gives rise to power exchange between the guided modes.

4) \mathcal{L}_2 is the contribution of the coupling between guided and radiating modes, which gives rise to power leakage from the guided modes to the radiating ones (effective diffusion) and addition of frequency-dependent phases on the guided mode amplitudes (effective dispersion). The effective diffusion and dispersion are produced by causal phenomena and they related to each other through Kramers-Konig relations [11].

5) \mathcal{L}_3 is the contribution of the coupling between guided and evanescent modes, which gives rise to additional phase terms on the guided mode amplitudes (effective dispersion). This term is the main effect when the waveguide supports only one propagating mode and the core boundaries are hard or soft so that there is no radiating mode [9].

6) If the generator \mathcal{L} is applied to a test function that depends only on the mode powers $(P_j)_{j=0}^{N-1}$, with $P_j = |\mathbf{a}_j|^2$, then the result is a function that depends only on $(P_j)_{j=0}^{N-1}$. Thus, the mode powers $(P_j(z))_{j=0}^{N-1}$ define a Markov process, with infinitesimal generator defined by (6.1) below.

7) The radiation mode amplitudes remain constant on $L^2((0, k^2))^2$, equipped with the weak topology, as $\varepsilon \rightarrow 0$. However, this does not describe the power $\sum_{t \in \{e, o\}} \int_0^{k^2} |a_{t, \gamma}^\varepsilon|^2 d\gamma$ transported by the radiation modes, because the convergence does not hold in the strong topology of $L^2((0, k^2))^2$ so we do not have $\sum_{t \in \{e, o\}} \int_0^{k^2} |a_{t, \gamma}^\varepsilon|^2 d\gamma \rightarrow \sum_{t \in \{e, o\}} \int_0^{k^2} |\mathbf{a}_{t, \gamma}|^2 d\gamma$ as $\varepsilon \rightarrow 0$.

8) When $N=1$, then the generator is

$$\begin{aligned} \mathcal{L} = & \frac{\Gamma_{00}^1}{2} (2\mathbf{a}_0 \overline{\mathbf{a}_0} \partial_{\mathbf{a}_0} \partial_{\overline{\mathbf{a}_0}} - \mathbf{a}_0 \mathbf{a}_0 \partial_{\mathbf{a}_0} \partial_{\mathbf{a}_0} - \overline{\mathbf{a}_0} \overline{\mathbf{a}_0} \partial_{\overline{\mathbf{a}_0}} \partial_{\overline{\mathbf{a}_0}} - \mathbf{a}_0 \partial_{\mathbf{a}_0} - \overline{\mathbf{a}_0} \partial_{\overline{\mathbf{a}_0}}) \\ & - \frac{\Lambda_0}{2} (\mathbf{a}_0 \partial_{\mathbf{a}_0} + \overline{\mathbf{a}_0} \partial_{\overline{\mathbf{a}_0}}) + \frac{i}{2} (\kappa_0 - \Lambda_0^s) (\mathbf{a}_0 \partial_{\mathbf{a}_0} - \overline{\mathbf{a}_0} \partial_{\overline{\mathbf{a}_0}}). \end{aligned} \quad (5.12)$$

This shows that \mathbf{a}_0 (the amplitude of the unique guided mode) has the same distribution as

$$\mathbf{a}_0(z) = a_{0,s} \exp\left(\frac{i}{2}(\kappa_0 - \Lambda_0^s)z + i\sqrt{\Gamma_{00}^1}W_z^1 - \frac{\Lambda_0}{2}z\right),$$

where W_z^1 is a standard Brownian motion. The mode amplitude experiences a random phase modulation and a deterministic damping, which both depend on frequency and two-point statistics of the medium perturbations [9].

9) When $N \geq 2$, the limit process $(\mathbf{a}_j(z))_{j=0}^{N-1}$ can be identified as the solution of a system of stochastic differential equations driven by Brownian motions.

COROLLARY 5.1. *Let $(W_j^1)_{j=0}^{N-1}$ be a N -dimensional correlated Brownian motion with covariance function*

$$\mathbb{E}[W_{j,z}^1 W_{j',z'}^1] = \Gamma_{jj'}^1 z \wedge z'.$$

Let $(W_{jl})_{0 \leq j < l \leq N-1}$ and $(\tilde{W}_{jl})_{0 \leq j < l \leq N-1}$ be independent standard Brownian motions. Set $W_{lj} = W_{jl}$ and $\tilde{W}_{lj} = -\tilde{W}_{jl}$ for $j < l$.

Then the limit Markov process $(\mathbf{a}_j(z))_{j=0}^{N-1}$ has the same distribution as the unique solution of

$$d\mathbf{a}_j = i\mathbf{a}_j \circ dW_{j,z}^1 + \sum_{l \neq j} \frac{\sqrt{\Gamma_{jl}}}{\sqrt{2}} \mathbf{a}_l \circ (idW_{jl,z} - d\tilde{W}_{jl,z}) + \frac{1}{2} (i\Gamma_{jj}^s - \Lambda_j - i\Lambda_j^s + i\kappa_j) \mathbf{a}_j dz,$$

starting from $\mathbf{a}_j(z=0) = a_{j,s}$, $j=0, \dots, N-1$, or, in Itô's form:

$$\begin{aligned} d\mathbf{a}_j &= i\mathbf{a}_j dW_{j,z}^1 + \sum_{l \neq j} \frac{\sqrt{\Gamma_{jl}}}{\sqrt{2}} \mathbf{a}_l (idW_{jl,z} - d\tilde{W}_{jl,z}) \\ &\quad + \frac{1}{2} (\Gamma_{jj} + i\Gamma_{jj}^s - \Gamma_{jj}^1 - \Lambda_j - i\Lambda_j^s + i\kappa_j) \mathbf{a}_j dz. \end{aligned}$$

The proof of the corollary is a straightforward application of Itô's formula.

6. The effective Markovian dynamics for the mode powers From the result for the complex mode amplitudes we get the following result.

COROLLARY 6.1. *The process $(|a_j^\varepsilon(z)|^2)_{j=0}^{N-1}$ converges towards a Markov process $\mathbf{P}(z) = (P_j(z))_{j=0}^{N-1}$ whose infinitesimal generator $\mathcal{L}_{\mathbf{P}}$ writes:*

$$\mathcal{L}_{\mathbf{P}} = \sum_{j \neq l} \Gamma_{jl} \left[P_l P_j \left(\frac{\partial}{\partial P_j} - \frac{\partial}{\partial P_l} \right) \frac{\partial}{\partial P_j} + (P_l - P_j) \frac{\partial}{\partial P_j} \right] - \sum_{j=0}^{N-1} \Lambda_j P_j \frac{\partial}{\partial P_j}, \quad (6.1)$$

where Γ_{jl} is defined by (5.7) and Λ_j is defined by (5.11).

The coefficients Γ_{jl} describe the effective mode coupling between guided modes due to random scattering. The coefficients Λ_j are effective mode-dependent dissipation coefficients and they come from the coupling between guided and radiative modes due to random scattering.

From the form of the generator $\mathcal{L}_{\mathbf{P}}$, we can establish that the n th-order moments of the mode powers satisfy closed equations. We will apply this to compute the first moments of \mathbf{P} , as well as its second moments later in Section 8.

Using (6.1) we find that the mean mode powers

$$Q_j(z) = \mathbb{E}[P_j(z)] \quad (6.2)$$

satisfy the closed system of equations

$$\partial_z Q_j = -\Lambda_j Q_j + \sum_{l=0}^{N-1} \Gamma_{jl} (Q_l - Q_j), \quad (6.3)$$

starting from $Q_j(0) = |a_{j,s}|^2$. The form of these coupled-mode equations is well-known [7] although the mode-dependent attenuation terms Λ_j are usually introduced heuristically. The solution explicitly writes:

$$\mathbf{Q}(z) = \exp(\mathbf{A}z) \mathbf{Q}(0), \quad (6.4)$$

with the matrix \mathbf{A} defined by (δ_{jl} is the Kronecker symbol):

$$\mathbf{A} := (\Gamma_{jl} - \Lambda_j \delta_{jl})_{j,l=0}^{N-1}. \quad (6.5)$$

We can also remark that (6.3) with $\Lambda_j = 0$ can be interpreted as the Kolmogorov equation associated to a random walk on the finite space $\{0, \dots, N-1\}$. If we denote by $(J_z)_{z \geq 0}$ the Markov process on the state space $\{0, \dots, N-1\}$ with infinitesimal generator $\mathbf{\Gamma}$, then Feynman-Kac formula gives the following probabilistic representation of the mean mode powers $Q_j(z)$:

$$Q_j(z) = \mathbb{E} \left[|a_{J_z, s}|^2 \exp \left(- \int_0^z \Lambda_{J_{z'}} dz' \right) \middle| J_0 = j \right].$$

This representation makes it possible to anticipate the result derived below in the continuum approximation (when the number of modes becomes large), namely that the Q_j 's can be approximated by the solution of a diffusion equation, because the normalized random walk $(J_z/N)_{z \geq 0}$ can be approximated in distribution by a diffusion process on $[0, 1]$.

7. Long-range behavior of the mean mode powers From now on we assume that the symmetric matrix $\mathbf{\Gamma}$ defined by Γ_{jl} given by (5.7) for $j \neq l$, $\Gamma_{jj} = -\sum_{l' \neq j} \Gamma_{jl'}$, is irreducible. We consider the matrix $\mathbf{A} = \mathbf{\Gamma} - \mathbf{\Phi}$, with $\Phi_{jl} = \Lambda_j \delta_{jl}$. By Perron-Frobenius theorem, the first eigenvalue of \mathbf{A} is simple and nonpositive (we denote it by $-\lambda$) and the components of the corresponding unit eigenvector \mathbf{V} have all the same sign (so we can assume that they are positive). By (6.4) we get the following result.

PROPOSITION 7.1. *The mean mode powers (6.2) satisfy*

$$Q_j(z) \stackrel{z \rightarrow +\infty}{\simeq} c_V V_j \exp(-\lambda z) (1 + o(1)), \quad (7.1)$$

where $(-\lambda, \mathbf{V})$ is the first eigenvalue/eigenvector of \mathbf{A} and

$$c_V = \sum_{l=0}^{N-1} V_l |a_{l,s}|^2. \quad (7.2)$$

In the following we discuss special cases where explicit expressions can be obtained.

No effective dissipation. If there is no effective dissipation $\mathbf{\Phi} = \mathbf{0}$, then the first eigenvalue/eigenvector $(-\lambda^{(0)}, \mathbf{V}^{(0)})$ of the matrix $\mathbf{\Gamma}$ is

$$\lambda^{(0)} = 0, \quad \mathbf{V}^{(0)} = (1/\sqrt{N})_{j=0}^{N-1}, \quad (7.3)$$

which gives the standard equipartition result [6, 8, 10]:

$$Q_j(z) \xrightarrow{z \rightarrow +\infty} \frac{1}{N} \sum_{l=0}^{N-1} |a_{l,s}|^2, \quad \forall j = 0, \dots, N-1.$$

The total input power $\sum_{l=0}^{N-1} |a_{l,s}|^2$ becomes equipartitioned amongst all guided modes.

Weak effective dissipation. We next consider the case when the effective dissipation is weak, that is to say, the matrix $\mathbf{\Phi}$ is much smaller than the matrix $\mathbf{\Gamma}$, with a typical ratio of the order of θ . We then assume that $\Lambda_j = \theta \Lambda_j^{(1)}$, with $\theta \ll 1$. Then we can write $\mathbf{\Phi} = \theta \mathbf{\Phi}^{(1)}$ with $\Phi_{jl}^{(1)} = \Lambda_j^{(1)} \delta_{jl}$ and $\mathbf{\Gamma} = \mathbf{\Gamma}^{(0)}$ and the first eigenvalue/eigenvector $(-\lambda, \mathbf{V})$ of the matrix $\mathbf{\Gamma} - \mathbf{\Phi}$ can be expanded as

$$\lambda = \lambda^{(0)} + \theta \lambda^{(1)} + \theta^2 \lambda^{(2)} + O(\theta^3), \quad \mathbf{V} = \mathbf{V}^{(0)} + \theta \mathbf{V}^{(1)} + O(\theta^2)$$

with $(\lambda^{(0)}, \mathbf{V}^{(0)})$ given by (7.3),

$$\lambda^{(1)} = \mathbf{V}^{(0)T} \mathbf{\Phi}^{(1)} \mathbf{V}^{(0)} = \frac{1}{N} \sum_{j=0}^{N-1} \Lambda_j^{(1)}, \quad (7.4)$$

$$\lambda^{(2)} = \mathbf{V}^{(0)T} \mathbf{\Phi}^{(1)} \mathbf{V}^{(1)}, \quad (7.5)$$

and $\mathbf{V}^{(1)}$ is solution of $\mathbf{\Gamma}^{(0)} \mathbf{V}^{(1)} = (\mathbf{\Phi}^{(1)} - \lambda^{(1)}) \mathbf{V}^{(0)}$ and is orthogonal to $\mathbf{V}^{(0)}$. If, for instance, $\Gamma_{jl} \equiv \Gamma > 0$ for all $j \neq l$, then

$$\mathbf{V}^{(1)} = -\frac{1}{\Gamma N^{3/2}} (\Lambda_j^{(1)} - \lambda^{(1)})_{j=0}^{N-1} \quad (7.6)$$

and

$$\lambda^{(2)} = -\frac{1}{\Gamma N^2} \sum_{j=0}^{N-1} (\Lambda_j^{(1)} - \lambda^{(1)})^2. \quad (7.7)$$

Eqs. (7.3)-(7.4) show that, when coupling is stronger than dissipation, then the effective damping of the mean mode powers is approximately the arithmetic average of the effective mode-dependent damping coefficients. Eqs. (7.3)-(7.6) show that the distribution of the mean mode powers is approximately equipartitioned, with reduced allocations for the modes with the strongest damping coefficients.

Weak coupling. We next consider the case when the coupling is weak, that is to say, the matrix $\mathbf{\Gamma}$ is much smaller than the matrix $\mathbf{\Phi}$, with a typical ratio of the order of θ . We then assume that $\Gamma_{jl} = \theta \Gamma_{jl}^{(1)}$, with $\theta \ll 1$. We also assume that the dissipation coefficients have a unique minimum

$$\Lambda_{j_*} = \min_{j=0, \dots, N-1} (\Lambda_j), \quad \Lambda_j > \Lambda_{j_*} \quad \forall j \neq j_*. \quad (7.8)$$

Then we can write $\mathbf{\Phi} = \mathbf{\Phi}^{(0)}$ and $\mathbf{\Gamma} = \theta \mathbf{\Gamma}^{(1)}$ and the first eigenvalue/eigenvector $(-\lambda, \mathbf{V})$ of the matrix $\mathbf{\Gamma} - \mathbf{\Phi}$ can be expanded as

$$\lambda = \lambda^{(0)} + \theta \lambda^{(1)} + \theta^2 \lambda^{(2)} + O(\theta^3), \quad \mathbf{V} = \mathbf{V}^{(0)} + \theta \mathbf{V}^{(1)} + O(\theta^2),$$

with

$$\lambda^{(0)} = \Lambda_{j_*}, \quad (7.9)$$

$$\mathbf{V}_j^{(0)} = \delta_{jj_*}, \quad (7.10)$$

$$\lambda^{(1)} = -\mathbf{V}^{(0)T} \mathbf{\Gamma}^{(1)} \mathbf{V}^{(0)} = -\Gamma_{j_* j_*}^{(1)} = \sum_{j \neq j_*} \Gamma_{jj_*}^{(1)}, \quad (7.11)$$

$$\mathbf{V}_j^{(1)} = \frac{\Gamma_{jj_*}^{(1)}}{\Lambda_j - \Lambda_{j_*}} \quad \forall j \neq j_*, \quad \mathbf{V}_{j_*}^{(1)} = 0, \quad (7.12)$$

$$\lambda^{(2)} = -\mathbf{V}^{(0)T} \mathbf{\Gamma}^{(1)} \mathbf{V}^{(1)} = -\sum_{j \neq j_*} \frac{(\Gamma_{jj_*}^{(1)})^2}{\Lambda_j - \Lambda_{j_*}}. \quad (7.13)$$

Eq. (7.9) shows that, when coupling is weaker than dissipation, then the effective damping of the mean mode powers is approximately the minimum of the effective mode-dependent damping coefficients. The distribution of the mean mode powers is, moreover, concentrated on the mode corresponding to the minimal damping coefficient.

Continuum approximation. Here we want to address situations where the coupling between guided modes is via nearest neighbors and the number of modes is large.

The number of modes becomes large when $(n^2 - 1)k^2 d^2 \gg 1$ (see (A.8)). In other words, the number of modes is large when the frequency is large because it is proportional to the ratio of the waveguide diameter over the wavelength.

For type II perturbations, coupling via nearest neighbors happens when the fluctuations of the boundaries are smooth so that the Fourier transform of \mathcal{R}_{II} decays fast and the correlation radius is larger than the core diameter (which is much larger than wavelength). Under such circumstances, we have $\beta_j - \beta_{j+l} \simeq \frac{\pi\sqrt{n^2-1}}{nd} \frac{j}{N} l$ for any $l \geq 1$ (see Appendix A), $|\hat{\mathcal{R}}_{\text{II}}(\beta_j - \beta_{j+1})| \gg |\hat{\mathcal{R}}_{\text{II}}(\beta_j - \beta_{j+l})|$ for any $l \geq 2$, and we can approximate the matrix $\mathbf{\Gamma}$ for all $l \neq j$ by:

$$\Gamma_{jl} = \begin{cases} \gamma_{\min(l,j)} & \text{if } |j-l|=1, \\ 0 & \text{if } |j-l| \geq 2, \end{cases} \quad (7.14)$$

with

$$\gamma_j = \frac{k^2(n^2-1)^2 d^2}{2\beta_j\beta_{j+1}} [\phi_j \phi_{j+1}] \left(\frac{d}{2}\right)^2 \hat{\mathcal{R}}_{\text{II}}(\beta_j - \beta_{j+1}). \quad (7.15)$$

For type I perturbations, coupling via nearest neighbors happens under similar conditions. Indeed, let us assume that $\mathcal{R}_{\text{I}}(x, x', z)$ can be factorized as

$$\mathcal{R}_{\text{I}}(x, x', z) = \mathcal{R}_{\text{I},c}(x, x') \mathcal{R}_{\text{I},1}(z),$$

then for all $l \neq j$:

$$\Gamma_{jl} = \frac{k^2}{4\beta_j\beta_l} \hat{\mathcal{R}}_{\text{I},1}(\beta_j - \beta_l) \iint_{[-d/2, d/2]^2} \mathcal{R}_{\text{I},c}(x, x') \phi_j \phi_l(x) \phi_j \phi_l(x') dx dx',$$

where $\hat{\mathcal{R}}_{\text{I},1}$ is the Fourier transform of $\mathcal{R}_{\text{I},1}$. Again, if the fluctuations of the index of refraction are smooth so that $\hat{\mathcal{R}}_{\text{I},1}$ decays fast and the longitudinal correlation radius is larger than the core diameter, then we can approximate Γ_{jl} by (7.14) with

$$\gamma_j = \frac{k^2}{4\beta_j\beta_{j+1}} \hat{\mathcal{R}}_{\text{I},1}(\beta_j - \beta_{j+1}) \iint_{[-d/2, d/2]^2} \mathcal{R}_{\text{I},c}(x, x') \phi_j \phi_{j+1}(x) \phi_j \phi_{j+1}(x') dx dx'. \quad (7.16)$$

Similarly, we find that Λ_j can be approximated by

$$\Lambda_j = \begin{cases} \Lambda_{N-1} & \text{if } j = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Other circumstances can lead to the same conclusions. For instance the band-limiting idealization hypothesis in [13] gives the same result, and it is based on the behavior of the transverse covariance function $\mathcal{R}_{\text{I},c}$.

The coupled mode power equations then read

$$\partial_z Q_{N-1}^{(N)} = -\Lambda_{N-1}^{(N)} Q_{N-1}^{(N)} + \gamma_{N-2}^{(N)} (Q_{N-2}^{(N)} - Q_{N-1}^{(N)}), \quad (7.17)$$

$$\partial_z Q_j^{(N)} = \gamma_{j-1}^{(N)} (Q_{j-1}^{(N)} - Q_j^{(N)}) + \gamma_j^{(N)} (Q_{j+1}^{(N)} - Q_j^{(N)}) \text{ for } 1 \leq j \leq N-2, \quad (7.18)$$

$$\partial_z Q_0^{(N)} = \gamma_0^{(N)} (Q_1^{(N)} - Q_0^{(N)}). \quad (7.19)$$

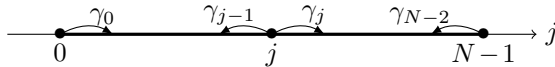


FIG. 7.1. The transition rates of the jump Markov process $(J_z)_{z \geq 0}$ on the state space $\{0 \leq j \leq N-1\}$. The absorption is concentrated on the point $j = N-1$.

The superscript (N) is added to remember that all coefficients depend on N . We have nearest-neighbor coupling: The j th mode can exchange power with the l th mode only if they are neighbors, that is, if they satisfy $|j-l| \leq 1$.

We note that the terms involving $\gamma_j^{(N)}$ in (7.17-7.19) define the infinitesimal generator of a random walk on the finite set $\{0 \leq j \leq N-1\}$ (see Figure 7.1). As shown in [13], following the ideas developed in [20, Chapter 11], if $\gamma_j^{(N)}$ converges in the sense that $\gamma_{\lfloor xN \rfloor}^{(N)} \rightarrow \gamma_\infty(x)$ for any $x \in (0,1)$, where γ_∞ is smooth and positive, then, for any function φ in $L^2(0,1)$, the function

$$Q_\varphi^{(N)}(z, u) = \sum_{j=0}^{N-1} Q_j^{(N)}(z) \varphi(\lfloor j/N \rfloor),$$

where $Q_j^{(N)}$ is the solution of (7.17-7.19) starting from $Q_j^{(N)}(z=0) = \delta_{j, \lfloor uN \rfloor}$, satisfies

$$\lim_{N \rightarrow \infty} Q_\varphi^{(N)}(z, u) = Q_\varphi(z, u),$$

where Q_φ is the solution of the partial differential equation:

$$\partial_z Q_\varphi = \mathcal{H}_1 Q_\varphi, \quad \mathcal{H}_1 = \partial_u (\gamma_\infty(u) \partial_u \cdot), \quad (7.20)$$

with the mixed Neumann-Dirichlet boundary conditions

$$\partial_u Q_\varphi(z, 0) = 0, \quad Q_\varphi(z, 1) = 0, \quad Q_\varphi(0, u) = \varphi(u). \quad (7.21)$$

For type I perturbations we have from (7.16):

$$\begin{aligned} \gamma_\infty(u) &= \frac{1}{4(n^2-1)d^2(\frac{n^2}{n^2-1}-u^2)} \hat{\mathcal{R}}_{I,1}\left(\frac{\pi\sqrt{n^2-1}}{nd}u\right) \\ &\times \iint_{[-d/2, d/2]^2} \mathcal{R}_{I,c}(x, x') \sin(\pi x/d) \sin(\pi x'/d) dx dx'. \end{aligned}$$

For type II perturbations, we have from (7.15):

$$\gamma_\infty(u) = \frac{2(n^2-1)u^4}{(\frac{n^2}{n^2-1}-u^2)} \hat{\mathcal{R}}_{II}\left(\frac{\pi\sqrt{n^2-1}}{nd}u\right).$$

As a consequence of this result we get the following result.

PROPOSITION 7.2. *In the continuum approximation, the first eigenvalue $\lambda^{(N)}$ of the matrix $\mathbf{A}^{(N)}$ converges to*

$$\lambda = \inf_{\varphi \in \mathcal{D}_1} \int_0^1 \gamma_\infty(u) \varphi'(u)^2 du \quad (7.22)$$

as $N \rightarrow \infty$, with

$$\mathcal{D}_1 = \left\{ \varphi \in \mathcal{C}^\infty([0,1]), \int_0^1 \varphi(u)^2 du = 1, \varphi'(0) = 0, \varphi(1) = 0 \right\}. \quad (7.23)$$

Moreover, λ is a simple eigenvalue of the operator \mathcal{H}_1 , the corresponding eigenvector φ is smooth and unique (up to a multiplication by -1) and it can be chosen so as to satisfy $\varphi(u) > 0$ for $u \in [0,1]$ (the proof is similar as the one proposed in [13] for the Pekeris waveguide and it is based on a modified version of Krein-Rutman theorem). The eigenvector φ gives the asymptotic mode distribution for large propagation distance:

$$Q_j^{(N)}(z) \stackrel{z \rightarrow +\infty}{\simeq} c_V \varphi(j/N) \exp(-\lambda z), \quad (7.24)$$

with $c_V = \sum_{l=0}^{N-1} |a_{l,s}|^2 \varphi(l/N) / N$. We, therefore, observe an exponential decay of the mean power transported by the guided modes and a form of equipartition of the mean mode powers, but not with the uniform distribution, but with the distribution proportional to the eigenvector φ .

If γ_∞ is constant, then $\lambda = \pi^2 \gamma_\infty / 4$ and the eigenvector is $\varphi(u) = \sqrt{2} \cos(\pi u / 2)$. This happens in particular for type I perturbations when $0 < n - 1 \ll 1$, so that, for all $u \in (0,1)$,

$$\gamma_\infty(u) \simeq \frac{1}{4d^2} \hat{\mathcal{R}}_{I,1}(0) \iint_{[-d/2, d/2]^2} \mathcal{R}_{I,c}(x, x') \sin(\pi x / d) \sin(\pi x' / d) dx dx'.$$

8. Fluctuation analysis By (6.1) we find that the second-order moments of the mode powers

$$R_{jl}(z) = \mathbb{E}[P_j(z)P_l(z)], \quad j, l = 0, \dots, N-1, \quad (8.1)$$

satisfy the closed equations

$$\partial_z R_{jj} = -2\Lambda_j R_{jj} + \sum_{n \neq j} \Gamma_{jn} (4R_{jn} - 2R_{jj}), \quad (8.2)$$

$$\begin{aligned} \partial_z R_{jl} = & -(2\Gamma_{jl} + \Lambda_j + \Lambda_l) R_{jl} + \sum_{n \neq l} \Gamma_{ln} (R_{jn} - R_{jl}) \\ & + \sum_{n \neq j} \Gamma_{jn} (R_{nl} - R_{jl}), \quad j \neq l. \end{aligned} \quad (8.3)$$

This system has the same form as the one found in the literature dedicated to coupled mode theory [6, 7]. The initial conditions are $R_{jl}(0) = |a_{j,s}|^2 |a_{l,s}|^2$. Let us introduce $\mathbf{S} = (S_{jl})_{0 \leq j \leq l \leq N-1}$ defined by

$$S_{jl} = \begin{cases} R_{jl} + R_{lj} = 2R_{jl} & \text{if } j < l, \\ R_{jj} & \text{if } j = l. \end{cases} \quad (8.4)$$

The S_{jl} 's satisfy the system

$$\partial_z S_{jl} = -(\Psi \mathbf{S})_{jl} + (\Theta \mathbf{S})_{jl}, \quad (8.5)$$

$$(\Psi \mathbf{S})_{jl} = (\Lambda_j + \Lambda_l) S_{jl}, \quad (8.6)$$

$$\begin{aligned} (\Theta \mathbf{S})_{jl} = & 2\Gamma_{jl} \mathbf{1}_{j \neq l} (S_{jj} + S_{ll} - 2S_{jl}) \\ & + \sum_{n \notin \{j, l\}} [\Gamma_{ln} (S_{jn} - S_{jl}) + \Gamma_{jn} (S_{nl} - S_{jl})], \end{aligned} \quad (8.7)$$

with the convention that whenever S_{jl} occurs with $j > l$, it is replaced by S_{lj} . The operator Θ is the infinitesimal generator of a random Markov process $(J_z, L_z)_{z \geq 0}$ that is a random walk on the discrete triangle $\{(j, l) \in \mathbb{N}^2, 0 \leq j \leq l \leq N-1\}$. Using Feynmac-Kac formula we get the following probabilistic representation of S_{jl} :

$$S_{jl}(z) = \mathbb{E} \left[|a_{J_z, s}|^2 |a_{L_z, s}|^2 \exp \left(- \int_0^z \Lambda_{J_{z'}} + \Lambda_{L_{z'}} dz' \right) \middle| J_0 = j, L_0 = l \right].$$

We can anticipate that, in the continuum limit, the Markov process $(J_z/N, L_z/N)_{z \geq 0}$ behaves as a diffusion process on the triangle $\{(u, v) \in \mathbb{R}^2, 0 \leq u \leq v \leq 1\}$, and, therefore, S_{jl} satisfies a diffusion equation on this triangle.

Long-range behavior of the second-order moments of the mode powers.

Eq. (8.5) has the form $\partial_z \mathbf{S} = (\Theta - \Psi) \mathbf{S}$. The linear operator Ψ is diagonal and the linear operator Θ is self-adjoint: for any \mathbf{T} and $\tilde{\mathbf{T}}$, we have

$$\begin{aligned} \sum_{j \leq l} (\Theta \mathbf{T})_{jl} \tilde{T}_{jl} &= - \sum_{j \leq l} \Theta_{jl, jl} T_{jl} \tilde{T}_{jl} + \sum_{j < l, n \notin \{j, l\}} (\Gamma_{ln} T_{jn} \tilde{T}_{jl} + \Gamma_{jn} T_{nl} \tilde{T}_{jl}) \\ &\quad + \sum_{j \neq n} (\Gamma_{jn} T_{jn} \tilde{T}_{jj} + \Gamma_{jn} T_{nj} \tilde{T}_{jj}) + 2 \sum_{j < l} (\Gamma_{jl} T_{jj} \tilde{T}_{jl} + \Gamma_{jl} T_{jj} \tilde{T}_{jl}) \\ &= \sum_{j \leq l} T_{jl} (\Theta \tilde{\mathbf{T}})_{jl}, \end{aligned}$$

because $2 \sum_{j < l} = \sum_{j \neq l}$. As a consequence, $\Theta - \Psi$ can be diagonalized and as a consequence of Perron-Frobenius theorem we get the following result.

PROPOSITION 8.1. *The second-order moments of the mode powers satisfy*

$$\mathbf{S}(z) \stackrel{z \rightarrow +\infty}{\simeq} c_W \mathbf{W} \exp(-\mu z) (1 + o(1)), \quad (8.8)$$

where $(-\mu, \mathbf{W})$ is the first eigenvalue/eigenvector of $\Theta - \Psi$ and c_W is the projection of the initial conditions on the first eigenvector \mathbf{W}

$$c_W = \sum_{j, l=0}^{N-1} W_{jl} |a_{j, s}|^2 |a_{l, s}|^2, \quad (8.9)$$

with the convention that whenever W_{jl} occurs with $j > l$, it is replaced by W_{lj} .

We next address special cases.

No effective dissipation. If there is no effective dissipation, then the first eigenvalue/eigenvector $(-\mu^{(0)}, \mathbf{W}^{(0)})$ of the matrix Θ is $\mathbf{W}^{(0)} = (c_N)_{0 \leq j \leq l \leq N-1}$, $\mu^{(0)} = 0$, with $c_N = \sqrt{2}/\sqrt{N(N+1)}$. We have $\mathbf{S}(z) \stackrel{z \rightarrow +\infty}{\simeq} c_W \mathbf{W}^{(0)}$. As $\sum_{j \leq l} S_{jl}(z) = \sum_{j, l} R_{jl}(z) = (\sum_{j=0}^{N-1} |a_{j, s}|^2)^2$, we deduce

$$S_{jl}(z) \stackrel{z \rightarrow +\infty}{\simeq} \left(\sum_{l'=0}^{N-1} |a_{l', s}|^2 \right)^2 \frac{2}{N(N+1)},$$

and

$$R_{jl}(z) \stackrel{z \rightarrow +\infty}{\simeq} \left(\sum_{l'=0}^{N-1} |a_{l', s}|^2 \right)^2 \frac{1 + \delta_{jl}}{N(N+1)}.$$

This means that, when $N \gg 1$, the mode powers P_j become uncorrelated and their marginal distributions have the same moments as exponential distributions. In other words, the mode powers behave as the square moduli of independent and identically distributed complex Gaussian variables.

Weak effective dissipation. We next consider the case when the effective dissipation is weak, say $\Lambda_j = \theta \Lambda_j^{(1)}$ with $\theta \ll 1$. Then we can write $\Psi = \theta \Psi^{(1)}$ and $\Theta = \Theta^{(0)}$ and the first eigenvalue/eigenvector $(-\mu, \mathbf{W})$ of the matrix $\Theta - \Psi$ can be expanded as

$$\mathbf{W} = \mathbf{W}^{(0)} + \theta \mathbf{W}^{(1)} + O(\theta^2), \quad \mu = \theta \mu^{(1)} + \theta^2 \mu^{(2)} + O(\theta^3),$$

with

$$\begin{aligned} \mu^{(1)} &= \mathbf{W}^{(0)T} \Psi^{(1)} \mathbf{W}^{(0)} = \frac{2}{N} \sum_{j=0}^{N-1} \Lambda_j^{(1)} = 2\lambda^{(1)}, \\ \mu^{(2)} &= \mathbf{W}^{(1)T} \Theta^{(0)} \mathbf{W}^{(1)}, \end{aligned} \tag{8.10}$$

and $\mathbf{W}^{(1)}$ is solution of $\Theta^{(0)} \mathbf{W}^{(1)} = (\Psi^{(1)} - \mu^{(1)}) \mathbf{W}^{(0)}$ and is orthogonal to $\mathbf{W}^{(0)}$. If, for instance, $\Gamma_{jl} \equiv \Gamma > 0$ for all $j \neq l$, then

$$W_{jl}^{(1)} = -\frac{cN}{\Gamma N} (\Lambda_j^{(1)} + \Lambda_l^{(1)} - 2\lambda^{(1)}), \quad j \leq l,$$

and

$$\begin{aligned} \mu^{(2)} &= \mathbf{W}^{(1)T} \Psi^{(1)} \mathbf{W}^{(0)} \\ &= -\frac{2(N+2)}{N^2(N+1)\Gamma} \sum_{j=0}^{N-1} (\Lambda_j^{(1)} - \lambda^{(1)})^2. \end{aligned}$$

Note that

$$\begin{aligned} \mu - 2\lambda &= \theta^2 (\mu^{(2)} - 2\lambda^{(2)}) + O(\theta^3) \\ &= -\frac{2\theta^2}{N^2(N+1)\Gamma} \sum_{j=0}^{N-1} (\Lambda_j^{(1)} - \lambda^{(1)})^2 + O(\theta^3) \end{aligned} \tag{8.11}$$

is negative-valued as soon as there exist j, j' such that $\Lambda_j^{(1)} \neq \Lambda_{j'}^{(1)}$.

Weak coupling. We next consider the case when the coupling is weak, say $\Gamma_{jl} = \theta \Gamma_{jl}^{(1)}$, with $\theta \ll 1$. We again assume that the dissipation coefficients have a unique minimum (7.8). Then we can write $\Psi = \Psi^{(0)}$ and $\Theta = \theta \Theta^{(1)}$ and the first eigenvalue/eigenvector $(-\mu, \mathbf{W})$ of the matrix $\Theta - \Psi$ can then be expanded as

$$\mathbf{W} = \mathbf{W}^{(0)} + \theta \mathbf{W}^{(1)} + O(\theta^2), \quad \mu = \mu^{(0)} + \theta \mu^{(1)} + \theta^2 \mu^{(2)} + O(\theta^3),$$

with

$$\mu^{(0)} = 2\Lambda_{j_*}, \tag{8.12}$$

$$W_{jl}^{(0)} = \delta_{jj_*} \delta_{lj_*}, \tag{8.13}$$

$$\mu^{(1)} = -\mathbf{W}^{(0)T} \Theta^{(1)} \mathbf{W}^{(0)} = 2 \sum_{j \neq j_*} \Gamma_{jj_*}^{(1)}, \tag{8.14}$$

$\mathbf{W}^{(1)}$ is solution of $(\mu^{(0)} - \Phi^{(0)})\mathbf{W}^{(1)} = (-\Theta^{(1)} - \mu^{(1)})\mathbf{W}^{(0)}$ and is orthogonal to $\mathbf{W}^{(0)}$,

$$W_{jl}^{(1)} = \begin{cases} \frac{2\Gamma_{j_*l}}{\Lambda_j - \Lambda_{j_*}} & \text{if } j = j_*, l > j_*, \\ \frac{2\Gamma_{jj_*}}{\Lambda_j - \Lambda_{j_*}} & \text{if } j < j_*, l = j_*, \\ 0 & \text{otherwise,} \end{cases} \quad (8.15)$$

$$\mu^{(2)} = -\mathbf{W}^{(0)T} \Theta^{(1)} \mathbf{W}^{(1)} = -4 \sum_{j \neq j_*} \frac{(\Gamma_{jj_*}^{(1)})^2}{\Lambda_j - \Lambda_{j_*}}.$$

Note that

$$\mu - 2\lambda = -2\theta^2 \sum_{j \neq j_*} \frac{(\Gamma_{jj_*}^{(1)})^2}{\Lambda_j - \Lambda_{j_*}} + O(\theta^3) \quad (8.16)$$

is negative-valued (we have assumed the irreducibility of $\Gamma^{(1)}$, hence at least one of the $\Gamma_{jj_*}^{(1)}$ is non-zero).

Continuum approximation. Here we address the situations where the coupling between guided modes is via nearest neighbors and the number of modes is large. When Γ_{jl} is of the form (7.14), the system (8.5) for S_{jl} reads

$$\begin{aligned} \partial_z S_{jl} = & \delta_{jl} [2\gamma_j(S_{jj+1} - S_{jj})\mathbf{1}_{j \leq N-2} + 2\gamma_{j-1}(S_{j-1j} - S_{jj})\mathbf{1}_{j \geq 1}] \\ & + \delta_{lj+1} [2\gamma_j(S_{jj} + S_{j+1j+1} - 2S_{jj+1}) + \gamma_{j-1}(S_{j-1j+1} - S_{jj+1})\mathbf{1}_{j \geq 1} \\ & + \gamma_{j+1}(S_{jj+2} - S_{jj+1})\mathbf{1}_{j \leq N-3}] \\ & + \mathbf{1}_{j \leq l-2} [\gamma_{l-1}(S_{jl-1} - S_{jl})\mathbf{1}_{l \geq 1} + \gamma_l(S_{jl+1} - S_{jl})\mathbf{1}_{l \leq N-2} \\ & + \gamma_{j-1}(S_{j-1l} - S_{jl})\mathbf{1}_{j \geq 1} + \gamma_j(S_{j+1l} - S_{jl})] \\ & - \Lambda_{N-1}(\delta_{jN-1} + \delta_{lN-1})S_{jl}. \end{aligned} \quad (8.17)$$

Note that the terms involving γ_j define the infinitesimal generator of a random walk $(J_z, L_z)_{z \geq 0}$ on the finite set $D_N = \{0 \leq j \leq l \leq N-1\}$ (see Figure 8.1).

We proceed as in [13] to determine the asymptotic behavior of S_{jl} when $N \rightarrow +\infty$. We denote S_{jl} by $S_{jl}^{(N)}$ to keep track of the dependence with respect to N . We introduce the triangle $D = \{(u, v) \in \mathbb{R}^2, 0 < u < v < 1\}$. We get that, for any function ψ in $L^2(D)$, the function

$$S_\psi^{(N)}(z, u, v) = \sum_{0 \leq j \leq l \leq N-1} S_{jl}^{(N)}(z) \psi(\lfloor j/N \rfloor, \lfloor l/N \rfloor),$$

where $S_{jl}^{(N)}$ is the solution of (8.17) starting from $S_{jl}^{(N)}(z=0) = \delta_{j, \lfloor uN \rfloor} \delta_{l, \lfloor vN \rfloor}$, satisfies

$$\lim_{N \rightarrow \infty} S_\psi^{(N)}(z, u, v) = S_\psi(z, u, v),$$

where S_ψ is the solution of the partial differential equation:

$$\partial_z S_\psi = \mathcal{H}_2 S_\psi, \quad \mathcal{H}_2 = \partial_u (\gamma_\infty(u) \partial_u \cdot) + \partial_v (\gamma_\infty(v) \partial_v \cdot), \quad (8.18)$$

with the boundary condition (Dirichlet on the face $\{u=1\}$ of the triangle D , Neumann on the faces $\{v=0\}$ and $\{u=v\}$, see Figure 8.2):

$$\partial_u S_\psi(z, 0, v) = 0, \quad S_\psi(z, u, 1) = 0, \quad (\partial_u - \partial_v) S_\psi(z, u, v) |_{u=v} = 0, \quad (8.19)$$

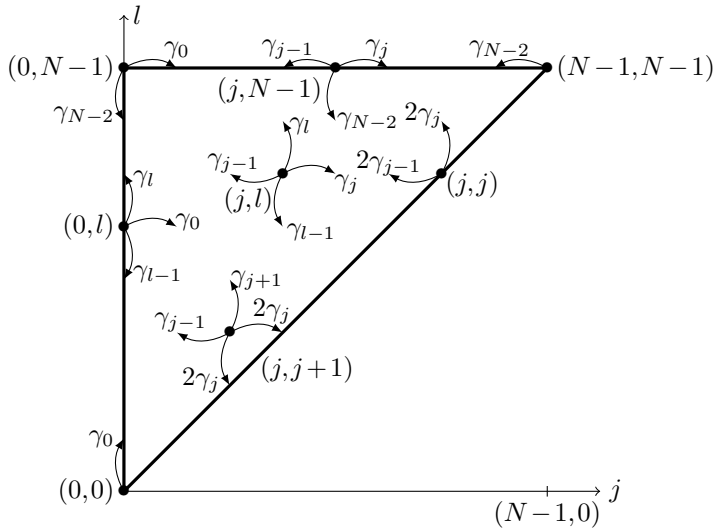


FIG. 8.1. The transition rates of the jump Markov process $(J_z, L_z)_{z \geq 0}$ on the state space $D_N = \{0 \leq j \leq l \leq N-1\}$. The absorption is concentrated on the line $(j, N-1)_{j=0}^{N-1}$.

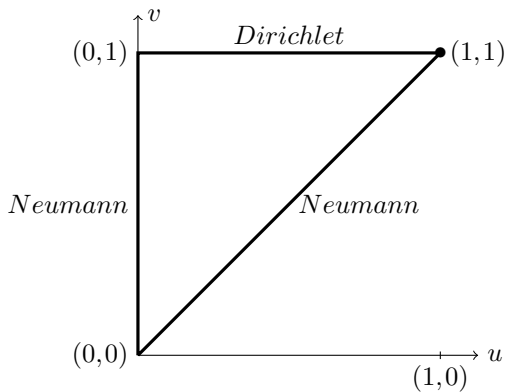


FIG. 8.2. The domain of the continuum approximation with its boundary conditions.

and the initial condition $S_\psi(0, u, v) = \psi(u, v)$.

Consequently, we get the following result.

PROPOSITION 8.2. In the continuum approximation, the first eigenvalue $\mu^{(N)}$ of $\Theta^{(N)} - \Psi^{(N)}$ converges as $N \rightarrow \infty$ to

$$\mu = \inf_{\psi \in \mathcal{D}_2} \iint_D \gamma_\infty(u) [\partial_u \psi(u, v)]^2 + \gamma_\infty(v) [\partial_v \psi(u, v)]^2 dudv, \tag{8.20}$$

with

$$\mathcal{D}_2 = \left\{ \psi \in \mathcal{C}^\infty(\bar{D}), \iint_D \psi(u,v)^2 dudv = 1, \right. \\ \left. \partial_u \psi(0,v) = 0, \psi(u,1) = 0, (\partial_u - \partial_v)\psi(u,v)|_{u=v} = 0 \right\}. \quad (8.21)$$

Equivalently,

$$\mu = \inf_{\check{\psi} \in \check{\mathcal{D}}_2} \iint_{(0,1)^2} \gamma_\infty(u) [\partial_u \check{\psi}(u,v)]^2 + \gamma_\infty(v) [\partial_v \check{\psi}(u,v)]^2 dudv, \quad (8.22)$$

with

$$\check{\mathcal{D}}_2 = \left\{ \check{\psi} \in \mathcal{C}^\infty([0,1]^2), \iint_{(0,1)^2} \check{\psi}(u,v)^2 dudv = 1, \right. \\ \left. \partial_u \check{\psi}(0,v) = 0, \partial_v \check{\psi}(u,0) = 0, \check{\psi}(u,1) = 0, \check{\psi}(1,v) = 0 \right\}. \quad (8.23)$$

Proof of (8.22). We denote by $\check{\mu}$ the right-hand side of (8.22). We can show as in [13] that $\check{\mu}$ is a simple eigenvalue of the operator \mathcal{H}_2 on $(0,1)^2$ with Dirichlet boundary conditions on $\{v=1\}$ and $\{u=1\}$ and Neumann boundary conditions on $\{u=0\}$ and $\{v=0\}$ and that the corresponding eigenvector $\check{\psi}$ is smooth and unique up to a multiplication by -1 . Moreover, the function $\check{\psi}_2 : (u,v) \in (0,1)^2 \mapsto \check{\psi}(v,u)$ also satisfies $\mathcal{H}_2 \check{\psi}_2 = \check{\mu} \check{\psi}_2$ with the same Dirichlet/Neumann boundary conditions. By uniqueness, we get that $\check{\psi}$ is symmetric: $\check{\psi}(u,v) = \check{\psi}(v,u)$, so it satisfies $(\partial_u - \partial_v)\check{\psi}(u,v)|_{u=v} = 0$. Therefore $\psi_2 : (u,v) \in D \mapsto \sqrt{2}\check{\psi}(u,v)$ is an eigenvector of \mathcal{H}_2 on D with the boundary conditions $\partial_u \psi_2(0,v) = 0$, $\psi_2(u,1) = 0$, $(\partial_u - \partial_v)\psi_2(u,v)|_{u=v} = 0$ with the eigenvalue $\check{\mu}$, so we must have $\mu \leq \check{\mu}$.

We can show as in [13] that μ is a simple eigenvalue of the operator \mathcal{H}_2 on D with the Dirichlet/Neumann boundary conditions $\partial_u \psi(0,v) = 0$, $\psi(u,1) = 0$, $(\partial_u - \partial_v)\psi(u,v)|_{u=v} = 0$, and that the corresponding eigenvector ψ is smooth. If we use $\check{\psi}(u,v) = \psi(u,v)\mathbf{1}_{u \leq v} + \psi(v,u)\mathbf{1}_{u > v}$ as a test function in (8.22), then we find that $\check{\mu} \leq \mu$. \square

If φ is the eigenvector of \mathcal{H}_1 with the boundary conditions $\varphi'(0) = 0$, $\varphi(1) = 0$ with eigenvalue λ , then $\check{\psi} : (u,v) \in (0,1)^2 \mapsto \varphi(u)\varphi(v)$ is an eigenvector of \mathcal{H}_2 on $(0,1)^2$ with the boundary conditions $\partial_u \check{\psi}(0,v) = 0$, $\partial_v \check{\psi}(u,0) = 0$, $\check{\psi}(u,1) = 0$, $\check{\psi}(1,v) = 0$ with the eigenvalue 2λ . This shows that $\mu \leq 2\lambda$. We know that we must have $\mu \geq 2\lambda$ because $\mu^{(N)} \geq 2\lambda^{(N)}$ for any N , therefore we get the following result.

PROPOSITION 8.3. *In the continuum approximation, we have*

$$\mu = 2\lambda. \quad (8.24)$$

By uniqueness this implies that the eigenvector $\check{\psi}$ of \mathcal{H}_2 on $(0,1)^2$ associated to μ is $\check{\psi} : (u,v) \in (0,1)^2 \mapsto \varphi(u)\varphi(v)$. This in turn implies that the eigenvector ψ of \mathcal{H}_2 on D associated to $\mu = 2\lambda$ is $\psi : (u,v) \in D \mapsto \sqrt{2}\varphi(u)\varphi(v)$. As a result we get

$$S_{jl}(z) \stackrel{z \rightarrow +\infty}{\simeq} 2c_W \varphi(j/l)\varphi(l/N) \exp(-2\lambda z),$$

with $c_W = \sum_{j,l=0}^{N-1} \varphi(j/N)\varphi(l/N)|a_{j,s}|^2|a_{l,s}|^2/N^2 = c_V^2$, $c_V = \sum_{j=0}^{N-1} \varphi(j/N)|a_{j,s}|^2/N$, and therefore

$$R_{jl}(z) \stackrel{z \rightarrow +\infty}{\simeq} c_V^2(1 + \delta_{jl})\varphi(j/l)\varphi(l/N) \exp(-2\lambda z). \quad (8.25)$$

This result is the key to show that we will not observe any exponential growth of the relative intensity fluctuations in the continuum approximation.

Exponential growth of the intensity fluctuations. It is a general feature that, for any matrix $\mathbf{\Gamma}$ and effective dissipation coefficients Λ_j , we have $\mu - 2\lambda \leq 0$ (this is a consequence of Cauchy-Schwarz inequality: the square of the mean mode power cannot grow faster than the mean square mode power). The first two moments of the pointwise intensity $|p(x, z)|^2$ for large z are

$$\mathbb{E}[|p(x, z)|^2] \stackrel{z \rightarrow \infty}{\simeq} \sum_{j=0}^{N-1} \frac{\phi_j(x)^2}{\beta_j} c_V V_j e^{-\lambda z}, \quad (8.26)$$

$$\mathbb{E}[|p(x, z)|^4] \stackrel{z \rightarrow \infty}{\simeq} \sum_{j, l=0}^{N-1} \frac{\phi_j(x)^2 \phi_l(x)^2}{\beta_j \beta_l} c_W W_{jl} e^{-\mu z}. \quad (8.27)$$

Without dissipation we have the following result for the relative fluctuations of the pointwise intensity:

$$\frac{\mathbb{E}[|p(x, z)|^4]}{\mathbb{E}[|p(x, z)|^2]^2} \stackrel{z \rightarrow \infty}{\rightarrow} \frac{2N}{N+1},$$

which is equal to 2 when $N \gg 1$.

With dissipation

$$\frac{\mathbb{E}[|p(x, z)|^4]}{\mathbb{E}[|p(x, z)|^2]^2} \stackrel{z \rightarrow \infty}{\sim} \exp(-(\mu - 2\lambda)z), \quad (8.28)$$

that grows exponentially with the propagation distance. With weak dissipation,

$$\frac{\mathbb{E}[|p(x, z)|^4]}{\mathbb{E}[|p(x, z)|^2]^2} \stackrel{z \rightarrow \infty}{\simeq} \frac{2N}{N+1} \exp(-(\mu - 2\lambda)z) (1 + o(1)), \quad (8.29)$$

because the first eigenvectors \mathbf{V} and \mathbf{W} are close to the ones of the non-dissipative case. Note, however, that the exponential growth happens only for very long distances, because $|\mu - 2\lambda|$ is very small as shown above. Eq. (8.11) gives the expression of the exponential growth rate when dissipation is weak and $\Gamma_{jl} \equiv \Gamma$ for $j \neq l$: the growth rate increases when the effective modal dissipation coefficients become different from each other and decreases when the number of modes increases. The analysis in the continuum approximation confirms that the exponential growth rate of the relative intensity fluctuations vanishes when the number of modes goes to infinity. More exactly, in the continuum approximation, when the number of modes becomes large, we have $\mu = 2\lambda$ and (8.25) holds. Therefore there is no exponential growth of the fluctuations and we have

$$\frac{\mathbb{E}[|p(x, z)|^4]}{\mathbb{E}[|p(x, z)|^2]^2} \stackrel{z \rightarrow \infty}{\simeq} 2, \quad (8.30)$$

which corresponds to a relative variance (or scintillation index) equal to one. We recover the standard result that the wavefield, in the point of view of the fourth-order moments, behaves as a Gaussian process with relative variance (scintillation index) equal to one [12].

9. Conclusion In this paper we have reviewed the asymptotic theory of wave propagation in random waveguides. We have recovered standard results about the first two moments of the mode amplitudes: the mean amplitudes decay exponentially and the mean powers satisfy a coupled mode equation. The fourth-order moment analysis also reveals that the fluctuation of the mode powers grow exponentially with the propagation distance. We have carefully studied the exponential growth rates of the relative variances. We have shown that, when the number of guided modes increases, the exponential growth rates vanish and the scintillation index (the relative variance of the intensity fluctuations) becomes equal to one, as observed in open medium in the random paraxial regime [12]. These results show that incoherent imaging in a random waveguide (such as a Pekeris waveguide in underwater acoustics) is challenging because the estimation of the second-order moments of the wavefield is extremely difficult. More generally, they show that the predictions of the coupled mode equations (which describe the evolutions of the statistical second-order moments of the wavefield, such as Eq. (6.3)) are not easy to check experimentally when the number of guided modes is not very large.

Appendix A. Wave mode decomposition. Let us introduce the Helmholtz operator

$$\mathcal{H} = \partial_x^2 + k^2 n^{(0)}(x). \quad (\text{A.1})$$

The Helmholtz operator \mathcal{H} is self-adjoint with respect to the standard scalar product defined on $L^2(\mathbb{R})$ by:

$$(\phi_1, \phi_2)_{L^2} := \int_{\mathbb{R}} \overline{\phi_1(x)} \phi_2(x) dx.$$

The Helmholtz operator has a spectrum of the form (3.1) where the N modal wavenumbers β_j are positive and $k^2 < \beta_{N-1}^2 < \dots < \beta_0^2 < n^2 k^2$.

Discrete spectrum. The j th eigenvector associated to the eigenvalue β_j^2 is even for even j :

$$\phi_j(x) = \begin{cases} A_j \cos(\sigma_j x/d) & \text{if } 0 \leq |x| \leq d/2 \\ A_j \cos(\sigma_j/2) \exp(-\zeta_j(|x|/d - 1/2)) & \text{if } |x| \geq d/2 \end{cases} \quad (\text{A.2})$$

and odd for odd j

$$\phi_j(x) = \begin{cases} A_j \sin(\sigma_j x/d) & \text{if } 0 \leq |x| \leq d/2 \\ A_j \sin(\sigma_j/2) \operatorname{sgn}(x) \exp(-\zeta_j(|x|/d - 1/2)) & \text{if } |x| \geq d/2 \end{cases} \quad (\text{A.3})$$

where

$$\sigma_j = \sqrt{n^2 k^2 - \beta_j^2} d, \quad \zeta_j = \sqrt{\beta_j^2 - k^2} d, \quad (\text{A.4})$$

and

$$A_j^2 = \begin{cases} \frac{1/d}{\left(\frac{1}{2} + \frac{\sin(\sigma_j)}{2\sigma_j}\right) + \frac{\cos^2(\sigma_j/2)}{\zeta_j}} & \text{for even } j \\ \frac{1/d}{\left(\frac{1}{2} - \frac{\sin(\sigma_j)}{2\sigma_j}\right) + \frac{\sin^2(\sigma_j/2)}{\zeta_j}} & \text{for odd } j \end{cases} \quad (\text{A.5})$$

For even j the σ_j 's are the solutions in $(0, \sqrt{n^2 - 1}kd)$ of

$$\tan(\sigma/2) = \frac{\sqrt{(n^2 - 1)k^2 d^2 - \sigma^2}}{\sigma}. \quad (\text{A.6})$$

For odd j the σ_j 's are the solutions in $(0, \sqrt{n^2 - 1}kd)$ of

$$\tan(\sigma/2) = -\frac{\sigma}{\sqrt{(n^2 - 1)k^2 d^2 - \sigma^2}}, \quad (\text{A.7})$$

and we denote by N the number of solutions. We have $\sigma_j \in (j\pi, (j+1)\pi)$ and

$$N = \lfloor \sqrt{n^2 - 1}kd/\pi \rfloor. \quad (\text{A.8})$$

Continuous spectrum. For $\gamma \in (-\infty, k^2)$, there are two improper eigenvectors (one is even and the other one is odd) and they have the form:

$$\phi_{e,\gamma}(x) = \begin{cases} A_{e,\gamma} \cos(\eta_\gamma x/d) & \text{if } 0 \leq |x| \leq d/2 \\ A_{e,\gamma} [\cos(\eta_\gamma/2) \cos(\xi_\gamma(|x|/d - 1/2))] \\ -\eta_\gamma/\xi_\gamma \sin(\eta_\gamma/2) \sin(\xi_\gamma(|x|/d - 1/2))] & \text{if } |x| \geq d/2 \end{cases} \quad (\text{A.9})$$

$$\phi_{o,\gamma}(x) = \begin{cases} A_{o,\gamma} \sin(\eta_\gamma x/d) & \text{if } 0 \leq |x| \leq d/2 \\ A_{o,\gamma} \operatorname{sgn}(x) [\sin(\eta_\gamma/2) \cos(\xi_\gamma(|x|/d - 1/2))] \\ +\eta_\gamma/\xi_\gamma \cos(\eta_\gamma/2) \sin(\xi_\gamma(|x|/d - 1/2))] & \text{if } |x| \geq d/2 \end{cases} \quad (\text{A.10})$$

where

$$\eta_\gamma = \sqrt{n^2 k^2 - \gamma}d, \quad \xi_\gamma = \sqrt{k^2 - \gamma}d, \quad (\text{A.11})$$

and

$$A_{e,\gamma}^2 = \frac{\xi_\gamma d}{2\pi(\xi_\gamma^2 \cos^2(\eta_\gamma/2) + \eta_\gamma^2 \sin^2(\eta_\gamma/2))}, \quad (\text{A.12})$$

$$A_{o,\gamma}^2 = \frac{\xi_\gamma d}{2\pi(\xi_\gamma^2 \sin^2(\eta_\gamma/2) + \eta_\gamma^2 \cos^2(\eta_\gamma/2))}. \quad (\text{A.13})$$

We remark that $\phi_{t,\gamma}$ does not belong to $L^2(\mathbb{R})$, but $(\phi_{t,\gamma}, \phi)_{L^2}$ can be defined for any test function $\phi \in L^2(\mathbb{R})$ as

$$(\phi_{t,\gamma}, \phi)_{L^2} = \lim_{M \rightarrow +\infty} \int_{-M}^M \phi_{t,\gamma}(x) \phi(x) dx, \quad (\text{A.14})$$

where the limit holds (as a function in γ) in $L^2((-\infty, k^2))$.

Completeness. We have for any $\phi \in L^2(\mathbb{R})$:

$$(\phi, \phi)_{L^2} = \sum_{j=0}^{N-1} |(\phi_j, \phi)_{L^2}|^2 + \sum_{t \in \{e, o\}} \int_{-\infty}^{k^2} |(\phi_{t,\gamma}, \phi)_{L^2}|^2 d\gamma. \quad (\text{A.15})$$

The map which assigns to every element of $\phi \in L^2(\mathbb{R})$ the coefficients of its spectral decomposition

$$\phi \mapsto \left((\phi_j, \phi)_{L^2}, j=0, \dots, N-1, (\phi_{t,\gamma}, \phi)_{L^2}, t \in \{e, o\}, \gamma \in (-\infty, k^2) \right)$$

is an isometry from $L^2(\mathbb{R})$ onto $\mathbb{C}^N \times L^2((-\infty, k^2))^2$. This means that any function $\phi \in L^2(\mathbb{R})$ can be expanded on the set of the eigenfunctions of \mathcal{H} .

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