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Option pricing under fast-varying long-memory stochastic volatility

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Abstract

Recent empirical studies suggest that the volatility of an underlying price process may have correlations that decay slowly under certain market conditions. In this paper, the volatility is modeled as a stationary process with long-range correlation properties in order to capture such a situation, and we consider European option pricing. This means that the volatility process is neither a Markov process nor a martingale. However, by exploiting the fact that the price process is still a semimartingale and accordingly using the martingale method, we can obtain an analytical expression for the option price in the regime where the volatility process is fast mean reverting. The volatility process is modeled as a smooth and bounded function of a fractional Ornstein–Uhlenbeck process. We give the expression for the implied volatility, which has a fractional term structure.

KEYWORDS

fractional long-range correlation, mean reversion, Ornstein–Uhlenbeck process, stochastic volatility

JEL CLASSIFICATION:

91G80, 60H10, 60G22, 60K37

1 | INTRODUCTION

1.1 | Stochastic volatility and the implied surface

Under many market scenarios, the assumption that volatility is constant, as in the standard Black–Scholes model, is not realistic. Practically, this reflects itself in an implied volatility that depends on the pricing parameters. This means that in order to match observed prices, the volatility that one needs to use in the Black–Scholes option pricing formula depends on time to maturity and log-moneyness, with moneyness being the strike price over the current price of the underlying. The implied volatility is a convenient way to parameterize the price of a financial contract relative to a particular underlying. It gives insight about how the market deviates from the ideal Black–Scholes situation. After calibration of an implied volatility model to liquid contracts, this model can be used for pricing less liquid contracts written on the same underlying. It is therefore of interest to identify a consistent parameterization of the implied volatility that corresponds to an underlying model for stochastic volatility fluctuations. As in Garnier and Sølna (2017), a main objective is to construct a stochastic volatility model that is a stationary process and that makes it possible to consider general times to maturity. For background on stochastic volatility models, we refer the reader to the books and surveys by Fouque, Papanicolaou, Sircar, and Sølna (2011), Gatheral (2006), Ghysels, Harvey, and Renaud (1995), Gulisashvili (2012), Henry-Labordère (2009), and Rebonato (2004) (see the references therein). We also refer the reader to our paper on fractional stochastic volatility (fSV), Garnier and Sølna (2017), for further references on the recent literature on the class of volatility models we consider here.

Empirical studies suggest that volatility may exhibit a “multiscale” character with long-range correlations, as in Bollerslev, Osterrieder, Sizova, and Tauchen (2013), Breidt, Crato, and De Lima (1998), Chronopoulou and Viens (2012b), Cont (2001, 2005), Engle and Patton (2001), and Oh, Kim, and Eom (2008). It means that correlations decay as a power law in time offset, while they would decay as an exponential function if stochastic volatility were Markovian. Here, we seek to identify parametric forms for the implied volatility consistent with such long-range correlations. In our recent paper, Garnier and Sølna (2017), we considered this question within the context where the magnitude of the volatility fluctuations is small. Here, we consider the situation where the magnitude of the volatility fluctuations is of the same order as the mean volatility. Indeed, empirical studies show that the volatility fluctuations may be quite large: Breidt et al. (1998), Cont (2001), and Engle and Patton (2001). While in Garnier and Sølna (2017) the volatility fluctuations were small, leading to a (regular) perturbative situation, here the situation is different in that it is the fast mean reversion (fast relative to the diffusion time of the underlying) that allows us to push through an asymptotic analysis. The presence of long-range correlations in this context gives a novel, singular perturbation situation. The analysis becomes significantly more complex. In particular, the detailed analysis of the covariation process is an important ingredient. We consider here option pricing, but the approach set forth is general and will be useful in other financial contexts as well.

It follows from our analysis that the form of the implied volatility surface is similar to the one obtained in the Markovian case. This confirms the robustness of the implied volatility parametric model with respect to the underlying price dynamics. There are, however, central differences. In particular, the long-range correlations produce a volatility covariance that is not integrable, which, in turn, gives an implied volatility surface that is a *random field*, whose statistics can be described in detail. Moreover, in the long-range case, the implied volatility has a fractional behavior as a function of time to maturity. The empirical study in Fouque et al. (2003) shows that in order to fit well the implied volatility, it is appropriate to consider a two-time scale model with one slow and one fast volatility factor. In Garnier and Sølna (2017), we considered a slow factor, which is closely associated with a small fluctuation factor. Here, we consider a fast factor with large fluctuations. Taken together, we have a generalization of the two-factor model of Fouque et al. (2003, 2011) for processes with long-range correlations. This leads to a fractional term structure of the implied volatility. It was shown in Fouque, Papanicolaou, Sircar, and Sølna (2004) that such a term structure may be useful for fitting the implied volatility under certain market conditions.

1.2 | Long memory and fast mean reversion

As mentioned above, the asymptotic regime considered in this paper is the situation where the volatility is fast mean reverting. We denote its time scale by ε, the small parameter in our model. The volatility then decorrelates on the time scale ε.

Stochastic volatility models are most often set with a volatility driving process that has mean zero and mixing properties. This means that the random values of the volatility driving process at times t and $t + \Delta t$, which are Z_t^ε and $Z_{t+\Delta t}^\varepsilon$, become rapidly uncorrelated when $\Delta t \to \infty$; that is, the autocovariance function $C^\varepsilon(\Delta t) = \mathbb{E}[Z_t^\varepsilon Z_{t+\Delta t}^\varepsilon]$ decays rapidly to zero as $\Delta t \to \infty$. More precisely, we say that the volatility driving process is mixing if its autocovariance function decays fast enough at infinity, so that it is absolutely integrable

$$\int_0^\infty |C^\varepsilon(t)| dt < \infty . \tag{1}$$

In this case, we may associate the process with the finite correlation time $t_c = 2\int_0^\infty C^\varepsilon(t)dt / C^\varepsilon(0)$, which is of order ε.

Stochastic volatility models with long-range correlation properties have recently attracted a lot of attention, as more and more data collected under various situations confirm that this situation can be encountered in many different markets. Qualitatively, the long-range correlation property means that the random process has long memory (in contrast to a mixing process). This means that the correlation between the random values Z_t^ε and $Z_{t+\Delta t}^\varepsilon$ taken at two times separated by Δt is not completely negligible even for large Δt. More precisely, we say that the random process Z_t^ε has the H-long-range correlation property if its autocovariance function satisfies

$$C^\varepsilon(t) \overset{|t|\to\infty}{\simeq} r_H \left| \frac{t}{\varepsilon} \right|^{2H-2} , \tag{2}$$

where $r_H > 0$ and $H \in (1/2, 1)$. We refer to H as the Hurst exponent. Here, the correlation time ε is the critical time scale beyond which the power law behavior (2) is valid. Note that the autocovariance function is not integrable as $2H - 2 \in (-1, 0)$, which means that a random process with the H-long-range correlation property is not mixing. As we describe in more detail below, a common approach for modeling long-range dependence is by using fractional Brownian motion (fBm) processes as introduced in Mandelbrot and Van Ness (1968).

Long-memory stochastic volatility models are easy to introduce, but difficult to analyze. This is largely due to the fact that the volatility process is neither a Markov process nor a semimartingale. It is, however, important to note that the price process is still a semimartingale, and the problem formulation does not entail arbitrage (Mendes, Oliveira, & Rodrigues, 2015), as has been argued for some models whose price process itself is driven by fractional processes, as in Bjork and Hult (2005), Rogers (1997), and Shiryaev (1998). A main motivation for long memory is to be able to fit observed implied volatilities. One common challenge regarding the fitting of implied volatility surfaces is to capture a strong moneyness dependence for short time to maturity without creating artificial behavior for long time to maturity. Another typical challenge is to retain a strong parametric dependence for long maturities despite averaging effects that occur in this regime, as discussed in Bollerslev and Mikkelsen (1999), Bollerslev et al. (2013), Comte et al. (2012), and Sundaresan (2000). We remark that models involving jumps have been promoted as one approach to meet these challenges by Carr and Wu (2003) and Mijatovic and Tankov (2016). Recent works show that stochastic volatility models with long-range dependence also provide a promising framework for meeting such challenges. Approaches based on using fractional noises in the description of the stochastic volatility process were used by Comte and Renault (1998) and Comte et al. (2012). Such stochastic volatility models with long-range dependence can capture the steepness of long-term volatility smiles without overemphasizing the short-run persistence. In order to get explicit results for the implied volatility, a number of asymptotic regimes have been considered. Chief among them has been the regime of short time to maturity. The model presented in Comte et al. (2012) was recently revisited in Guennoun, Jacquier, and Roome (in press), where short and long times to maturity asymptotics are analyzed using large deviations theory. In Alòs et al. (2007), the authors use Malliavin calculus to decompose the option price as the sum of the classic Black–Scholes formula price and a term due to the volatility of the volatility. In the Black–Scholes formula, they use a volatility parameter that is equal to the root mean square future average volatility plus a term due to the leverage effect (i.e., the correlation between the underlying return and its changes in volatility). Their model is a fractional version of the Bates model (Bates, 1996). They find that the implied volatility flattens in the long-range-dependent case in the limit of short time to maturity. In Forde and Zhang (2015), the authors use large deviation principles to compute the short-time-to-maturity asymptotic form of the implied volatility. They consider the leverage effect and obtain results that are consistent with those in Alòs et al. (2007). They consider stochastic volatility models driven by fBms, which are analyzed using rough path theory. They also consider long time asymptotics for some fractional processes. Short-time-to-maturity asymptotic results were also recently presented in Gulisashvili, Viens, and Zhang (2018) in a context of long-range processes. In Bayer, Friz, and Gatheral (2016), the authors consider the rough Bergomi model, or "rBergomi" model, and discuss the form of the associated implied volatility term structure. In Fukasawa (2011), the author discusses how small volatility fluctuations with long-range dependence impact the implied volatility as an application of the general theory he sets forth. In that paper, as well as in Alòs et al. (2007), the authors use a model where time 0 plays a special role, and hence the modeling is not completely satisfactory, because it leads to a nonstationary volatility model. On the other hand, in Garnier and Sølna (2017), which deals with small volatility fluctuations, the authors use a formulation with a stationary model. This is also the case in the recent paper by Fukasawa (2017), which considers short-time asymptotics in the rough volatility case, with $H < 1/2$. This distinction is important: If the volatility factor is an fBm emanating from the origin, then the implied volatility surface is identified conditioned on the present value of the volatility factor only. In our paper, we use a stationary model so that the implied volatility surface depends on the path of the volatility factor until the present, reflecting the non-Markovian nature of fBm. We discuss in detail in Section 6 the consequences of this for interpretating the implied volatility surface as a random field. Recently, pricing approximations in the regime of small fractional volatility fluctuations were presented in Alòs and Yang (2017). In terms of computation of prices for general maturities and fractional volatility fluctuations, so far mainly numerical approximations have been available. Here, we present an asymptotic regime based on fast mean reversion that gives explicit price approximations. Together, the results of Garnier and Sølna (2017) and the current paper make it possible to construct a fractional, two-time-scale stochastic volatility model, which gives enough flexibility to fit both the short and long times to maturity parts of the implied volatility surface.

Let us note that we consider here the long-range correlation case where $H > 1/2$ as opposed to the rough volatility case where $H < 1/2$. Indeed, both regimes have been identified from the empirical perspective. We refer the reader, for instance, to Gatheral, Jaisson, and Rosenbaum (2018) for observations of rough volatility, and to Chronopoulou and Viens (2012a) for cases of long-range volatility. A long-range, mean-reverting volatility situation is reported in Jensen (2016) in a discrete modeling framework. Long-range volatility situations are also reported for currencies in Walther, Klein, Thu, and Piontek (2017), for commodities in Charfeddine (2014), and for equity indices in Chia et al. (2015). Analysis of electricity markets data typically gives $H < 1/2$, as reported in Simonsen (2002), Rypdal and Lovsletten (2013), and Bennedsen (2015). We believe that both the rough and the long-range cases are important and can be observed depending on the specific market and regime. Even though the "rough" case with $H < 1/2$ may be the most common situation, the understanding of the situation where $H > 1/2$ may be of particular importance for pricing and hedging. In this paper, we only consider the analytic aspects of our model. The fitting with respect to specific data is beyond the scope of this paper and will be presented in future work.

The fractional model we set forth here produces typical "stylized facts," such as heavy tails of returns, volatility clustering, mean reversion, and long memory or volatility persistence. Additionally, here, we incorporate the leverage effect. This term was coined by Black (1976), referring to stock price movements that are correlated (typically negatively) with volatility, as falling stock prices may imply more uncertainty, and hence volatility. Note, however, that the model for the implied volatility surface derived below is linear in log-moneyness. This may seem somewhat restrictive from the point of view of fitting, because, in many cases, a strong skew in log-moneyness may be observed in certain markets. This has particularly been the case for stock markets, but relatively less so in other markets, such as fixed income markets. Nevertheless, if one considers higher order approximations, then they also generate skew effects. A number of modeling issues not considered here, such as transaction costs, bid–ask spreads, and liquidity, may also affect the skew shape. Note also that, for simplicity, we do not incorporate a nonzero interest rate or a market price for risk aspects.

1.3 | Rapid clustering, long memory, and the implied surface

Next, we will summarize the main result of the paper from the point of view of calibration, that is, the form of the implied volatility surface in the context of a stochastic volatility modeled by a fast process with long-range correlation properties. We will first summarize some aspects of the modeling.

We consider a continuous-time stochastic volatility model that is a smooth function of a Gaussian long-range process. Explicitly, we model the fSV as a smooth function of a fractional Ornstein–Uhlenbeck (fOU) process. The fOU process is a classic model for a stationary process with a fractional long-range correlation structure. This process can be expressed in terms of an integral of an fBm process. The distribution of an fBm process is characterized in terms of the Hurst exponent $H \in (0, 1)$. The fBm process is locally Hölder continuous of exponent H' for all $H' < H$, and this property is inherited by the fOU process. The fBm process, W_t^H, is also self-similar in that

$$\left\{W_{\alpha t}^H, t \in \mathbb{R}\right\} \overset{dist.}{=} \left\{\alpha^H W_t^H, t \in \mathbb{R}\right\} \quad \text{for all} \quad \alpha > 0. \tag{3}$$

The self-similarity property is inherited approximately by the fOU process on time scales smaller than the mean-reversion time of the fOU process, which we denote by ε below. In this sense, we may refer to the fOU process as a multiscale process on short time scales. The case $H \in (1/2, 1)$ that we address in this paper gives an fOU process that is long-range. This regime corresponds to a persistent process where consecutive increments of the fBm are positively correlated. The stronger, positive correlation for the consecutive increments of the associated fBm process with increasing H values gives a smoother process whose autocovariance function decays slowly. For more details regarding the fBm and fOU processes, we refer the reader to Biagini, Hu, Øksendal, and Zhang (2008), Coutin (2007), Doukhan, Oppenheim, and Taqqu (2003), Mandelbrot and Van Ness (1968), Cheridito, Kawaguchi, and Maejima (2003), and Kaarakka and Salminen (2011).

The volatility driving process is the ε-scaled fOU process defined by

$$Z_t^\varepsilon = \varepsilon^{-H} \int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H. \tag{4}$$

It is a zero-mean, stationary Gaussian process that exhibits long-range correlations for the Hurst exponent $H \in (1/2, 1)$. It is important to note that this is a process whose "natural time scale" is ε, in the sense that the mean-reversion time, or time before the process reaches its equilibrium distribution, is of the order of ε. It is also important to note that the decay of the correlations (on the ε time scale) is polynomial rather than exponential, as in the standard Ornstein–Uhlenbeck process. Explicitly, the correlation of the process between times t and $t + \Delta t$ decays as $(\Delta t/\varepsilon)^{2H-2}$, while the variance of the process is independent of ε.

In this paper, we consider a stochastic volatility model that is a smooth function of the rapidly varying fOU process with Hurst coefficient $H \in (1/2, 1)$. It is given by

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{5}$$

where F is a smooth, positive, one-to-one bounded function with bounded derivatives, and with an additional technical condition that is given in equation (26). The process σ_t^ε inherits the long-range correlation properties of the fOU Z_t^ε.

The main result, in Section 5, is an expression for the implied volatility of the European call option for strike K, maturity T, and current time t,

$$I_t = \mathbb{E}\left[\frac{1}{T-t}\int_t^T (\sigma_s^\varepsilon)^2 ds \,\middle|\, \mathcal{F}_t\right]^{1/2} + \overline{\sigma} a_F \left[\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2} \log\left(\frac{K}{X_t}\right)\right]. \tag{6}$$

Here

$$a_F = \varepsilon^{1-H} \frac{\widetilde{\sigma}\rho \langle FF'\rangle \bar{\tau}^H}{2^{3/2}\overline{\sigma}\Gamma(H+3/2)}, \tag{7}$$

$\tau = T - t$ is time to maturity, ρ the correlation between the Brownian motion driving the fBm and the Brownian motion driving the underlying, and

$$\bar{\tau} = \frac{2}{\overline{\sigma}^2} \tag{8}$$

is the characteristic diffusion time. Furthermore, we have

$$\overline{\sigma}^2 = \left\langle F^2 \right\rangle = \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z)^2 p(z) dz,$$

$$\widetilde{\sigma} = \langle F \rangle = \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z) p(z) dz,$$

$$\left\langle FF' \right\rangle = \int_{\mathbb{R}} F(\sigma_{\mathrm{ou}} z) F'(\sigma_{\mathrm{ou}} z) p(z) dz,$$

where $\sigma_{\mathrm{ou}}^2 = 1/(2\sin(\pi H))$ and $p(z)$ is the probability density function (pdf) of the standard normal distribution. In other words, we form moments of the volatility function averaged with respect to the invariant distribution of the fOU process Z_t^ε.

The first term in equation (6) is indeed the expected effective volatility until maturity conditioned on the present. The second term is a skewness term that is nonzero only when the volatility process and the underlying are correlated so that ρ is nonzero. Note that the exponent of the fractional term structure depends on the Hurst exponent, which determines the smoothness and the decorrelation rate of the volatility driving process Z_t^ε. The smoother the process, the larger the implied volatility for long times to maturity.

In the fast case presented here with large and fast volatility fluctuations, the implied volatility explodes in the regime of short time to maturity. Indeed, short time to maturity means that the time to maturity is smaller than the diffusion time (8), but larger than the mean-reversion time ε. Therefore, short time to maturity involves large volatility fluctuations over a short maturity horizon resulting in a moneyness correction that explodes and dominates the pure maturity term. In the context of short or long times to maturity, the conditional expected effective volatility gives a small contribution, and we have for short times to maturity and $K \neq X_t$

$$I_t \sim \overline{\sigma} a_F \left[\left(\frac{\tau}{\overline{\tau}} \right)^{H-3/2} \log \left(\frac{K}{X_t} \right) \right], \tag{9}$$

and for long times to maturity

$$I_t \sim \overline{\sigma} a_F \left(\frac{\tau}{\overline{\tau}} \right)^{H-1/2}. \tag{10}$$

We note here that the fractional scaling in the skewness term in equation (6) is exactly the fractional scaling that corresponds to the case of long time to maturity and small volatility fluctuations given in Garnier and Sølna (2017). That means that with long times to maturity, we have a situation reminiscent of the one we have here with rapid volatility fluctuations. Here, however, the volatility fluctuations are large compared to the small volatility fluctuations considered in Garnier and Sølna (2017).

We remark also that the case with a mixing volatility, and hence integrable correlation function for the volatility fluctuations, would correspond to $H \searrow 1/2$. Note, however, that our derivation is valid only for $H \in (1/2, 1)$. If we consider the formula (39) for σ_ϕ that determines the variance of the first term in equation (6), we observe that it vanishes when $H \searrow 1/2$, which shows that the first term in equation (6) becomes deterministic. In the mixing case, the first-order correction to the implied volatility is deterministic, while the nonintegrability of the volatility covariance function makes it a stochastic process in the general, long-range case with a variance that goes to zero as $H \searrow 1/2$. Indeed, in the limit case $H \searrow 1/2$, we get a result similar to the one obtained in Fouque, Papanicolaou, and Sircar (2000, section 5.2.5) that deals with the mixing case. Explicitly, we consider the mixing case where the volatility driving process is an ordinary Ornstein–Uhlenbeck process; moreover, the interest rate and market price of volatility risk are zero. Then, Fouque et al. (2000, equation (5.55)) give the implied volatility in terms of a coefficient V_3 defined in Fouque et al. (2000, section 5.2.5),

$$I_t = \overline{\sigma} - V_3 \left[\frac{1}{2\overline{\sigma}} + \frac{1}{\overline{\sigma}^3 \tau} \log \left(\frac{K}{X_t} \right) \right], \tag{11}$$

which has the same form as the formal limit of (6) as $H \searrow 1/2$. The averaging expression giving the coefficient V_3 does not, however, correspond to the interpretation we arrive at here by the formal limit $H \searrow 1/2$. That is because the singular perturbation situation we consider is, in fact, "singular" at $H = 1/2$, and the ordering of important terms changes. Nevertheless, it is important from the calibration point of view that we have continuity of the implied volatility parameterization and its form at $H = 1/2$, providing robustness to the asymptotic framework.

In Section 6, we give the complete statistical description of the stochastic correction coefficient, which determines the random component of the price correction and the implied volatility (the first term in equation (6)). It is a random function of the maturity T and the current time t with Gaussian statistics and with a covariance function that we describe in detail. This covariance function has interesting and nontrivial, self-similar properties, and this function is important in order to construct and characterize estimators of the implied volatility surface.

1.4 | Outline

The outline of the paper is as follows. In Section 2, we describe the fOU process and derive some fundamental a priori bounds. In Section 3, we describe the stochastic volatility model. In Section 4, we derive the expression for the price in the fast mean-reverting fractional case. The derivation is based on the martingale method. That is, we make an ansatz for the price as a process that has the correct payoff and whose leading-order term is a martingale. Then, indeed, this process is the leading-order expression for the price with an error that is of the order of the nonmartingale part. This approach involves introducing correctors so that the nonmartingale part is pushed to a small term; we give the resulting decomposition in Section 4. Based on the expression for the price, we derive the associated implied volatility in Section 5 and present our concluding remarks in Section 7. We give a convenient Hermite decomposition of the volatility in Appendix A. A number of the technical lemmas are proved in Appendix B.

2 | THE RAPID FRACTIONAL ORNSTEIN–UHLENBECK PROCESS

We use a rapid fOU process as the volatility factor and describe here how this process can be represented in terms of an fBm. Because fBm can be expressed in terms of ordinary Brownian motion, we also arrive at an expression for the rapid fOU process as a filtered version of Brownian motion.

An fBm is a zero-mean Gaussian process $(W_t^H)_{t\in\mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \tag{12}$$

where σ_H is a positive constant. We use the following moving-average stochastic integral representation of the fBm (Mandelbrot & Van Ness, 1968)

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dW_s, \tag{13}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion over $\mathbb{R}$. Then, $(W_t^H)_{t\in\mathbb{R}}$ is indeed a zero-mean Gaussian process with the covariance (12), and we have

$$\begin{aligned}\sigma_H^2 &= \frac{1}{\Gamma(H+\frac{1}{2})^2}\left[\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right] \\ &= \frac{1}{\Gamma(2H+1)\sin(\pi H)}. \end{aligned} \tag{14}$$

We introduce the ε-scaled fOU as

$$Z_t^\varepsilon = \varepsilon^{-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H = \varepsilon^{-H} W_t^H - \varepsilon^{-1-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} W_s^H ds. \tag{15}$$

Thus, the fOU process is, in fact, an fBm with a restoring force toward zero. It is a zero-mean, stationary Gaussian process, with variance

$$\mathbb{E}[(Z_t^\varepsilon)^2] = \sigma_{\rm ou}^2, \text{ with } \sigma_{\rm ou}^2 = \frac{1}{2}\Gamma(2H+1)\sigma_H^2 = \frac{1}{2\sin(\pi H)}, \tag{16}$$

which is independent of ε, and covariance

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_{\rm ou}^2 C_Z\left(\frac{s}{\varepsilon}\right),$$

which is a function of s/ε only, with

$$\begin{aligned} C_Z(s) &= \frac{1}{\Gamma(2H+1)}\left[\frac{1}{2}\int_{\mathbb{R}} e^{-|v|}|s+v|^{2H}dv - |s|^{2H}\right] \\ &= \frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(sx)\frac{x^{1-2H}}{1+x^2}dx. \end{aligned}$$

This shows that ε is the natural scale of variation of the fOU Z_t^ε. Note that the random process Z_t^ε is neither a martingale nor a Markov process. For $H \in (1/2, 1)$, it possesses long-range correlation properties

$$C_Z(s) = \frac{1}{\Gamma(2H-1)}s^{2H-2} + o\left(s^{2H-2}\right), \qquad s \gg 1. \tag{17}$$

This shows that the correlation function is nonintegrable at infinity. In this paper, we focus on the case $H \in (1/2, 1)$.

We remark that if $H = 1/2$, then the standard Ornstein–Uhlenbeck process (synthesized with a standard Brownian motion) is a stationary Gaussian Markov process with an exponential correlation, and hence a mixing process. It is possible to simulate paths of the fOU process using the Cholesky method (see Figure 1), or other well-known methods described in Omre, Sølna, and Tjelmeland (1993) and Bardet, Lang, Oppenheim, Philippe, and Taqqu (2003).

Using equations (13) and (15), we arrive at the moving-average integral representation of the scaled fOU as

$$Z_t^\varepsilon = \sigma_{\rm ou}\int_{-\infty}^t \mathcal{K}^\varepsilon(t-s)dW_s, \tag{18}$$

where

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}\left(\frac{t}{\varepsilon}\right), \qquad \mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})\sigma_{\rm ou}}\left[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}}e^{-s}ds\right]. \tag{19}$$

The main properties of the kernel $\mathcal{K}$ in our context are the following (valid for any $H \in (1/2, 1)$):

- $\mathcal{K}$ is nonnegative-valued, $\mathcal{K} \in L^2(0, \infty)$ with $\int_0^\infty \mathcal{K}^2(u)du = 1$, but $\mathcal{K} \notin L^1(0, \infty)$,
- for short times $t \ll 1$

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})\sigma_{\rm ou}}\left(t^{H-\frac{1}{2}} + O\left(t^{H+\frac{1}{2}}\right)\right), \tag{20}$$

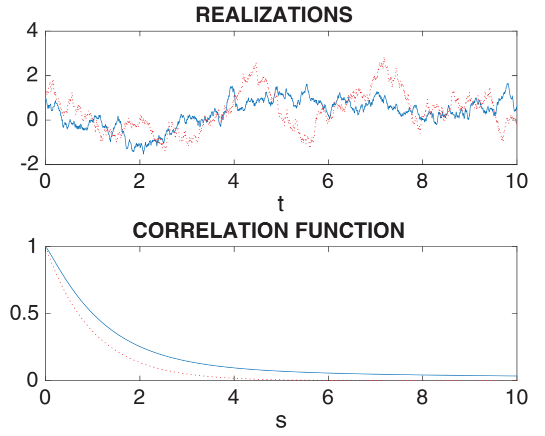

FIGURE 1 The top plot shows a realization, Z_t^ε, $t \in (0, 10)$, of the fOU process with Hurst index $H = 0.6$ and correlation time $\varepsilon = 1$ (blue solid line) and a realization of the standard Ornstein–Uhlenbeck process with $H = 1/2$ and $\varepsilon = 1$ (red dashed line). The trajectories are more regular when H is larger. The bottom plot shows the corresponding correlation functions, $C_Z(s)$, and the "heavy" tail of the blue solid line of the case $H = 0.6$ gives the long-range property [Color figure can be viewed at wileyonlinelibrary.com]

– for long times $t \gg 1$

$$\mathcal{K}(t) = \frac{1}{\Gamma(H - \frac{1}{2})\sigma_{\rm ou}} \left(t^{H-\frac{3}{2}} + O\left(t^{H-\frac{5}{2}} \right) \right), \tag{21}$$

and, in particular, $\mathcal{K}(t) - \frac{1}{\sigma_{\rm ou}\Gamma(H-\frac{1}{2})} t^{H-\frac{3}{2}} \in L^1(0, \infty)$.

3 | THE STOCHASTIC VOLATILITY MODEL

The price of the risky asset follows the stochastic differential equation

$$dX_t = \sigma_t^\varepsilon X_t dW_t^*, \tag{22}$$

where the stochastic volatility is

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{23}$$

and Z_t^ε is the scaled fOU introduced in the previous section, which is adapted to the Brownian motion W_t. Moreover, W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1-\rho^2} B_t, \tag{24}$$

where the Brownian motion B_t is independent of W_t.

The function F is assumed to be one-to-one, positive-valued, smooth, bounded, and with bounded derivatives. Accordingly, the filtration $\mathcal{F}_t$ generated by (B_t, W_t) is also the one generated by X_t. Indeed, it is equivalent to the filtration generated by (W_t^*, W_t), or (W_t^*, Z_t^ε). Because F is one-to-one, it is equivalent to the filtration generated by (W_t^*, σ_t). Because F is positive valued, it is equivalent to the filtration generated by $(W_t^*, (\sigma_t^\varepsilon)^2)$, or X_t.

We denote the Hermite coefficients of the volatility function F with respect to the invariant distribution of the fOU process by C_k,

$$C_k = \int_{\mathbb{R}} H_k(z) F^2(\sigma_{\rm ou} z) p(z) dz, \qquad H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2}, \tag{25}$$

with $p(z) = \exp(-z^2/2)/\sqrt{2\pi}$. We use these in Appendix A to derive some technical lemmas. Indeed, there is a technical reason requiring that F satisfies the following condition: There exists some $\alpha > 2$ such that

$$\sum_{k=0}^{\infty} \frac{\alpha^k C_k^2}{k!} < \infty. \tag{26}$$

As discussed above, the volatility driving process Z_t^ε possesses long-range correlation properties. As we now show, the volatility process σ_t^ε itself inherits this property.

Lemma 3.1. *We denote, for $j = 1, 2$,*

$$\left\langle F^j \right\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^j p(z) dz, \qquad \left\langle F'^j \right\rangle = \int_{\mathbb{R}} F'(\sigma_{\rm ou} z)^j p(z) dz, \tag{27}$$

where $p(z)$ is the pdf of the standard normal distribution.

1. *The process σ_t^ε is a stationary random process with mean $\mathbb{E}[\sigma_t^\varepsilon] =< F >$ and variance $\mathrm{Var}(\sigma_t^\varepsilon) = < F^2 > - < F >^2$, independently of ε.*
2. *The covariance function of the process σ_t^ε is of the form*

$$\mathrm{Cov}\left(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon\right) = \left(\left\langle F^2 \right\rangle - \langle F \rangle^2\right) \mathcal{C}_\sigma\left(\frac{s}{\varepsilon}\right), \tag{28}$$

where the correlation function $\mathcal{C}_\sigma$ satisfies $\mathcal{C}_\sigma(0) = 1$ and

$$\mathcal{C}_\sigma(s) = \frac{1}{\Gamma(2H-1)} \frac{\sigma_{\rm ou}^2 \langle F' \rangle^2}{\left\langle F^2 \right\rangle - \langle F \rangle^2} s^{2H-2} + o\left(s^{2H-2}\right), \quad \textit{for } s \gg 1. \tag{29}$$

Consequently, the process σ_t^ε possesses long-range correlation properties (i.e., its correlation function is not integrable at infinity).

Proof. The fact that σ_t^ε is a stationary random process with mean $< F >$ is straightforward in view of the definition (23) of σ_t^ε.

For any t, s, the vector $\sigma_{\rm ou}^{-1}(Z_t^\varepsilon, Z_{t+s}^\varepsilon)$ is a Gaussian random vector with mean (0,0) and 2×2 covariance matrix

$$\mathbf{C}^\varepsilon = \begin{pmatrix} 1 & \mathcal{C}_Z(s/\varepsilon) \\ \mathcal{C}_Z(s/\varepsilon) & 1 \end{pmatrix}.$$

Therefore, denoting $F_c(z) = F(\sigma_{\rm ou} z) - < F >$, the covariance function of the process σ_t^ε is

$$\mathrm{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) = \mathbb{E}\left[F_c(\sigma_{\rm ou}^{-1} Z_t^\varepsilon) F_c(\sigma_{\rm ou}^{-1} Z_{t+s}^\varepsilon)\right]$$

$$= \frac{1}{2\pi\sqrt{\det \mathbf{C}^\varepsilon}} \iint_{\mathbb{R}^2} F_c(z_1)F_c(z_2)\exp\left(-\frac{(z_1,z_2)\mathbf{C}^{\varepsilon -1}(z_1,z_2)^T}{2}\right) dz_1 dz_2$$
$$= \Psi\left(C_Z\left(\frac{s}{\varepsilon}\right)\right),$$

with

$$\Psi(C) = \frac{1}{2\pi\sqrt{1-C^2}} \iint_{\mathbb{R}^2} F_c(z_1)F_c(z_2)\exp\left(-\frac{z_1^2+z_2^2-2Cz_1z_2}{2(1-C^2)}\right) dz_1 dz_2 .$$

This shows that $\mathrm{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon)$ is a function of s/ε only. Moreover, the function Ψ can be expanded in powers of C for small C,

$$\Psi(C) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} F_c(z_1)F_c(z_2)\exp\left(-\frac{z_1^2+z_2^2}{2}\right) dz_1 dz_2$$
$$+C\frac{1}{2\pi} \iint_{\mathbb{R}^2} z_1 z_2 F_c(z_1)F_c(z_2)\exp\left(-\frac{z_1^2+z_2^2}{2}\right) dz_1 dz_2 + O(C^2), \qquad C \ll 1,$$

which gives with (17) the form (29) of the correlation function for σ_t^ε. □

4 | THE OPTION PRICE

Our aim is to compute the option price defined as the martingale

$$M_t = \mathbb{E}\left[h(X_T)|\mathcal{F}_t\right], \tag{30}$$

where h is a smooth function. Weaker assumptions are, in fact, possible for h, as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than h.

We introduce the operator

$$\mathcal{L}_{\mathrm{BS}}(\sigma) = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_x^2, \tag{31}$$

that is, the standard Black–Scholes operator at zero interest rate and (constant) volatility σ.

We next exploit the fact that the price process is a martingale to obtain an approximation, by constructing an explicit function $Q_t^\varepsilon(x)$, so that $Q_T^\varepsilon(x) = h(x)$ and $Q_t^\varepsilon(X_t)$ is a martingale up to first-order terms. Then, $Q_t^\varepsilon(X_t)$ gives the approximation for M_t to this order.

The following proposition gives the first-order correction to the expression for the martingale M_t in the regime of small ε.

Proposition 4.1. *When ε is small, we have*

$$M_t = Q_t^\varepsilon(X_t) + o(\varepsilon^{1-H}), \tag{32}$$

where

$$Q_t^\varepsilon(x) = Q_t^{(0)}(x) + \left(x^2\partial_x^2 Q_t^{(0)}(x)\right)\phi_t^\varepsilon + \varepsilon^{1-H}\widetilde{\sigma}\rho Q_t^{(1)}(x). \tag{33}$$

The function $Q_t^{(0)}(x)$ is deterministic and given by the Black–Scholes formula with constant volatility $\overline{\sigma}$,

$$\mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x). \tag{34}$$

The parameters $\overline{\sigma}^2$ and $\widetilde{\sigma}$ are deterministic and given by

$$\overline{\sigma}^2 = \langle F^2 \rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^2 p(z)dz, \qquad \widetilde{\sigma} = \langle F \rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)p(z)dz, \tag{35}$$

where $p(z)$ is the pdf of the standard normal distribution. The random component ϕ_t^ε is given by

$$\phi_t^\varepsilon = \mathbb{E}\left[\frac{1}{2}\int_t^T \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) ds \,\middle|\, \mathcal{F}_t\right]. \tag{36}$$

The function $Q_t^{(1)}(x)$ is the deterministic correction

$$Q_t^{(1)}(x) = x\partial_x \left(x^2\partial_x^2 Q_t^{(0)}(x)\right) D_t, \tag{37}$$

with D_t defined by

$$D_t = \overline{D}(T-t)^{H+\frac{1}{2}}, \qquad \overline{D} = \frac{\langle FF'\rangle}{\Gamma(H+\frac{3}{2})} = \frac{1}{\Gamma(H+\frac{3}{2})}\int_{\mathbb{R}} FF'(\sigma_{\rm ou} z)p(z)dz. \tag{38}$$

As shown in Lemma B.3 (first item), as $\varepsilon \to 0$, the zero-mean random variable $\varepsilon^{H-1}\phi_t^\varepsilon$ has a variance that converges to $\sigma_\phi^2 (T-t)^{2H}$, with

$$\sigma_\phi^2 = \langle FF'\rangle^2 \left(\frac{1}{\Gamma(2H+1)\sin(\pi H)} - \frac{1}{2H\Gamma(H+\frac{1}{2})^2}\right). \tag{39}$$

Moreover, it converges in distribution to a Gaussian random variable with mean zero and variance $\sigma_\phi^2(T-t)^{2H}$. This shows that the two corrective terms in (33) are of the same order ε^{1-H}, but the first one is random, zero-mean, and approximately Gaussian distributed, while the second one is deterministic.

Proof. For any smooth function $q_t(x)$, we have by Itô's formula

$$\begin{aligned} dq_t(X_t) &= \partial_t q_t(X_t)dt + \left(x\partial_x q_t\right)(X_t)\sigma_t^\varepsilon dW_t^* + \frac{1}{2}\left(x^2\partial_x^2 q_t\right)(X_t)(\sigma_t^\varepsilon)^2 dt \\ &= \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)q_t(X_t)dt + \left(x\partial_x q_t\right)(X_t)\sigma_t^\varepsilon dW_t^*, \end{aligned}$$

where the last term is a martingale. Therefore, by (34), we have

$$dQ_t^{(0)}(X_t) = \frac{1}{2}\left((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\right)\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)dt + dN_t^{(0)}, \tag{40}$$

where $N_t^{(0)}$ is a martingale

$$dN_t^{(0)} = \left(x\partial_x\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon dW_t^*.$$

Note also that in equation (40) (and below), we use the notation

$$\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t) = \left(\left(x^2\partial_x^2\right) Q_t^{(0)}(x)\right)\Big|_{x=X_t}.$$

Let ϕ_t^ε be defined by (36). We have

$$\phi_t^\varepsilon = \psi_t^\varepsilon - \frac{1}{2}\int_0^t \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) ds,$$

where the martingale ψ_t^ε is defined by

$$\psi_t^\varepsilon = \mathbb{E}\left[\frac{1}{2}\int_0^T \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) ds \,\Big|\, \mathcal{F}_t\right]. \tag{41}$$

We can write

$$\frac{1}{2}\left((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\right)\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)dt = \left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)d\psi_t^\varepsilon - \left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)d\phi_t^\varepsilon.$$

By Itô's formula,

$$\begin{aligned} d\left[\phi_t^\varepsilon\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)\right] &= \left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)d\phi_t^\varepsilon + \left(x\partial_x\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* \\ &+\mathcal{L}_{\mathrm{BS}}(\sigma_t^\varepsilon)\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &+\left(x\partial_x\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\phi^\varepsilon, W^*\right\rangle_t. \end{aligned}$$

Because $\mathcal{L}_{\mathrm{BS}}(\sigma_t^\varepsilon) = \mathcal{L}_{\mathrm{BS}}(\overline{\sigma}) + \frac{1}{2}((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2)(x^2\partial_x^2)$ and $\mathcal{L}_{\mathrm{BS}}(\overline{\sigma})(x^2\partial_x^2)Q_t^{(0)}(x) = 0$, this gives

$$\begin{aligned} d\left[\phi_t^\varepsilon\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)\right] &= -\frac{1}{2}\left((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\right)\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)dt \\ &+\frac{1}{2}\left((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\right)\left(x^2\partial_x^2\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &+\left(x\partial_x\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\phi^\varepsilon, W^*\right\rangle_t \\ &+\left(x\partial_x\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)d\psi_t^\varepsilon. \end{aligned}$$

We have $<\phi^\varepsilon, W^*>_t=<\psi^\varepsilon, W^*>_t= \rho < \psi^\varepsilon, W>_t$, and therefore

$$\begin{aligned} d\left[(\phi_t^\varepsilon\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)\right] &= -\frac{1}{2}\left((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\right)\left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)dt \\ &+\frac{1}{2}\left((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\right)\left(x^2\partial_x^2\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &+\rho\left(x\partial_x\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\psi^\varepsilon, W\right\rangle_t \\ &+dN_t^{(1)}, \end{aligned}$$

where $N_t^{(1)}$ is a martingale

$$dN_t^{(1)} = \left(x\partial_x\left(x^2\partial_x^2\right)\right) Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \left(x^2\partial_x^2\right) Q_t^{(0)}(X_t)d\psi_t^\varepsilon.$$

Therefore,

$$
\begin{aligned}
d\left[Q_t^{(0)}(X_t)+\phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(X_t)\right] &= \frac{1}{2}\left((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\right)\left(x^2\partial_x^2\left(x^2\partial_x^2\right)\right)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\
&+\rho\left(x\partial_x\left(x^2\partial_x^2\right)\right)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle\psi^\varepsilon,W\right\rangle_t \\
&+dN_t^{(0)}+dN_t^{(1)}.
\end{aligned}
\tag{42}
$$

The deterministic function $Q_t^{(1)}$ defined by (37) satisfies

$$
\mathcal{L}_{\mathrm{BS}}(\overline{\sigma})Q_t^{(1)}(x)=-\left(x\partial_x\left(x^2\partial_x^2Q_t^{(0)}(x)\right)\right)\theta_t,\qquad Q_T^{(1)}(x)=0,
$$

where $\theta_t=-dD_t/dt$ is such that

$$
d\left\langle\psi^\varepsilon,W\right\rangle_t=\left(\varepsilon^{1-H}\theta_t+\widetilde{\theta}_t^\varepsilon\right)dt,
$$

as shown in Lemmas B.1–B.2 with $\widetilde{\theta}_t^\varepsilon$ characterized in equation (B.9). By applying Itô's formula, we obtain

$$
\begin{aligned}
dQ_t^{(1)}(X_t) &= \mathcal{L}_{\mathrm{BS}}(\sigma_t^\varepsilon)Q_t^{(1)}(X_t)dt+\left(x\partial_x\right)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^* \\
&= \mathcal{L}_{\mathrm{BS}}(\overline{\sigma})Q_t^{(1)}(X_t)dt+\frac{1}{2}\left((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\right)\left(x^2\partial_x^2\right)Q_t^{(1)}(X_t)dt \\
&\quad+\left(x\partial_x\right)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^* \\
&= \frac{1}{2}\left((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\right)\left(x^2\partial_x^2\right)Q_t^{(1)}(X_t)dt-\left(x\partial_x\left(x^2\partial_x^2\right)\right)Q_t^{(0)}(X_t)\theta_tdt+dN_t^{(2)},
\end{aligned}
$$

where $N_t^{(2)}$ is a martingale

$$
dN_t^{(2)}=\left(x\partial_x\right)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^*.
$$

Therefore,

$$
\begin{aligned}
&d\left[Q_t^{(0)}(X_t)+\phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(X_t)+\varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(X_t)\right] \\
&=\frac{1}{2}\left((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\right)\left(x^2\partial_x^2\left(x^2\partial_x^2\right)\right)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt+\frac{\varepsilon^{1-H}}{2}\rho\widetilde{\sigma}\left((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\right)\left(x^2\partial_x^2\right)Q_t^{(1)}(X_t)dt \\
&\quad+\varepsilon^{1-H}\rho\left(x\partial_x\left(x^2\partial_x^2\right)\right)Q_t^{(0)}(X_t)(\sigma_t^\varepsilon-\widetilde{\sigma})\theta_tdt+\rho\left(x\partial_x\left(x^2\partial_x^2\right)\right)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\widetilde{\theta}_t^\varepsilon dt \\
&\quad+dN_t^{(0)}+dN_t^{(1)}+\varepsilon^{1-H}\rho\widetilde{\sigma}dN_t^{(2)}.
\end{aligned}
\tag{43}
$$

We next show that the first four terms of the right-hand side are smaller than ε^{1-H}. We introduce, for any $t\in[0,T]$,

$$
R_{t,T}^{(1)}=\int_t^T\frac{1}{2}\left(x^2\partial_x^2\left(x^2\partial_x^2\right)\right)Q_s^{(0)}(X_s)\left((\sigma_s^\varepsilon)^2-\overline{\sigma}^2\right)\phi_s^\varepsilon ds,
\tag{44}
$$

$$R^{(2)}_{t,T} = \int_t^T \frac{\varepsilon^{1-H}}{2} \rho\widetilde{\sigma} \left(x^2\partial_x^2\right) Q_s^{(1)}(X_s) \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) ds, \tag{45}$$

$$R^{(3)}_{t,T} = \int_t^T \varepsilon^{1-H} \rho \left(x\partial_x \left(x^2\partial_x^2\right)\right) Q_s^{(0)}(X_s)\theta_s(\sigma_s^\varepsilon - \widetilde{\sigma}) ds, \tag{46}$$

$$R^{(4)}_{t,T} = \int_t^T \rho \left(x\partial_x \left(x^2\partial_x^2\right)\right) Q_s^{(0)}(X_s)\sigma_s^\varepsilon \widetilde{\theta}_s^\varepsilon ds. \tag{47}$$

We show that, for $j = 1, 2, 3, 4$,

$$\lim_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(R^{(j)}_{t,T})^2\right]^{1/2} = 0. \tag{48}$$

Step 1: *Proof of (48) for* $j = 1$.

We denote

$$Y_s^{(1)} = \left(x^2\partial_x^2 \left(x^2\partial_x^2\right)\right) Q_s^{(0)}(X_s)$$

and

$$\gamma_t^\varepsilon = \frac{1}{2}\int_0^t \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) \phi_s^\varepsilon ds, \tag{49}$$

so that we can write

$$R^{(1)}_{t,T} = \int_t^T Y_s^{(1)} \frac{d\gamma_s^\varepsilon}{ds} ds.$$

Note that $Y_s^{(1)}$ is a bounded semimartingale with bounded quadratic variations, so that its mean square increments $\mathbb{E}[(Y_s^{(1)} - Y_{s'}^{(1)})^2]$ are uniformly bounded by $K|s - s'|$. Let N be a positive integer. We denote $t_k = t + (T - t)k/N$. We have

$$R^{(1)}_{t,T} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Y_s^{(1)} \frac{d\gamma_s^\varepsilon}{ds} ds = R^{(1,a)}_{t,T} + R^{(1,b)}_{t,T},$$

$$R^{(1,a)}_{t,T} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Y_{t_k}^{(1)} \frac{d\gamma_s^\varepsilon}{ds} ds = \sum_{k=0}^{N-1} Y_{t_k}^{(1)} \left(\gamma_{t_{k+1}}^\varepsilon - \gamma_{t_k}^\varepsilon\right),$$

$$R^{(1,b)}_{t,T} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left(Y_s^{(1)} - Y_{t_k}^{(1)}\right) \frac{d\gamma_s^\varepsilon}{ds} ds.$$

Note that we obtain by Minkowski's inequality

$$\mathbb{E}\left[(R^{(1,a)}_{t,T})^2\right]^{1/2} \le 2\sum_{k=0}^{N} \|Y^{(1)}\|_\infty \mathbb{E}[(\gamma_{t_k}^\varepsilon)^2]^{1/2} \le 2(N+1)\|Y^{(1)}\|_\infty \sup_{s\in[0,T]} \mathbb{E}[(\gamma_s^\varepsilon)^2]^{1/2},$$

so that, by Lemma B.4, we have, for any fixed N,

$$\lim_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(R^{(1,a)}_{t,T})^2\right]^{1/2} = 0.$$

On the other hand,

$$\mathbb{E}\left[(R_{t,T}^{(1,b)})^2\right]^{1/2} \le \|F\|_\infty^2 \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[\left(Y_s^{(1)} - Y_{t_k}^{(1)}\right)^4]^{1/4} \mathbb{E}[(\phi_s^\varepsilon)^4]^{1/4} ds$$

$$\le K \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (s-t_k)^{1/2} ds \sup_{s\in[0,T]} \mathbb{E}[(\phi_s^\varepsilon)^4]^{1/4}$$

$$\le \frac{K'}{\sqrt{N}} \sup_{s\in[0,T]} \mathbb{E}[(\phi_s^\varepsilon)^4]^{1/4}.$$

Therefore, by Lemma B.3 (fourth item), we get

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(R_{t,T}^{(1)})^2\right]^{1/2} \le \limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(R_{t,T}^{(1,b)})^2\right]^{1/2} \le \frac{K'}{\sqrt{N}}.$$

Because this is true for any N, we get the desired result.

***Step 2*:** *Proof of (48) for* $j=2$.

We denote

$$Y_s^{(2)} = \rho\widetilde{\sigma}\left(x^2\partial_x^2\right) Q_s^{(1)}(X_s)$$

and

$$\kappa_t^\varepsilon = \frac{\varepsilon^{1-H}}{2} \int_0^t \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) ds, \tag{50}$$

so that we can write

$$R_{t,T}^{(2)} = \int_t^T Y_s^{(2)} \frac{d\kappa_s^\varepsilon}{ds} ds.$$

Note that $Y_s^{(2)}$ is a bounded semimartingale with bounded quadratic variations. Let N be a positive integer. We denote as above $t_k = t + (T-t)k/N$. We then have

$$R_{t,T}^{(2)} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Y_s^{(2)} \frac{d\kappa_s^\varepsilon}{ds} ds = R_{t,T}^{(2,a)} + R_{t,T}^{(2,b)},$$

$$R_{t,T}^{(2,a)} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Y_{t_k}^{(2)} \frac{d\kappa_s^\varepsilon}{ds} ds = \sum_{k=0}^{N-1} Y_{t_k}^{(2)} \left(\kappa_{t_{k+1}}^\varepsilon - \kappa_{t_k}^\varepsilon\right),$$

$$R_{t,T}^{(2,b)} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left(Y_s^{(2)} - Y_{t_k}^{(2)}\right) \frac{d\kappa_s^\varepsilon}{ds} ds.$$

Then, on the one hand,

$$\mathbb{E}\left[(R_{t,T}^{(2,a)})^2\right]^{1/2} \le 2\sum_{k=0}^{N} \|Y^{(2)}\|_\infty \mathbb{E}[(\kappa_{t_k}^\varepsilon)^2]^{1/2} \le 2(N+1)\|Y^{(2)}\|_\infty \sup_{s\in[0,T]} \mathbb{E}[(\kappa_s^\varepsilon)^2]^{1/2},$$

so that, by Lemma B.6, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(R_{t,T}^{(2,a)})^2\right]^{1/2} = 0.$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left[(R_{t,T}^{(2,b)})^2\right]^{1/2} &\le \varepsilon^{1-H}\|F\|_\infty^2 \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \mathbb{E}[\left(Y_s^{(2)}-Y_{t_k}^{(2)}\right)^2]^{1/2} ds \\ &\le K\varepsilon^{1-H}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(s-t_k)^{1/2} ds \\ &\le \frac{K'\varepsilon^{1-H}}{\sqrt{N}}.\end{aligned}$$

Therefore, we get

$$\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\left[(R_{t,T}^{(2)})^2\right]^{1/2} \le \limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\left[(R_{t,T}^{(2,b)})^2\right]^{1/2} \le \frac{K'}{\sqrt{N}}.$$

Because this is true for any N, we get the desired result.

Step 3: *Proof of (48) for* $j = 3$.

This proof follows the same lines as the proof of Step 2 with

$$\eta_t^\varepsilon = \varepsilon^{1-H}\int_0^t \left(\sigma_s^\varepsilon - \widetilde{\sigma}\right) ds, \tag{51}$$

instead of κ_t^ε, and using the fact that θ_t is bounded. We then get the desired result by Lemma B.5.

Step 4: *Proof of (48) for* $j = 4$.

We have

$$\mathbb{E}\left[(R_{t,T}^{(4)})^2\right]^{1/2} \le K\int_t^T \mathbb{E}\left[(\widetilde{\theta}_s^\varepsilon)^2\right]^{1/2} ds \le K'\sup_{s\in[0,T]}\mathbb{E}\left[(\widetilde{\theta}_s^\varepsilon)^2\right]^{1/2}.$$

By Lemma B.2, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\left[(R_{t,T}^{(4)})^2\right]^{1/2} = 0.$$

We can now complete the proof of Proposition 4.1. In (33), we introduced the approximation

$$Q_t^\varepsilon(x) = Q_t^{(0)}(x) + \phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x).$$

We then have

$$Q_T^\varepsilon(x) = h(x),$$

because $Q_T^{(0)}(x) = h(x)$, $\phi_T^\varepsilon = 0$, and $Q_T^{(1)}(x) = 0$. Let us denote

$$R_{t,T} = R_{t,T}^{(1)} + R_{t,T}^{(2)} + R_{t,T}^{(3)} + R_{t,T}^{(4)}, \tag{52}$$

$$N_t = \int_0^t dN_s^{(0)} + dN_s^{(1)} + \varepsilon^{1-H}\rho\widetilde{\sigma}dN_s^{(2)}. \tag{53}$$

By (43), we have

$$Q_T^\varepsilon(X_T) - Q_t^\varepsilon(X_t) = R_{t,T} + N_T - N_t.$$

Therefore,

$$
\begin{aligned}
M_t &= \mathbb{E}\left[h(X_T)|\mathcal{F}_t\right] = \mathbb{E}\left[Q_T^\varepsilon(X_T)|\mathcal{F}_t\right] = Q_t^\varepsilon(X_t) + \mathbb{E}\left[R_{t,T}|\mathcal{F}_t\right] + \mathbb{E}\left[N_T - N_t|\mathcal{F}_t\right] \\
&= Q_t^\varepsilon(X_t) + \mathbb{E}\left[R_{t,T}|\mathcal{F}_t\right],
\end{aligned} \tag{54}
$$

which gives the desired result, because $\mathbb{E}[R_{t,T}|\mathcal{F}_t]$ is of order $o(\varepsilon^{1-H})$ in L^2. □

5 | CALL PRICE CORRECTION AND IMPLIED VOLATILITY

We denote the Black–Scholes call price, with current time t, maturity T, strike K, underlying value x, and volatility σ, by $C_{\rm BS}(t,x;K,T;\sigma)$, so that $Q_t^{(0)}$ in equation (34) is

$$
Q_t^{(0)}(x) = C_{\rm BS}(t,x;K,T;\overline{\sigma}).
$$

Indeed, $C_{\rm BS}$ gives an explicit formula for the price when volatility is constant. In the case with stochastic volatility as considered here, no explicit pricing formula exists. As shown in equation (33), however, we can get an asymptotic expression for the price in the case with the stochastic volatility model (5) as a correction to $Q_t^{(0)}(x)$, the Black–Scholes price evaluated at the effective, or "homogenized," volatility $\bar{\sigma}$. Here, we show that this corrected price takes on a rather simple, generic form in the two parameters: relative time to maturity and moneyness. This representation then leads to a simple representation for the implied volatility, as we show below. The long-range character of the volatility fluctuations indeed has a strong impact on the form of the implied volatility, and this observation is important in a calibration context.

We denote the time to maturity by $\tau = T - t$, and we introduce the characteristic diffusion time $\bar{\tau} = 2/\overline{\sigma}^2$ and the dimensionless effective skewness factor

$$
a_F = \varepsilon^{1-H}\frac{\rho\widetilde{\sigma}\overline{D}\bar{\tau}^H}{2^{3/2}\overline{\sigma}} = \varepsilon^{1-H}\frac{\widetilde{\sigma}\rho\left\langle FF'\right\rangle\bar{\tau}^H}{2^{3/2}\overline{\sigma}\Gamma(H+3/2)}, \tag{55}
$$

with $\overline{\sigma}$, $\widetilde{\sigma}$, and $\overline{D}$ given in Proposition 4.1 and the correlation ρ introduced in equation (24).

Lemma 5.1. *The price correction in equation (33), normalized by the strike K, can be written in the form*

$$
\begin{aligned}
&\frac{1}{K}\left(\phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x)\right) \\
&= \left(\frac{e^{-d_1^2/2}\frac{x}{K}}{\sqrt{\pi}}\right)\left\{\frac{\phi_t^\varepsilon}{2}\left(\frac{\tau}{\bar{\tau}}\right)^{-1/2} + a_F\left[\left(\frac{\tau}{\bar{\tau}}\right)^H + \left(\frac{\tau}{\bar{\tau}}\right)^{H-1}\log\left(\frac{K}{x}\right)\right]\right\},
\end{aligned} \tag{56}
$$

with

$$
d_1 = \sqrt{\frac{\bar{\tau}}{2\tau}}\left[\frac{\tau}{\bar{\tau}} - \log\left(\frac{K}{x}\right)\right]. \tag{57}
$$

Here, the dimensionless random and deterministic correction coefficients are small,

$$
\phi_t^\varepsilon = O\left(\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^H\right), \qquad a_F = O\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}, \tag{58}
$$

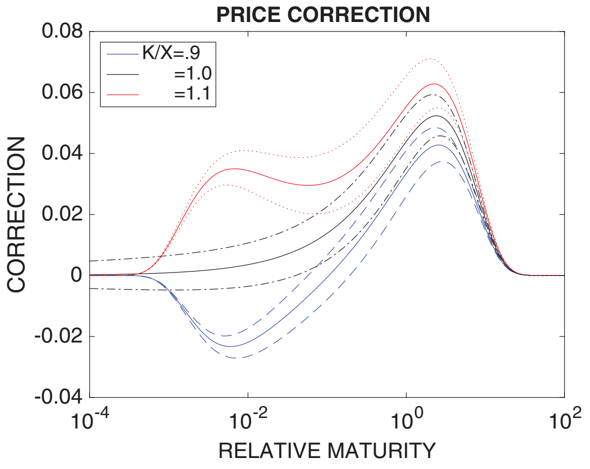

FIGURE 2 Price correction as a function of the relative time to maturity $\tau/\bar{\tau}$. The three solid lines correspond (from bottom to top) to the mean price correction for $K/X = 0.9$, 1.0, and 1.1, respectively. The dashed/dotted lines correspond to the mean ± 1 standard deviation. Here, $H = 0.6$, $a_F = 0.1$, and $((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi) = 0.04$ [Color figure can be viewed at wileyonlinelibrary.com]

where we used the fact that ϕ_t^ε as defined in Proposition 4.1 is centered and with standard deviation

$$\mathrm{Var}\left(\phi_t^\varepsilon\right)^{1/2} = \left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^{H}\left(\bar{\tau}\sigma_\phi\right) + o(\varepsilon^{1-H}), \tag{59}$$

with σ_ϕ defined by equation (39) (see also equation (B.14) in Lemma B.3). We comment in more detail about the statistical structure of ϕ_t^ε in the next section.

It follows from the above that the normalized price correction depends on the two parameters—the moneyness K/x and the relative time to maturity $\tau/\bar{\tau}$—and exhibits a term structure in fractional powers of relative time to maturity.

In Figure 2, we show the relative price correction in equation (56) as a function of relative time to maturity $\tau/\bar{\tau}$ for three values of the moneyness K/x. The solid lines plot the mean relative price correction, and the dashed lines give the mean plus/minus one standard deviation. We use here $H = 0.6$, $a_F = 0.1$, and $((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi) = 0.04$. The mean relative price correction is largest for a midrange of times to maturity. For very short times to maturity relative to the characteristic diffusion time, the effect of the volatility fluctuations is small, while for long times, the rapid mean reversion "averages" out the effect of the fluctuations. Note, however, that at the money, the random component of the price correction decays slowly as

$$\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2},$$

as $\tau \to 0$, while "around the money" with the moneyness K/x being different from one, the decay has the form

$$\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2}\exp\left(-\frac{\bar{\tau}|\log(K/x)|^2}{4\tau}\right).$$

This reflects the fact that the vega is diverging in this limit so that the sensitivity to volatility fluctuations becomes strong. We remark that this would affect calibration schemes using at-the-money data. Moreover, results regarding short-time asymptotics for the coherent-implied volatility become questionable in this context as the dominating contribution comes from the random component of the price

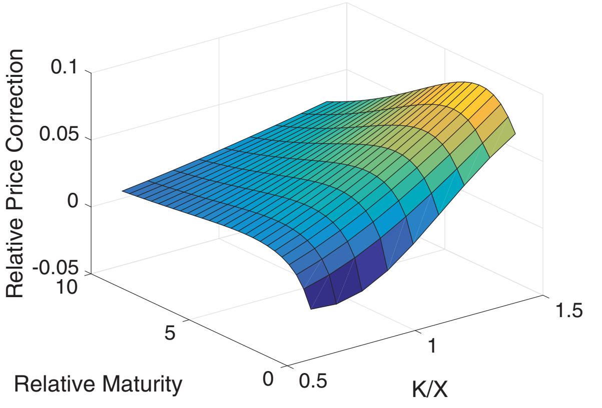

FIGURE 3 The price correction surface as a function of the relative time to maturity $\tau/\bar{\tau}$ and the moneyness K/X. The parameters are like those in Figure 2 [Color figure can be viewed at wileyonlinelibrary.com]

correction. Note also that the parameters are not calibrated to market data; this will be considered in another publication.

In Figure 3, we show the price correction surface as a function of the relative time to maturity $\tau/\bar{\tau}$ and the moneyness K/x. The figure shows that the price correction is large when the time to maturity is of the order of the characteristic diffusion time.

We next present the proof of Lemma 5.1.

Proof. For the European call option with payoff $h(x) = (x - K)_+$, we have

$$C_{\rm BS}(t,x;K,T;\sigma) = x\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) + \frac{\sigma\sqrt{T-t}}{2}\right) - K\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) - \frac{\sigma\sqrt{T-t}}{2}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. We then have, in particular, the "Greek" relationships for the call price

$$\partial_\sigma C_{\rm BS} = (T-t)\overline{\sigma} x^2 \partial_x^2 C_{\rm BS}, \qquad x\partial_x\partial_\sigma C_{\rm BS} = \left(\frac{1}{2} + \frac{\log\frac{K}{x}}{\overline{\sigma}^2(T-t)}\right)\partial_\sigma C_{\rm BS}.$$

We then get

$$x^2\partial_x^2 Q_t^{(0)}(x) = \frac{1}{\overline{\sigma}(T-t)}\partial_{\bar{\sigma}} C_{\rm BS}(t,x;K,T;\overline{\sigma}), \tag{60}$$

$$x\partial_x x^2\partial_x^2 Q_t^{(0)}(x) = \left[\frac{1}{2\overline{\sigma}(T-t)} + \frac{\log\frac{K}{x}}{\overline{\sigma}^3(T-t)^2}\right]\partial_{\bar{\sigma}} C_{\rm BS}(t,x;K,T;\overline{\sigma}), \tag{61}$$

where the "vega" is given by

$$\partial_\sigma C_{\rm BS}(t,x;K,T;\overline{\sigma}) = \frac{xe^{-d_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}}, \qquad d_1 = \frac{\frac{1}{2}\sigma^2(T-t) - \log\frac{K}{x}}{\sigma\sqrt{T-t}}. \tag{62}$$

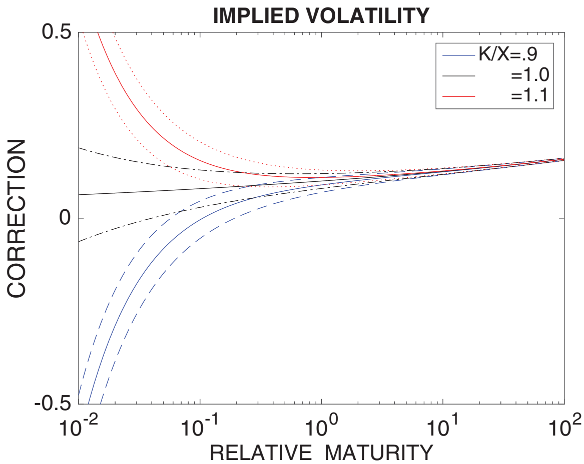

FIGURE 4 The implied volatility correction as a function of the relative time to maturity $\tau/\bar{\tau}$. The three solid lines correspond (from bottom to top) to the mean implied volatility correction for $K/X = 0.9$, 1.0, and 1.1, respectively. The dashed/dotted lines correspond to the mean ± 1 standard deviation [Color figure can be viewed at wileyonlinelibrary.com]

Then, with $Q_t^{(1)}(x)$ given in equation (37), we can identify the form of the price correction as

$$\begin{aligned}
&\phi_t^\varepsilon \left(x^2\partial_x^2\right) Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x)\\
&= \phi_t^\varepsilon \left(x^2\partial_x^2\right) Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}D(t)x\partial_x x^2\partial_x^2 Q_t^{(0)}(x)\\
&= \phi_t^\varepsilon \left(\frac{xe^{-d_1^2/2}}{\overline{\sigma}\sqrt{2\pi(T-t)}}\right) + \varepsilon^{1-H}\left(\frac{x\rho\widetilde{\sigma}\overline{D}e^{-d_1^2/2}}{\sqrt{2\pi}}\right)\left[\frac{(T-t)^H}{2\overline{\sigma}} + \frac{\log\frac{K}{x}}{\overline{\sigma}^3(T-t)^{1-H}}\right],
\end{aligned} \tag{63}$$

which, in turn, gives (56). □

We next consider the implied volatility associated with the price correction. For the stochastic volatility model in equation (5), we want to identify the implied volatility I_t so that in terms of the corrected price in Lemma 4.1, we have

$$C_{\mathrm{BS}}(t, x; K, T; I_t) = Q_t^{(0)}(x) + \phi_t^\varepsilon \left(x^2\partial_x^2\right) Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x). \tag{64}$$

We define the relative implied volatility correction δI_t by

$$I_t = \overline{\sigma}(1 + \delta I_t). \tag{65}$$

Lemma 5.2. *The relative implied volatility correction has the form*

$$\delta I_t = \frac{\phi_t^\varepsilon}{2}\left(\frac{\tau}{\bar{\tau}}\right)^{-1} + a_F\left[\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2}\log\left(\frac{K}{X_t}\right)\right] + o(\varepsilon^{1-H}), \tag{66}$$

where ϕ_t^ε is defined by (36) and a_F by (55).

In Figure 4, we show the implied volatility correction in equation (66) as a function of relative time to maturity $\tau/\bar{\tau}$ for three values of the moneyness K/x. We again used $H = 0.6, a_F = 0.1$, and $((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi) = 0.04$. Note that due to the form of the "vega" (which is the sensitivity of the price to the volatility), the form of the implied volatility surface is very different from that of the price

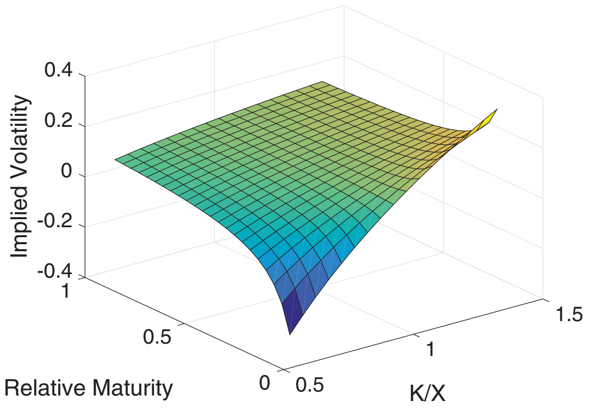

FIGURE 5 The mean implied volatility correction surface as a function of the relative time to maturity $\tau/\bar{\tau}$ and the moneyness K/X. The parameters are like those in Figure 4 [Color figure can be viewed at wileyonlinelibrary.com]

correction. In Figure 5, we show the implied volatility correction surface as a function of the relative time to maturity $\tau/\bar{\tau}$ and the moneyness K/x.

Proof. We find by using equations (63) and (62) that the implied volatility is given by

$$I_t = \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} + \varepsilon^{1-H}\widetilde{\sigma}\rho D_t\left[\frac{1}{2\overline{\sigma}(T-t)} + \frac{\log\frac{K}{X_t}}{\overline{\sigma}^3(T-t)^2}\right] + o(\varepsilon^{1-H}). \tag{67}$$

Because D_t is deterministic and given by (38), we can then write

$$\begin{aligned} I_t = {} & \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} \\ & + \varepsilon^{1-H}\frac{\widetilde{\sigma}\rho\langle FF'\rangle}{\overline{\sigma}\Gamma(H+\frac{3}{2})}\left[\frac{1}{2}(T-t)^{H-\frac{1}{2}} + \frac{\log\frac{K}{X_t}}{\overline{\sigma}^2(T-t)^{\frac{3}{2}-H}}\right] + o(\varepsilon^{1-H}), \end{aligned} \tag{68}$$

and the lemma follows. □

The first two terms in equation (68) can be combined and rewritten as (up to terms of order $o(\varepsilon^{1-H})$)

$$\overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} = \mathbb{E}\left[\frac{1}{T-t}\int_t^T(\sigma_s^\varepsilon)^2ds\middle|\mathcal{F}_t\right]^{1/2} + o(\varepsilon^{1-H}). \tag{69}$$

Because D_t is deterministic and given by (38), we can then write

$$\begin{aligned} I_t = {} & \mathbb{E}\left[\frac{1}{T-t}\int_t^T(\sigma_s^\varepsilon)^2ds\middle|\mathcal{F}_t\right]^{1/2} \\ & + \overline{\sigma}a_F\left[\left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2}\log\left(\frac{K}{X_t}\right)\right] + o\left(\varepsilon^{1-H}\right), \end{aligned} \tag{70}$$

so that the implied volatility is the expected effective volatility over the remaining time horizon conditioned on the present and with an added skewness correction.

In view of equation (59), when the time to maturity is short, the fourth term (in $\tau^{H-\frac{3}{2}}$) dominates in (66). We remark here that this is related to the fact that the small parameter in our problem is the mean-reversion time, so that for any time to maturity of order one in this regime, the volatility has enough time to fluctuate and mean revert, giving a price correction as in Lemma 5.1. Moreover, because the "vega," $\partial_\sigma C_{\rm BS}$, is small away from the money (see equation (62)), we get a strong moneyness dependence, and the implied volatility blows up when the time to maturity goes to zero.

When the time to maturity is long, the third term (in $\tau^{H-\frac{1}{2}}$) dominates in (66). The long-range dependence gives smooth volatility fluctuations, which gives an implied volatility that blows up when the time to maturity goes to infinity. The current value of the underlying is less important in this long-time-to-maturity regime.

6 | THE *t*-*T* PROCESS AND THE STOCHASTIC IMPLIED SURFACE

We introduced in equation (36) the stochastic correction coefficient $\phi_t^\varepsilon \equiv \phi_{t,T}^\varepsilon$, which gives the random component of the price correction and the implied volatility. Note that we explicitly display here the dependence on maturity T. If the volatility process had been a Markovian process, then the correction would have been deterministic, as in Fouque et al. (2011). The presence of long-range memory in the volatility process means that information from the past (volatility path) must be carried forward, and this makes the price correction relative to the price at the homogenized volatility a stochastic process; this is also the case for the implied volatility.

Here, we discuss the statistical structure of the random field, which describes the implied volatility surface in the scaling regime that we consider. The implied volatility is the central quantity in typical calibration processes. To design efficient estimators for both the coherent and incoherent parts of the implied volatility, as well as to characterize the resulting estimation precision, it is important to understand the statistical fluctuations of the observed implied surface. We give a precise characterization of these fluctuations below. The fluctuations of the implied volatility for long times to maturity (relative to $\bar{\tau}$) become strong when the Hurst exponent is large, because the large Hurst exponent gives strong temporal coherence and large correction to the anticipated volatility. On the other hand, for short times to maturity, the fluctuations become large when the Hurst exponent is small, because the small Hust exponent gives a rough process with large fluctuations even over very small intervals. It is also interesting to note that the correlation structure of the implied volatility surface, in fact, encodes information about the long-range character of the underlying stochastic volatility. Observing, for instance, at-the-money implied volatility fluctuations as a function of current time for fixed time to maturity gives information that makes it possible to estimate the Hurst exponent and to check for the consistency of the modeling framework. In Livieri, Mouti, Pallavicini, and Rosenbaum (2017), observed at-the-money implied volatility was used to estimate the Hurst exponent. The authors found a coefficient that was slightly larger than the corresponding estimates using historical data and explained this discrepancy in terms of a smoothing effect due to the remaining time to maturity. To construct and interpret estimators of this kind, a model for the implied surface as a random field relating it to the underlying volatility parameters is clearly essential.

In order to understand the implied volatility random field, note first that it follows from Lemma B.3 that as $\varepsilon \to 0$, the random process $\varepsilon^{H-1}\phi_{t,T}^\varepsilon/[\sigma_\phi(T-t)^H]$, $t<T$, converges in distribution (in the sense of finite-dimensional distributions) to a Gaussian stochastic process $\psi_{t,T}$, $t<T$, *the normalized t-T correction process*, with mean zero, variance one, and covariance $\mathbb{E}[\psi_{t,T}\psi_{t',T'}] = C_\phi(t,t';T,T')$

for any $t \in [0,T)$, $t' \in [0,T')$. The four-parameter function C_ϕ is given by equation (B.16). We will discuss next in more detail the t-T process $\psi_{t,T}$, a two-parameter process of current time t and maturity T. This process is defined on $0 < t < T$; it is a nonstationary Gaussian process, and it is scaled to have constant unit variance. As we see below, close to maturity $t \approx T$, the process is strongly affected by the presence of the maturity boundary.

Let us first consider the case of a fixed maturity T and introduce the process

$$\psi_0(t;T) = \psi_{t,T}, \qquad t \in [0,T]. \tag{71}$$

When the times are short relative to the time to maturity, that is, for $|t-t'| \ll T-t$, it follows from equation (B.16) that the process $(\psi_0(t;T))_{t\in[0,T]}$ decorrelates as

$$\mathbb{E}\left[\psi_0(t;T)\psi_0(t';T)\right] \sim 1 - \frac{|t-t'|}{2(T-t)},$$

which means that it decorrelates as a Markovian process for short times. More generally, the autocovariance function of $(\psi_0(t;T))_{t\in[0,T]}$ is

$$\mathbb{E}\left[\psi_0(t;T)\psi_0(t';T)\right] = C(\Delta_0(t,t';T)),$$

$$C(\Delta) = \frac{\int_0^\infty du \left[\left(u + \frac{|\Delta|+1}{\sqrt{1-\Delta^2}}\right)^{H-\frac12} - u^{H-\frac12}\right]\left[\left(u + \frac{|\Delta|+1}{\sqrt{1-\Delta^2}}\right)^{H-\frac12} - \left(u + \frac{2|\Delta|}{\sqrt{1-\Delta^2}}\right)^{H-\frac12}\right]}{\int_0^\infty du \left[(1+u)^{H-\frac12} - u^{H-\frac12}\right]^2},$$

with

$$\Delta_0(t,t';T) = \frac{t'-t}{|2T-(t+t')|}, \tag{72}$$

which shows that the correlation function of the process $(\psi_0(t;T))_{t\in[0,T]}$ depends only on this relative separation, giving a situation with a canonical relative decorrelation that depends only on the times to maturity $\tau = T-t$, $\tau' = T-t'$. Therefore, we introduce the process $(\psi_1(\tau;T))_{\tau\in[0,T]}$ defined by

$$\psi_1(\tau;T) = \psi_{T-\tau,T}, \qquad \tau \in [0,T]. \tag{73}$$

The process $(\psi_1(\tau;T))_{\tau\in[0,T]}$ is Gaussian with mean zero and autocovariance function

$$\mathbb{E}\left[\psi_1(\tau;T)\psi_1(\tau';T)\right] = C(\Delta_1(\tau,\tau')),$$

with C as above and

$$\Delta_1(\tau,\tau') = \frac{\tau-\tau'}{|\tau+\tau'|}. \tag{74}$$

For $|\tau-\tau'| \ll \tau$, the process decorrelates on the time scale τ so that the process fluctuations become more rapid close to maturity. Close to maturity, the price fluctuations become small. When we magnify them, however, we see fluctuations on small time scales when the time to maturity is short, which reflects the self-similarity of the driving volatility factor. In Figure 6, we show the correlation function $\Delta_1 \mapsto C(\Delta_1)$ as a function of the relative separation time $\Delta_1 \in [-1,1]$ and for $H = 0.6$. The process decorrelates as a Markovian process for short times; indeed, as one of the times to maturity goes to zero (relative to the other time to maturity), the correlation goes rapidly to zero.

Note that it follows from the expression (74) for Δ_1 that it is scale invariant, in that $\Delta_1(a\tau, a\tau') = \Delta_1(\tau,\tau')$ for $a > 0$, giving rapid fluctuations for short times to maturity. The process indeed has a

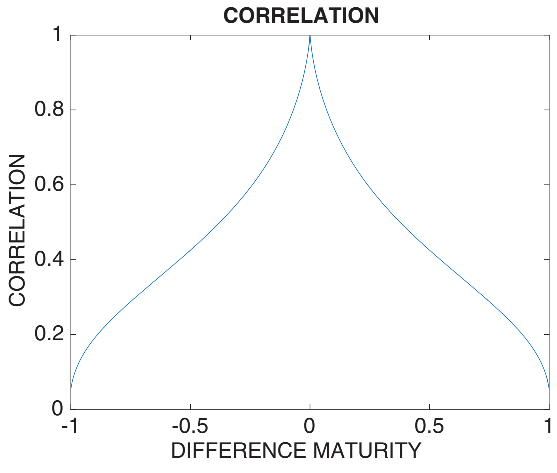

FIGURE 6 Autocovariance function of the *t*-*T* process $\psi_1(\tau;1)$ as a function of the relative time to maturity separation $\Delta_1 = (\tau-\tau')/|\tau+\tau'|$ with $H=0.6$. The correlation decays approximately linearly at the origin and rapidly as one of the times to maturity goes to zero [Color figure can be viewed at wileyonlinelibrary.com]

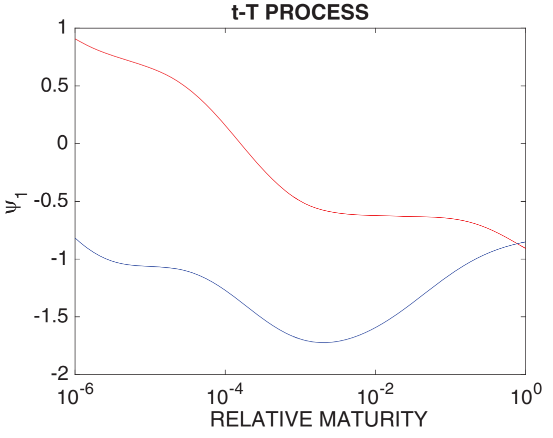

FIGURE 7 Realizations of the process $\psi_1(\tau;1)$ as a function of the time to maturity τ for fixed maturity $T=1$ with $H=0.6$ [Color figure can be viewed at wileyonlinelibrary.com]

self-similar property. We have in distribution

$$\big(\psi_1(\tau;1)\big)_{\tau\in[0,1]} \sim \big(\psi_1(\tau T;T)\big)_{\tau\in[0,1]},$$

for any $T>0$. In Figure 7, we show two realizations of the process $\psi_1(\tau;1)$ as a function of time to maturity τ.

One can also investigate the structure of the *t*-*T* process for a fixed time to maturity τ, as a function of time t. Thus, if we observe the price for a given time to maturity, we would like to know how the price correction (and the implied volatility) would fluctuate with respect to the current time, or time translation. Accordingly, we consider the process

$$\psi_2(t;\tau) = \psi_{t,\tau+t}, \qquad t\geq 0, \tag{75}$$

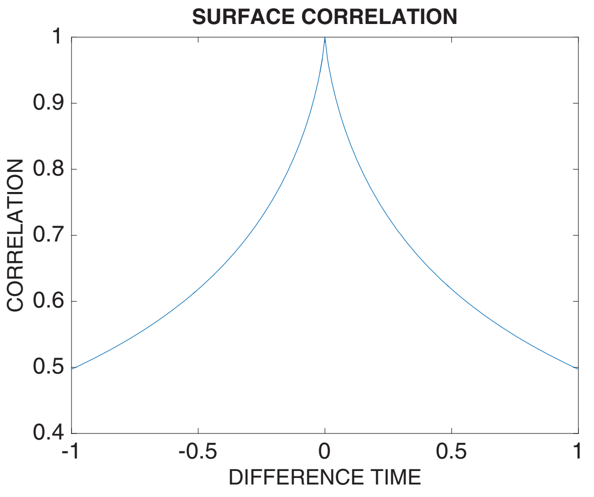

FIGURE 8 Autocovariance function of the t-T process $\psi_2(t;1)$ as a function of the time $t'-t$ for fixed time to maturity $\tau=1$ with $H=0.6$. On the short time scales, the process decorrelates as a Markovian process; on the long time scales, it exhibits long-range correlations [Color figure can be viewed at wileyonlinelibrary.com]

for fixed $\tau>0$. The process $(\psi_2(t;\tau))_{t\in[0,\infty)}$ is Gaussian with mean zero and autocovariance function

$$\mathbb{E}\left[\psi_2(t;\tau)\psi_2(t';\tau)\right]=C_2(\Delta_2(t,t';\tau)), \tag{76}$$

$$C_2(\Delta)=\frac{\int_0^\infty du\left[(u+1)^{H-\frac{1}{2}}-u^{H-\frac{1}{2}}\right]\left[(u+1+|\Delta|)^{H-\frac{1}{2}}-(u+|\Delta|)^{H-\frac{1}{2}}\right]}{\int_0^\infty du\left[(1+u)^{H-\frac{1}{2}}-u^{H-\frac{1}{2}}\right]^2},$$

with

$$\Delta_2(t,t';\tau)=\frac{t'-t}{\tau}. \tag{77}$$

The expression of Δ_2 shows that the coherence time of this process is proportional to the time to maturity τ. We see again that the rescaled implied volatility surface fluctuations are more rapid when they are close to maturity. We also see that on transects parallel to the maturity boundary in the t,T plane, these fluctuations are stationary. This is consistent with the fact that we have an underlying consistent model with a stationary volatility driving factor. The fluctuations, moreover, have a self-similar property. We have in distribution

$$\left(\psi_2(t;1)\right)_{t\in[0,\infty)}\sim\left(\psi_2(\tau t;\tau)\right)_{t\in[0,\infty)},$$

for any $\tau>0$. The autocovariance function of $(\psi_2(t;1))_{t\in[0,\infty)}$ is plotted in Figure 8. In the figure, one can see the rapid decay at the origin followed by a long-range behavior. This shows how the implied surface decorrelates as we move in time. In Figure 9, we show the autocorrelation function in a *log-log* plot with the dashed line corresponding to the correlation decay $|t'-t|^{2H-2}$. In Figure 10, we show two realizations of the process $\psi_2(t;1)$.

Finally, it is of interest to consider the case where we evaluate the stochastic correction factor as a function of time to maturity for the fixed current time t,

$$\psi_3(\tau;t)=\psi_{t,t+\tau},\qquad \tau\geq 0. \tag{78}$$

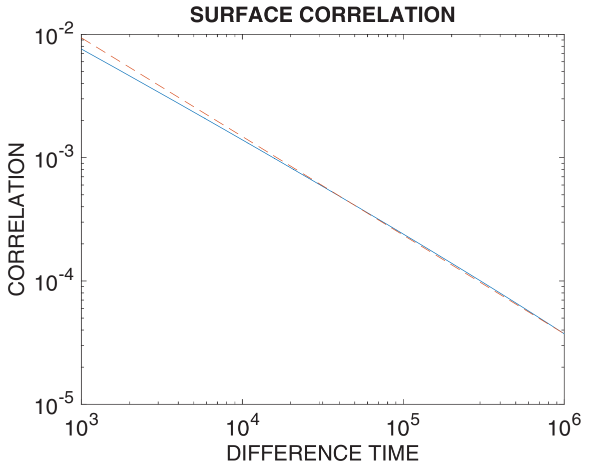

FIGURE 9 Autocovariance function of the t-T process $\psi_2(t;1)$ as in Figure 8, but on a *log-log* scale with the dashed line showing the decay $|t'-t|^{2H-2}$ [Color figure can be viewed at wileyonlinelibrary.com]

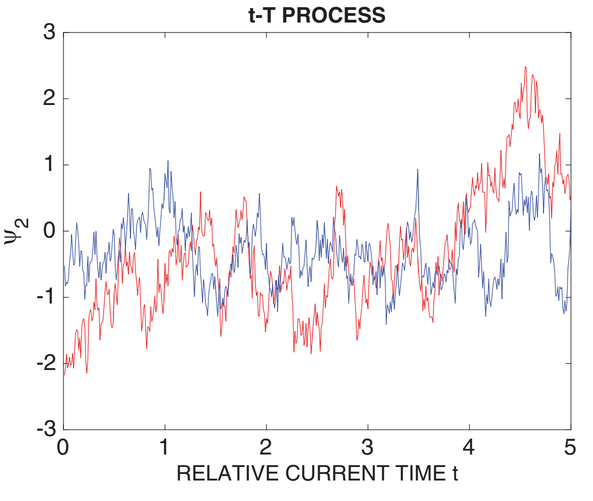

FIGURE 10 Realizations of the process $\psi_2(t;1)$ with $H=0.6$ [Color figure can be viewed at wileyonlinelibrary.com]

The process $(\psi_3(\tau;t))_{\tau\in[0,\infty)}$ is Gaussian with mean zero and autocovariance function

$$\mathbb{E}\left[\psi_3(\tau;t)\psi_3(\tau';t)\right] = C_3(\Delta_3(\tau,\tau')),$$

$$C_3(\Delta) = \frac{\int_0^\infty du\left[(u+1/\sqrt{1+|\Delta|})^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]\left[(u+\sqrt{1+|\Delta|})^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]}{\int_0^\infty du\left[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]^2},$$

with

$$\Delta_3(\tau,\tau') = \frac{\tau-\tau'}{\tau\wedge\tau'}. \tag{79}$$

This covariance function is plotted in Figure 11. Note that it follows from the expression (79) for Δ_3 that it is scale invariant in that $\Delta_3(a\tau,a\tau') = \Delta_3(\tau,\tau')$ for $a>0$, so that again, the process fluctuates

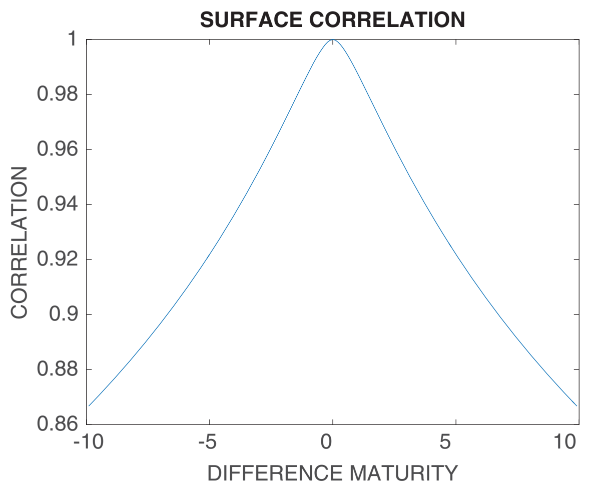

FIGURE 11 Autocovariance function of the *t-T* process $\psi_3(\tau;1)$ as a function of the relative time-to-maturity separation $\Delta_3 = (\tau - \tau')/(\tau \wedge \tau')$ with $H = 0.6$. Note that the correlation function exhibits slow decay [Color figure can be viewed at wileyonlinelibrary.com]

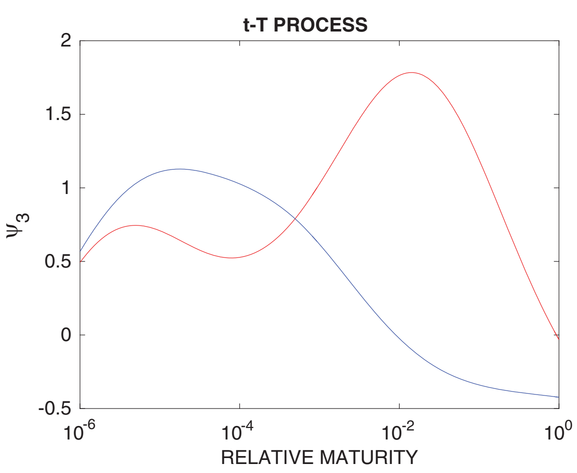

FIGURE 12 Realizations of the process $\psi_3(\tau;1)$ for fixed current time $t = 1$ and $H = 0.6$, with the smooth and slow decay of the correlations giving a smooth time-to-maturity dependence [Color figure can be viewed at wileyonlinelibrary.com]

more rapidly for small maturities. The distribution of the process $(\psi_3(\tau;t))_{\tau\in[0,\infty)}$ does not depend on t, and it has a self-similar property. For any $a > 0$, we have in distribution

$$\big(\psi_3(\tau;t)\big)_{\tau\in[0,\infty)} \sim \big(\psi_3(a\tau;t)\big)_{\tau\in[0,\infty)}.$$

In Figure 12 , we show two realizations of the process $(\psi_3(\tau;t))_{\tau\in[0,1)}$.

7 | CONCLUSION

We have considered a continuous time stochastic volatility model with long-range correlation properties. We have addressed the regime of fast mean reversion. This makes it possible to derive an explicit expression for the approximate European call option price and the implied volatility. Specifically, the volatility is a smooth function of an fOU process. Analyzing such a non-Markovian situation is challenging. To the best of our knowledge, we present the first analytical expression for the price approximation for general maturities when the volatility fluctuations are of order 1. So far, the price computations for such situations have been based on numerical approximations. The main result from the applied viewpoint is then the form of the fractional term structure that we obtain for the implied volatility surface. Indeed, we get an implied volatility that grows large with time to maturity while generating a strong skew for short times to maturity, which is consistent with common observations. We stress that in our formulation, the mean-reversion time is small compared to any fixed maturity as we consider a fast mean-reverting process. Let us note, finally, that we have considered the case of processes with long-range correlation properties with the Hurst exponent $H > 1/2$ explaining the growth of implied volatility for large maturity.

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APPENDIX A: HERMITE DECOMPOSITION OF THE STOCHASTIC VOLATILITY MODEL

We denote

$$\widetilde{F}(z) = F(\sigma_{\rm ou} z)^2. \tag{A.1}$$

Because $\mathbb{E}[\widetilde{F}(Z)^2] < \infty$ is finite when Z is a standard normal variable, the function $\widetilde{F}$ can be expanded in terms of the Hermite polynomials

$$H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2}, \tag{A.2}$$

and the series

$$\sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(z), \tag{A.3}$$

with

$$C_k = \mathbb{E}\left[H_k(Z)\widetilde{F}(Z)\right] = \int_{\mathbb{R}} H_k(z)\widetilde{F}(z)p(z)dz, \tag{A.4}$$

converges in $L^2(\mathbb{R}, p(z)dz)$ to $\widetilde{F}(z)$. The Hermite polynomials satisfy

$$\mathbb{E}[H_k(Z)H_j(Z)] = \int_{\mathbb{R}} H_k(z)H_j(z)p(z)dz = \delta_{kj}k!,$$

and we have $\sum_{k=0}^{\infty} \frac{C_k^2}{k!} = \mathbb{E}[\widetilde{F}(Z)^2] < \infty$. Note that $C_0 =< F^2 >$.

Lemma A.1. *If there exists $\alpha > 2$ such that the function $\widetilde{F}$ defined by (A.1) satisfies*

$$\sum_{k=0}^{\infty} \frac{\alpha^k C_k^2}{k!} < \infty, \tag{A.5}$$

then the random process

$$I_t^\varepsilon = \int_0^t F^2(Z_s^\varepsilon) - \left\langle F^2 \right\rangle ds \tag{A.6}$$

satisfies

$$\sup_{t\in[0,T]} \mathbb{E}[(I_t^\varepsilon)^4] \le K\varepsilon^{4-4H}, \tag{A.7}$$

for some constant K.

Proof. Denoting $\widetilde{Z}_t^\varepsilon = \sigma_{\rm ou}^{-1} Z_t^\varepsilon$, which is a zero-mean Gaussian process with covariance function $\mathbb{E}[\widetilde{Z}_t^\varepsilon \widetilde{Z}_{t+s}^\varepsilon] = \mathcal{C}_Z(s/\varepsilon)$, we have

$$I_t^\varepsilon = \int_0^t \widetilde{F}(\widetilde{Z}_s^\varepsilon) - \left\langle F^2 \right\rangle ds = \sum_{m=1}^{\infty} C_m I_{t,m}^\varepsilon,$$

where

$$I_{t,m}^\varepsilon = \frac{1}{m!}\int_0^t H_m(\widetilde{Z}_s^\varepsilon)ds, \qquad m \ge 1.$$

From Taqqu (1978, lemma 2.2), the fourth-order moment of $I_{t,m}^\varepsilon$ can be expanded as

$$\mathbb{E}[(I_{t,m}^\varepsilon)^4] = \frac{1}{2^m(2m)!}\sum \int_0^t \cdots \int_0^t dt_1dt_2dt_3dt_4 \prod_{\ell=1}^{m} \mathcal{C}_Z\left(\frac{t_{i_\ell} - t_{j_\ell}}{\varepsilon}\right),$$

where the sum is over all indices $i_1, j_1, \dots, i_{2m}, j_{2m}$ such that:

(i) $i_1, j_1, \dots, i_{2m}, j_{2m} \in \{1,2,3,4\}$,

(ii) $i_1 \neq j_1, \dots, i_{2m} \neq j_{2m}$,

(iii) each number 1,2,3,4 appears exactly m times in $(i_1, j_1, \dots, i_{2m}, j_{2m})$.

The number N_{2m} of terms in this sum is therefore smaller than $(4m)!/m!^4$ (it would be exactly this cardinal without the second condition; therefore, it is smaller than this number).

Because $\mathcal{C}_Z(s) \leq 1 \wedge K|s|^{2H-2}$ for some constant K, we have, for any $t \in [0,T]$,

$$\mathbb{E}[(I^{\varepsilon}_{t,m})^4] \leq \frac{1}{2^{2m}(2m)!} \sum \int_0^T \cdots \int_0^T dt_1 dt_2 dt_3 dt_4 \prod_{\ell=1}^{2m} 1 \wedge K \left(\frac{|t_{i_\ell} - t_{j_\ell}|}{\varepsilon} \right)^{2H-2} .$$

For each term of the sum, we apply the change of variables $s_1 = t_{i_1}$, $s_2 = t_{j_1}$, $s_3 = t_{\min(\{1,2,3,4\}\setminus\{i_1,j_1\})}$, and $s_4 = t_{\max(\{1,2,3,4\}\setminus\{i_1,j_1\})}$. In the product, we keep the first term: $K(|s_1 - s_2|/\varepsilon)^{2H-2}$, and the first term that has s_3 in it: $K(|s_3 - s_j|/\varepsilon)^{2H-2}$, so that we can write, for any $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\left[(I^{\varepsilon}_{t,m})^4\right] &\leq \frac{N_{2m}K^2}{2^{2m}(2m)!} \int_0^T \cdots \int_0^T ds_1 ds_2 ds_3 ds_4 \left(\frac{|s_1 - s_2|}{\varepsilon} \right)^{2H-2} \left[\left(\frac{|s_3 - s_1|}{\varepsilon} \right)^{2H-2} \right. \\ &\quad \left. + \left(\frac{|s_3 - s_2|}{\varepsilon} \right)^{2H-2} + \left(\frac{|s_3 - s_4|}{\varepsilon} \right)^{2H-2} \right] \\ &\leq K' \frac{(4m)!}{2^{2m}(2m)!m!^4} \varepsilon^{4-4H},\end{aligned}$$

for some constant K' (which depends on H and T), because s^{2H-2} is integrable over $[0,T]$. By Stirling's formula, we obtain

$$\frac{(4m)!}{2^{2m}(2m)!m!^4} \simeq \frac{2^{2m}}{m!^2} \frac{1}{\sqrt{2\pi m}}.$$

Therefore, by Minkowski's inequality, we have, for any $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\left[\left(I^{\varepsilon}_t\right)^4\right)\Big]^{1/4} &\leq \sum_{m=1}^{\infty} |C_m| \mathbb{E}\left[\left(I^{\varepsilon}_m\right)^4\right)\Big]^{1/4} \leq K''\varepsilon^{1-H} \sum_{m=1}^{\infty} |C_m| \left(\frac{2^m}{m!} \right)^{1/2} \\ &\leq K''\varepsilon^{1-H} \left(\sum_{m=1}^{\infty} \frac{\alpha^m C_m^2}{m!} \right)^{1/2} \left(\sum_{m=1}^{\infty} \frac{2^m}{\alpha^m} \right)^{1/2},\end{aligned}$$

for some constant K'', which gives the desired result. □

The hypothesis (A.5) in Lemma A.1 requires some smoothness for the function $\widetilde{F}$. The following lemma gives a sufficient condition.

Lemma A.2. *If the function $\widetilde{F}$ defined by (A.1) is of the form*

$$\widetilde{F}(x) = \int_{-\infty}^{x} f(y)dy, \tag{A.8}$$

where the Fourier transform of the function f satisfies $|\hat{f}(\nu)| \leq C \exp(-\nu^2)$ for some $C > 0$; then there exists $K > 0$ such that, for any $k \geq 0$,

$$\frac{C_k^2}{k!} \leq K 3^{-k}. \tag{A.9}$$

The inequality (A.9) is sufficient to ensure that the hypothesis (A.5) is fulfilled. We may, for instance, consider

$$\widetilde{F}(x) = \int_{-\infty}^{x} e^{-y^2/4} dy \text{ or } \widetilde{F}(x) = \int_{-\infty}^{x} \operatorname{sinc}^2(y) dy. \tag{A.10}$$

Proof. The function $\widetilde{F}$ is of class C^∞, and we have, for any $k \geq 1$, using integration by parts,

$$C_k = \int_{\mathbb{R}} \widetilde{F}(z) H_k(z) p(z) dz = \int_{\mathbb{R}} \widetilde{F}^{(k)}(z) p(z) dz = \int_{\mathbb{R}} f^{(k-1)}(z) p(z) dz.$$

By Parseval formula, we have

$$C_k = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\nu^2/2} (i\nu)^{k-1} \hat{f}(\nu) d\nu.$$

Because $|\hat{f}(\nu)| \leq C \exp(-\nu^2)$, we obtain

$$|C_k| \leq C \int_{\mathbb{R}} e^{-3\nu^2/2} |\nu|^{k-1} d\nu = C \left(\frac{2}{3}\right)^{\frac{k}{2}} \int_0^\infty e^{-s} s^{\frac{k}{2}-1} ds = C \left(\frac{2}{3}\right)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right),$$

which gives the desired result using Stirling's formula $\Gamma(z) \sim z^{z-1/2} e^{-z} \sqrt{2\pi}$. □

APPENDIX B: TECHNICAL LEMMAS

We denote

$$G(z) = \frac{1}{2} \left(F(z)^2 - \overline{\sigma}^2 \right). \tag{B.1}$$

The martingale ψ_t^ε defined by (41) has the form

$$\psi_t^\varepsilon = \mathbb{E} \left[\int_0^T G(Z_s^\varepsilon) ds \middle| \mathcal{F}_t \right]. \tag{B.2}$$

Lemma B.1. $(\psi_t^\varepsilon)_{t \in [0,T]}$ *is a square-integrable martingale and*

$$d \left\langle \psi^\varepsilon, W \right\rangle_t = \vartheta_t^\varepsilon dt, \qquad \vartheta_t^\varepsilon = \sigma_{\rm ou} \int_t^T \mathbb{E} \left[G'(Z_s^\varepsilon) | \mathcal{F}_t \right] \mathcal{K}^\varepsilon(s-t) ds. \tag{B.3}$$

An alternative expression of the bracket $<\psi^\varepsilon, W>_t$ is given in (B.5) and (B.6).

Proof. For $t \leq s$, the conditional distribution of Z_s^ε given $\mathcal{F}_t$ is Gaussian with mean

$$\mathbb{E} \left[Z_s^\varepsilon | \mathcal{F}_t \right] = \sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u,$$

and deterministic variance given by

$$\text{Var} \left(Z_s^\varepsilon | \mathcal{F}_t \right) = (\sigma_{0,s-t}^\varepsilon)^2,$$

where we have defined, for any $0 \leq t \leq s \leq \infty$,

$$(\sigma_{t,s}^\varepsilon)^2 = \sigma_{\rm ou}^2 \int_t^s \mathcal{K}^\varepsilon(u)^2 du. \tag{B.4}$$

We thus have that the distribution of

$$\frac{1}{\sigma_{0,s-t}^\varepsilon} \left(\left(Z_s^\varepsilon - \sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u \right) \middle| \mathcal{F}_t \right)$$

is standard normal. Therefore, we have

$$\mathbb{E} \left[G(Z_s^\varepsilon) | \mathcal{F}_t \right] = \int_{\mathbb{R}} G \left(\sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u + \sigma_{0,s-t}^\varepsilon z \right) p(z) dz,$$

where $p(z)$ is the pdf of the standard normal distribution. As a random process in t, it is a continuous martingale. By Itô's formula, for any $t \leq s$,

$$
\begin{aligned}
\mathbb{E}\left[G(Z_s^\varepsilon)|\mathcal{F}_t\right] &= \int_{\mathbb{R}} G\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{0}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\right)p(z)dz \\
&+ \int_0^t \int_{\mathbb{R}} G'\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon du \\
&+\sigma_{\mathrm{ou}}\int_0^t \int_{\mathbb{R}} G'\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) p(z)dz\mathcal{K}^\varepsilon(s-u)dW_u \\
&+\frac{\sigma_{\mathrm{ou}}^2}{2}\int_0^t \int_{\mathbb{R}} G''\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) p(z)dz\mathcal{K}^\varepsilon(s-u)^2 du
\end{aligned}
$$

and

$$
\begin{aligned}
G(Z_s^\varepsilon) &= G\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{s}\mathcal{K}^\varepsilon(s-v)dW_v\right) \\
&= \int_{\mathbb{R}} G\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{s}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,0}^\varepsilon z\right)p(z)dz \\
&= \int_{\mathbb{R}} G\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{0}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\right)p(z)dz \\
&+ \int_0^s \int_{\mathbb{R}} G'\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon du \\
&+\sigma_{\mathrm{ou}}\int_0^s \int_{\mathbb{R}} G'\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) p(z)dz\mathcal{K}^\varepsilon(s-u)dW_u \\
&+\frac{\sigma_{\mathrm{ou}}^2}{2}\int_0^s \int_{\mathbb{R}} G''\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) p(z)dz\mathcal{K}^\varepsilon(s-u)^2 du.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\psi_t^\varepsilon &= \int_0^t G(Z_s^\varepsilon)ds + \int_t^T \mathbb{E}\left[G(Z_s^\varepsilon)|\mathcal{F}_t\right] ds \\
&= \left[\int_{\mathbb{R}}\int_0^T G\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{0}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\right)ds p(z)dz\right] \\
&+ \int_0^t \left[\int_u^T \int_{\mathbb{R}} G'\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon ds\right] du \\
&+\sigma_{\mathrm{ou}}\int_0^t \left[\int_u^T \int_{\mathbb{R}} G'\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) p(z)dz\mathcal{K}^\varepsilon(s-u)ds\right] dW_u \\
&+\frac{\sigma_{\mathrm{ou}}^2}{2}\int_0^t \left[\int_u^T \int_{\mathbb{R}} G''\left(\sigma_{\mathrm{ou}}\int_{-\infty}^{u}\mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\right) p(z)dz\mathcal{K}^\varepsilon(s-u)^2 ds\right] du.
\end{aligned}
$$

This gives

$$d\left\langle \psi^{\varepsilon}, W\right\rangle_t = \vartheta_t^{\varepsilon} dt, \tag{B.5}$$

with

$$\vartheta_t^{\varepsilon} = \sigma_{\rm ou} \int_t^T \int_{\mathbb{R}} G'\left(\sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^{\varepsilon}(s-v)dW_v + \sigma_{0,s-t}^{\varepsilon} z\right) p(z)dz\mathcal{K}^{\varepsilon}(s-t)ds, \tag{B.6}$$

which can also be written as stated in the lemma. □

The important properties of the random process $\vartheta_t^{\varepsilon}$ are stated in the following lemma.

Lemma B.2. *For any $t \in [0, T]$, we have*

$$\vartheta_t^{\varepsilon} = \varepsilon^{1-H}\theta_t + \widetilde{\theta}_t^{\varepsilon}, \tag{B.7}$$

where θ_t is deterministic and defined by

$$\theta_t = \overline{\theta}(T-t)^{H-\frac{1}{2}}, \qquad \overline{\theta} = \frac{\langle G' \rangle}{\Gamma(H+\frac{1}{2})}, \tag{B.8}$$

and $\widetilde{\theta}_t^{\varepsilon}$ is random, but smaller than ε^{1-H},

$$\limsup_{\varepsilon \to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(\widetilde{\theta}_t^{\varepsilon})^2\right]^{1/2} = 0. \tag{B.9}$$

Proof. Recall first from equation (19) that

$$\mathcal{K}^{\varepsilon}(t) = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}\left(\frac{t}{\varepsilon}\right), \qquad \mathcal{K}(t) = \frac{1}{\sigma_{\rm ou}\Gamma(H+\frac{1}{2})}\left[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}}e^{-s}ds\right].$$

The expectation of $\vartheta_t^{\varepsilon}$ is then equal to

$$\mathbb{E}\left[\vartheta_t^{\varepsilon}\right] = \sigma_{\rm ou}\left\langle G'\right\rangle \int_0^{T-t} \mathcal{K}^{\varepsilon}(s)ds = \sigma_{\rm ou}\left\langle G'\right\rangle \sqrt{\varepsilon}\int_0^{(T-t)/\varepsilon} \mathcal{K}(s)ds.$$

Therefore, the difference

$$\mathbb{E}\left[\vartheta_t^{\varepsilon}\right] - \varepsilon^{1-H}\theta_t = \sigma_{\rm ou}\left\langle G'\right\rangle \varepsilon^{1/2}\int_0^{(T-t)/\varepsilon} \mathcal{K}(s) - \frac{s^{H-\frac{3}{2}}}{\sigma_{\rm ou}\Gamma(H-\frac{1}{2})}ds$$

can be bounded by

$$\left|\mathbb{E}\left[\vartheta_t^{\varepsilon}\right] - \varepsilon^{1-H}\theta_t\right| \le C\varepsilon^{1/2}, \tag{B.10}$$

uniformly in $t \in [0, T]$, for some constant C, because $\mathcal{K}(s) - \frac{s^{H-\frac{3}{2}}}{\sigma_{\rm ou}\Gamma(H-\frac{1}{2})}$ is in L^1.

We have

$$\begin{aligned}
\mathrm{Var}(\vartheta_t^{\varepsilon}) &= \sigma_{\rm ou}^2 \int_t^T ds \int_t^T ds' \mathcal{K}^{\varepsilon}(s-t)\mathcal{K}^{\varepsilon}(s'-t)\mathrm{Cov}\left(\mathbb{E}\left[G'(Z_s^{\varepsilon})|\mathcal{F}_t\right], \mathbb{E}\left[G'(Z_{s'}^{\varepsilon})|\mathcal{F}_t\right]\right) \\
&\le \sigma_{\rm ou}^2 \left(\int_t^T ds\mathcal{K}^{\varepsilon}(s-t)\mathrm{Var}\left(\mathbb{E}\left[G'(Z_s^{\varepsilon})|\mathcal{F}_t\right]\right)^{1/2}\right)^2 \\
&= \sigma_{\rm ou}^2 \left(\int_0^{T-t} ds\mathcal{K}^{\varepsilon}(s)\mathrm{Var}\left(\mathbb{E}\left[G'(Z_s^{\varepsilon})|\mathcal{F}_0\right]\right)^{1/2}\right)^2.
\end{aligned}$$

The conditional distribution of Z_t^ε given $\mathcal{F}_0$ is Gaussian with mean

$$\mathbb{E}\left[Z_t^\varepsilon|\mathcal{F}_0\right] = \sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(t-u)dW_u$$

and variance

$$\mathrm{Var}\left(Z_t^\varepsilon|\mathcal{F}_0\right) = (\sigma_{0,t}^\varepsilon)^2 = \sigma_{\rm ou}^2\int_0^t \mathcal{K}^\varepsilon(u)^2du.$$

Therefore,

$$\mathrm{Var}\left(\mathbb{E}\left[G'(Z_t^\varepsilon)|\mathcal{F}_0\right]\right) = \mathrm{Var}\left(\int_{\mathbb{R}} G'\left(\mathbb{E}\left[Z_t^\varepsilon|\mathcal{F}_0\right] + \sigma_{0,t}^\varepsilon z\right)p(z)dz\right).$$

The random variable $\mathbb{E}[Z_t^\varepsilon|\mathcal{F}_0]$ is Gaussian with mean zero and variance

$$(\sigma_{t,\infty}^\varepsilon)^2 = \sigma_{\rm ou}^2\int_t^\infty \mathcal{K}^\varepsilon(u)^2du,$$

so that

$$\begin{aligned}
\mathrm{Var}\left(\mathbb{E}\left[G'(Z_t^\varepsilon)|\mathcal{F}_0\right]\right) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dzdz'p(z)p(z')\int_{\mathbb{R}}\int_{\mathbb{R}} dudu'p(u)p(u') \\
&\quad\times\left[G'\left(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z\right) - G'\left(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z\right)\right] \\
&\quad\times\left[G'\left(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z'\right) - G'\left(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z'\right)\right] \\
&\leq \|G''\|_\infty^2(\sigma_{t,\infty}^\varepsilon)^2\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dudu'p(u)p(u')(u-u')^2 \\
&= \|G''\|_\infty^2(\sigma_{t,\infty}^\varepsilon)^2.
\end{aligned} \tag{B.11}$$

Therefore,

$$\begin{aligned}
\mathrm{Var}(\vartheta_t^\varepsilon)^{1/2} &\leq \|G''\|_\infty\sigma_{\rm ou}^2\int_0^{T-t} ds\mathcal{K}^\varepsilon(s)\left(\int_s^\infty du\mathcal{K}^\varepsilon(u)^2\right)^{1/2} \\
&\leq \|G''\|_\infty\sigma_{\rm ou}^2\varepsilon^{1/2}\int_0^{(T-t)/\varepsilon} ds\mathcal{K}(s)\left(\int_s^\infty du\mathcal{K}(u)^2\right)^{1/2}.
\end{aligned}$$

Because $\mathcal{K}(s) \leq 1 \wedge Ks^{H-\frac{3}{2}}$, this gives

$$\mathrm{Var}(\vartheta_t^\varepsilon)^{1/2} \leq C\begin{cases}\varepsilon^{1/2} & \text{if } H < 3/4, \\ \varepsilon^{1/2}\ln(\varepsilon) & \text{if } H = 3/4, \\ \varepsilon^{2-2H} & \text{if } H > 3/4,\end{cases} \tag{B.12}$$

uniformly in $t \in [0, T]$, for some constant C. This completes the proof of the lemma. □

The random term ϕ_t^ε defined by (36) has the form

$$\phi_{t,T}^\varepsilon = \mathbb{E}\left[\int_t^T G(Z_s^\varepsilon)ds\,\middle|\,\mathcal{F}_t\right]. \tag{B.13}$$

Here, we write explicitly the argument T (maturity) as we compute the correlations of these random terms for different maturities.

Lemma B.3.

1. *For any $t \le T$, $\phi^\varepsilon_{t,T}$ is a zero-mean random variable with standard deviation of order ε^{1-H},*

$$\varepsilon^{2H-2}\mathbb{E}[(\phi^\varepsilon_{t,T})^2] \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_\phi^2 (T-t)^{2H}, \tag{B.14}$$

where σ_ϕ is defined by (39).

2. *The covariance function of $\phi^\varepsilon_{t,T}$ has the following limit for any $t \le T$, $t' \le T'$, with $t \le t'$:*

$$\varepsilon^{2H-2}\mathbb{E}[\phi^\varepsilon_{t,T}\phi^\varepsilon_{t',T'}] \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_\phi^2 (T-t)^{H}(T'-t')^H \mathcal{C}_\phi(t,t';T,T'), \tag{B.15}$$

where the limit correlation is

$$\mathcal{C}_\phi(t,t';T,T') = \frac{\int_0^\infty du \left[(u+r)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]\left[(u+s)^{H-\frac{1}{2}} - (u+q)^{H-\frac{1}{2}}\right]}{\int_0^\infty du \left[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]^2}, \tag{B.16}$$

with

$$q = \frac{t'-t}{\sqrt{(T-t)(T'-t')}}, \qquad r = \frac{\sqrt{T-t}}{\sqrt{T'-t'}}, \qquad s = \frac{T'-t}{\sqrt{(T-t)(T'-t')}}.$$

3. *As $\varepsilon \to 0$, the random process $\varepsilon^{H-1}\phi^\varepsilon_{t,T}$, $t \le T$, converges in distribution (in the sense of finite-dimensional distributions) to a Gaussian random process $\phi_{t,T}$, $t \le T$, with mean zero and covariance $\varepsilon^{2(H-1)}\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2 (T-t)^H (T'-t')^H \mathcal{C}_\phi(t,t';T,T')$ for any $t \in [0,T]$, $t' \in [0,T']$, with $t \le t'$.*

4. *The fourth-order moments of $\varepsilon^{H-1}\phi^\varepsilon_{t,T}$ are uniformly bounded: There exists a constant K_T independent of ε such that*

$$\sup_{t\in[0,T]} \mathbb{E}[(\phi^\varepsilon_{t,T})^4]^{1/4} \le K_T \varepsilon^{1-H}. \tag{B.17}$$

Note that the mean square increment of the limit process $\phi_{t,T}$ satisfies, for $t, t+h \in [0,T]$,

$$\begin{aligned}\mathbb{E}\left[(\phi_{t,T} - \phi_{t+h,T})^2\right] &= \frac{1}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty du \left[(T-t-h+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]^2 \\ &\quad - \left[(T-t+u)^{H-\frac{1}{2}} - (u+h)^{H-\frac{1}{2}}\right]^2 + \left[(u+h)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right]^2 \\ &= \frac{(T-t)^{2H-1}}{\Gamma(H+\frac{1}{2})^2} h + o(h), \quad h \to 0.\end{aligned} \tag{B.18}$$

This shows that the limit Gaussian process $\phi_{t,T}$ has the same local regularity (as a function of t) as a standard Brownian motion. We also have, for any $t < T \le T+h$,

$$\mathbb{E}\left[(\phi_{t,T+h} - \phi_{t,T})^2\right] = \frac{(T-t)^{2H-2}}{(2-2H)\Gamma(H-\frac{1}{2})^2} h^2 + o(h^2), \quad h \to 0. \tag{B.19}$$

This shows that the limit Gaussian process $\phi_{t,T}$ is smooth (mean square differentiable) as a function of the maturity T.

Proof. Let us fix $T_0 > 0$. For $t \in [0,T]$, $t' \in [0,T']$, with $T, T' \le T_0$, and $t \le t'$, the covariance of $\phi^\varepsilon_{t,T}$ is

$$\begin{aligned}\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T G(Z^\varepsilon_s)ds \middle| \mathcal{F}_t\right] \mathbb{E}\left[\int_{t'}^{T'} G(Z^\varepsilon_s)ds \middle| \mathcal{F}_{t'}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T G(Z^\varepsilon_s)ds \middle| \mathcal{F}_t\right] \mathbb{E}\left[\int_{t'}^{T'} G(Z^\varepsilon_s)ds \middle| \mathcal{F}_t\right]\right] \\ &= \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \mathrm{Cov}\left(\mathbb{E}\left[G\left(Z^\varepsilon_s\right)|\mathcal{F}_0\right], \mathbb{E}\left[G\left(Z^\varepsilon_{s'}\right)|\mathcal{F}_0\right]\right).\end{aligned}$$

Then, proceeding as in the proof of the previous lemma, we obtain

$$\mathrm{Var}(\phi^\varepsilon_{t,T}) \le \left(\int_0^{T-t} ds \mathrm{Var}\left(\mathbb{E}\left[G(Z^\varepsilon_s)|\mathcal{F}_0\right]\right)^{1/2}\right)^2 \le \|G'\|^2_\infty \left(\int_0^{T-t} ds \sigma^\varepsilon_{s,\infty}\right)^2.$$

Because $\mathcal{K}(s) \le 1 \wedge K s^{H-\frac{3}{2}}$, this gives

$$\mathrm{Var}(\phi^\varepsilon_{t,T}) \le C_{T_0} \varepsilon^{2-2H},$$

uniformly in $t \le T \le T_0$, for some constant C_{T_0}. More precisely, for $t \in [0,T]$, $t' \in [0,T']$, with $T, T' \le T_0$, and $t \le t'$, we have

$$\begin{aligned}\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) = \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z)p(z') \\ \times\mathbb{E}\left[G\left(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)dW_u + \sigma^\varepsilon_{0,s} z\right) G\left(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')dW_{u'} + \sigma^\varepsilon_{0,s'} z'\right)\right].\end{aligned}$$

Using the fact that $< G >= 0$, we can write

$$\begin{aligned}\mathrm{Cov}\left(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}\right) = \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z)p(z') \\ \times\mathbb{E}\left[\left(G\left(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)dW_u + \sigma^\varepsilon_{0,s} z\right) - G(\sigma_{\mathrm{ou}} z)\right)\right. \\ \left.\times\left(G\left(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')dW_{u'} + \sigma^\varepsilon_{0,s'} z'\right) - G(\sigma_{\mathrm{ou}} z')\right)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Cov}\left(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}\right) = \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z)p(z') G'(\sigma_{\mathrm{ou}} z) G'(\sigma_{\mathrm{ou}} z') \\ \times\mathbb{E}\left[\left(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)dW_u + \left(\sigma^\varepsilon_{0,s} - \sigma_{\mathrm{ou}}\right) z\right)\right. \\ \left.\times\left(\sigma_{\mathrm{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')dW_{u'} + \left(\sigma^\varepsilon_{0,s'} - \sigma_{\mathrm{ou}}\right) z'\right)\right] + V^\varepsilon_3,\end{aligned}$$

up to a term V_3^ε, which is of order ε^{3-3H},

$$
\begin{aligned}
V_3^\varepsilon \leq\; & 2\|G'\|_\infty \|G''\|_\infty \int_0^{T-t} ds \int_0^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z)p(z') \\
& \times \mathbb{E}\left[\left(\sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + \left(\sigma^\varepsilon_{0,s} - \sigma_{\rm ou}\right) z \right)^2 \right. \\
& \left. \times \left| \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + (\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}) z' \right| \right] \\
& \leq C\|G'\|_\infty \|G''\|_\infty \int_0^{T_0-t} ds \int_0^{T_0-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z)p(z') \\
& \times \left(\sigma_{\rm ou}^2 \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)^2 du + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})^2 z^2 \right) \\
& \times \left(\sigma_{\rm ou}^2 \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')^2 du' + \left(\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}\right)^2 z'^2 \right)^{1/2} \\
& \leq C'\|G'\|_\infty \|G''\|_\infty \left[\int_0^{T_0-t} ds \int_{\mathbb{R}} dz p(z) \left(\left(\sigma^\varepsilon_{s,\infty}\right)^2 + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})^2 z^2 \right) \right]^{3/2} \\
& \leq C'\|G'\|_\infty \|G''\|_\infty \left[\int_0^{T_0-t} ds \left(\sigma^\varepsilon_{s,\infty}\right)^2 + \left(\sigma^\varepsilon_{0,s} - \sigma_{\rm ou}\right)^2 \right]^{3/2}.
\end{aligned}
$$

Using $(\sigma^\varepsilon_{s,\infty})^2 + (\sigma^\varepsilon_{0,s})^2 = \sigma_{\rm ou}^2$ and

$$
\begin{aligned}
|\sigma_{\rm ou} - \sigma^\varepsilon_{0,s}| &= \sigma_{\rm ou} \left(1 - \left(\int_0^{s/\varepsilon} \mathcal{K}(u)^2 du \right)^{1/2} \right) = \sigma_{\rm ou} \left(1 - \left(1 - \int_{s/\varepsilon}^\infty \mathcal{K}(u)^2 du \right)^{1/2} \right) \\
&\leq \sigma_{\rm ou} \int_{s/\varepsilon}^\infty \mathcal{K}(u)^2 du \leq \sigma_{\rm ou} \left(1 \wedge K \left(\frac{s}{\varepsilon} \right)^{2H-2} \right),
\end{aligned}
\tag{B.20}
$$

where the fist inequality follows from $\sqrt{1-x} > 1-x$ for $0 \leq x \leq 1$, we get

$$
\begin{aligned}
V_3^\varepsilon &\leq C'\|G'\|_\infty \|G''\|_\infty \left[\int_0^{T_0-t} ds 2\sigma_{\rm ou}(\sigma_{\rm ou} - \sigma^\varepsilon_{0,s}) \right]^{3/2} \\
&\leq C''\|G'\|_\infty \|G''\|_\infty \varepsilon^{3-3H}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
{\rm Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) &= \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z)p(z') G'(\sigma_{\rm ou} z) G'(\sigma_{\rm ou} z') \\
&\quad \times \left(\sigma_{\rm ou}^2 \int_0^\infty \mathcal{K}^\varepsilon(s+u)\mathcal{K}^\varepsilon(s'+u) du + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})(\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}) z z' \right) + V_3^\varepsilon \\
&= V_1^\varepsilon \left\langle G' \right\rangle^2 + V_2^\varepsilon \sigma_{\rm ou}^2 \left\langle G'' \right\rangle^2 + V_3^\varepsilon,
\end{aligned}
$$

with

$$V_1^\varepsilon = \sigma_{\rm ou}^2 \int_0^\infty du \left(\int_0^{T-t} ds \mathcal{K}^\varepsilon(s+u) \right) \left(\int_{t'-t}^{T'-t} ds' \mathcal{K}^\varepsilon(s'+u) \right),$$

$$V_2^\varepsilon = \left(\int_0^{T-t} ds (\sigma_{0,s}^\varepsilon - \sigma_{\rm ou}) \right) \left(\int_{t'-t}^{T'-t} ds' (\sigma_{0,s'}^\varepsilon - \sigma_{\rm ou}) \right).$$

Using again (B.20), we find that

$$V_2^\varepsilon \leq C\varepsilon^{4-4H},$$

while

$$V_1^\varepsilon = \frac{1}{\Gamma(H+\frac{1}{2})^2} \int_0^\infty \left((T-t+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right) \times \left((T'-t+u)^{H-\frac{1}{2}} - (u+t'-t)^{H-\frac{1}{2}} \right) du \, \varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

Applying the change of variable

$$u \to (T-t)^{\frac{1}{2}} (T'-t')^{\frac{1}{2}} u$$

gives the first and second items of the lemma with

$$\sigma_\phi^2 = \frac{\langle G' \rangle^2}{\Gamma(H+\frac{1}{2})^2} \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right)^2 du,$$

which is equivalent to (39).

In order to prove the third item, we introduce

$$\check{\phi}_{t,T}^\varepsilon = \mathbb{E}\left[\int_t^T Z_s^\varepsilon ds \middle| \mathcal{F}_t \right], \tag{B.21}$$

which is a Gaussian random process with mean zero and covariance, for $t \in [0,T]$, $t' \in [0,T']$, with $t \leq t'$,

$$\begin{aligned}
{\rm Cov}\left(\check{\phi}_{t,T}^\varepsilon, \check{\phi}_{t',T'}^\varepsilon \right) &= \int_t^T ds \int_{t'}^{T'} ds' \mathbb{E}\left[\mathbb{E}[Z_s^\varepsilon | \mathcal{F}_t] \mathbb{E}[Z_s^\varepsilon | \mathcal{F}_{t'}] \right] \\
&= \int_t^T ds \int_{t'}^{T'} ds' \mathbb{E}\left[\mathbb{E}[Z_s^\varepsilon | \mathcal{F}_t] \mathbb{E}[Z_s^\varepsilon | \mathcal{F}_t] \right] \\
&= \sigma_{\rm ou}^2 \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \mathbb{E}\left[\left(\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u \right) \left(\int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u) dW_u \right) \right] \\
&= \sigma_{\rm ou}^2 \int_0^\infty du \left(\int_0^{T-t} ds \mathcal{K}^\varepsilon(s+u) \right) \left(\int_{t'-t}^{T'-t} ds' \mathcal{K}^\varepsilon(s'+u) \right).
\end{aligned}$$

Therefore, for $t_j \in [0,T_j]$, with $t_1 \leq \cdots \leq t_n$, the random vector $(\varepsilon^{H-1} < G' > \check{\phi}_{t_1,T_1}^\varepsilon, \dots, \varepsilon^{H-1} < G' > \check{\phi}_{t_n,T_n}^\varepsilon)$ converges to a Gaussian random vector with mean 0 and covariance matrix $(\sigma_\phi^2 (T_j - t_j)^H (T_l -$

$t_l)^H \mathcal{C}_\phi(t_j, t_l; T_j, T_l))_{j,l=1}^n$. In other words, the random process $(\varepsilon^{H-1} < G' > \check{\phi}^\varepsilon_{t,T})_{0 \le t \le T < \infty}$ converges in the sense of finite-dimensional distributions to a Gaussian process $(\phi_{t,T})_{0 \le t \le T < \infty}$ with mean 0 and covariance function $\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2 (T-t)^H (T'-t')^H \mathcal{C}_\phi(t, t'; T, T')$, for $t \in [0, T]$, $t' \in [0, T']$, with $t \le t'$.

Moreover, we have

$$\mathrm{Var}\left(\check{\phi}^\varepsilon_{t,T}\right) = \frac{1}{\Gamma(H+\frac{1}{2})^2} \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du \, (T-t)^{2H} \varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

Similarly,

$$\mathbb{E}\left[\check{\phi}^\varepsilon_{t,T} \phi^\varepsilon_{t,T}\right] = \frac{\langle G' \rangle}{\Gamma(H+\frac{1}{2})^2} \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du \, (T-t)^{2H} \varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

As a result,

$$\varepsilon^{2H-2} \mathbb{E}\left[(\phi^\varepsilon_{t,T} - \left\langle G' \right\rangle \check{\phi}^\varepsilon_{t,T})^2\right] \overset{\varepsilon \to 0}{\longrightarrow} 0,$$

and the random process $(\varepsilon^{H-1} < G' > \check{\phi}^\varepsilon_{t,T})_{0 \le t \le T < \infty}$ converges in the sense of finite-dimensional distributions to a Gaussian process $(\phi_{t,T})_{0 \le t \le T < \infty}$ with mean 0 and covariance function $\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2 (T-t)^H (T'-t')^H \mathcal{C}_\phi(t, t'; T, T')$ for $t \in [0, T]$, $t' \in [0, T']$, with $t \le t'$. This gives the third item of the lemma.

To prove the fourth item of the lemma, we note that

$$\phi^\varepsilon_{t,T} = \frac{1}{2} \mathbb{E}\left[I^\varepsilon_T | \mathcal{F}_t\right] - \frac{1}{2} I^\varepsilon_t,$$

where I^ε_t is defined by (A.6). Therefore,

$$\sup_{t \in [0,T]} \mathbb{E}\left[(\phi^\varepsilon_{t,T})^4\right] \le \sup_{t \in [0,T]} \mathbb{E}\left[(I^\varepsilon_t)^4\right],$$

and the result follows from Lemma A.1, equation (A.7). □

Lemma B.4. *Let us define, for any $t \in [0, T]$,*

$$\gamma^\varepsilon_t = \frac{1}{2} \int_0^t \left((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\right) \phi^\varepsilon_s ds, \tag{B.22}$$

as in (49). We have

$$\limsup_{\varepsilon \to 0} \varepsilon^{H-1} \sup_{t \in [0,T]} \mathbb{E}\left[(\gamma^\varepsilon_t)^2\right]^{1/2} = 0. \tag{B.23}$$

Proof. Let us define, for any $t \in [0, T]$,

$$\Gamma^\varepsilon_t = \int_t^T \left((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\right) \phi^\varepsilon_s ds. \tag{B.24}$$

By the definition (41) of ϕ^ε_s, we have

$$\Gamma^\varepsilon_t = 2 \int_t^T ds \int_s^T du \mathbb{E}\left[G(Z^\varepsilon_s) G(Z^\varepsilon_u) | \mathcal{F}_s\right].$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\left(\Gamma_t^\varepsilon\right)^2\right] &= 2\int_t^T ds \int_s^T du \int_s^T ds' \int_{s'}^T du' \mathbb{E}\left[\mathbb{E}\left[G(Z_s^\varepsilon)G(Z_u^\varepsilon)|\mathcal{F}_s\right]\mathbb{E}\left[G(Z_{s'}^\varepsilon)G(Z_{u'}^\varepsilon)|\mathcal{F}_{s'}\right]\right] \\
&= 2\int_t^T ds \int_s^T du \int_s^T ds' \int_{s'}^T du' \mathbb{E}\left[G(Z_s^\varepsilon)G(Z_u^\varepsilon)\mathbb{E}\left[G(Z_{s'}^\varepsilon)G(Z_{u'}^\varepsilon)\middle| \mathcal{F}_s\right]\right] \\
&= \int_t^T ds \int_s^T du \mathbb{E}\left[G(Z_s^\varepsilon)G(Z_u^\varepsilon)\mathbb{E}\left[\left(\int_s^T G(Z_{s'}^\varepsilon)ds'\right)^2\middle| \mathcal{F}_s\right]\right] \\
&= \int_t^T ds \mathbb{E}\left[G(Z_s^\varepsilon)\mathbb{E}\left[\int_s^T G(Z_u^\varepsilon)du\,|\mathcal{F}_s\right]\mathbb{E}\left[\left(\int_s^T G(Z_{s'}^\varepsilon)ds'\right)^2\middle| \mathcal{F}_s\right]\right] \\
&\leq \|G\|_\infty \int_t^T ds \mathbb{E}\left[\left|\mathbb{E}\left[\left(\int_s^T G(Z_{s'}^\varepsilon)ds'\right)^2\middle| \mathcal{F}_s\right]\right|^{3/2}\right] \\
&\leq \|G\|_\infty \int_t^T ds \mathbb{E}\left[\left|\int_s^T G(Z_{s'}^\varepsilon)ds'\right|^3\right] \\
&\leq \|G\|_\infty \int_t^T ds \mathbb{E}\left[\left(\int_s^T G(Z_{s'}^\varepsilon)ds'\right)^4\right]^{3/4},
\end{aligned}
$$

where in the first inequality, we use that

$$
\left|\mathbb{E}\left[\int_s^T G\left(Z_u^\varepsilon\right)du\middle| \mathcal{F}_s\right]\right| \leq \left|\mathbb{E}\left[\left(\int_s^T G\left(Z_u^\varepsilon\right)du\right)^2\middle| \mathcal{F}_s\right]\right|^{1/2},
$$

which follows from the conditional version of Jensen's inequality. It follows by Lemma A.1 that $\mathbb{E}[(\Gamma_t^\varepsilon)^2]$ is smaller than $K'\varepsilon^{3-3H}$ for some constant K'. This proves

$$
\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(\Gamma_t^\varepsilon)^2\right]^{1/2} = 0. \tag{B.25}
$$

Note that γ_t^ε defined by (49) is related to Γ_t^ε through

$$
\gamma_t^\varepsilon = 2\left(\Gamma_0^\varepsilon - \Gamma_t^\varepsilon\right).
$$

Therefore, equation (B.25) also implies

$$
\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(\gamma_t^\varepsilon)^2\right]^{1/2} = 0,
$$

which is the desired result. □

Lemma B.5. *Let us define, for any $t \in [0,T]$,*

$$
\eta_t^\varepsilon = \varepsilon^{1-H}\int_0^t \left(\sigma_s^\varepsilon - \widetilde{\sigma}\right)ds, \tag{B.26}
$$

as in (51). We have

$$
\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(\eta_t^\varepsilon)^2\right]^{1/2} = 0. \tag{B.27}
$$

Proof. By Lemma 3.1, we obtain

$$
\begin{aligned}
\mathbb{E}\left[(\eta_t^\varepsilon)^2\right] &= \varepsilon^{2-2H}\mathbb{E}\left[\left(\int_0^t \left(\sigma_s^\varepsilon - \widetilde{\sigma}\right) ds\right)^2\right] \\
&= \varepsilon^{2-2H}\int_0^t \int_0^t \mathrm{Cov}\left(F(Z_s^\varepsilon), F(Z_{s'}^\varepsilon)\right) ds ds' \\
&= \varepsilon^{2-2H}\left(\left\langle F^2\right\rangle - \langle F\rangle^2\right)\int_0^t \int_0^t \mathcal{C}_\sigma\left(\frac{s-s'}{\varepsilon}\right) ds ds' \\
&\leq K\varepsilon^{2-2H}\int_0^T \int_0^T \left(\frac{|s-s'|}{\varepsilon}\right)^{2H-2} ds ds' \\
&\leq K'\varepsilon^{4-4H},
\end{aligned}
$$

for some constants K, K', because s^{2H-2} is integrable over $(0, T)$, which gives the desired result. □

Lemma B.6. *Let us define, for any $t \in [0, T]$,*

$$
\kappa_t^\varepsilon = \frac{\varepsilon^{1-H}}{2}\int_0^t \left((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\right) ds, \tag{B.28}
$$

as in (50). We have

$$
\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\left[(\kappa_t^\varepsilon)^2\right]^{1/2} = 0. \tag{B.29}
$$

Proof. The proof is similar to the one in Lemma B.5. □