

Option Pricing under Fast-varying and Rough Stochastic Volatility

Josselin Garnier · Knut Sølna

Received: date / Accepted: date

Abstract Recent empirical studies suggest that the volatilities associated with financial time series exhibit short-range correlations. This entails that the volatility process is very rough and its autocorrelation exhibits sharp decay at the origin. Another classic stylistic feature often assumed for the volatility is that it is mean reverting. In this paper it is shown that the price impact of a rapidly mean reverting rough volatility model coincides with that associated with fast mean reverting Markov stochastic volatility models. This reconciles the empirical observation of rough volatility paths with the good fit of the implied volatility surface to models of fast mean reverting Markov volatilities. Moreover, the result conforms with recent numerical results regarding rough stochastic volatility models. It extends the scope of models for which the asymptotic results of fast mean reverting Markov volatilities are valid. The paper concludes with a general discussion of fractional volatility asymptotics and their interrelation. The regimes discussed there include fast and slow volatility factors with strong or small volatility fluctuations and with the limits not commuting in general. The notion of a characteristic term structure exponent is introduced, this exponent governs the implied volatility term structure in the various asymptotic regimes.

Keywords Stochastic volatility · Short-range correlation · Fractional Ornstein–Uhlenbeck process · Hurst exponent · Mean reversion

Mathematics Subject Classification (2000) 91G80 · 60H10 · 60G22 · 60K37

J. Garnier
Centre de Mathématiques Appliquées,
Ecole Polytechnique, 91128 Palaiseau Cedex, France
E-mail: josselin.garnier@polytechnique.edu

K. Solna
Department of Mathematics,
University of California, Irvine CA 92697
E-mail: ksolna@math.uci.edu

1 Introduction

The assumption that the volatility is constant, as in the standard Black–Scholes model, is not realistic. Indeed, practically, in order to match observed prices, one needs to use an implied volatility that depends on the pricing parameters. Therefore, a consistent parameterization of the implied volatility is needed so that, after calibration of the implied volatility model to liquid contracts, it can be used for pricing of less liquid contracts written on the same underlying. Stochastic volatility models have been introduced because they give such consistent parameterizations of the implied volatility. Here we will consider a specific class of stochastic volatility models and identify the associated parameterization of the price correction and associated implied volatility correction that follow from the volatility fluctuations.

Empirical studies suggest that the volatility may exhibit a “multiscale” character as in [Bollerslev et al. \(2013\)](#); [Breidt et al. \(1998\)](#); [Chronopoulou and Viens \(2012\)](#); [Cont \(2001, 2005\)](#); [Engle and Patton \(2001\)](#); [Oh et al. \(2008\)](#). That is, correlations that decay as a power law in offset rather than as an exponential function as in a Markov process. Recent empirical evidence in particular shows that stochastic volatility is often rough with rapidly decaying correlations at the origin (see [Gatheral et al. \(2016\)](#)). In [Funahashi and Kijima \(2017\)](#) it was shown numerically that the implied volatility correction for a rough fractional stochastic volatility model tends to the correction associated with the Markov case in the regime of long time to maturity. This is consistent with the analytic result derived in this paper where we consider a fast mean reverting rough volatility. In this paper, using the martingale method, we get an analytical expression for the option price and the corresponding implied volatility in the regime when the volatility process is fast mean reverting and rough. The main conclusion is that the corrections to the option prices and the corresponding implied volatilities have exactly the same forms as in the mixing case (when the stochastic volatility is Markov).

A main technical aspect of our derivation is a careful analysis of the form and properties of the covariation between the Brownian motion and the martingale process being the conditional square volatility shift over the time epoch of interest, see Eq. (33) below. It is important in this context that we incorporate leverage in our model so that these processes are correlated leading to a non-trivial covariation. Another main aspect of our modeling is that we model the stochastic volatility fluctuations as being stationary. In a number of recent works a model for the volatility has been introduced where the initial time plays a special role leading to a non-stationary process which is artificial from the modeling viewpoint. However, we show that in fact in the regime of fast mean reversion the asymptotic results of the non-stationary case coincide with those of the stationary case considered here since the volatility process then forgets its initial state.

A central aspect of our analysis is the notion of time scales and time scale separation. It is then important to identify a reference time scale. Here we will use the characteristic diffusion time $\bar{\tau} = 2/\bar{\sigma}^2$, as the reference time, where

$\bar{\sigma}$ is the effective volatility, see Eqs. (14) and (25) below. Then we consider a regime where the time to maturity and the characteristic diffusion time are of the same order of magnitude, while the mean reversion time of the volatility fluctuations is small relative to the characteristic diffusion time.

Note that various asymptotic frameworks can be considered in the context of option pricing and we discuss some of them in this paragraph. The choice of the appropriate asymptotic framework depends on the particular market and contract under consideration. One may consider an asymptotic framework where the volatility fluctuations have small amplitude, and the time to maturity, the characteristic diffusion time, and the mean reversion time for the volatility are of the same order of magnitude. This approach is sensitive to the time dynamic aspects of the volatility. Such an asymptotic situation was considered for instance in [Fouque et al. \(2011\)](#) in the mixing case and recently in [Garnier and Sølna \(2015\)](#) in the context of rough volatility factors. However, this asymptotic situation does not capture situations with large volatility fluctuations over the time scale of the time to maturity. In the asymptotic framework considered in this paper the volatility mean reversion time is small relative to the time to maturity. In such a framework with long time horizon for the contract we can also incorporate order one or strong volatility fluctuations over the time scale of the time to maturity. Such an asymptotic framework is also considered in [Fouque et al. \(2003, 2011\)](#) in the context of mixing processes. Note that as an option contract approaches maturity the sensitivity to the payoff function is enhanced leading to important and interesting issues. Indeed a number of recent works consider implied volatility asymptotics in a regime of short time to maturity, see for instance [Gulisashvili \(2015\)](#) and references therein. The case with contracts that are moreover close to the money are discussed in [Bayer et al \(2017\)](#) and [Fukasawa \(2017\)](#) for instance. Asymptotics in the context of large strikes is discussed in [Benaim et al \(2010\)](#) for instance. We finally remark that for some special models, like the Heston model (see [Heston \(1993\)](#)) it is possible to get explicit or semi-explicit option price formulas in a context where volatility fluctuations have large amplitude and the mean reversion time is of the same order as the time to maturity.

In this paragraph we discuss some special aspects of long- and short-range correlation properties in various asymptotic regimes. In [Garnier and Sølna \(2015\)](#) we consider the situation where the multiscale stochastic volatility has small amplitude, and we treat also the case where it has slow variations, which is similar from the analytic viewpoint, with the latter case corresponding to a maturity that is small relatively to the mean reversion time for the volatility fluctuations. The corrections to the option prices and the corresponding implied volatilities are identified there and the situation is then qualitatively different from the one considered here in that the correction to the price has a fractional behavior in time to maturity. The characteristic term structure exponent then reflects the roughness of the underlying volatility path, see Section 6 below. In [Garnier and Sølna \(2016\)](#) we consider the case with stochastic volatility fluctuations that are fast mean reverting and that have a standard

deviation of the same order as the mean, as we do here. However, in Garnier and Sølna (2016) the stochastic volatility fluctuations have long-range correlation properties so that the paths are smoother than those associated with the Markov case. Then the “persistence” of the volatility fluctuations leads to a fractional term structure and it also leads to a random component of the price correction that is adapted to the filtration generated by the price process and whose covariance structure can be identified in detail. As we explain below in our modeling the Hurst exponent H parameterizes the smoothness of the paths with $H < 1/2$ corresponding to short-range or rough paths considered here and $H > 1/2$ producing the long-range case. Indeed both regimes have been identified from the empirical perspective. We refer to for instance Gatheral et al. (2016) for observations of rough volatility, while in Chronopoulou and Viens (2012) cases of volatility with long-range correlation properties are reported. Long-range volatility situations have been reported for currencies in Walther et al. (2017), for commodities in Charfeddine (2014) and for equity index in Chia et al. (2015), while analysis of electricity markets data typically gives $H < 1/2$ as in Simonsen (2002); Rypdal and Lovsletten (2013); Bennedsen (2015). We believe that both the rough and the long-range cases are important and can be observed depending on the specific market and regime.

Taken together with the present paper the papers Garnier and Sølna (2015, 2016) give a generalization of the two-factor model of Fouque et al. (2003, 2011) to the case of processes with multiscale fluctuations, and for general smoothness of the volatility factor, that has either short-range or long-range correlation properties. Here we consider fractional volatility in the context of option pricing, see Fouque and Hu (2017) and Fouque and Hu (2017b) for applications to portfolio optimization.

We also remark that, albeit not treated in detail here, model calibration is an important issue for the practical use of the models. An important aspect of the asymptotic results derived in our paper is their use in calibration. They provide a tool for robust calibration because they identify the parameters, the group market parameters, that are important in affecting prices on contracts written on the underlying. Thus, the asymptotic results can be used as a tool to avoid overfitting and noisy parameter estimates. Indeed, the analysis of this paper shows that even at the level of the correction the Hurst parameter does not affect the price. Thus, for long dated contracts the calibration scheme should not aim to identify the Hurst exponent. We also comment that the asymptotic results identify generic parameters, parameters that are common for different contracts written on the same underlying. The results presented here mean that the same calibration scheme as that used in the Markovian case is appropriate. Calibration schemes for the Markovian case are considered in for instance Fouque et al. (2003, 2004, 2011). The framework considered there is to calibrate parameters from the implied volatility surface. We remark that estimates of volatility, spot volatility or proxies like VIX index for instance, also provide relevant information for calibration. Again here the asymptotic analysis provides an important tool because it identifies how basic aspects of the underlying affect the observables and how to connect information from

for instance the VIX index to pricing of financial contracts in the various asymptotic regimes. Here our focus is on fast mean reverting rough volatility and how it affects pricing and thus implied volatility, but the tools presented can also be used to analyze issues associated with using historical data and for instance the VIX index for calibration.

One may ask why the Hurst exponent does not affect the implied volatility, as we will show below, even at the order of the correction. The intuition for this is that the rough case with small Hurst exponent $H < 1/2$ is mainly important on short time scales as it is the roughness, the rapid decay of correlations at the origin, which primarily distinguishes the situation from the Markovian context. The correlations decay fast for large offsets so that the correlation function for the rough volatility process is integrable. Thus, on time scales long relative to the mean reverting time of the rough volatility process, from a “birds eye perspective”, the roughness is not felt and the process appears as a Markovian process. This is in contrast to the long-range case with $H > 1/2$ where the correlations persist for a long time and the correlation function has heavy tails and is not integrable. Then the effect is felt also for long times. Indeed, we show in Garnier and Sølna (2016) that with $H > 1/2$, and in the asymptotic framework considered there, the Hurst exponent indeed has a strong impact on the form of the prices and hence the implied volatility. In fact such a dichotomy in terms of behavior and analytical approach depending on $H \leq 1/2$ versus $H > 1/2$ can also be observed in modeling of physical systems. See for instance the recent paper Kalbasi et al (2017) which analyzes the behavior of a Lyapunov coefficient in the context of a driving fractional Brownian motion.

In Section 6 we summarize the form of the fractional term structure exponent as it depends on the smoothness of volatility fluctuations, fluctuation magnitude, and the time scale of mean reversion. Otherwise the outline of the paper is as follows: In Section 2 we introduce the volatility factor in terms of a fractional Ornstein–Uhlenbeck process and in Section 3 the full stochastic volatility model. In Section 4 we present the main result and its proof. A number of technical lemmas that are used in the proof are presented in the appendices.

2 The Rapid Fractional Ornstein–Uhlenbeck Process

We use a rapid fractional Ornstein–Uhlenbeck (fOU) process as the volatility factor and describe here how this process can be represented in terms of a fractional Brownian motion. Since fractional Brownian motion can be expressed in terms of ordinary Brownian motion we also arrive at an expression for the rapid fOU process as a filtered version of Brownian motion.

A fractional Brownian motion (fBM) is a zero-mean Gaussian process $(W_t^H)_{t \in \mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad (1)$$

where σ_H is a positive constant.

We use the following moving-average stochastic integral representation of the fBM (see [Mandelbrot and Van Ness \(1968\)](#)):

$$W_t^H = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s, \quad (2)$$

where $(W_t)_{t \in \mathbb{R}}$ is a standard Brownian motion over \mathbb{R} . Then indeed $(W_t^H)_{t \in \mathbb{R}}$ is a zero-mean Gaussian process with the covariance (1) and we have

$$\begin{aligned} \sigma_H^2 &= \frac{1}{\Gamma(H + \frac{1}{2})^2} \left[\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})^2 ds + \frac{1}{2H} \right] \\ &= \frac{1}{\Gamma(2H+1) \sin(\pi H)}. \end{aligned} \quad (3)$$

We introduce the ε -scaled fOU process as

$$Z_t^\varepsilon = \varepsilon^{-H} \int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H = \varepsilon^{-H} W_t^H - \varepsilon^{-1-H} \int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} W_s^H ds. \quad (4)$$

Thus, the fOU process is in fact a fractional Brownian motion with a restoring force towards zero. It is a zero-mean, stationary Gaussian process, with variance

$$\mathbb{E}[(Z_t^\varepsilon)^2] = \sigma_{\text{ou}}^2, \quad \text{with } \sigma_{\text{ou}}^2 = \frac{1}{2} \Gamma(2H+1) \sigma_H^2 = \frac{1}{2 \sin(\pi H)}, \quad (5)$$

that is independent of ε , and covariance:

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_{\text{ou}}^2 \mathcal{C}_Z\left(\frac{s}{\varepsilon}\right),$$

that is a function of s/ε only, with

$$\begin{aligned} \mathcal{C}_Z(s) &= \frac{1}{\Gamma(2H+1)} \left[\frac{1}{2} \int_{\mathbb{R}} e^{-|v|} |s+v|^{2H} dv - |s|^{2H} \right] \\ &= \frac{2 \sin(\pi H)}{\pi} \int_0^\infty \cos(sx) \frac{x^{1-2H}}{1+x^2} dx. \end{aligned} \quad (6)$$

This shows that ε is the natural scale of variation of the fOU Z_t^ε . Note that the random process Z_t^ε is not a martingale, nor a Markov process. For $H \in (0, 1/2)$ it possesses short-range correlation properties in the sense that its correlation function is rough at zero:

$$\mathcal{C}_Z(s) = 1 - \frac{1}{\Gamma(2H+1)} s^{2H} + o(s^{2H}), \quad s \ll 1, \quad (7)$$

while it is integrable and it decays as s^{2H-2} at infinity:

$$\mathcal{C}_Z(s) = \frac{1}{\Gamma(2H-1)} s^{2H-2} + o(s^{2H-2}), \quad s \gg 1. \quad (8)$$

Using Eqs. (2) and (4) we arrive at the moving-average integral representation of the scaled fOU as:

$$Z_t^\varepsilon = \sigma_{\text{ou}} \int_{-\infty}^t \mathcal{K}^\varepsilon(t-s) dW_s, \quad (9)$$

where

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \mathcal{K}\left(\frac{t}{\varepsilon}\right), \quad \mathcal{K}(t) = \frac{1}{\sigma_{\text{ou}} \Gamma(H + \frac{1}{2})} \left[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}} e^{-s} ds \right]. \quad (10)$$

The main properties of the kernel \mathcal{K} in our context are the following ones (valid for any $H \in (0, 1/2)$):

- (i) $\mathcal{K} \in L^2(0, \infty)$ with $\int_0^\infty \mathcal{K}^2(u) du = 1$ and $\mathcal{K} \in L^1(0, \infty)$.
- (ii) For small times $t \ll 1$:

$$\mathcal{K}(t) = \frac{1}{\sigma_{\text{ou}} \Gamma(H + \frac{1}{2})} \left(t^{H-\frac{1}{2}} + O(t^{H+\frac{1}{2}}) \right). \quad (11)$$

- (iii) For large times $t \gg 1$:

$$\mathcal{K}(t) = \frac{1}{\sigma_{\text{ou}} \Gamma(H - \frac{1}{2})} \left(t^{H-\frac{3}{2}} + O(t^{H-\frac{5}{2}}) \right). \quad (12)$$

Remark. The results presented in this paper can be generalized to any stochastic volatility model of the form (14) and (9) provided $\mathcal{K}^\varepsilon(t) = \mathcal{K}(t/\varepsilon)/\sqrt{\varepsilon}$ is such that the kernel \mathcal{K} satisfies the properties (i)-(ii)-(iii) up to multiplicative constants.

3 The Stochastic Volatility Model

The price of the risky asset follows the stochastic differential equation:

$$dX_t = \sigma_t^\varepsilon X_t dW_t^*. \quad (13)$$

The stochastic volatility is

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \quad (14)$$

where Z_t^ε is the scaled fOU with Hurst parameter $H \in (0, 1/2)$ introduced in the previous section which is adapted to the Brownian motion W_t . Moreover, W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1 - \rho^2} B_t, \quad (15)$$

where the Brownian motion B_t is independent of W_t .

The function F is assumed to be one-to-one, positive-valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration \mathcal{F}_t generated by (B_t, W_t) is also the one generated by X_t . Indeed, it is equivalent to the one generated by (W_t^*, W_t) , or (W_t^*, Z_t^ε) . Since F is one-to-one, it is

equivalent to the one generated by $(W_t^*, \sigma_t^\varepsilon)$. Since F is positive-valued, it is equivalent to the one generated by $(W_t^*, (\sigma_t^\varepsilon)^2)$, or X_t .

As we have discussed above, the volatility driving process Z_t^ε has short-range correlation properties. As we now show the volatility process σ_t^ε inherits this property.

Lemma 1 *We denote, for $j = 1, 2$:*

$$\langle F^j \rangle = \int_{\mathbb{R}} F(\sigma_{\text{ou}} z)^j p(z) dz, \quad \langle F'^j \rangle = \int_{\mathbb{R}} F'(\sigma_{\text{ou}} z)^j p(z) dz, \quad (16)$$

where $p(z)$ is the pdf of the standard normal distribution.

1. The process σ_t^ε is a stationary random process with mean $\mathbb{E}[\sigma_t^\varepsilon] = \langle F \rangle$ and variance $\text{Var}(\sigma_t^\varepsilon) = \langle F^2 \rangle - \langle F \rangle^2$, independently of ε .
2. The covariance function of the process σ_t^ε is of the form

$$\text{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) = (\langle F^2 \rangle - \langle F \rangle^2) \mathcal{C}_\sigma\left(\frac{s}{\varepsilon}\right), \quad (17)$$

where the correlation function \mathcal{C}_σ satisfies $\mathcal{C}_\sigma(0) = 1$ and

$$\mathcal{C}_\sigma(s) = 1 - \frac{1}{\Gamma(2H+1)} \frac{\sigma_{\text{ou}}^2 \langle F'^2 \rangle}{\langle F^2 \rangle - \langle F \rangle^2} s^{2H} + o(s^{2H}), \quad \text{for } s \ll 1, \quad (18)$$

$$\mathcal{C}_\sigma(s) = \frac{1}{\Gamma(2H-1)} \frac{\sigma_{\text{ou}}^2 \langle F' \rangle^2}{\langle F^2 \rangle - \langle F \rangle^2} s^{2H-2} + o(s^{2H-2}), \quad \text{for } s \gg 1. \quad (19)$$

Consequently, the process σ_t^ε has short-range correlation properties and its covariance function is integrable.

Proof The fact that σ_t^ε is a stationary random process with mean $\langle F \rangle$ is straightforward in view of the definition (14) of σ_t^ε .

For any t, s , the vector $\sigma_{\text{ou}}^{-1}(Z_t^\varepsilon, Z_{t+s}^\varepsilon)$ is a Gaussian random vector with mean $(0, 0)$ and 2×2 covariance matrix:

$$\mathbf{C}^\varepsilon = \begin{pmatrix} 1 & \mathcal{C}_Z(s/\varepsilon) \\ \mathcal{C}_Z(s/\varepsilon) & 1 \end{pmatrix}.$$

Therefore, denoting $F_c(z) = F(\sigma_{\text{ou}} z) - \langle F \rangle$, the covariance function of the process σ_t^ε is

$$\begin{aligned} \text{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) &= \mathbb{E}[F_c(\sigma_{\text{ou}}^{-1} Z_t^\varepsilon) F_c(\sigma_{\text{ou}}^{-1} Z_{t+s}^\varepsilon)] \\ &= \frac{1}{2\pi \sqrt{\det \mathbf{C}^\varepsilon}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\left(-\frac{1}{2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^T \mathbf{C}^{\varepsilon-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) dz_1 dz_2 \\ &= \Psi\left(\mathcal{C}_Z\left(\frac{s}{\varepsilon}\right)\right), \end{aligned}$$

with

$$\Psi(C) = \frac{1}{2\pi \sqrt{1-C^2}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\left(-\frac{z_1^2 + z_2^2 - 2Cz_1 z_2}{2(1-C^2)}\right) dz_1 dz_2.$$

This shows that $\text{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon)$ is a function of s/ε only.

The function Ψ can be expanded in powers of $1 - C$ for C close to one:

$$\begin{aligned} \Psi(C) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} F_c\left(z \frac{\sqrt{1+C}}{\sqrt{2}} + \zeta \frac{\sqrt{1-C}}{\sqrt{2}}\right) F_c\left(z \frac{\sqrt{1+C}}{\sqrt{2}} - \zeta \frac{\sqrt{1-C}}{\sqrt{2}}\right) \\ &\quad \times \exp\left(-\frac{z^2}{2} - \frac{\zeta^2}{2}\right) dz d\zeta \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_c(z)^2 \exp\left(-\frac{z^2}{2}\right) dz \\ &\quad + (1-C) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F'_c(z)^2 \exp\left(-\frac{z^2}{2}\right) dz + O_{C \rightarrow 1}((1-C)^2), \end{aligned}$$

which gives with (7) the form (18) of the correlation function for σ_t^ε .

The function Ψ can be expanded in powers of C for small C :

$$\begin{aligned} \Psi(C) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) dz_1 dz_2 \\ &\quad - \frac{C}{2\pi} \iint_{\mathbb{R}^2} z_1 z_2 F_c(z_1) F_c(z_2) \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) dz_1 dz_2 + O_{C \rightarrow 0}(C^2), \end{aligned}$$

which gives with (8) the form (19) of the correlation function for σ_t^ε .

4 The Option Price

We aim at computing the option price defined as the martingale

$$M_t = \mathbb{E}[h(X_T) | \mathcal{F}_t], \quad (20)$$

where h is a smooth payoff function and $t \leq T$. In fact weaker assumptions are possible for h , as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than h , as is discussed in (Garnier and Sølna, 2015, Section 4).

We introduce the operator

$$\mathcal{L}_{\text{BS}}(\sigma) = \partial_t + \frac{1}{2} \sigma^2 x^2 \partial_x^2, \quad (21)$$

that is, the standard Black–Scholes operator at zero interest rate and constant volatility σ .

We next exploit the fact that the price process is a martingale to obtain an approximation, via constructing an explicit function $Q_t^\varepsilon(x)$ so that $Q_T^\varepsilon(x) = h(x)$ and so that $Q_t^\varepsilon(X_t)$ is a martingale to first-order corrected terms. Then, indeed $Q_t^\varepsilon(X_t)$ gives the approximation for the option price M_t to this order.

The following proposition gives the first-order correction to the expression for the martingale M_t in the regime where ε is small.

Proposition 1 *We have*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E} [|M_t - Q_t^\varepsilon(X_t)|^2]^{1/2} = 0, \quad (22)$$

where

$$Q_t^\varepsilon(x) = Q_t^{(0)}(x) + \varepsilon^{1/2} \rho Q_t^{(1)}(x), \quad (23)$$

$Q_t^{(0)}(x)$ is deterministic and given by the Black–Scholes formula with constant volatility $\bar{\sigma}$,

$$\mathcal{L}_{\text{BS}}(\bar{\sigma})Q_t^{(0)}(x) = 0, \quad Q_T^{(0)}(x) = h(x), \quad (24)$$

with

$$\bar{\sigma}^2 = \langle F^2 \rangle = \int_{\mathbb{R}} F(\sigma_{\text{ou}} z)^2 p(z) dz, \quad (25)$$

$p(z)$ is the pdf of the standard normal distribution,

$Q_t^{(1)}(x)$ is the deterministic correction

$$Q_t^{(1)}(x) = (T - t) \bar{D}(x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(x), \quad (26)$$

\bar{D} is the coefficient defined by

$$\bar{D} = \sigma_{\text{ou}} \int_0^\infty \left[\iint_{\mathbb{R}^2} F(\sigma_{\text{ou}} z) (FF')(\sigma_{\text{ou}} z') p_{\mathcal{C}_Z(s)}(z, z') dz dz' \right] \mathcal{K}(s) ds, \quad (27)$$

$p_{\mathcal{C}}(z, z')$ is the pdf of the bivariate normal distribution with mean zero and covariance matrix $\begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix}$, and $\mathcal{C}_Z(s)$ is given by (6).

This proposition shows that the result is similar to the mixing (Markov) case addressed in [Fouque et al. \(2000, 2011\)](#). In the fast-varying framework, the short-range correlation property of the stochastic volatility is not visible to leading order nor in the first correction. This is in contrast to the slowly-varying case addressed in [Garnier and Sølna \(2015\)](#) and this is the main result of this paper.

4.1 The Case of Riemann-Liouville Fractional Brownian Motion

A common approach for modeling with fractional processes is to use the Riemann-Liouville fractional Brownian motion defined by:

$$W_t^{H,0} = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t - s)^{H - \frac{1}{2}} dW_s^0, \quad (28)$$

where W^0 is a standard Brownian motion. From a mathematical perspective this is convenient as compared to the fractional Brownian motion in (2) because one does not have to deal with the integral for negative times and the associated compensator. However, from the modeling perspective it has the disadvantage that the time zero plays a special role and the process does not

have stationary increments. By modeling the driving process as in (2) we obtain on the other hand a time homogeneous process. In Fukasawa (2017) a somewhat different approach to modeling with time-homogeneous fractional processes is used by using a representation of fractional Brownian motion introduced by Muralev (2011). In any case, we can use the representation in (28) to define a fOU process analogous to (4) by

$$\begin{aligned} Z_t^{\varepsilon,0} &= Z_0 e^{-t/\varepsilon} + \varepsilon^{-H} \int_0^t e^{-\frac{t-s}{\varepsilon}} dW_s^{H,0} \\ &= Z_0 e^{-t/\varepsilon} + \varepsilon^{-H} W_t^{H,0} - \varepsilon^{-H-1} \int_0^t e^{-\frac{t-s}{\varepsilon}} W_s^{H,0} ds, \end{aligned} \quad (29)$$

where Z_0 is considered as a constant. In the modeling context considered here, from the point of view of the process covariance, the time epoch of negative times is in fact quickly forgotten so that for any $t > 0$, $s \geq 0$:

$$\lim_{\varepsilon \rightarrow 0} \text{Cov}(Z_t^{\varepsilon,0}, Z_{t+\varepsilon s}^{\varepsilon,0}) = \lim_{\varepsilon \rightarrow 0} \text{Cov}(Z_t^\varepsilon, Z_{t+\varepsilon s}^\varepsilon) = \sigma_{\text{ou}}^2 \mathcal{C}_Z(s),$$

and in fact the covariances only differ for a time epoch of duration ε after time zero. The consequence is that Proposition 1 holds true when Z_t^ε is replaced by $Z_t^{\varepsilon,0}$. We discuss this in more detail in Appendix B.

4.2 Sketch of Proof of Proposition 1

Our objective is to construct an approximation $Q_t^\varepsilon(X_t)$ for the price. Note that a natural first choice is to choose the approximation as $Q_t^{(0)}(X_t)$, that is the Black–Scholes price at the effective volatility. In order to construct a higher-order approximation we look for a correction, denoted by $\Delta Q_t^\varepsilon(X_t)$, so that

$$Q_t^{(0)}(X_t) + \Delta Q_t^\varepsilon(X_t) = \text{martingale} + \text{“small terms”},$$

and with $\Delta Q_T^\varepsilon(X_T) = 0$ so that the corrected approximation has the correct payoff. Indeed, the corrected approximation now differs from the exact price only by the magnitude of the “small terms” because the exact price is a martingale. Moreover, the volatility fluctuation process $(\sigma_s^\varepsilon)^2 - \bar{\sigma}^2$ drives the difference between the exact price and $Q_t^{(0)}(X_t)$. To identify the form of the price correction $\Delta Q_t^\varepsilon(X_t)$ and to prove the smallness of the resulting (non-martingale) error terms the introduction of the martingale defined in terms of the residual volatility fluctuations is useful. That is why we introduce the process

$$\psi_t^\varepsilon = \mathbb{E} \left[\frac{1}{2} \int_0^T ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds \middle| \mathcal{F}_t \right]. \quad (30)$$

This martingale is zero in the constant volatility case. It is important to understand its properties and those of its covariation process with respect to the

underlying driving Brownian motion in order to prove the accuracy of the approximation and to control the error terms. These properties are given in terms of the original technical lemmas in Appendix A. These lemmas could be useful also for the asymptotic analysis of other quantities than those considered here, but defined in terms of underlyings modeled as in this paper.

4.3 Proof of Proposition 1

For any smooth function $q_t(x)$, we have by Itô's formula

$$\begin{aligned} dq_t(X_t) &= \partial_t q_t(X_t) dt + (x \partial_x) q_t(X_t) \sigma_t^\varepsilon dW_t^* + \frac{1}{2} (x^2 \partial_x^2) q_t(X_t) (\sigma_t^\varepsilon)^2 dt \\ &= \mathcal{L}_{\text{BS}}(\sigma_t^\varepsilon) q_t(X_t) dt + (x \partial_x) q_t(X_t) \sigma_t^\varepsilon dW_t^*, \end{aligned}$$

the last term being a martingale. Here and below $(x \partial_x) q_t(X_t)$ stands for $x \partial_x q_t(x)$ evaluated at $x = X_t$. Therefore, by (24), we have

$$dQ_t^{(0)}(X_t) = \frac{1}{2} ((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2) (x^2 \partial_x^2) Q_t^{(0)}(X_t) dt + dN_t^{(0)}, \quad (31)$$

with $N_t^{(0)}$ a martingale:

$$dN_t^{(0)} = (x \partial_x) Q_t^{(0)}(X_t) \sigma_t^\varepsilon dW_t^*.$$

Let ϕ_t^ε be defined by

$$\phi_t^\varepsilon = \mathbb{E} \left[\frac{1}{2} \int_t^T ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds \middle| \mathcal{F}_t \right]. \quad (32)$$

We have

$$\phi_t^\varepsilon = \psi_t^\varepsilon - \frac{1}{2} \int_0^t ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds,$$

where the martingale ψ_t^ε is defined by

$$\psi_t^\varepsilon = \mathbb{E} \left[\frac{1}{2} \int_0^T ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds \middle| \mathcal{F}_t \right]. \quad (33)$$

We can write

$$\frac{1}{2} ((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2) (x^2 \partial_x^2) Q_t^{(0)}(X_t) dt = (x^2 \partial_x^2) Q_t^{(0)}(X_t) d\psi_t^\varepsilon - (x^2 \partial_x^2) Q_t^{(0)}(X_t) d\phi_t^\varepsilon.$$

By Itô's formula:

$$\begin{aligned} d[\phi_t^\varepsilon (x^2 \partial_x^2) Q_t^{(0)}(X_t)] &= (x^2 \partial_x^2) Q_t^{(0)}(X_t) d\phi_t^\varepsilon + (x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(X_t) \sigma_t^\varepsilon \phi_t^\varepsilon dW_t^* \\ &\quad + \mathcal{L}_{\text{BS}}(\sigma_t^\varepsilon) (x^2 \partial_x^2) Q_t^{(0)}(X_t) \phi_t^\varepsilon dt \\ &\quad + (x \partial_x (x^2 \partial_x^2)) Q_t^{(0)}(X_t) \sigma_t^\varepsilon d \langle \phi^\varepsilon, W^* \rangle_t. \end{aligned}$$

Since $\mathcal{L}_{\text{BS}}(\sigma_t^\varepsilon) = \mathcal{L}_{\text{BS}}(\bar{\sigma}) + \frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2)$ and $\mathcal{L}_{\text{BS}}(\bar{\sigma})(x^2 \partial_x^2)Q_t^{(0)}(x) = 0$, this gives

$$\begin{aligned} d[\phi_t^\varepsilon(x^2 \partial_x^2)Q_t^{(0)}(X_t)] &= -\frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2)Q_t^{(0)}(X_t)dt \\ &\quad + \frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2(x^2 \partial_x^2))Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &\quad + (x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\langle \phi^\varepsilon, W^* \rangle_t \\ &\quad + (x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\varepsilon \phi_t^\varepsilon dW_t^* + (x^2 \partial_x^2)Q_t^{(0)}(X_t)d\psi_t^\varepsilon. \end{aligned}$$

We have $\langle \phi^\varepsilon, W^* \rangle_t = \langle \psi^\varepsilon, W^* \rangle_t = \rho \langle \psi^\varepsilon, W \rangle_t$ and therefore

$$\begin{aligned} d[(\phi_t^\varepsilon(x^2 \partial_x^2)Q_t^{(0)}(X_t))] &= -\frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2)Q_t^{(0)}(X_t)dt \\ &\quad + \frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2(x^2 \partial_x^2))Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &\quad + \rho(x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\langle \psi^\varepsilon, W \rangle_t \\ &\quad + dN_t^{(1)}, \end{aligned}$$

where $N_t^{(1)}$ is a martingale,

$$dN_t^{(1)} = (x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\varepsilon \phi_t^\varepsilon dW_t^* + (x^2 \partial_x^2)Q_t^{(0)}(X_t)d\psi_t^\varepsilon.$$

Therefore

$$\begin{aligned} d[Q_t^{(0)}(X_t) + \phi_t^\varepsilon(x^2 \partial_x^2)Q_t^{(0)}(X_t)] &= \frac{1}{2}(x^2 \partial_x^2(x^2 \partial_x^2))Q_t^{(0)}(X_t)((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)\phi_t^\varepsilon dt \\ &\quad + \rho(x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(X_t)\sigma_t^\varepsilon \vartheta_t^\varepsilon dt \\ &\quad + dN_t^{(0)} + dN_t^{(1)}. \end{aligned} \quad (34)$$

Here, we have introduced the covariation increments

$$d\langle \psi^\varepsilon, W \rangle_t = \vartheta_t^\varepsilon dt, \quad (35)$$

defined in Lemma 2.

The deterministic function $Q_t^{(1)}$ defined by (26) satisfies

$$\mathcal{L}_{\text{BS}}(\bar{\sigma})Q_t^{(1)}(x) = -\bar{D}(x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(x), \quad Q_T^{(1)}(x) = 0.$$

Applying Itô's formula

$$\begin{aligned} dQ_t^{(1)}(X_t) &= \mathcal{L}_{\text{BS}}(\sigma_t^\varepsilon)Q_t^{(1)}(X_t)dt + (x \partial_x Q_t^{(1)})(X_t)\sigma_t^\varepsilon dW_t^* \\ &= \mathcal{L}_{\text{BS}}(\bar{\sigma})Q_t^{(1)}(X_t)dt + \frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2)Q_t^{(1)}(X_t)dt \\ &\quad + (x \partial_x)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^* \\ &= \frac{1}{2}((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)(x^2 \partial_x^2)Q_t^{(1)}(X_t)dt - (x \partial_x(x^2 \partial_x^2))Q_t^{(0)}(X_t)\bar{D}dt \\ &\quad + dN_t^{(2)}, \end{aligned}$$

where $N_t^{(2)}$ is a martingale,

$$dN_t^{(2)} = (x\partial_x)Q_t^{(1)}(X_t)\sigma_t^\varepsilon dW_t^*.$$

Therefore

$$\begin{aligned} & d[Q_t^{(0)}(X_t) + \phi_t^\varepsilon(x^2\partial_x^2)Q_t^{(0)}(X_t) + \varepsilon^{1/2}\rho Q_t^{(1)}(X_t)] \\ &= \frac{1}{2}(x^2\partial_x^2(x^2\partial_x^2))Q_t^{(0)}(X_t)((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)\phi_t^\varepsilon dt \\ & \quad + \frac{\varepsilon^{1/2}}{2}\rho(x^2\partial_x^2)Q_t^{(1)}(X_t)((\sigma_t^\varepsilon)^2 - \bar{\sigma}^2)dt \\ & \quad + \rho(x\partial_x(x^2\partial_x^2))Q_t^{(0)}(X_t)(\sigma_t^\varepsilon\vartheta_t^\varepsilon - \varepsilon^{1/2}\bar{D})dt \\ & \quad + dN_t^{(0)} + dN_t^{(1)} + \varepsilon^{1/2}\rho dN_t^{(2)}. \end{aligned} \quad (36)$$

We next show that the first three terms of the right-hand side of (36) are smaller than $\varepsilon^{1/2}$. We introduce for any $t \in [0, T]$:

$$R_{t,T}^{(1)} = \int_t^T \frac{1}{2}(x^2\partial_x^2(x^2\partial_x^2))Q_s^{(0)}(X_s)((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2)\phi_s^\varepsilon ds, \quad (37)$$

$$R_{t,T}^{(2)} = \int_t^T \frac{\varepsilon^{1/2}}{2}\rho(x^2\partial_x^2)Q_s^{(1)}(X_s)((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2)ds, \quad (38)$$

$$R_{t,T}^{(3)} = \int_t^T \rho(x\partial_x(x^2\partial_x^2))Q_s^{(0)}(X_s)(\vartheta_s^\varepsilon\sigma_s^\varepsilon - \varepsilon^{1/2}\bar{D})ds. \quad (39)$$

We will show that, for $j = 1, 2, 3$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(R_{t,T}^{(j)})^2]^{1/2} = 0. \quad (40)$$

Step 1: Proof of (40) for $j = 1$.

Since $Q^{(0)}$ is smooth and bounded and F is bounded, there exists C such that

$$\sup_{t \in [0, T]} \mathbb{E}[(R_{t,T}^{(1)})^2] \leq CT \int_0^T \mathbb{E}[(\phi_s^\varepsilon)^2] ds.$$

By Lemma 5 we get the desired result.

Step 2: Proof of (40) for $j = 2$.

We denote

$$Y_s^{(2)} = \rho(x^2\partial_x^2)Q_s^{(1)}(X_s)$$

and

$$\kappa_t^\varepsilon = \frac{\varepsilon^{1/2}}{2} \int_0^t ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds, \quad (41)$$

so that

$$R_{t,T}^{(2)} = \int_t^T Y_s^{(2)} \frac{d\kappa_s^\varepsilon}{ds} ds.$$

Note that $Y_s^{(2)}$ is a bounded semimartingale with bounded quadratic variations. Let N be a positive integer. We denote $t_k = t + (T - t)k/N$. We then have

$$\begin{aligned} R_{t,T}^{(2)} &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Y_s^{(2)} \frac{d\kappa_s^\varepsilon}{ds} ds = R_{t,T}^{(2,a)} + R_{t,T}^{(2,b)}, \\ R_{t,T}^{(2,a)} &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} Y_{t_k}^{(2)} \frac{d\kappa_s^\varepsilon}{ds} ds = \sum_{k=0}^{N-1} Y_{t_k}^{(2)} (\kappa_{t_{k+1}}^\varepsilon - \kappa_{t_k}^\varepsilon), \\ R_{t,T}^{(2,b)} &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (Y_s^{(2)} - Y_{t_k}^{(2)}) \frac{d\kappa_s^\varepsilon}{ds} ds. \end{aligned}$$

Then, on the one hand

$$\begin{aligned} \mathbb{E}[(R_{t,T}^{(2,a)})^2]^{1/2} &\leq 2 \sum_{k=0}^{N-1} \|Y^{(2)}\|_\infty \mathbb{E}[(\kappa_{t_k}^\varepsilon)^2]^{1/2} \\ &\leq 2(N+1) \|Y^{(2)}\|_\infty \sup_{s \in [0, T]} \mathbb{E}[(\kappa_s^\varepsilon)^2]^{1/2}, \end{aligned}$$

so that, by Lemma 6,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(R_{t,T}^{(2,a)})^2]^{1/2} = 0.$$

On the other hand

$$\begin{aligned} \mathbb{E}[(R_{t,T}^{(2,b)})^2]^{1/2} &\leq \varepsilon^{1/2} \|F\|_\infty^2 \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E}[(Y_s^{(2)} - Y_{t_k}^{(2)})^2]^{1/2} ds \\ &\leq K \varepsilon^{1/2} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (s - t_k)^{1/2} ds = \frac{2KT^{3/2} \varepsilon^{1/2}}{3\sqrt{N}}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(R_{t,T}^{(2)})^2]^{1/2} &\leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(R_{t,T}^{(2,b)})^2]^{1/2} \\ &\leq \frac{2KT^{3/2}}{3\sqrt{N}}. \end{aligned}$$

Since this is true for any N , we get the desired result.

Step 3: Proof of (40) for $j = 3$.

We repeat the same arguments as in the previous step. It remains to show that

$$\tilde{\kappa}_t^\varepsilon = \int_0^t (\vartheta_s^\varepsilon \sigma_s^\varepsilon - \varepsilon^{1/2} \bar{D}) ds$$

satisfies

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \sup_{t \in [0, T]} \mathbb{E}[(\tilde{\kappa}_t^\varepsilon)^2]^{1/2} = 0.$$

Since $(a + b)^2 \leq 2a^2 + 2b^2$, we have

$$\mathbb{E}[(\tilde{\kappa}_t^\varepsilon)^2] \leq 2 \int_0^t ds \int_0^t ds' \text{Cov}(\vartheta_s^\varepsilon \sigma_s^\varepsilon, \vartheta_{s'}^\varepsilon \sigma_{s'}^\varepsilon) + 2 \left(\int_0^t (\mathbb{E}[\vartheta_s^\varepsilon \sigma_s^\varepsilon] - \varepsilon^{1/2} \overline{D}) ds \right)^2.$$

By Lemma 3, items 1 and 3, and dominated convergence theorem, the first term of the right-hand side is $o(\varepsilon)$ uniformly in $t \in [0, T]$. By Lemma 3, item 2, the second term of the right-hand side is $o(\varepsilon)$ uniformly in $t \in [0, T]$. This gives the desired result.

We can now complete the proof of Proposition 1. We introduce the approximation:

$$\tilde{Q}_t^\varepsilon(x) = Q_t^{(0)}(x) + \phi_t^\varepsilon(x^2 \partial_x^2) Q_t^{(0)}(x) + \varepsilon^{1/2} \rho Q_t^{(1)}(x).$$

We then have

$$\tilde{Q}_T^\varepsilon(x) = h(x),$$

because $Q_T^{(0)}(x) = h(x)$, $\phi_T^\varepsilon = 0$, and $Q_T^{(1)}(x) = 0$. Let us denote

$$R_{t,T} = R_{t,T}^{(1)} + R_{t,T}^{(2)} + R_{t,T}^{(3)}, \quad (42)$$

$$N_t = \int_0^t dN_s^{(0)} + dN_s^{(1)} + \varepsilon^{1/2} \rho dN_s^{(2)}. \quad (43)$$

By (36) we have

$$\tilde{Q}_T^\varepsilon(X_T) - \tilde{Q}_t^\varepsilon(X_t) = R_{t,T} + N_T - N_t.$$

Therefore

$$\begin{aligned} M_t &= \mathbb{E}[h(X_T) | \mathcal{F}_t] = \mathbb{E}[\tilde{Q}_T^\varepsilon(X_T) | \mathcal{F}_t] \\ &= \tilde{Q}_t^\varepsilon(X_t) + \mathbb{E}[R_{t,T} | \mathcal{F}_t] + \mathbb{E}[N_T - N_t | \mathcal{F}_t] \\ &= \tilde{Q}_t^\varepsilon(X_t) + \mathbb{E}[R_{t,T} | \mathcal{F}_t], \end{aligned} \quad (44)$$

which gives the desired result since $\mathbb{E}[R_{t,T} | \mathcal{F}_t]$ and ϕ_t^ε are uniformly of order $o(\varepsilon^{1/2})$ in L^2 (see (40) for $R_{t,T}$ and see Lemma 5 for ϕ_t^ε).

5 The Implied Volatility

We now compute and discuss the implied volatility associated with the price approximation given in Proposition 1. This implied volatility is the volatility that when used in the constant volatility Black–Scholes pricing formula gives the same price as the approximation, to the order of the approximation. The

implied volatility in the context of the European option introduced in the previous section is then given by

$$I_t = \bar{\sigma} + \varepsilon^{1/2} \rho \bar{D} \left[\frac{1}{2\bar{\sigma}} + \frac{\log(K/X_t)}{\bar{\sigma}^3(T-t)} \right] + o(\varepsilon^{1/2}). \quad (45)$$

The expression (45) is in agreement with the one obtained in (Fouque et al., 2000, Eq. (5.55)) with a stochastic volatility that is an ordinary Ornstein–Uhlenbeck process, that is, a Markovian process with correlations decaying exponentially fast. See for instance Fouque et al. (2003, 2004, 2011) and references therein for data calibration examples.

6 A Brief Review on Fractional Stochastic Volatility Asymptotics

This paper together with Garnier and Sølna (2015, 2016) discuss different fractional stochastic volatility models with $H < 1/2$ or $H > 1/2$. We summarize here some main aspects.

6.1 Characteristic Term Structure Exponent

We write the implied volatility associated with a European Call Option for strike K , maturity T , current time t , and current value for the underlying X_t (as in Eq. (45)) in the general form:

$$I_t = \sigma_{t,T} + \Delta\sigma \left[\left(\frac{\tau}{\bar{\tau}} \right)^{\zeta(H)} + \left(\frac{\tau}{\bar{\tau}} \right)^{\zeta(H)-1} \log \left(\frac{K}{X_t} \right) \right], \quad (46)$$

$$\sigma_{t,T} = \mathbb{E} \left[\frac{1}{T-t} \int_t^T (\sigma_s)^2 ds \middle| \mathcal{F}_t \right]^{1/2} = \bar{\sigma} + \tilde{\sigma}_{t,T}, \quad (47)$$

where σ_s is the volatility path, $\bar{\tau}$ is the characteristic diffusion time defined by

$$\bar{\tau} = \frac{2}{\bar{\sigma}^2}, \quad (48)$$

and $\tau = T-t$ is the time to maturity. Note that in the regimes that we consider we assume that the time to maturity is of the same order as the characteristic diffusion time. We refer to $\zeta(H)$ as the characteristic term structure exponent. Note that $\tilde{\sigma}_{t,T}$ is the price path predicted volatility correction relative to the time horizon and is a stochastic process adapted to the filtration generated by the underlying price process. This is a correction term that reflects the multiscale nature of the volatility fluctuations and the memory aspect of this process. The second correction term in Eq. (46) involving $\Delta\sigma$ is a skewness correction and vanish in the case $\rho = 0$. As mentioned above it is natural to let the characteristic diffusion time be the reference time scale, if we denote the mean reversion time of the volatility fluctuations by τ_{mr} then we have considered two main multiscale asymptotic regimes in the context of the characteristic term structure exponent:

- Slow mean reverting volatility fluctuations, $\tau_{\text{mr}} \gg \bar{\tau}$, (see [Garnier and Sølna \(2015\)](#)). In this case:

$$\zeta(H) = H + \frac{1}{2}. \quad (49)$$

- Fast mean reverting volatility fluctuations, $\tau_{\text{mr}} \ll \bar{\tau}$, (see [Garnier and Sølna \(2016\)](#) for $H \in (1/2, 1)$ and this paper for $H \in (0, 1/2)$). In this case:

$$\zeta(H) = \max\left(H - \frac{1}{2}, 0\right). \quad (50)$$

Thus, we see that in the case of fast mean reversion leading to a singular perturbation expansion we have a fractional characteristic term structure exponent only in the case $H > 1/2$ when we have long-range correlation properties. While in the slow mean reversion case leading to a regular perturbation expansion we have a fractional characteristic term structure exponent for all values of the Hurst exponent H .

In [Garnier and Sølna \(2015\)](#) we considered also the case of small volatility fluctuations whose mean reversion time is of the same order as the characteristic diffusion time. This leads to an asymptotic regime where the characteristic term structure exponent is replaced by a more general characteristic term structure factor of the form (assuming a fOU volatility factor):

$$\left(\frac{\tau}{\bar{\tau}}\right)^{\zeta(H)} \rightarrow \mathcal{A}\left(\frac{\tau}{\bar{\tau}}, \frac{\tau}{\tau_{\text{mr}}}\right) = \left(\frac{\tau}{\bar{\tau}}\right)^{H+1/2} \left\{1 - \int_0^{\tau/\tau_{\text{mr}}} e^{-v} \left(1 - \frac{v}{\tau/\tau_{\text{mr}}}\right)^{H+\frac{3}{2}} dv\right\}.$$

We then have in a subsequent limit of either slow ($\tau_{\text{mr}} \gg \tau$) or fast ($\tau_{\text{mr}} \ll \tau$) mean reversion:

$$\mathcal{A}\left(\frac{\tau}{\bar{\tau}}, \frac{\tau}{\tau_{\text{mr}}}\right) \propto \begin{cases} \left(\frac{\tau}{\bar{\tau}}\right)^{H+1/2} & \text{for } \tau \ll \tau_{\text{mr}}, \\ \left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} & \text{for } \tau \gg \tau_{\text{mr}}, \end{cases} \quad (51)$$

where we have a fractional term structure for all values of H . It follows that the characteristic term structure exponent is consistent with the result (49) obtained in the slow mean reverting limit. It is also consistent with the result (50) obtained in the fast mean reverting limit, but only in the case $H \in (1/2, 1)$. There is no contradiction in the case $H \in (0, 1/2)$ because there is no fundamental reason that would justify that the limits “small amplitude” and “fast mean reversion” are exchangeable. This means that the prediction (51) for $H \in (0, 1/2)$ in the limit “small amplitude” and then “fast mean reversion” does not capture the leading-order contribution of the limit “fast mean reversion” that is independent of time to maturity, but is negligible for small-amplitude volatility fluctuations. Note that when the standard deviation of the volatility fluctuations is of the same order as the mean volatility and the time to maturity is of the same order as the mean reversion time, then the implied volatility reflects the particular structure of the model, see for instance the analysis of the Heston ([Heston \(1993\)](#)) model in [Alòs and Yang \(2014\)](#). Note also that with a model for how the implied volatility depends

on the Hurst exponent we can actually estimate the Hurst exponent based on recordings of the implied volatility. An example with estimation of the Hurst exponent based on a spot volatility proxy deriving from implied volatility is in [Livieri et al. \(2017\)](#) and yields a rough volatility regime. A calibration example for H using VIX futures is in [Jacquier et al \(2017\)](#) and yields again a rough volatility regime.

6.2 Flapping of the Implied Surface

Regarding the price path predicted volatility correction $\tilde{\sigma}_{t,T}$ which depends on the price history we have the following picture in the regime of fast mean reversion:

- Rough volatility fluctuations, $H < 1/2$:

$$\tilde{\sigma}_{t,T} = o(\Delta\sigma). \quad (52)$$

- Smooth volatility fluctuations, $H > 1/2$:

$$\tilde{\sigma}_{t,T} = O(\Delta\sigma). \quad (53)$$

In the case of smooth volatility fluctuations we discuss in detail in [Garnier and Sølna \(2016\)](#) the statistical structure of the “t-T” process $\tilde{\sigma}_{t,T}$. We remark that indeed in the scaling addressed in this paper we have $\Delta\sigma/\bar{\sigma} \ll 1$. In the regime of slow mean reversion as discussed in [Garnier and Sølna \(2015\)](#) we have that $\tilde{\sigma}_{t,T} = O(\bar{\sigma})$ since then the current level of volatility plays a central role.

7 Conclusion

We have considered rough fractional stochastic volatility models. Such modeling is motivated by a number of recent empirical findings that the volatility is not well modeled by a Markov process with exponentially decaying correlations and certainly not by a constant. Rather it should be modeled as a stochastic process with correlations that are rapidly decaying at the origin, qualitatively faster than the decay that can be associated with a Markov process. In general such models are challenging to use since the volatility factor is not a Markov process nor a martingale so we do not have a pricing partial differential equation. However, here we consider the situation where the volatility is fast mean reverting in the sense that its mean reversion time is short relative to the characteristic diffusion time of the price process. In this regime the pricing problem and associated implied volatility surface can be reduced to a parametric form corresponding to that of the Markovian case. An important aspect of our modeling is that we model the rough stochastic volatility as being a stationary process. Many if not most papers on this subject have hitherto used a non-stationary framework where the “time zero” plays a special role.

In the case of processes with memory of the past, rather than being Markov, we consider this aspect to be crucial from the modeling viewpoint. Indeed, in the general case the history (in principle observable from the underlying price path) impacts the implied volatility. However, in the regime of fast mean reversion the impact of the price history becomes lower order relative to the leading correction associated with the stochastic volatility which is explicit and which is identified in this paper.

It is important to note that this picture in fact breaks down in the case of long-range stochastic volatility when the volatility factor paths are smoother than in the Markovian case and when their correlations decay slower than in the Markovian case. It also breaks down in the asymptotic context when the mean reversion time is of the same order as the characteristic diffusion time of the price process, but volatility fluctuations have small standard deviation compared to the mean. In these cases both with short- (rough volatility) and long-range correlation properties the structure of the model for the price correction and the implied volatility changes and leads to a picture with a fractional term structure. These results are derived in Garnier and Sølna (2015, 2016) and summarized in Section 6 herein.

These observations then serve to partly explain why parameterizations for the implied surface deriving from a Markovian modeling have been successful in capturing the implied volatility surface despite empirical observations that refute the Markovian framework. Finally, this analytic result confirms the results of the recent paper Funahashi and Kijima (2017) when the price associated with a rough stochastic volatility model was computed numerically.

Acknowledgements

This research has been supported in part by Centre Cournot, Fondation Cournot, and Université Paris Saclay (chaire D'Alembert).

A Technical Lemmas

We denote

$$G(z) = \frac{1}{2}(F(z)^2 - \bar{\sigma}^2). \quad (54)$$

The martingale ψ_t^ε defined by (33) has the form

$$\psi_t^\varepsilon = \mathbb{E} \left[\int_0^T G(Z_s^\varepsilon) ds \middle| \mathcal{F}_t \right]. \quad (55)$$

Lemma 2 $(\psi_t^\varepsilon)_{t \in [0, T]}$ is a square-integrable martingale and

$$d \langle \psi^\varepsilon, W \rangle_t = \vartheta_t^\varepsilon dt, \quad \vartheta_t^\varepsilon = \sigma_{\text{ou}} \int_t^T \mathbb{E} [G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathcal{K}^\varepsilon(s-t) ds. \quad (56)$$

Proof See Lemma B.1 in Garnier and Sølna (2016).

The important properties of the random process ϑ_t^ε are stated in the following lemma.

Lemma 3 1. *The exists a constant K_T such that, for any $t \in [0, T]$, we have almost surely*

$$|\sigma_t^\varepsilon \vartheta_t^\varepsilon| \leq K_T \varepsilon^{1/2}. \quad (57)$$

2. *For any $t \in [0, T]$, we have*

$$\mathbb{E}[\sigma_t^\varepsilon \vartheta_t^\varepsilon] = \varepsilon^{1/2} \bar{D} + \tilde{D}_t^\varepsilon, \quad (58)$$

where \bar{D} is the deterministic constant (27) and \tilde{D}_t^ε is smaller than $\varepsilon^{1/2}$:

$$\sup_{\varepsilon \in (0, 1]} \sup_{t \in [0, T]} \varepsilon^{-1/2} |\tilde{D}_t^\varepsilon| < \infty, \quad (59)$$

and

$$\forall t \in [0, T], \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} |\tilde{D}_t^\varepsilon| = 0. \quad (60)$$

3. *For any $0 \leq t < t' < T$, we have*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} |\text{Cov}(\sigma_t^\varepsilon \vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon)| = 0. \quad (61)$$

Proof Using the expression (56) of ϑ_t^ε :

$$|\vartheta_t^\varepsilon \sigma_t^\varepsilon| \leq \sigma_{\text{ou}} \|F\|_\infty \|G'\|_\infty \int_0^\infty |\mathcal{K}^\varepsilon(s)| ds$$

The proof of the first item follows from the fact that $\mathcal{K}^\varepsilon(t) = \mathcal{K}(t/\varepsilon)/\sqrt{\varepsilon}$, $\mathcal{K} \in L^1(0, \infty)$.

The expectation of $\sigma_t^\varepsilon \vartheta_t^\varepsilon$ is equal to

$$\begin{aligned} \mathbb{E}[\sigma_t^\varepsilon \vartheta_t^\varepsilon] &= \sigma_{\text{ou}} \int_t^T \mathbb{E}[F(Z_t^\varepsilon) G'(Z_s^\varepsilon)] \mathcal{K}^\varepsilon(s-t) ds \\ &= \sigma_{\text{ou}} \varepsilon^{1/2} \int_0^{(T-t)/\varepsilon} \mathbb{E}[F(Z_0^\varepsilon) G'(Z_{\varepsilon s}^\varepsilon)] \mathcal{K}(s) ds \\ &= \sigma_{\text{ou}} \varepsilon^{1/2} \int_0^{(T-t)/\varepsilon} \left[\iint_{\mathbb{R}^2} F(\sigma_{\text{ou}} z) G'(\sigma_{\text{ou}} z') p_{\mathcal{C}_{Z(s)}}(z, z') dz dz' \right] \mathcal{K}(s) ds, \end{aligned}$$

with $p_{\mathcal{C}}$ defined in Proposition 1.

Therefore the difference

$$\mathbb{E}[\sigma_t^\varepsilon \vartheta_t^\varepsilon] - \varepsilon^{1/2} \bar{D} = \sigma_{\text{ou}} \varepsilon^{1/2} \int_{(T-t)/\varepsilon}^\infty \mathbb{E}[F(Z_0^\varepsilon) G'(Z_{\varepsilon s}^\varepsilon)] \mathcal{K}(s) ds$$

can be bounded by

$$|\mathbb{E}[\sigma_t^\varepsilon \vartheta_t^\varepsilon] - \varepsilon^{1/2} \bar{D}| \leq \|F\|_\infty \|G'\|_\infty \sigma_{\text{ou}} \varepsilon^{1/2} \int_{(T-t)/\varepsilon}^\infty |\mathcal{K}(s)| ds, \quad (62)$$

which gives the second item since $\mathcal{K} \in L^1(0, \infty)$.

Let us consider $0 \leq t \leq t' \leq T$. We have

$$\begin{aligned} \mathbb{E}[\sigma_t^\varepsilon \vartheta_t^\varepsilon \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon] &= \sigma_{\text{ou}}^2 \int_t^T ds \mathcal{K}^\varepsilon(s-t) \int_{t'}^T ds' \mathcal{K}^\varepsilon(s'-t') \\ &\quad \times \mathbb{E} \left[\mathbb{E}[F(Z_t^\varepsilon) G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[F(Z_{t'}^\varepsilon) G'(Z_{s'}^\varepsilon) | \mathcal{F}_{t'}] \right], \end{aligned}$$

so we can write

$$\begin{aligned} \text{Cov}(\sigma_t^\varepsilon \vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon) &= \sigma_{\text{ou}}^2 \int_t^T ds \mathcal{K}^\varepsilon(s-t) \int_{t'}^T ds' \mathcal{K}^\varepsilon(s'-t') \\ &\quad \times \left(\mathbb{E} \left[\mathbb{E}[F(Z_t^\varepsilon) G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[F(Z_{t'}^\varepsilon) G'(Z_{s'}^\varepsilon) | \mathcal{F}_{t'}] \right] \right. \\ &\quad \left. - \mathbb{E} \left[\mathbb{E}[F(Z_t^\varepsilon) G'(Z_s^\varepsilon) | \mathcal{F}_t] \mathbb{E}[F(Z_{t'}^\varepsilon) G'(Z_{s'}^\varepsilon) | \mathcal{F}_{t'}] \right] \right), \end{aligned}$$

and therefore

$$\begin{aligned} |\text{Cov}(\sigma_t^\varepsilon \vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon)| &\leq \sigma_{\text{ou}}^2 \|F\|_\infty \|G'\|_\infty \int_t^T ds |\mathcal{K}^\varepsilon(s-t)| \int_{t'}^T ds' |\mathcal{K}^\varepsilon(s'-t')| \\ &\quad \times \mathbb{E} \left[\left(\mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t] - \mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)] \right)^2 \right]^{1/2}. \end{aligned}$$

We can write for any $\tau > t$:

$$Z_\tau^\varepsilon = A_{t\tau}^\varepsilon + B_{t\tau}^\varepsilon, \quad A_{t\tau}^\varepsilon = \sigma_{\text{ou}} \int_{-\infty}^t \mathcal{K}^\varepsilon(\tau-u) dW_u, \quad B_{t\tau}^\varepsilon = \sigma_{\text{ou}} \int_t^\tau \mathcal{K}^\varepsilon(\tau-u) dW_u,$$

where $A_{t\tau}^\varepsilon$ is \mathcal{F}_t adapted while $B_{t\tau}^\varepsilon$ is independent from \mathcal{F}_t . Therefore ($s' \geq t' \geq t$)

$$\begin{aligned} &\mathbb{E} \left[\left(\mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t] - \mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)] \right)^2 \right] \\ &= \mathbb{E} \left[\mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t]^2 \right] - \mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)]^2 \\ &= \mathbb{E} \left[F(A_{tt'}^\varepsilon + B_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + B_{ts'}^\varepsilon)F(A_{tt'}^\varepsilon + \tilde{B}_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + \tilde{B}_{ts'}^\varepsilon) \right. \\ &\quad \left. - F(A_{tt'}^\varepsilon + B_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + B_{ts'}^\varepsilon)F(\tilde{A}_{tt'}^\varepsilon + \tilde{B}_{tt'}^\varepsilon)G'(\tilde{A}_{ts'}^\varepsilon + \tilde{B}_{ts'}^\varepsilon) \right], \end{aligned}$$

where $(\tilde{A}_{tt'}^\varepsilon, \tilde{B}_{tt'}^\varepsilon, \tilde{A}_{ts'}^\varepsilon, \tilde{B}_{ts'}^\varepsilon)$ is an independent copy of $(A_{tt'}^\varepsilon, B_{tt'}^\varepsilon, A_{ts'}^\varepsilon, B_{ts'}^\varepsilon)$. We can then write

$$\begin{aligned} &\mathbb{E} \left[\left(\mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t] - \mathbb{E}[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)] \right)^2 \right] \\ &\leq \|F\|_\infty \|G'\|_\infty \mathbb{E} \left[\left(F(A_{tt'}^\varepsilon + \tilde{B}_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + \tilde{B}_{ts'}^\varepsilon) - F(\tilde{A}_{tt'}^\varepsilon + \tilde{B}_{tt'}^\varepsilon)G'(\tilde{A}_{ts'}^\varepsilon + \tilde{B}_{ts'}^\varepsilon) \right)^2 \right]^{1/2} \\ &\leq C \left(\mathbb{E}[(A_{tt'}^\varepsilon - \tilde{A}_{tt'}^\varepsilon)^2]^{1/2} + \mathbb{E}[(A_{ts'}^\varepsilon - \tilde{A}_{ts'}^\varepsilon)^2]^{1/2} \right) \\ &\leq 2C \left(\mathbb{E}[(A_{tt'}^\varepsilon)^2]^{1/2} + \mathbb{E}[(A_{ts'}^\varepsilon)^2]^{1/2} \right) \\ &\leq 2C \left[\left(\sigma_{\text{ou}}^2 \int_{-\infty}^t \mathcal{K}^\varepsilon(t-u)^2 du \right)^{1/2} + \left(\sigma_{\text{ou}}^2 \int_{-\infty}^t \mathcal{K}^\varepsilon(s'-u)^2 du \right)^{1/2} \right] \\ &\leq 4C \sigma_{t'-t, \infty}^\varepsilon \leq C_1 (1 \wedge (\varepsilon/(t'-t))^{1-H}), \end{aligned}$$

where we used Lemma 7 in the last inequality. Then, using the fact that $\mathcal{K} \in L^1$, this gives

$$\begin{aligned} |\text{Cov}(\sigma_t^\varepsilon \vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon)| &\leq C_2 \int_t^T ds |\mathcal{K}^\varepsilon(s-t)| \int_{t'}^T ds' |\mathcal{K}^\varepsilon(s'-t')| (1 \wedge (\varepsilon/(t'-t)))^{(1-H)/2} \\ &\leq C_3 \varepsilon (1 \wedge (\varepsilon/(t'-t)))^{(1-H)/2}, \end{aligned}$$

which proves the third item.

Lemma 4 *For any smooth function f with bounded derivative, we have*

$$\text{Var}(\mathbb{E}[f(Z_t^\varepsilon)|\mathcal{F}_0]) \leq \|f'\|_\infty^2 (\sigma_{t, \infty}^\varepsilon)^2. \quad (63)$$

Proof The conditional distribution of Z_t^ε given \mathcal{F}_0 is Gaussian with mean

$$\mathbb{E}[Z_t^\varepsilon|\mathcal{F}_0] = \sigma_{\text{ou}} \int_{-\infty}^0 \mathcal{K}^\varepsilon(t-u) dW_u$$

and variance

$$\text{Var}(Z_t^\varepsilon|\mathcal{F}_0) = (\sigma_{0,t}^\varepsilon)^2 = \sigma_{\text{ou}}^2 \int_0^t \mathcal{K}^\varepsilon(u)^2 du.$$

Therefore

$$\text{Var}(\mathbb{E}[f(Z_t^\varepsilon)|\mathcal{F}_0]) = \text{Var}\left(\int_{\mathbb{R}} f(\mathbb{E}[Z_t^\varepsilon|\mathcal{F}_0] + \sigma_{0,t}^\varepsilon z)p(z)dz\right),$$

where $p(z)$ is the pdf of the standard normal distribution. The random variable $\mathbb{E}[Z_t^\varepsilon|\mathcal{F}_0]$ is Gaussian with mean zero and variance $(\sigma_{t,\infty}^\varepsilon)^2$ so that

$$\begin{aligned} \text{Var}(\mathbb{E}[f(Z_t^\varepsilon)|\mathcal{F}_0]) &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} dz dz' p(z)p(z') \int_{\mathbb{R}} \int_{\mathbb{R}} du du' p(u)p(u') \\ &\quad \times \left[f(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z) - f(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z) \right] \\ &\quad \times \left[f(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z') - f(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z') \right] \\ &\leq \|f'\|_\infty^2 (\sigma_{t,\infty}^\varepsilon)^2 \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} du du' p(u)p(u') (u - u')^2 \\ &= \|f'\|_\infty^2 (\sigma_{t,\infty}^\varepsilon)^2, \end{aligned}$$

which is the desired result.

The random term ϕ_t^ε defined by (32) has the form

$$\phi_t^\varepsilon = \mathbb{E}\left[\int_t^T G(Z_s^\varepsilon)ds|\mathcal{F}_t\right], \quad (64)$$

with G defined in (54).

Lemma 5 For any $t \leq T$, ϕ_t^ε is a zero-mean random variable with standard deviation of order ε^{1-H} :

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in [0,T]} \varepsilon^{2H-2} \mathbb{E}[(\phi_t^\varepsilon)^2] < \infty. \quad (65)$$

Proof For $t \in [0, T]$ the second moment of ϕ_t^ε is:

$$\begin{aligned} \mathbb{E}[(\phi_t^\varepsilon)^2] &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T G(Z_s^\varepsilon)ds|\mathcal{F}_t\right]^2\right] \\ &= \int_0^{T-t} ds \int_0^{T-t} ds' \text{Cov}(\mathbb{E}[G(Z_s^\varepsilon)|\mathcal{F}_0], \mathbb{E}[G(Z_{s'}^\varepsilon)|\mathcal{F}_0]). \end{aligned}$$

We have by Lemma 4

$$\mathbb{E}[(\phi_t^\varepsilon)^2] \leq \left(\int_0^{T-t} ds \text{Var}(\mathbb{E}[G(Z_s^\varepsilon)|\mathcal{F}_0])^{1/2}\right)^2 \leq \|G'\|_\infty^2 \left(\int_0^{T-t} ds \sigma_{s,\infty}^\varepsilon\right)^2.$$

In view of Lemma 7 we then have

$$\mathbb{E}[(\phi_t^\varepsilon)^2] \leq C_T (\varepsilon + \varepsilon^{1-H})^2 \leq 4C_T \varepsilon^{2-2H},$$

uniformly in $t \leq T$ and $\varepsilon \in (0, 1]$ for some constant C_T .

Lemma 6 Let us define for any $t \in [0, T]$:

$$\kappa_t^\varepsilon = \frac{\varepsilon^{1/2}}{2} \int_0^t ((\sigma_s^\varepsilon)^2 - \bar{\sigma}^2) ds = \varepsilon^{1/2} \int_0^t G(Z_s^\varepsilon) ds, \quad (66)$$

as in (41). We have

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0,T]} \varepsilon^{-1/2} \mathbb{E}[(\kappa_t^\varepsilon)^2]^{1/2} = 0. \quad (67)$$

Proof Since the expectation $\mathbb{E}[G(Z_0^\varepsilon)] = 0$, we have

$$\mathbb{E}[(\kappa_t^\varepsilon)^2] = \varepsilon \mathbb{E} \left[\left(\int_0^t G(Z_s^\varepsilon) ds \right)^2 \right] = 2\varepsilon \int_0^t ds (t-s) \text{Cov}(G(Z_s^\varepsilon), G(Z_0^\varepsilon)) ds.$$

We have moreover

$$\begin{aligned} |\text{Cov}(G(Z_s^\varepsilon), G(Z_0^\varepsilon))| &= |\mathbb{E}[(\mathbb{E}[G(Z_s^\varepsilon)|\mathcal{F}_0] - \mathbb{E}[G(Z_s^\varepsilon)])G(Z_0^\varepsilon)]| \\ &\leq \|G\|_\infty \text{Var}(\mathbb{E}[G(Z_s^\varepsilon)|\mathcal{F}_0])^{1/2}. \end{aligned}$$

By Lemma 4 we obtain

$$|\text{Cov}(G(Z_s^\varepsilon), G(Z_0^\varepsilon))| \leq \|G\|_\infty \|G'\|_\infty \sigma_{s,\infty}^\varepsilon.$$

In view of Lemma 7 we then have

$$\mathbb{E}[(\kappa_t^\varepsilon)^2] \leq C_T \varepsilon (\varepsilon + \varepsilon^{1-H}) \leq 2C_T \varepsilon^{2-H},$$

uniformly in $t \in [0, T]$ and $\varepsilon \in (0, 1]$, which gives the desired result.

Lemma 7 *Define*

$$\sigma_{t,\infty}^\varepsilon = \sigma_{\text{ou}} \left(\int_t^\infty \mathcal{K}^\varepsilon(s)^2 ds \right)^{1/2}, \quad (68)$$

Then there exists $C > 0$ such that

$$\sigma_{t,\infty}^\varepsilon \leq C(1 \wedge (\varepsilon/t)^{1-H}). \quad (69)$$

Proof This follows from $|\mathcal{K}(s)| \leq Ks^{H-\frac{3}{2}}$ for $s \geq 1$ and $\mathcal{K} \in L^2$.

B An Alternative Model

In Comte and Renault (1998); Funahashi and Kijima (2017) the authors consider a stochastic volatility model that is a kind of fractional Orstein-Uhlenbeck process, but they consider the following representation of the fractional Brownian motion:

$$W_t^{H,0} = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} dW_s^0, \quad (70)$$

where $(W_t^0)_{t \in \mathbb{R}^+}$ is a standard Brownian motion over \mathbb{R}^+ . $(W_t^{H,0})_{t \in \mathbb{R}^+}$ is a zero-mean self-similar Gaussian process, in the sense that $(\alpha^H W_{t/\alpha}^{H,0})_{t \in \mathbb{R}^+}$ and $(W_t^{H,0})_{t \in \mathbb{R}^+}$ have the same distribution, but it is not stationary, nor does it have stationary increments. Its variance is

$$\mathbb{E}[(W_t^{H,0})^2] = \frac{1}{2H\Gamma(H + \frac{1}{2})^2} t^{2H},$$

while the variance of its increment is (for $s > 0$):

$$\mathbb{E}[(W_{t+s}^{H,0} - W_t^{H,0})^2] = \frac{1}{\Gamma(H + \frac{1}{2})^2} \left[\int_0^{t/s} ((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}})^2 du + \frac{1}{2H} \right] s^{2H},$$

which has the following behavior

$$\mathbb{E}[(W_{t+s}^{H,0} - W_t^{H,0})^2] \xrightarrow{t \rightarrow +\infty} \frac{1}{\Gamma(2H+1) \sin(\pi H)} s^{2H}.$$

This model is special because time zero plays a special role, and we think it is desirable to deal with the stationary situation addressed in this paper. However, it turns out that the two models give the same result in the fast-varying case. Indeed, the modified fOU process corresponding to (70) is (to be compared with (4)):

$$\begin{aligned} Z_t^{\varepsilon,0} &= Z_0 e^{-t/\varepsilon} + \varepsilon^{-H} \int_0^t e^{-\frac{t-s}{\varepsilon}} dW_s^{H,0} \\ &= Z_0 e^{-t/\varepsilon} + \varepsilon^{-H} W_t^{H,0} - \varepsilon^{-H-1} \int_0^t e^{-\frac{t-s}{\varepsilon}} W_s^{H,0} ds, \end{aligned} \quad (71)$$

where Z_0 is considered as a constant as in Comte and Renault (1998); Funahashi and Kijima (2017). In terms of the Brownian motion W_t^0 this reads:

$$Z_t^{\varepsilon,0} = Z_0 e^{-t/\varepsilon} + \sigma_{\text{ou}} \int_0^t \mathcal{K}^\varepsilon(t-s) dW_s^0, \quad (72)$$

where \mathcal{K}^ε is defined in (10). It is a Gaussian process with the following covariance ($t, s \geq 0$):

$$\text{Cov}(Z_t^{\varepsilon,0}, Z_{t+s}^{\varepsilon,0}) = \sigma_{\text{ou}}^2 \mathcal{C}_{t/\varepsilon}^0\left(\frac{s}{\varepsilon}\right),$$

that is a function of t/ε and s/ε with

$$\mathcal{C}_t^0(s) = \frac{\int_0^t \mathcal{K}(u)\mathcal{K}(u+s)du}{\int_0^\infty \mathcal{K}(u)^2 du}.$$

Note that

$$\mathcal{C}_t^0(s) \xrightarrow{t \rightarrow +\infty} \frac{\int_0^\infty \mathcal{K}(u)\mathcal{K}(u+s)du}{\int_0^\infty \mathcal{K}(u)^2 du} = \mathcal{C}_Z(s),$$

with \mathcal{C}_Z defined by (6). In other words, except for a small period of time just after time 0 which is of duration of the order of ε , the modified process has the same behavior as the one introduced in this paper. One can then check the detailed calculations carried out in this paper and find that Proposition 1 still holds true with the modified model $Z_t^{\varepsilon,0}$.

References

- E. Alòs and Y. Yang, A closed-form option pricing approximation formula for a fractional Heston model, Economics Working Papers 1446, Department of Economics and Business, Universitat Pompeu Fabra. [18](#)
- C. Bayer, P. K. Friz, A. Gulisashvili, B. Horvath, and B. Stemper, Short-time near-the-money skew in rough fractional volatility models, arXiv:1703.05132. [3](#)
- S. Benaim, P. Friz, and R. Lee, On Black–Scholes Implied Volatility at Extreme Strikes, *Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling*, (2010). [3](#)
- M. Bennedsen, Rough electricity: a new fractal multi-factor model of electricity spot prices. Working paper (2015), available at <http://ssrn.com/abstract=2636829>. [4](#)
- T. Bollerslev, D. Osterrieder, N. Sizova, and G. Tauchen, Risk and return: Long-run relations, fractional cointegration, and return predictability, *Journal of Financial Economics* **108** (2013), pp. 409–424. [2](#)
- F. J. Breidt, N. Crato, and P. De Lima, The detection and estimation of long-memory in stochastic volatility, *Journal of Econometrics* **83** (1998), pp. 325–348. [2](#)
- L. Charfeddine, True or spurious long memory in volatility: Further evidence on the energy futures markets, *Energy Policy* **71** (2014), pp. 76–93. [4](#)
- K. C. Chia, A. Bahar, I. L. Kane, C.-M. Ting, and H. A. Rahman, Estimation of stochastic volatility with long memory for index prices of FTSE Bursa Malaysia KLCI, *AIP Conference Proceedings* 1643, 73 (2015). [4](#)

- A. Chronopoulou and F. G. Viens, Estimation and pricing under long-memory stochastic volatility, *Annals of Finance* **8** (2012), pp. 379–403. [2](#), [4](#)
- F. Comte and E. Renault, Long memory in continuous-time stochastic volatility models, *Mathematical Finance* **8** (1998), pp. 291–323. [24](#), [25](#)
- R. Cont, Empirical properties of asset returns: Stylized facts and statistical issues, *Quant. Finance* **1** (2001), pp. 1–14. [2](#)
- R. Cont, Long range dependence in financial markets, in *Fractals in Engineering*, edited by J. Lévy Véhel and E. Lutton, Springer, London, 2005, pp. 159–179. [2](#)
- R. F. Engle and A. J. Patton, What good is a volatility model?, *Quantitative Finance* **1** (2001), pp. 237–245. [2](#)
- J.-P. Fouque, G. Papanicolaou, and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, 2000. [10](#), [17](#)
- J.-P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Sølna, *Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives*, Cambridge University Press, Cambridge, 2011. [3](#), [4](#), [10](#), [17](#)
- J.-P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Sølna, Timing the smile, *The Wilmott Magazine*, March 2004. [4](#), [17](#)
- J.-P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Sølna, Short time scales in S&P500 volatility, *The Journal of Computational Finance* **6** (2003), pp. 1–24. [3](#), [4](#), [17](#)
- J.-P. Fouque and R. Hu, Optimal portfolio under fractional stochastic environment, arXiv:1703.06969. [4](#)
- J.-P. Fouque and R. Hu, Optimal portfolio under fast mean-reverting fractional stochastic environment, arXiv:1706.03139. [4](#)
- M. Fukasawa, Short-time at-the-money skew and rough fractional volatility, *Quantitative Finance* **17** (2017), pp. 189–198. [3](#), [11](#)
- H. Funahashi and M. Kijima, Does the Hurst index matter for option prices under fractional volatility? *Ann Finance* **13** (2017), pp. 55–74. [2](#), [20](#), [24](#), [25](#)
- J. Garnier and K. Sølna, Correction to Black–Scholes formula due to fractional stochastic volatility, *SIAM J. Finan. Math.* **8** (2017), pp. 560–588. [3](#), [4](#), [9](#), [10](#), [17](#), [18](#), [19](#), [20](#)
- J. Garnier and K. Sølna, Option pricing under fast-varying long-memory stochastic volatility, to appear in *Mathematical Finance*, arXiv:1604.00105. [3](#), [4](#), [5](#), [17](#), [18](#), [19](#), [20](#)
- J. Gatheral, T. Jaisson, and M. Rosenbaum, Volatility is rough, arXiv:1410.3394. [2](#), [4](#)
- A. Gulisashvili, F. Viens, and X. Zhang, Small-time asymptotics for Gaussian self-similar stochastic volatility models, arXiv:1505.05256. [3](#)
- S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *The Review of Financial Studies* **6** (1993), pp. 327–343. [3](#), [18](#)
- A. Jacquier, C. Martini, and A. Muguruza, On VIX futures in the rough Bergomi model, arXiv:1701.04260. [19](#)
- K. Kalbasi, T. S. Mountford, and F. G. Viens, Anderson polymer in a fractional Brownian environment: Asymptotic behavior of the partition function, *J. Theor. Probab.* (2017). <https://doi.org/10.1007/s10959-017-0756-2>. [5](#)
- G. Livieri, S. Mouti, A. Pallavicini, and M. Rosenbaum, Rough volatility: evidence from option prices, arxiv:1702.02777. [19](#)
- B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Review* **10** (1968), pp. 422–437. [6](#)
- A. A. Muravlev, Representation of fractional Brownian motion in terms of an infinite-dimensional Ornstein–Uhlenbeck process, *Russ. Math. Surv.* **66** (2011), pp. 439–441. [11](#)
- G. Oh, S. Kim, and C. Eom, Long-term memory and volatility clustering in high-frequency price changes, *Physica A: Statistical Mechanics and its Applications*, **387** (2008), pp. 1247–1254. [2](#)
- M. Rypdal and O. Løvsletten, Modeling electricity spot prices using mean-reverting multifractal processes, *Physica A* **392** (2013), pp. 194–207. [4](#)
- I. Simonsen, Measuring anti-correlations in the nordic electricity spot market by wavelets, *Physica A* **233** (2002), pp. 597–606. [4](#)
- T. Walther, T. Klein, H.P. Thu, and K. Piontek, True or spurious long memory in European non-EMU currencies, *Research in International Business and Finance* **40** (2017), pp. 217–230. [4](#)