Stochastics and Dynamics

Apparent Attenuation of Shear Waves Propagating through a Randomly Stratified Anisotropic Medium

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Waves propagating through heterogeneous media experience scattering that can convert a coherent pulse into small incoherent fluctuations. This may appear as attenuation for the transmitted front pulse. The classic O'Doherty-Anstey theory describes such a transformation for scalar waves in finely layered media. Recent observations for seismic waves in the earth suggest that this theory can explain a significant component of seismic attenuation. An important question to answer is then how the O'Doherty-Anstey theory generalizes to seismic waves when several wave modes, possibly with the same velocity, interact. An important aspect of the O'Doherty-Anstey theory is the statistical stability property, which means that the transmitted front pulse is actually deterministic and depends only on the statistics of the medium but not on the particular medium realization when the medium is modeled as a random process. It is shown in this paper that this property generalizes in the case of elastic waves in a nontrivial way: the energy of the transmitted front pulse, but not the pulse shape itself, is statistically stable. This result is based on a separation of scales technique and a diffusion-approximation theorem that characterize the transmitted front pulse as the solution of a stochastic partial differential equation driven by two Brownian motions.

Keywords: Waves in random media; diffusion-approximation; seismic attenuation; anisotropy.

AMS Subject Classification: 60H15, 35R60, 74J20

1. Introduction

In this paper we analyze the propagation of shear waves in a randomly stratified transverse anisotropic elastic medium. Our motivation is twofold:
- it has been known for a long time in the geophysical literature that the laminated structure of the earth can explain apparent attenuation of seismic waves due to multiple scattering [13, 16]. More recently it was suggested that, even in a medium

where the density and velocities are constant, anisotropic heterogeneities could induce elastic attenuation, and therefore it could be responsible for a significant part of the observed attenuation of seismic waves [15].
- it is explained in the mathematical literature that a pulse traveling through a randomly laminated medium undergoes two transformations: a random time shift and a deterministic deformation.
The result that the transmitted pulse shape is deterministic (up to a random time shift) is called pulse stabilization in the literature. It was predicted in geophysics by O'Doherty and Anstey [13]. It was proved for the scalar wave equation using a time-domain integral equation approach in [2, 3] and using a frequency-domain approach in [1, 5, 6]. The martingale representation established in [7] allowed to revisit and clarify the pulse stabilization in the frequency-domain approach. Finally the frequency-domain approach was used again in [11] to extend the result to a general hyperbolic system. However, an hypothesis is required in the case of the general hyperbolic system, namely that the wave mode velocities are distinct [11]. This hypothesis is not fulfilled in the situation with anisotropic heterogeneities and elastic waves addressed in this paper.

We therefore ask the question to what extent the O'Doherty-Anstey theory generalizes when we consider anisotropic elastic waves corresponding to a situation with wave velocities that coincide. It turns out that such wave scattering gives rise to more complex effects: there is no pulse stabilization and the transmitted front pulse exhibits a random deformation and not a deterministic one. Only a careful stochastic and multiscale analysis reveal this behavior. The random shape of the transmitted front pulse can be analyzed by diffusion-approximation theorems and its statistical distribution can be characterized in terms of a stochastic partial differential equation driven by two Brownian motions (Proposition 6.1 and Corollary 6.1). However the pulse stabilization generalizes in the sense that the energy of the deformed front pulse is deterministic and exhibits a deterministic apparent attenuation that is related to the anisotropy level and to the correlation length of the fluctuations of the symmetry axis (Proposition 7.1). Thus, there is coherent energy stabilization, but not pulse stabilization. This corroborates the physical conjecture that scattering may partly explain observed seismic attenuation.

The paper is organized as follows: In Section 2 we introduce the anisotropic elastic wave equation. In Section 3 we introduce the mode decomposition for the wave field. We describe the scaling regime addressed in this paper in Section 4. We apply a diffusion-approximation theorem in Section 5 in order to describe the asymptotic statistical distribution of the transmitted front pulse in Section 6. We characterize the stability of the energy of the transmitted front pulse in Section 7.

2. Formulation

We consider linear elastic waves propagating in a three-dimensional anisotropic medium. The spatial coordinates are denoted by $\boldsymbol{x} = (x_i)_{i=1}^3$. The medium can be characterized by its density ρ and its elastic tensor $\mathbf{c}$. They are $\boldsymbol{x}$-dependent quantities when the medium is heterogeneous. The elastic tensor $\mathbf{c} = (c_{ijkl})_{i,j,k,l=1}^3$ in a transverse anisotropic medium has five independent components A, N, F, C, L and it can be written in the form [4, 18]

$$\begin{aligned} c_{ijkl} = & (A-2N)\delta_{ij}\delta_{kl} + N\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) \\ & +(F-A+2N)\big(\delta_{ij}s_k s_l + \delta_{kl}s_i s_j\big) \\ & +(L-N)\big(\delta_{ik}s_j s_l + \delta_{il}s_j s_k + \delta_{jk}s_i s_l + \delta_{jl}s_i s_k\big) \\ & +(A+C-2F-4L)s_i s_j s_k s_l, \end{aligned} \tag{2.1}$$

where $\boldsymbol{s} = (s_i)_{i=1}^3$ is the unit vector representing the orientation of the symmetry axis and δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$ and 0 otherwise). Furthermore, the five parameters can be expressed as $A = C_{11}$, $C = C_{33}$, $L = C_{44}$, $N = C_{66}$, and $F = C_{13}$, in Voigt's abbreviated notation, when the symmetry axis is parallel to the x_1-axis [4, 18].

In this paper we assume that the anisotropy is heterogeneous, and that the symmetry axis vector $\boldsymbol{s}$ is in the plane $(x_1 x_2)$ with an angle $\psi(x_3)$ with the x_1-axis:

$$s_1 = \cos\psi(x_3), \qquad s_2 = \sin\psi(x_3), \qquad s_3 = 0. \tag{2.2}$$

All other parameters (A, N, F, C, L) are assumed to be constant (i.e. $\boldsymbol{x}$-independent). This is a classical model in the geophysics literature [15, 16].

By Hook's law the elastic tensor $\mathbf{c}(\boldsymbol{x})$ relates the symmetric stress tensor $\boldsymbol{\sigma}(t, \boldsymbol{x})$, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^3$, to the symmetric strain tensor $\boldsymbol{\varepsilon}(t, \boldsymbol{x})$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1}^3$:

$$\sigma_{ij} = \sum_{k,l=1}^3 c_{ijkl}\varepsilon_{kl}, \tag{2.3}$$

where the strain tensor is defined by

$$\varepsilon_{kl} = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big) \tag{2.4}$$

in terms of the displacement vector $\boldsymbol{u}(t, \boldsymbol{x})$, $\boldsymbol{u} = (u_i)_{i=1}^3$. The conservation of the linear momentum reads [9]

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sum_{k=1}^3 \frac{\partial \sigma_{ki}}{\partial x_k}, \qquad i = 1, \ldots, 3, \tag{2.5}$$

where ρ is the density of the medium (assumed to be constant). We will consider a shear plane wave propagating along the x_3-direction, perpendicular to the laminated structure. The wave equation for this shear wave is then:

$$\rho\frac{\partial^2}{\partial t^2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{\partial}{\partial x_3}\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix}, \tag{2.6}$$

with

$$\sigma_{13} = \big[N + (L-N)\cos^2\psi(x_3)\big]\frac{\partial u_1}{\partial x_3} + \big[(L-N)\sin\psi(x_3)\cos\psi(x_3)\big]\frac{\partial u_2}{\partial x_3}, \tag{2.7}$$

$$\sigma_{23} = \big[N + (L-N)\sin^2\psi(x_3)\big]\frac{\partial u_2}{\partial x_3} + \big[(L-N)\sin\psi(x_3)\cos\psi(x_3)\big]\frac{\partial u_1}{\partial x_3}. \tag{2.8}$$

In absence of anisotropy $L = N = K$ (where K is the constant bulk modulus of the medium), we get two independent scalar wave equations:

$$\rho\frac{\partial^2 u_i}{\partial t^2} - K\frac{\partial^2 u_i}{\partial x_3^2} = 0, \qquad i = 1, 2, \tag{2.9}$$

which correspond to two shear waves propagating with the velocity $c = \sqrt{K/\rho}$.

In this paper we address the anisotropic case $L \neq N$ and assume that the angle $\psi(x_3)$ is randomly varying and can be modeled by a stationary and ergodic process whose stationary distribution is uniform over $[0, 2\pi]$. Precise assumptions to ensure that we can apply diffusion-approximation theorems are that $\psi(x_3)$ is a Markov process on the torus with generator satisfying the Fredholm alternative [6] or that $\psi(x_3)$ is a ϕ-mixing process with $\phi \in L^{1/2}$ [10] .

3. Mode Decomposition

From now on the longitudinal spatial variable x_3 is denoted by z. We define the Fourier transform of a function $f(t)$ by:

$$\hat f(\omega) = \int f(t)e^{i\omega t}dt, \qquad f(t) = \frac{1}{2\pi}\int \hat f(\omega)e^{-i\omega t}d\omega.$$

We introduce the two-dimensional vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$:

$$\hat{\boldsymbol{u}}(z,\omega) = \begin{pmatrix}\hat u_1\\ \hat u_2\end{pmatrix},$$

$$\hat{\boldsymbol{v}}(z,\omega) = \frac{1}{2i\omega}\Big[(N+L)\begin{pmatrix}1&0\\0&1\end{pmatrix} + (L-N)\begin{pmatrix}\cos 2\psi & \sin 2\psi\\ \sin 2\psi & \cos 2\psi\end{pmatrix}\Big]\frac{\partial}{\partial z}\begin{pmatrix}\hat u_1\\ \hat u_2\end{pmatrix}.$$

The four-dimensional vector $(\hat u_1, \hat u_2, \hat v_1, \hat v_2)^t$ satisfies the linear system

$$\frac{\partial}{\partial z}\begin{pmatrix}\hat{\boldsymbol{u}}\\ \hat{\boldsymbol{v}}\end{pmatrix} = i\omega\begin{pmatrix}\mathbf{0} & \frac{1}{K}\mathbf{I} + \frac{\kappa}{K}\mathbf{J}(z)\\ \rho\mathbf{I} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{u}}\\ \hat{\boldsymbol{v}}\end{pmatrix}, \tag{3.1}$$

where the superscript ^{t} stands for transpose, $\mathbf{0}$, $\mathbf{I}$, and $\mathbf{J}(z)$ are the 2×2 matrices

$$\mathbf{0} = \begin{pmatrix}0&0\\0&0\end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \mathbf{J}(z) = \begin{pmatrix}\cos 2\psi(z) & \sin 2\psi(z)\\ \sin 2\psi(z) & -\cos 2\psi(z)\end{pmatrix}, \tag{3.2}$$

and

$$\frac{1}{K} = \frac{N+L}{2LN}, \qquad \kappa = \frac{N-L}{N+L}. \tag{3.3}$$

K is the bulk modulus and κ is the anisotropy coefficient of the medium. Note that the orthogonal operator $\mathbf{J}(z)$ can be interpreted as the composition of a reflection operator and a rotation operator.

Let us introduce the impedance and velocity parameters

$$\zeta = \sqrt{K\rho}, \qquad c = \frac{\zeta}{\rho} = \sqrt{\frac{K}{\rho}}, \tag{3.4}$$

and the mode decomposition

$$\hat{\boldsymbol{a}}(z,\omega) = e^{-i\frac{\omega}{c}z}\big(\zeta^{1/2}\hat{\boldsymbol{u}}(z,\omega) + \zeta^{-1/2}\hat{\boldsymbol{v}}(z,\omega)\big), \tag{3.5}$$

$$\hat{\boldsymbol{b}}(z,\omega) = e^{i\frac{\omega}{c}z}\big(-\zeta^{1/2}\hat{\boldsymbol{u}}(z,\omega) + \zeta^{-1/2}\hat{\boldsymbol{v}}(z,\omega)\big), \tag{3.6}$$

so that

$$\hat{\boldsymbol{u}}(z,\omega) = \frac{1}{2}\zeta^{-1/2}\big(\hat{\boldsymbol{a}}(z,\omega)e^{i\frac{\omega}{c}z} - \hat{\boldsymbol{b}}(z,\omega)e^{-i\frac{\omega}{c}z}\big), \tag{3.7}$$

$$\hat{\boldsymbol{v}}(z,\omega) = \frac{1}{2}\zeta^{1/2}\big(\hat{\boldsymbol{a}}(z,\omega)e^{i\frac{\omega}{c}z} + \hat{\boldsymbol{b}}(z,\omega)e^{-i\frac{\omega}{c}z}\big). \tag{3.8}$$

In absence of anisotropy $\kappa = 0$ the complex mode amplitudes $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ are constant in z. Here $\hat{\boldsymbol{a}}$, resp. $\hat{\boldsymbol{b}}$, are the amplitudes of the wave modes that propagate towards increasing, resp. decreasing, z.
In the presence of anisotropy $\kappa \neq 0$ the complex mode amplitudes satisfy the linear system:

$$\frac{\partial}{\partial z}\begin{pmatrix}\hat{\boldsymbol{a}}\\ \hat{\boldsymbol{b}}\end{pmatrix} = i\frac{\omega\kappa}{2c}\begin{pmatrix}\mathbf{J}(z) & \mathbf{J}(z)e^{-i\frac{2\omega}{c}z}\\ -\mathbf{J}(z)e^{i\frac{2\omega}{c}z} & -\mathbf{J}(z)\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{a}}\\ \hat{\boldsymbol{b}}\end{pmatrix}. \tag{3.9}$$

The complex mode amplitudes also satisfy the following boundary conditions:

$$\hat{\boldsymbol{a}}(z=0,\omega) = \hat{\boldsymbol{f}}(\omega) = \begin{pmatrix}\hat{f}_1(\omega)\\ \hat{f}_2(\omega)\end{pmatrix}, \qquad \hat{\boldsymbol{b}}(z=L,\omega) = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{3.10}$$

which correspond to a shear wave incoming at $z = 0$ and radiation condition at $z = L$.

4. The Weakly Anisotropic Regime

We assume from now on that anisotropy is weak and that the correlation length of the fluctuations of the symmetry axis and the typical wavelength of the incoming wave are much smaller than the propagation distance L. We introduce the small dimensionless parameter ε to characterize this scaling regime:

$$\kappa = \varepsilon\tilde{\kappa}, \qquad \psi(z) = \tilde{\psi}\Big(\frac{z}{\varepsilon^2}\Big), \qquad f(t) = \tilde{f}\Big(\frac{t}{\varepsilon^2}\Big),$$

and we drop the tilde in the following. We introduce the scaled Fourier transform for a function $f(t)$:

$$\hat{f}^\varepsilon(\omega) = \frac{1}{\varepsilon^2}\int f(t) e^{i\frac{\omega}{\varepsilon^2}t} dt, \qquad f(t) = \frac{1}{2\pi}\int \hat{f}^\varepsilon(\omega) e^{-i\frac{\omega}{\varepsilon^2}t} d\omega.$$

The transmitted wave at $z = L$ observed around the expected arrival time L/c is

$$\begin{aligned} \boldsymbol{u}_{\rm tr}^\varepsilon(\tau) &= \boldsymbol{u}\Big(z = L, t = \frac{L}{c} + \varepsilon^2\tau\Big) \\ &= \frac{1}{2\pi}\int \hat{\boldsymbol{u}}^\varepsilon(z = L,\omega) e^{-i\frac{\omega L}{\varepsilon^2 c}} e^{-i\omega\tau} d\omega \\ &= \frac{1}{4\pi\zeta^{1/2}}\int \hat{\boldsymbol{a}}^\varepsilon(z = L,\omega) e^{-i\omega\tau} d\omega, \end{aligned} \tag{4.1}$$

where the complex mode amplitudes $\hat{\boldsymbol{a}}^\varepsilon$ and $\hat{\boldsymbol{b}}^\varepsilon$ satisfy

$$\frac{\partial}{\partial z}\begin{pmatrix}\hat{\boldsymbol{a}}^\varepsilon \\ \hat{\boldsymbol{b}}^\varepsilon\end{pmatrix} = i\frac{\omega\kappa}{2\varepsilon c}\begin{pmatrix} \mathbf{J}(\frac{z}{\varepsilon^2}) & \mathbf{J}(\frac{z}{\varepsilon^2}) e^{-i\frac{2\omega}{\varepsilon^2 c}z} \\ -\mathbf{J}(\frac{z}{\varepsilon^2}) e^{i\frac{2\omega}{\varepsilon^2 c}z} & -\mathbf{J}(\frac{z}{\varepsilon^2}) \end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{a}}^\varepsilon \\ \hat{\boldsymbol{b}}^\varepsilon\end{pmatrix}, \tag{4.2}$$

with the boundary conditions

$$\hat{\boldsymbol{a}}^\varepsilon(z = 0,\omega) = \hat{\boldsymbol{f}}(\omega), \qquad \hat{\boldsymbol{b}}^\varepsilon(z = L,\omega) = \begin{pmatrix}0\\0\end{pmatrix}. \tag{4.3}$$

Using (4.2) we find that the complex mode amplitudes $\hat{\boldsymbol{a}}^\varepsilon$ and $\hat{\boldsymbol{b}}^\varepsilon$ satisfy

$$\frac{\partial}{\partial z}\big(\|\hat{\boldsymbol{a}}^\varepsilon(z,\omega)\|^2 - \|\hat{\boldsymbol{b}}^\varepsilon(z,\omega)\|^2\big) = 0, \tag{4.4}$$

where $\|\hat{\boldsymbol{a}}\| = \sqrt{|\hat{a}_1|^2 + |\hat{a}_2|^2}$ is the Euclidean norm, which shows, using also the boundary conditions (4.3), that

$$\|\hat{\boldsymbol{a}}^\varepsilon(L,\omega)\|^2 + \|\hat{\boldsymbol{b}}^\varepsilon(0,\omega)\|^2 = \|\hat{\boldsymbol{f}}(\omega)\|^2. \tag{4.5}$$

This can be be interpreted as a conservation of energy relation: the sum of the square moduli of the transmitted ($\hat{\boldsymbol{a}}^\varepsilon(L,\omega)$) and reflected ($\hat{\boldsymbol{b}}^\varepsilon(0,\omega)$) wave mode amplitudes is equal to the sum of the square moduli of the incident ($\hat{\boldsymbol{f}}(\omega)$) wave mode amplitudes.

We introduce the 4×4 propagator matrix $\mathbf{P}^\varepsilon(z,\omega)$ that satisfies

$$\frac{\partial}{\partial z}\mathbf{P}^\varepsilon = i\frac{\omega\kappa}{2\varepsilon c}\begin{pmatrix} \mathbf{J}(\frac{z}{\varepsilon^2}) & \mathbf{J}(\frac{z}{\varepsilon^2}) e^{-i\frac{2\omega}{\varepsilon^2 c}z} \\ -\mathbf{J}(\frac{z}{\varepsilon^2}) e^{i\frac{2\omega}{\varepsilon^2 c}z} & -\mathbf{J}(\frac{z}{\varepsilon^2}) \end{pmatrix}\mathbf{P}^\varepsilon, \tag{4.6}$$

with the initial condition:

$$\mathbf{P}^\varepsilon(z = 0,\omega) = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}. \tag{4.7}$$

Note that $\det \mathbf{P}^\varepsilon(z,\omega)=1$ for any z as the trace of the matrix in the right-hand side of (4.6) is zero. Moreover, from the symmetry properties of the evolution equation (4.6) it is easy to check that the propagator matrix has the form

$$\mathbf{P}^\varepsilon(z,\omega)=\begin{pmatrix}\mathbf{A}^\varepsilon(z,\omega) & \overline{\mathbf{B}^\varepsilon(z,\omega)}\\ \mathbf{B}^\varepsilon(z,\omega) & \overline{\mathbf{A}^\varepsilon(z,\omega)}\end{pmatrix}, \tag{4.8}$$

where the 2×2 matrices $\mathbf{A}^\varepsilon(z,\omega)$ and $\mathbf{B}^\varepsilon(z,\omega)$ satisfy:

$$\frac{\partial}{\partial z}\mathbf{A}^\varepsilon = i\frac{\omega\kappa}{2\varepsilon c}\mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{A}^\varepsilon(z,\omega)+i\frac{\omega\kappa}{2\varepsilon c}e^{-i\frac{2\omega}{\varepsilon^2 c}z}\mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{B}^\varepsilon(z,\omega), \tag{4.9}$$

$$\frac{\partial}{\partial z}\mathbf{B}^\varepsilon = -i\frac{\omega\kappa}{2\varepsilon c}\mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{B}^\varepsilon(z,\omega)-i\frac{\omega\kappa}{2\varepsilon c}e^{i\frac{2\omega}{\varepsilon^2 c}z}\mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{A}^\varepsilon(z,\omega), \tag{4.10}$$

starting from $\mathbf{A}^\varepsilon(z=0,\omega)=\mathbf{I}$ and $\mathbf{B}^\varepsilon(z=0,\omega)=\mathbf{0}$. Note that, for any z the matrix $\mathbf{A}^\varepsilon(z,\omega)$ is invertible. Indeed, if there exist L and $\hat{\boldsymbol{f}}^0(\omega)$ such that $\mathbf{A}^\varepsilon(L,\omega)\hat{\boldsymbol{f}}^0(\omega)=(0,0)^t$, then we have

$$\mathbf{P}^\varepsilon(L,\omega)\begin{pmatrix}\hat{\boldsymbol{f}}^0(\omega)\\(0,0)^t\end{pmatrix}=\begin{pmatrix}(0,0)^t\\ \mathbf{B}^\varepsilon(L,\omega)\hat{\boldsymbol{f}}^0(\omega)\end{pmatrix}.$$

However (4.4) implies $\|\hat{\boldsymbol{f}}^0(\omega)\|^2-0=0-\|\mathbf{B}^\varepsilon(L,\omega)\hat{\boldsymbol{f}}^0(\omega)\|^2$, which imposes $\hat{\boldsymbol{f}}^0(\omega)=(0,0)^t$.

By linearity of the evolution equation (4.2) the complex mode amplitudes satisfy

$$\begin{pmatrix}\hat{\boldsymbol{a}}^\varepsilon(L,\omega)\\ \hat{\boldsymbol{b}}^\varepsilon(L,\omega)\end{pmatrix}=\begin{pmatrix}\mathbf{A}^\varepsilon(L,\omega) & \overline{\mathbf{B}^\varepsilon(L,\omega)}\\ \mathbf{B}^\varepsilon(L,\omega) & \overline{\mathbf{A}^\varepsilon(L,\omega)}\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{a}}^\varepsilon(0,\omega)\\ \hat{\boldsymbol{b}}^\varepsilon(0,\omega)\end{pmatrix}.$$

From the boundary conditions (4.3) we get the expressions of the transmitted complex mode amplitudes

$$\hat{\boldsymbol{a}}^\varepsilon(z=L,\omega)=\check{\mathbf{T}}^\varepsilon(L,\omega)\hat{\boldsymbol{f}}(\omega), \tag{4.11}$$

where the transmission matrix $\check{\mathbf{T}}^\varepsilon$ is defined by

$$\check{\mathbf{T}}^\varepsilon(z,\omega)=\mathbf{A}^\varepsilon(z,\omega)-\overline{\mathbf{B}^\varepsilon(z,\omega)}\,\overline{(\mathbf{A}^\varepsilon)^{-1}(z,\omega)}\mathbf{B}^\varepsilon(z,\omega), \tag{4.12}$$

for general z and corresponds to the transmission operator over the slab segment $(0,z)$ for a wave incoming from the left (see Figure 1):

$$\begin{pmatrix}\check{\mathbf{T}}^\varepsilon(z,\omega)\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{A}^\varepsilon(z,\omega) & \overline{\mathbf{B}^\varepsilon(z,\omega)}\\ \mathbf{B}^\varepsilon(z,\omega) & \overline{\mathbf{A}^\varepsilon(z,\omega)}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \check{\mathbf{R}}^\varepsilon(z,\omega)\end{pmatrix}. \tag{4.13}$$

By (4.9-4.10) this matrix satisfies

$$\frac{\partial\check{\mathbf{T}}^\varepsilon}{\partial z}=i\frac{\omega\kappa}{2\varepsilon c}\mathbf{J}(\frac{z}{\varepsilon^2})\check{\mathbf{T}}^\varepsilon+i\frac{\omega\kappa}{2\varepsilon c}e^{i\frac{2\omega}{\varepsilon^2 c}z}\overline{\mathbf{B}^\varepsilon}\,\overline{(\mathbf{A}^\varepsilon)^{-1}}\mathbf{J}(\frac{z}{\varepsilon^2})\check{\mathbf{T}}^\varepsilon, \tag{4.14}$$

starting from $\check{\mathbf{T}}^\varepsilon(z=0,\omega)=\mathbf{I}$. Note that, by (4.5), the elements of $\check{\mathbf{T}}^\varepsilon(z,\omega)$ are uniformly bounded by one.

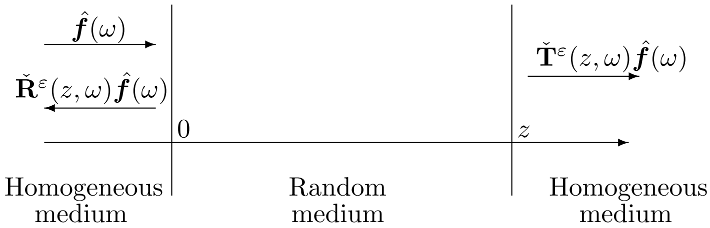

Fig. 1. Reflection and transmission matrices of a slab segment $(0, z)$, corresponding to a wave incoming from the left half-space.

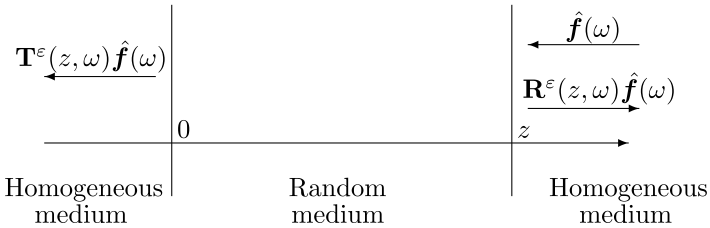

Fig. 2. Auxiliary reflection and transmission matrices of the slab segment $(0, z)$, corresponding to a wave incoming from the right half-space.

We can also introduce an auxiliary pair of 2×2 reflection and transmission matrices for the slab segment $(0, z)$ and a wave incoming from the right half-space (see Figure 2):

$$\mathbf{R}^\varepsilon(z,\omega) = \overline{\mathbf{B}^\varepsilon(z,\omega)}\,\overline{(\mathbf{A}^\varepsilon)^{-1}(z,\omega)}, \qquad \mathbf{T}^\varepsilon(z,\omega) = \overline{(\mathbf{A}^\varepsilon)^{-1}(z,\omega)}. \tag{4.15}$$

They are such that

$$\begin{pmatrix} \mathbf{R}^\varepsilon(z,\omega) \\ \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^\varepsilon(z,\omega) & \overline{\mathbf{B}^\varepsilon(z,\omega)} \\ \mathbf{B}^\varepsilon(z,\omega) & \overline{\mathbf{A}^\varepsilon(z,\omega)} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{T}^\varepsilon(z,\omega) \end{pmatrix}. \tag{4.16}$$

By (4.9-4.10) these matrices satisfy the Riccati equations

$$\begin{aligned} \frac{\partial \mathbf{R}^\varepsilon}{\partial z} &= i\frac{\omega\kappa}{2\varepsilon c}\big(\mathbf{R}^\varepsilon\mathbf{J}(\frac{z}{\varepsilon^2}) + \mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{R}^\varepsilon\big) + i\frac{\omega\kappa}{2\varepsilon c}e^{i\frac{2\omega}{\varepsilon^2 c}z}\mathbf{R}^\varepsilon\mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{R}^\varepsilon \\ &\quad + i\frac{\omega\kappa}{2\varepsilon c}e^{-i\frac{2\omega}{\varepsilon^2 c}z}\mathbf{J}(\frac{z}{\varepsilon^2}), \end{aligned} \tag{4.17}$$

$$\frac{\partial \mathbf{T}^\varepsilon}{\partial z} = i\frac{\omega\kappa}{2\varepsilon c}\mathbf{T}^\varepsilon\mathbf{J}(\frac{z}{\varepsilon^2}) + i\frac{\omega\kappa}{2\varepsilon c}e^{i\frac{2\omega}{\varepsilon^2 c}z}\mathbf{T}^\varepsilon\mathbf{J}(\frac{z}{\varepsilon^2})\mathbf{R}^\varepsilon, \tag{4.18}$$

starting from the initial conditions

$$\mathbf{R}^\varepsilon(z=0,\omega) = \mathbf{0}, \qquad \mathbf{T}^\varepsilon(z=0,\omega) = \mathbf{I}.$$

Note that, by the analogue of (4.5), the elements of $\mathbf{R}^\varepsilon(z,\omega)$ and $\mathbf{T}^\varepsilon(z,\omega)$ are uniformly bounded by one.

The transpose matrix $(\mathbf{R}^\varepsilon)^t(z,\omega)$ satisfies the same equation (4.17) as $\mathbf{R}^\varepsilon(z,\omega)$ because $\mathbf{J}$ is symmetric. By uniqueness of the solution, we have $(\mathbf{R}^\varepsilon)^t(z,\omega) =$

$\mathbf{R}^\varepsilon(z,\omega)$. Moreover, the transpose matrix $(\mathbf{T}^\varepsilon)^t(z,\omega)$ satisfies:

$$\frac{\partial (\mathbf{T}^\varepsilon)^t}{\partial z} = i\frac{\omega\kappa}{2\varepsilon c}\mathbf{J}(\frac{z}{\varepsilon^2})(\mathbf{T}^\varepsilon)^t + i\frac{\omega\kappa}{2\varepsilon c}e^{i\frac{2\omega}{\varepsilon^2 c}z}\mathbf{R}^\varepsilon\mathbf{J}(\frac{z}{\varepsilon^2})(\mathbf{T}^\varepsilon)^t,$$

starting from $(\mathbf{T}^\varepsilon)^t(z=0,\omega) = \mathbf{I}$, and by uniqueness (compare with (4.14)) we have

$$\check{\mathbf{T}}^\varepsilon(z,\omega) = (\mathbf{T}^\varepsilon)^t(z,\omega). \tag{4.19}$$

5. The Diffusion Approximation Theorem

Let us fix the frequency ω. Applying the diffusion-approximation result of [6] (which is a refined form of [14]) to the random equations (4.17-4.18) we obtain that $(\mathbf{R}^\varepsilon(z,\omega), \mathbf{T}^\varepsilon(z,\omega))$ converge in distribution to $(\mathbf{R}(z), \mathbf{T}(z))$ as $\varepsilon \to 0$ that is solution to the matrix-valued stochastic differential equation

$$\begin{aligned} d\mathbf{R} = & i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big[(\boldsymbol{\sigma}_3\mathbf{R}+\mathbf{R}\boldsymbol{\sigma}_3)\circ dW_z^1 + (\boldsymbol{\sigma}_1\mathbf{R}+\mathbf{R}\boldsymbol{\sigma}_1)\circ dW_z^2\big] \\ & + \frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\big[(\boldsymbol{\sigma}_3-\mathbf{R}\boldsymbol{\sigma}_3\mathbf{R})\circ dW_z^3 + (\boldsymbol{\sigma}_1-\mathbf{R}\boldsymbol{\sigma}_1\mathbf{R})\circ dW_z^4\big] \\ & + i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\big[(\boldsymbol{\sigma}_3+\mathbf{R}\boldsymbol{\sigma}_3\mathbf{R})\circ dW_z^5 + (\boldsymbol{\sigma}_1+\mathbf{R}\boldsymbol{\sigma}_1\mathbf{R})\circ dW_z^6\big] \\ & - i\frac{\omega^2\kappa^2}{c^2}\gamma_s(2\omega)\mathbf{R}dz, \end{aligned} \tag{5.1}$$

$$\begin{aligned} d\mathbf{T} = & i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big[\mathbf{T}\boldsymbol{\sigma}_3\circ dW_z^1 + \mathbf{T}\boldsymbol{\sigma}_1\circ dW_z^2\big] \\ & - \frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\big[\mathbf{T}\boldsymbol{\sigma}_3\mathbf{R}\circ dW_z^3 + \mathbf{T}\boldsymbol{\sigma}_1\mathbf{R}\circ dW_z^4\big] \\ & + i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\big[\mathbf{T}\boldsymbol{\sigma}_3\mathbf{R}\circ dW_z^5 + \mathbf{T}\boldsymbol{\sigma}_1\mathbf{R}\circ dW_z^6\big] \\ & - i\frac{\omega^2\kappa^2}{2c^2}\gamma_s(2\omega)\mathbf{T}dz, \end{aligned} \tag{5.2}$$

starting from $\mathbf{R}(z=0) = \mathbf{0}$, $\mathbf{T}(z=0) = \mathbf{I}$, where $\boldsymbol{\sigma}_j$, $j=1,2,3$, are the Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.3}$$

the W_z^j, $j=1,\dots,6$, are independent Brownian motions, and

$$\gamma_c(2\omega) = \int_0^\infty \mathbb{E}\big[\cos 2\psi(0)\cos 2\psi(z)\big]\cos\big(2\frac{\omega z}{c}\big)dz, \tag{5.4}$$

$$\gamma_s(2\omega) = \int_0^\infty \mathbb{E}\big[\cos 2\psi(0)\cos 2\psi(z)\big]\sin\big(2\frac{\omega z}{c}\big)dz. \tag{5.5}$$

Note that $\gamma_c(2\omega)$ is non-negative valued as it is equal to half the power spectral density of the stationary random process $\cos 2\psi(z)$. In Appendix A we show that the limit processes $\mathbf{R}$ and $\mathbf{T}$ can also be characterized as the reflexion and transmission operators of the limit propagator matrix $\mathbf{P} = \lim_{\varepsilon\to 0}\mathbf{P}^\varepsilon$.

This diffusion-approximation result is the basis of the forthcoming analysis. It holds for any fixed frequency ω, however, below we will need to characterize the ω-dependence of the Brownian motions as the correlations between the transmission matrices at different frequencies play a fundamental role.

6. The Convergence of the Front Pulse

The following proposition shows that there is no pulse stabilization but the transmitted front pulse converges in distribution to a random shape that can be identified as the solution of a stochastic partial differential equation driven by two Brownian motions.

Proposition 6.1. *If $\hat{\boldsymbol{f}} \in L^1$, then the transmitted wave $(\boldsymbol{u}_{\rm tr}^\varepsilon(\tau))_{\tau\in\mathbb{R}}$ converges in distribution as $\varepsilon \to 0$ in the space of continuous functions $\mathcal{C}(\mathbb{R},\mathbb{R}^2)$ to $(\boldsymbol{u}_{\rm tr}(\tau))_{\tau\in\mathbb{R}}$ given by*

$$\boldsymbol{u}_{\rm tr}(\tau) = \frac{1}{4\pi\zeta^{1/2}} \int \check{\mathbf{T}}(L,\omega)\hat{\boldsymbol{f}}(\omega)e^{-i\omega\tau}d\omega, \tag{6.1}$$

where $\check{\mathbf{T}}(z,\omega)$ is the solution of

$$\begin{aligned} d\check{\mathbf{T}}(z,\omega) = {} & i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big[\boldsymbol{\sigma}_3\check{\mathbf{T}}(z,\omega)\circ dW_z^1 + \boldsymbol{\sigma}_1\check{\mathbf{T}}(z,\omega)\circ dW_z^2\big] \\ & -\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(2\omega)+i\gamma_s(2\omega)\big]\check{\mathbf{T}}(z,\omega)dz, \end{aligned} \tag{6.2}$$

starting from $\check{\mathbf{T}}(z=0,\omega)=\mathbf{I}$.

Here W^1 and W^2 are two independent Brownian motions and the spectral parameters γ_c and γ_s are defined by (5.4-5.5). The space $\mathcal{C}(\mathbb{R},\mathbb{R}^2)$ is equipped with the topology of the supremum norm over the compact subsets.

Proof. We have by (4.1), (4.11), and (4.19):

$$\boldsymbol{u}_{\rm tr}^\varepsilon(\tau) = \frac{1}{4\pi\zeta^{1/2}} \int (\mathbf{T}^\varepsilon)^t(L,\omega)\hat{\boldsymbol{f}}(\omega)e^{-i\omega\tau}d\omega. \tag{6.3}$$

We first establish the tightness of the process $(\boldsymbol{u}_{\rm tr}^\varepsilon(\tau))_{\tau\in\mathbb{R}}$ in the space of continuous functions. It is bounded by

$$\sup_{\tau\in\mathbb{R}} \|\boldsymbol{u}_{\rm tr}^\varepsilon(\tau)\| \leq \frac{1}{\pi\zeta^{1/2}}\int \|\hat{\boldsymbol{f}}(\omega)\|d\omega,$$

because $|T_{i,j}^\varepsilon(L,\omega)| \leq 1$. Here and below the norm $\|\cdot\|$ is again the Euclidean norm. Furthermore the modulus of continuity

$$M^\varepsilon(\delta) = \sup_{|\tau_1-\tau_2|\leq\delta} \|\boldsymbol{u}_{\rm tr}^\varepsilon(\tau_1) - \boldsymbol{u}_{\rm tr}^\varepsilon(\tau_2)\|$$

can be bounded uniformly in ε by the deterministic quantity

$$M^\varepsilon(\delta) \leq \frac{1}{\pi\zeta^{1/2}} \int \sup_{|\tau_1-\tau_2|\leq\delta} \big|1-e^{i\omega(\tau_2-\tau_1)}\big|\|\hat{\boldsymbol{f}}(\omega)\|d\omega.$$

This quantity goes to zero as $\delta \to 0$ by dominated convergence theorem, which proves the tightness (in the space of the continuous functions).

It remains then to show that the finite-dimensional distributions of $(\boldsymbol{u}^\varepsilon_{\rm tr}(\tau))_{\tau\in\mathbb{R}}$ converge to those of $(\boldsymbol{u}_{\rm tr}(\tau))_{\tau\in\mathbb{R}}$ defined by (6.1). Since $\boldsymbol{u}^\varepsilon_{\rm tr}(\tau)$ is bounded, it is sufficient to show that, for any n, $(j_l)_{l=1}^n \in \{1,2\}^n$, $(\tau_l)_{l=1}^n \in \mathbb{R}^n$, we have

$$\mathbb{E}\Big[\prod_{l=1}^n u^\varepsilon_{{\rm tr},j_l}(\tau_l)\Big] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\Big[\prod_{l=1}^n u_{{\rm tr},j_l}(\tau_l)\Big].$$

By (6.3) we have

$$\mathbb{E}\Big[\prod_{l=1}^n u^\varepsilon_{{\rm tr},j_l}(\tau_l)\Big] = \frac{1}{(4\pi)^n \zeta^{n/2}} \sum_{i_1,\ldots,i_n=1}^2 \int\cdots\int \mathbb{E}\Big[\prod_{l=1}^n T^\varepsilon_{i_l,j_l}(L,\omega_l)\Big] \times \prod_{l=1}^n \hat{f}_{i_l}(\omega_l) e^{-i\omega_l\tau_l} d\omega_1\cdots d\omega_n, \qquad (6.4)$$

and this shows that we need to study the convergence of the moments of products of transmission coefficients at distinct frequencies.

The diffusion-approximation result stated in the previous section can be extended to the multi-frequency case. Let $(\omega_k)_{k=1}^n$ be n *distinct* frequencies. Applying the diffusion-approximation result of [6] to the random equations (4.17-4.18) we obtain that

$$\big(\mathbf{R}^\varepsilon(z,\omega_1),\ldots,\mathbf{R}^\varepsilon(z,\omega_n),\mathbf{T}^\varepsilon(z,\omega_1),\ldots,\mathbf{T}^\varepsilon(z,\omega_n)\big)$$

converge in distribution to

$$\big(\mathbf{R}^{(1)}(z),\ldots,\mathbf{R}^{(n)}(z),\mathbf{T}^{(1)}(z),\ldots,\mathbf{T}^{(n)}(z)\big)$$

as $\varepsilon \to 0$ that is solution to the matrix-valued stochastic differential equation

$$\begin{aligned}
d\mathbf{R}^{(k)} = & i\frac{\omega_k\kappa}{2c}\sqrt{2\gamma_c(0)}\big[(\boldsymbol{\sigma}_3\mathbf{R}^{(k)} + \mathbf{R}^{(k)}\boldsymbol{\sigma}_3)\circ dW_z^1 \\
& \qquad + (\boldsymbol{\sigma}_1\mathbf{R}^{(k)} + \mathbf{R}^{(k)}\boldsymbol{\sigma}_1)\circ dW_z^2\big] \\
& + \frac{\omega_k\kappa}{2c}\sqrt{\gamma_c(2\omega_k)}\big[(\boldsymbol{\sigma}_3 - \mathbf{R}^{(k)}\boldsymbol{\sigma}_3\mathbf{R}^{(k)})\circ dW_z^{3(k)} \\
& \qquad + (\boldsymbol{\sigma}_1 - \mathbf{R}^{(k)}\boldsymbol{\sigma}_1\mathbf{R}^{(k)})\circ dW_z^{4(k)}\big] \\
& + i\frac{\omega_k\kappa}{2c}\sqrt{\gamma_c(2\omega_k)}\big[(\boldsymbol{\sigma}_3 + \mathbf{R}^{(k)}\boldsymbol{\sigma}_3\mathbf{R}^{(k)})\circ dW_z^{5(k)} \\
& \qquad + (\boldsymbol{\sigma}_1 + \mathbf{R}^{(k)}\boldsymbol{\sigma}_1\mathbf{R}^{(k)})\circ dW_z^{6(k)}\big] \\
& - i\frac{\omega_k^2\kappa^2}{c^2}\gamma_s(2\omega_k)\mathbf{R}^{(k)}dz, \qquad (6.5)\\
d\mathbf{T}^{(k)} = & i\frac{\omega_k\kappa}{2c}\sqrt{2\gamma_c(0)}\big[\mathbf{T}^{(k)}\boldsymbol{\sigma}_3\circ dW_z^1 + \mathbf{T}^{(k)}\boldsymbol{\sigma}_1\circ dW_z^2\big] \\
& - \frac{\omega_k\kappa}{2c}\sqrt{\gamma_c(2\omega_k)}\big[\mathbf{T}^{(k)}\boldsymbol{\sigma}_3\mathbf{R}^{(k)}\circ dW_z^{3(k)} + \mathbf{T}^{(k)}\boldsymbol{\sigma}_1\mathbf{R}^{(k)}\circ dW_z^{4(k)}\big] \\
& + i\frac{\omega_k\kappa}{2c}\sqrt{\gamma_c(2\omega_k)}\big[\mathbf{T}^{(k)}\boldsymbol{\sigma}_3\mathbf{R}^{(k)}\circ dW_z^{5(k)} + \mathbf{T}^{(k)}\boldsymbol{\sigma}_1\mathbf{R}^{(k)}\circ dW_z^{6(k)}\big] \\
& - i\frac{\omega_k^2\kappa^2}{2c^2}\gamma_s(2\omega_k)\mathbf{T}^{(k)}dz, \qquad (6.6)
\end{aligned}$$

where W_z^1, W_z^2, $W_z^{j(k)}$, $j = 3,\dots,6$, $k = 1,\dots,n$, are independent Brownian motions. Note that the evolution equations for the different frequency components $(\mathbf{R}^{(k)},\mathbf{T}^{(k)})$ share the same Brownian motions W_z^1 and W_z^2 and have independent Brownian motions $W_z^{j(k)}$, $j = 3,\dots,6$. This comes from the fact that the terms in the right-hand sides of (4.17-4.18) that have rapid phases become independent in the limit $\varepsilon \to 0$, while the terms that do not have rapid phases stay correlated.

The equation (6.6) can be written in Itô's form as

$$\begin{aligned}
d\mathbf{T}^{(k)} = & i\frac{\omega_k\kappa}{2c}\sqrt{2\gamma_c(0)}\big[\mathbf{T}^{(k)}\boldsymbol{\sigma}_3 dW_z^1 + \mathbf{T}^{(k)}\boldsymbol{\sigma}_1 dW_z^2\big] \\
& - \frac{\omega_k\kappa}{2c}\sqrt{\gamma_c(2\omega_k)}\big[\mathbf{T}^{(k)}\boldsymbol{\sigma}_3\mathbf{R}^{(k)} dW_z^{3(k)} + \mathbf{T}^{(k)}\boldsymbol{\sigma}_1\mathbf{R}^{(k)} dW_z^{4(k)}\big] \\
& + i\frac{\omega_k\kappa}{2c}\sqrt{\gamma_c(2\omega_k)}\big[\mathbf{T}^{(k)}\boldsymbol{\sigma}_3\mathbf{R}^{(k)} dW_z^{5(k)} + \mathbf{T}^{(k)}\boldsymbol{\sigma}_1\mathbf{R}^{(k)} dW_z^{6(k)}\big] \\
& - \frac{\omega_k^2\kappa^2}{2c^2}\big[\gamma_c(0) + \gamma_c(2\omega_k) + i\gamma_s(2\omega_k)\big]\mathbf{T}^{(k)}dz.
\end{aligned}$$

Applying Itô's formula, for any $i_1,\dots,i_n,j_1,\dots,j_n \in \{1,2\}$, we have

$$\begin{aligned}
\frac{\partial}{\partial z}\mathbb{E}\Big[\prod_{l=1}^n T_{i_l,j_l}^{(l)}\Big] = & -\frac{\gamma_c(0)\kappa^2}{4c^2}\sum_{k\neq k'=1}^n \omega_k\omega_{k'}(-1)^{j_k+j_{k'}}\mathbb{E}\Big[\prod_{l=1}^n T_{i_l,j_l}^{(l)}\Big] \\
& - \frac{\gamma_c(0)\kappa^2}{4c^2}\sum_{k\neq k'=1}^n \omega_k\omega_{k'}\mathbb{E}\Big[\prod_{l\neq k,k'}^n T_{i_l,j_l}^{(l)} T_{i_k,3-j_k}^{(k)} T_{i_{k'},3-j_{k'}}^{(k')}\Big] \\
& - \frac{\kappa^2}{2c^2}\sum_{k=1}^n \omega_k^2\big[\gamma_c(0) + \gamma_c(2\omega_k) + i\gamma_s(2\omega_k)\big]\mathbb{E}\Big[\prod_{l=1}^n T_{i_l,j_l}^{(l)}\Big], \qquad (6.7)
\end{aligned}$$

starting from $\mathbb{E}\Big[\prod_{l=1}^n T^{(l)}_{i_l,j_l}(z=0)\Big] = \prod_{l=1}^n \mathbf{1}_{i_l}(j_l)$.

Then we observe that

$$\Big(\mathbb{E}\Big[\prod_{k=1}^n \widetilde{T}_{i_k,j_k}(z,\omega_k)\Big]\Big)_{i_1,\ldots,i_n,j_1,\ldots,j_n\in\{1,2\}}$$

satisfies the same closed linear system (6.7) as

$$\Big(\mathbb{E}\Big[\prod_{k=1}^n T^{(k)}_{i_k,j_k}(z)\Big]\Big)_{i_1,\ldots,i_n,j_1,\ldots,j_n\in\{1,2\}},$$

where $\widetilde{\mathbf{T}}(z,\omega)$ is the solution of

$$\begin{aligned} d\widetilde{\mathbf{T}}(z,\omega) = {} & i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big[\widetilde{\mathbf{T}}(z,\omega)\boldsymbol{\sigma}_3 dW_z^1 + \widetilde{\mathbf{T}}(z,\omega)\boldsymbol{\sigma}_1 dW_z^2\big] \\ & -\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(0)+\gamma_c(2\omega)+i\gamma_s(2\omega)\big]\widetilde{\mathbf{T}}(z,\omega)dz, \end{aligned}$$

starting from $\widetilde{\mathbf{T}}(z=0,\omega)=\mathbf{I}$, or in Stratonovich form:

$$\begin{aligned} d\widetilde{\mathbf{T}}(z,\omega) = {} & i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big[\widetilde{\mathbf{T}}(z,\omega)\boldsymbol{\sigma}_3 \circ dW_z^1 + \widetilde{\mathbf{T}}(z,\omega)\boldsymbol{\sigma}_1 \circ dW_z^2\big] \\ & -\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(2\omega)+i\gamma_s(2\omega)\big]\widetilde{\mathbf{T}}(z,\omega)dz. \end{aligned}$$

The system (6.7) has a unique solution. Therefore, for any distinct frequencies $(\omega_k)_{k=1}^n$, we have

$$\mathbb{E}\Big[\prod_{k=1}^n T^\varepsilon_{i_k,j_k}(z,\omega_k)\Big] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\Big[\prod_{k=1}^n \widetilde{T}_{i_k,j_k}(z,\omega_k)\Big].$$

Note that this property holds only for the product of transmission coefficients at distinct frequencies, it does not hold for $\mathbb{E}[|T^\varepsilon_{1,1}(z,\omega)|^2]$ for instance.

The matrix $\check{\mathbf{T}}(z,\omega) = (\widetilde{\mathbf{T}})^t(z,\omega)$ is the solution of (6.2) and it satisfies for any distinct frequencies $(\omega_k)_{k=1}^n$ and any set of indices $i_1,\ldots,i_n,j_1,\ldots,j_n\in\{1,2\}$:

$$\mathbb{E}\Big[\prod_{k=1}^n T^\varepsilon_{i_k,j_k}(z,\omega_k)\Big] \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\Big[\prod_{k=1}^n \check{T}_{j_k,i_k}(z,\omega_k)\Big]. \tag{6.8}$$

Substituting (6.8) into the expression (6.4) of the moment gives the desired convergence result. □

We can now discuss the form of the transmitted front pulse. The limit transmission matrix $\check{\mathbf{T}}(L,\omega)$ can be written in the form

$$\check{\mathbf{T}}(L,\omega) = \check{\mathbf{U}}(L,\omega)\exp\Big(-\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(2\omega)+i\gamma_s(2\omega)\big]L\Big), \tag{6.9}$$

where $\check{\mathbf{U}}(z,\omega)$ is a unitary matrix

$$\check{\mathbf{U}}(z,\omega) = \begin{pmatrix} \alpha(z,\omega) & -\overline{\beta(z,\omega)} \\ \beta(z,\omega) & \overline{\alpha(z,\omega)} \end{pmatrix} \tag{6.10}$$

with (α, β) the solution of

$$d\alpha(z,\omega) = i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big(\alpha(z,\omega)\circ dW_z^1 + \beta(z,\omega)\circ dW_z^2\big), \tag{6.11}$$

$$d\beta(z,\omega) = i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\big(-\beta(z,\omega)\circ dW_z^1 + \alpha(z,\omega)\circ dW_z^2\big), \tag{6.12}$$

starting from $\alpha(z=0,\omega)=1$ and $\beta(z=0,\omega)=0$.

The exponential term in (6.9) describes a deterministic deformation of the transmitted pulse

$$\boldsymbol{u}_{\rm tr}(\tau) = \frac{1}{4\pi\zeta^{1/2}}\int \check{\mathbf{U}}(L,\omega)\hat{\boldsymbol{f}}(\omega)\exp\Big(-\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(2\omega)+i\gamma_s(2\omega)\big]L\Big)e^{-i\omega\tau}d\omega,$$

that is due to the coupling between forward and backward wave modes. The unitary matrix $\check{\mathbf{U}}(L,\omega)$ in (6.9) describes a random deformation of the transmitted pulse that is due to the coupling between forward wave modes. Contrarily to the usual scalar case, this term cannot be reduced to a phase term of the form $\exp(i\omega T_L)$, that would give a random time shift T_L but no deformation. Here the presence of the unitary matrix deforms the pulse and can give the shapes shown in Figure 3. However, since the matrix $\check{\mathbf{U}}(L,\omega)$ is unitary, it does not modify the energy and therefore the energy of the transmitted coherent wave is deterministic in the regime $\varepsilon \to 0$ as explained in the next section.

Moreover, the unitary matrix $\check{\mathbf{U}}(L,\omega)$ has a frequency coherence ω_c that is

$$\omega_c = \frac{c}{\kappa\sqrt{\gamma_c(0)L}}, \tag{6.13}$$

as explained in the following Lemma 6.1. As a result, if the bandwidth of the incoming pulse is much smaller than ω_c, then we have

$$\boldsymbol{u}_{\rm tr}(\tau) = \frac{\check{\mathbf{U}}(L,\omega_0)}{4\pi\zeta^{1/2}}\int_0^\infty \hat{\boldsymbol{f}}(\omega)\exp\Big(-\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(2\omega)+i\gamma_s(2\omega)\big]L\Big)e^{-i\omega\tau}d\omega + c.c.,$$

where ω_0 is the carrier frequency of the incoming pulse, which shows that the transmitted pulse shape experiences a deterministic deformation, but the transmitted polarization state is random.

Lemma 6.1. *For any ω, ω_0, we have*

$$\mathbb{E}\big[\|\check{\mathbf{U}}^{-1}(L,\omega_0)\check{\mathbf{U}}(L,\omega)-\mathbf{I}\|^2\big] = 4\Big[1-\exp\Big(-\frac{(\omega-\omega_0)^2}{2\omega_c^2}\Big)\Big].$$

Here $\|\cdot\|$ stands for the Frobenius norm.

Note that

$$\mathbb{E}\big[\|\check{\mathbf{U}}(L,\omega)-\check{\mathbf{U}}(L,\omega_0)\|^2\big] = \mathbb{E}\big[\|\check{\mathbf{U}}^{-1}(L,\omega_0)\check{\mathbf{U}}(L,\omega)-\mathbf{I}\|^2\big] \le 2\frac{(\omega-\omega_0)^2}{\omega_c^2},$$

that is why we say that ω_c is the frequency coherence of the matrix $\check{\mathbf{U}}(L,\omega)$.

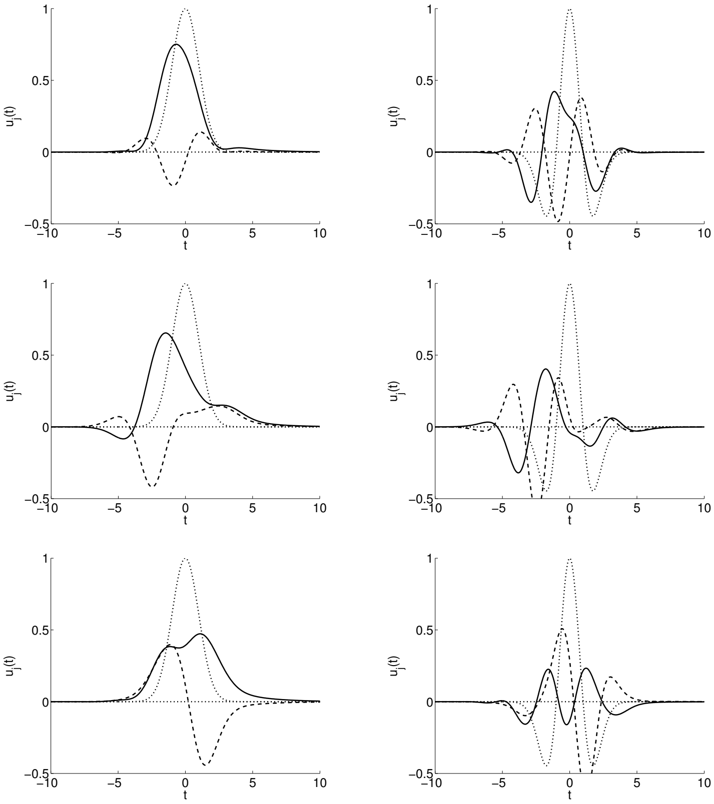

Fig. 3. Incoming pulse shapes $f_j(t)$, $j = 1, 2$ (dotted lines) and transmitted pulse shapes $u_{\mathrm{tr},j}(t)$, $j = 1$ (solid lines) and $j = 2$ (dashed lines). Here $f_1(t) = \exp(-t^2/2)$ (left column, Gaussian pulse) and $f_1(t) = (1-t^2)\exp(-t^2/2)$ (right column, Ricker pulse), $f_2(t) = 0$, $\mathbb{E}[\cos 2\psi(0)\cos 2\psi(z)] = \exp(-|z|/l_c)$, $l_c = 1$, $\kappa = 0.05$, $L = 50$, $c = 1$. This covariance function corresponds for instance to the case where $\psi(z)$ is stepwise constant over intervals that are independent and identically distributed according to an exponential distribution with mean l_c, and the values taken by $\psi(z)$ over each interval are independent and identically distributed according to a uniform distribution over $[0, 2\pi]$. The three rows correspond to three different realizations of the random medium.

Proof. We denote

$$\widetilde{\mathbf{V}}(z) = \check{\mathbf{U}}(z,\omega_0)\check{\mathbf{U}}^{-1}(z,\omega) - \mathbf{I}.$$

It satisfies:

$$d\widetilde{\mathbf{V}} = i\frac{\sqrt{2\gamma_c(0)}\kappa}{2c}\Big[(\omega_0-\omega)(\boldsymbol{\sigma}_3 dW_z^1 + \boldsymbol{\sigma}_1 dW_z^2) + (\omega_0\boldsymbol{\sigma}_3\widetilde{\mathbf{V}} - \omega\widetilde{\mathbf{V}}\boldsymbol{\sigma}_3)\circ dW_z^1 \\ +(\omega_0\boldsymbol{\sigma}_1\widetilde{\mathbf{V}} - \omega\widetilde{\mathbf{V}}\boldsymbol{\sigma}_1)\circ dW_z^2\Big],$$

starting from $\widetilde{\mathbf{V}}(z=0)=\mathbf{0}$, or in Itô's form:

$$d\widetilde{\mathbf{V}} = i\frac{\sqrt{2\gamma_c(0)}\kappa}{2c}\Big[(\omega_0-\omega)(\boldsymbol{\sigma}_3 dW_z^1+\boldsymbol{\sigma}_1 dW_z^2)+(\omega_0\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}-\omega\widetilde{\mathbf{V}}\boldsymbol{\sigma}_3)dW_z^1 \\ +(\omega_0\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}-\omega\widetilde{\mathbf{V}}\boldsymbol{\sigma}_1)dW_z^2\Big] \\ -\frac{\gamma_c(0)\kappa^2}{2c^2}\big[(\omega-\omega_0)^2\mathbf{I}+(\omega^2+\omega_0^2)\widetilde{\mathbf{V}}-\omega\omega_0(\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}\boldsymbol{\sigma}_3+\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}\boldsymbol{\sigma}_1\big]dz.$$

Using the fact that $\mathrm{Tr}(\boldsymbol{\sigma}_j\widetilde{\mathbf{V}}\boldsymbol{\sigma}_j)=\mathrm{Tr}(\boldsymbol{\sigma}_j^2\widetilde{\mathbf{V}})=\mathrm{Tr}(\widetilde{\mathbf{V}})$ and $\mathrm{Tr}(\mathbf{I})=2$, we find that $\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}})\big]$ is the solution to the ordinary differential equation

$$\frac{d}{dz}\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}})\big]=-\frac{\gamma_c(0)\kappa^2(\omega-\omega_0)^2}{2c^2}\big(2+\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}})\big]\big),$$

which gives

$$\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}}(z))\big]=2\exp\Big(-\frac{\gamma_c(0)\kappa^2(\omega-\omega_0)^2 z}{2c^2}\Big)-2. \tag{6.14}$$

Denoting by † the conjugate transpose, we also have

$$d\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger = i\frac{\sqrt{2\gamma_c(0)}\kappa}{2c}\Big[(\omega_0-\omega)\big(\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\boldsymbol{\sigma}_3\big)\circ dW_z^1+(\omega_0-\omega)\big(\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\boldsymbol{\sigma}_1\big)\circ dW_z^2 \\ +\omega_0\big(\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger\boldsymbol{\sigma}_3\big)\circ dW_z^1+\omega_0\big(\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger\boldsymbol{\sigma}_1\big)\circ dW_z^2\Big],$$

or in Itô's form:

$$d\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger = i\frac{\sqrt{2\gamma_c(0)}\kappa}{2c}\Big[(\omega_0-\omega)\big(\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\boldsymbol{\sigma}_3\big)dW_z^1+(\omega_0-\omega)\big(\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\boldsymbol{\sigma}_1\big)dW_z^2 \\ +\omega_0\big(\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger\boldsymbol{\sigma}_3\big)dW_z^1+\omega_0\big(\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger-\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger\boldsymbol{\sigma}_1\big)dW_z^2\Big] \\ -\frac{\gamma_c(0)\kappa^2}{2c^2}\big[-2(\omega-\omega_0)^2\mathbf{I}+(\omega_0^2-\omega^2)\big(\widetilde{\mathbf{V}}+\widetilde{\mathbf{V}}^\dagger\big)-\omega_0(\omega_0-\omega)\boldsymbol{\sigma}_3\big(\widetilde{\mathbf{V}}+\widetilde{\mathbf{V}}^\dagger\big)\boldsymbol{\sigma}_3 \\ -\omega_0(\omega_0-\omega)\boldsymbol{\sigma}_1\big(\widetilde{\mathbf{V}}+\widetilde{\mathbf{V}}^\dagger\big)\boldsymbol{\sigma}_1+2\omega_0^2\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger-\omega_0^2\boldsymbol{\sigma}_1\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger\boldsymbol{\sigma}_1-\omega_0^2\boldsymbol{\sigma}_3\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger\boldsymbol{\sigma}_3\big]dz.$$

We find that $\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger)\big]$ is the solution to the ordinary differential equation

$$\frac{d}{dz}\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger)\big]=\frac{\gamma_c(0)\kappa^2(\omega-\omega_0)^2}{c^2}\big(2+\mathrm{Re}\big\{\big[\mathrm{Tr}(\widetilde{\mathbf{V}})\big]\big\}\big),$$

which gives (using (6.14)):

$$\mathbb{E}\big[\mathrm{Tr}(\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger(z))\big]=4-4\exp\Big(-\frac{\gamma_c(0)\kappa^2(\omega-\omega_0)^2 z}{2c^2}\Big). \tag{6.15}$$

Since $\mathrm{Tr}(\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\dagger(L))=\|\widetilde{\mathbf{V}}(L)\|^2$ and $\check{\mathbf{U}}(L,\omega)$ and $\check{\mathbf{U}}(L,\omega_0)$ are unitary:

$$\|\check{\mathbf{U}}^{-1}(L,\omega_0)\check{\mathbf{U}}(L,\omega)-\mathbf{I}\|^2=\|\check{\mathbf{U}}(L,\omega_0)\check{\mathbf{U}}^{-1}(L,\omega)-\mathbf{I}\|^2=\|\widetilde{\mathbf{V}}(L)\|^2,$$

we finally get the desired result. □

By taking an inverse Fourier transform, we find an alternative (and equivalent) way to characterize the transmitted front pulse (6.1), as described in the following corollary.

Corollary 6.1. *The limit (6.1) of the transmitted front pulse is given by*

$$\boldsymbol{u}_{\rm tr}(\tau) = \boldsymbol{a}(L,\tau), \tag{6.16}$$

where $\boldsymbol{a}$ *is the two-dimensional vector field solution to the stochastic partial differential equation*

$$d\boldsymbol{a} = -\sqrt{\frac{\gamma_c(0)}{2}}\kappa\boldsymbol{\sigma}_3\frac{\partial \boldsymbol{a}}{\partial t}\circ dW_z^1 - \sqrt{\frac{\gamma_c(0)}{2}}\kappa\boldsymbol{\sigma}_1\frac{\partial \boldsymbol{a}}{\partial t}\circ dW_z^2, \tag{6.17}$$

starting from

$$\boldsymbol{a}(z=0,\tau) = \frac{1}{4\pi\zeta^{1/2}}\int \hat{\boldsymbol{f}}(\omega)\exp\Big(-\frac{\omega^2\kappa^2}{2c^2}\big[\gamma_c(2\omega)+i\gamma_s(2\omega)\big]L\Big)e^{-i\omega\tau}d\tau.$$

We finally mention that there is no other wave that is transmitted through the random medium with an amplitude of order one. This result is similar to the one of the standard scalar case [6] . In other words, if $t_0 \neq 0$, then

$$\begin{aligned}\boldsymbol{u}_{t_0}^\varepsilon(\tau) &= \boldsymbol{u}\Big(z=L, t=\frac{L}{c}+t_0+\varepsilon^2\tau\Big)\\ &= \frac{1}{4\pi\zeta^{1/2}}\int (\mathbf{T}^\varepsilon)^t(L,\omega)\hat{\boldsymbol{f}}(\omega)e^{-i\omega\tau-i\omega\frac{t_0}{\varepsilon^2}}d\omega\end{aligned}$$

converges in distribution to zero as $\varepsilon \to 0$. This can be shown by an analysis similar to that for Proposition 6.1 which shows that the process $(\boldsymbol{u}_{t_0}^\varepsilon(\tau))_{\tau\in\mathbb{R}}$ (as a continuous function in τ) vanishes as $\varepsilon \to 0$ due to the fast phase $\exp(-i\omega t_0/\varepsilon^2)$ in its integral representation. This implies that the incoherent waves that exit after the front pulse (the so-called coda waves in the geophysics literature) have small amplitude in the limit $\varepsilon \to 0$ (but not vanishing energy, as shown in [6] in the scalar case).

7. Stability of the Transmitted Coherent Energy

In this section we assume that $\hat{\boldsymbol{f}} \in L^1 \cap L^2$. We recall the definitions of energy density and energy flux in Appendix B. The incoming energy flux is

$$\mathcal{F}_{\rm inc}^\varepsilon = -K\varepsilon^2\sum_{k=1}^2\int \frac{\partial u_{{\rm inc},k}^\varepsilon}{\partial t}\frac{\partial u_{{\rm inc},k}^\varepsilon}{\partial z}(0,t)dt, \tag{7.1}$$

where

$$\boldsymbol{u}_{\rm inc}^\varepsilon(z,t) = \frac{1}{4\pi\zeta^{1/2}}\int \hat{\boldsymbol{f}}(\omega)e^{i\frac{\omega}{c}\frac{z-ct}{\varepsilon^2}}d\omega. \tag{7.2}$$

Here we have multiplied the flux by ε^2 so that it is independent of ε:

$$\mathcal{F}_{\rm inc}^\varepsilon = \mathcal{F}_{\rm inc} = \frac{1}{8\pi}\int \omega^2\|\hat{\boldsymbol{f}}(\omega)\|^2d\omega. \tag{7.3}$$

The total transmitted energy flux is

$$\mathcal{F}^\varepsilon_{\rm tr} = -K\varepsilon^2 \sum_{k=1}^2 \int \frac{\partial u^\varepsilon_k}{\partial t}\frac{\partial u^\varepsilon_k}{\partial z}(L,t)dt. \tag{7.4}$$

But in fact we are only interested in the energy of the transmitted front pulse, not in the energy of the coda or small wave fluctuations that arrive after. The coherent transmitted energy flux can be defined by

$$\mathcal{F}^\varepsilon_{\rm coh} = -K\varepsilon^2 \sum_{k=1}^2 \int \psi\Big(\frac{t-\frac{L}{c}}{\varepsilon^2}\Big)\frac{\partial u^\varepsilon_k}{\partial t}\frac{\partial u^\varepsilon_k}{\partial z}(L,t)dt, \tag{7.5}$$

where ψ is a smooth cut-off function, that is nonnegative, compactly supported, and such that $\psi(0)=1$. This means that we only collect the transmitted energy around the expected arrival time L/c over a time window whose duration is of the order of the initial pulse width.

Proposition 7.1. *The coherent transmitted energy flux $\mathcal{F}^\varepsilon_{\rm coh}$ converges in distribution as $\varepsilon\to 0$ to $\mathcal{F}_{\rm coh}$ given by*

$$\begin{aligned}\mathcal{F}_{\rm coh} = \frac{1}{16\pi^2}\iint &\exp\Big(-\frac{\omega_2^2\kappa^2}{2c^2}\big[\gamma_c(2\omega_2)+i\gamma_s(2\omega_2)\big]L\Big)\\ &\times\exp\Big(-\frac{\omega_1^2\kappa^2}{2c^2}\big[\gamma_c(2\omega_1)-i\gamma_s(2\omega_1)\big]L\Big)\\ &\times\check{\mathbf{U}}(L,\omega_2)\hat{\boldsymbol{f}}(\omega_2)\cdot\overline{\check{\mathbf{U}}(L,\omega_1)\hat{\boldsymbol{f}}(\omega_1)}\omega_1\omega_2\hat{\psi}(\omega_1-\omega_2)d\omega_1 d\omega_2. \end{aligned}\tag{7.6}$$

Let us assume that the initial pulse $\boldsymbol{f}(t)$ has carrier frequency ω_0 and bandwidth B. If the width T_ψ of the cut-off function ψ satisfies $T_\psi B\gg 1$ and $T_\psi\omega_c\gg 1$, where ω_c is the coherence frequency defined by (6.13), then

$$\mathcal{F}_{\rm coh} = \frac{1}{8\pi}\int \mathcal{D}(L,\omega)\omega^2\|\hat{\boldsymbol{f}}(\omega)\|^2 d\omega \tag{7.7}$$

is deterministic, with the damping coefficient

$$\begin{aligned}\mathcal{D}(L,\omega) = \frac{1}{2\pi}\int &\exp\Big(-\frac{(\omega-h/2)^2\kappa^2}{2c^2}\big[\gamma_c(2\omega-h)+i\gamma_s(2\omega-h)\big]L\Big)\\ &\times\exp\Big(-\frac{(\omega+h/2)^2\kappa^2}{2c^2}\big[\gamma_c(2\omega+h)-i\gamma_s(2\omega+h)\big]L\Big)\hat{\psi}(h)dh.\end{aligned}\tag{7.8}$$

If, additionally, the scaled spectral coefficients γ_c and γ_s in (7.8) are almost constant over frequency bands with width $1/T_\psi$, then

$$\mathcal{D}(L,\omega) = \exp\Big(-\frac{\omega^2\kappa^2}{c^2}\gamma_c(2\omega)L\Big). \tag{7.9}$$

If, additionally, $B\ll\omega_0$ and the spectral coefficients are almost constant over the frequency band $[\omega_0-B,\omega_0+B]$, then

$$\mathcal{F}_{\rm coh} = \mathcal{F}_{\rm inc}\exp\Big(-\frac{\omega_0^2\kappa^2}{c^2}\gamma_c(2\omega_0)L\Big), \tag{7.10}$$

and this exhibits a deterministic apparent attenuation of the transmitted coherent wave, whose decay rate

$$\frac{\omega_0^2 \kappa^2}{c^2} \gamma_c(2\omega_0) \tag{7.11}$$

depends on the anisotropy κ and on the power spectral density γ_c of the fluctuations of the anisotropy axis.

Proof. We have

$$\begin{aligned}
\mathcal{F}_{\rm coh}^\varepsilon &= \frac{1}{16\pi^2} \sum_{k=1}^2 \iint \hat{a}_k^\varepsilon(L,\omega_2) \overline{\hat{a}_k^\varepsilon(L,\omega_1)} \omega_1 \omega_2 \hat{\psi}(\omega_1 - \omega_2) d\omega_1 d\omega_2 \\
&= \frac{1}{16\pi^2} \sum_{i,j,k=1}^2 \iint T_{i,k}^\varepsilon(L,\omega_2) \overline{T_{j,k}^\varepsilon(L,\omega_1)} \\
&\qquad \times \hat{f}_i(\omega_2) \overline{\hat{f}_j(\omega_1)} \omega_1 \omega_2 \hat{\psi}(\omega_1 - \omega_2) d\omega_1 d\omega_2 .
\end{aligned}$$

This quantity is bounded uniformly by $\mathcal{F}_{\rm inc}$ and its moments converge to those of (7.6), which gives the first item of the proposition.

The second item can be obtained as follows. If $T_\psi \omega_c \gg 1$, then (7.6) can be simplified into

$$\begin{aligned}
\mathcal{F}_{\rm coh} = \frac{1}{16\pi^2} \iint & \exp\Big(- \frac{\omega_2^2 \kappa^2}{2c^2} \big[\gamma_c(2\omega_2) + i \gamma_s(2\omega_2) \big] L \Big) \\
& \times \exp\Big(- \frac{\omega_1^2 \kappa^2}{2c^2} \big[\gamma_c(2\omega_1) - i \gamma_s(2\omega_1) \big] L \Big) \\
& \times \hat{\boldsymbol{f}}(\omega_2) \cdot \overline{\hat{\boldsymbol{f}}(\omega_1)} \omega_1 \omega_2 \hat{\psi}(\omega_1 - \omega_2) d\omega_1 d\omega_2 ,
\end{aligned}$$

because $\check{\mathbf{U}}$ is unitary. If $T_\psi B \gg 1$, then this becomes (7.8). □

Remark. In the geophysics literature the so-called Q-factor (or anelastic attenuation factor, or seismic quality factor) quantifies the attenuation of elastic waves [17]. It measures the relative energy loss per oscillation cycle, so that the energy then decays as $\exp(-(\omega_0/c)(z/Q))$ as a function of the propagation distance z. In our context, we find that the Q factor due to wave scattering by the anisotropy axis fluctuations is

$$\frac{1}{Q} = \frac{\omega_0}{c} \kappa^2 \gamma_c(2\omega_0), \tag{7.12}$$

where γ_c is defined by (5.4).

8. Conclusion

When a shear wave propagates through a finely layered weakly anisotropic medium, the transmitted wave consists of a coherent front pulse followed by small incoherent waves. The coherent front pulse arrives first in a time window around the expected arrival time (computed with the homogeneous background velocity) and its duration is of the order of the initial pulse width. The shape of the coherent front pulse is randomly perturbed by scattering, but its energy is deterministic and exhibits an exponential decay with the propagation distance. The decay rate depends on the power spectral density of the fluctuations of the anisotropy axis. Following the coherent front pulse, incoherent wave components may also carry energy, but they have small amplitude and arrive over a long time interval after the arrival of the coherent front pulse.

These results show that pulse stabilization is not a universal behavior. The transmitted pulse shape can be random contrarily to the prediction of the classical O'Dohertey-Anstey theory. This happens when (at least) two transmitted wave modes have equal (or very close) velocities. This result has been shown in this paper in the context of shear wave propagation through a laminated anisotropic medium. It could be extended to more general hyperbolic systems, for instance electromagnetic waves [8].

This paper also confirms that diffusion-approximation theorems applied to the wave equation in the frequency domain are a powerful tool to analyze wave propagation in random media. The key issue, as revealed in this paper, is the correlation properties of the propagator (or transmission operator) as a function of the frequency.

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Appendix A. The Diffusion Approximation Theorem for the Propagator Matrix

The evolution equation for the propagator matrix can be expanded as

$$\begin{aligned}
\frac{\partial}{\partial z}\mathbf{P}^\varepsilon = & \, i\frac{\omega\kappa}{2\varepsilon c}\cos 2\psi\big(\frac{z}{\varepsilon^2}\big)\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & -\boldsymbol{\sigma}_3\end{pmatrix}\mathbf{P}^\varepsilon \\
& + i\frac{\omega\kappa}{2\varepsilon c}\sin 2\psi\big(\frac{z}{\varepsilon^2}\big)\begin{pmatrix}\boldsymbol{\sigma}_1 & \mathbf{0}\\ \mathbf{0} & -\boldsymbol{\sigma}_1\end{pmatrix}\mathbf{P}^\varepsilon \\
& + \frac{\omega\kappa}{2\varepsilon c}\cos 2\psi\big(\frac{z}{\varepsilon^2}\big)\sin\big(2\frac{\omega z}{\varepsilon^2 c}\big)\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_3\\ \boldsymbol{\sigma}_3 & \mathbf{0}\end{pmatrix}\mathbf{P}^\varepsilon \\
& + \frac{\omega\kappa}{2\varepsilon c}\sin 2\psi\big(\frac{z}{\varepsilon^2}\big)\sin\big(2\frac{\omega z}{\varepsilon^2 c}\big)\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1 & \mathbf{0}\end{pmatrix}\mathbf{P}^\varepsilon \\
& + i\frac{\omega\kappa}{2\varepsilon c}\cos 2\psi\big(\frac{z}{\varepsilon^2}\big)\cos\big(2\frac{\omega z}{\varepsilon^2 c}\big)\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_3\\ -\boldsymbol{\sigma}_3 & \mathbf{0}\end{pmatrix}\mathbf{P}^\varepsilon \\
& + i\frac{\omega\kappa}{2\varepsilon c}\sin 2\psi\big(\frac{z}{\varepsilon^2}\big)\cos\big(2\frac{\omega z}{\varepsilon^2 c}\big)\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_1\\ -\boldsymbol{\sigma}_1 & \mathbf{0}\end{pmatrix}\mathbf{P}^\varepsilon . \qquad \text{(A.1)}
\end{aligned}$$

Applying the result of [6] about limits of random matrix equations we obtain that $\mathbf{P}^\varepsilon(z,\omega)$ converge in distribution to $\mathbf{P}(z,\omega)$ as $\varepsilon \to 0$ that is solution to the matrix-valued stochastic differential equation

$$\begin{aligned}
d\mathbf{P} = & \, i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & -\boldsymbol{\sigma}_3\end{pmatrix}\mathbf{P}\circ dW_z^1 + i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\begin{pmatrix}\boldsymbol{\sigma}_1 & \mathbf{0}\\ \mathbf{0} & -\boldsymbol{\sigma}_1\end{pmatrix}\mathbf{P}\circ dW_z^2 \\
& + \frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_3\\ \boldsymbol{\sigma}_3 & \mathbf{0}\end{pmatrix}\mathbf{P}\circ dW_z^3 + \frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_1 & \mathbf{0}\end{pmatrix}\mathbf{P}\circ dW_z^4 \\
& + i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_3\\ -\boldsymbol{\sigma}_3 & \mathbf{0}\end{pmatrix}\mathbf{P}\circ dW_z^5 + i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}_1\\ -\boldsymbol{\sigma}_1 & \mathbf{0}\end{pmatrix}\mathbf{P}\circ dW_z^6 \\
& + i\frac{\omega^2\kappa^2}{2c^2}\gamma_s(2\omega)\begin{pmatrix}-\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\mathbf{P}dz, \qquad \text{(A.2)}
\end{aligned}$$

where the W_z^j, $j = 1,\dots,6$, are independent standard Brownian motions. The propagator matrix is of the form

$$\mathbf{P}(z,\omega) = \begin{pmatrix}\mathbf{A}(z,\omega) & \overline{\mathbf{B}(z,\omega)}\\ \mathbf{B}(z,\omega) & \overline{\mathbf{A}(z,\omega)}\end{pmatrix}, \qquad \text{(A.3)}$$

where the matrices $\mathbf{A}(z,\omega)$ and $\mathbf{B}(z,\omega)$ are the limits in distribution of $\mathbf{A}^\varepsilon(z,\omega)$ and $\mathbf{B}^\varepsilon(z,\omega)$ in (4.8) and they satisfy

$$\begin{aligned} d\mathbf{A} =& \, i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\boldsymbol{\sigma}_3\mathbf{A}\circ dW_z^1 + i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\boldsymbol{\sigma}_1\mathbf{A}\circ dW_z^2 \\ &+\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_3\mathbf{B}\circ dW_z^3 + \frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_1\mathbf{B}\circ dW_z^4 \\ &+i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_3\mathbf{B}\circ dW_z^5 + i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_1\mathbf{B}\circ dW_z^6 \\ &-i\frac{\omega^2\kappa^2}{2c^2}\gamma_s(2\omega)\mathbf{A}dz, \end{aligned} \tag{A.4}$$

$$\begin{aligned} d\mathbf{B} =& -i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\boldsymbol{\sigma}_3\mathbf{B}\circ dW_z^1 - i\frac{\omega\kappa}{2c}\sqrt{2\gamma_c(0)}\boldsymbol{\sigma}_1\mathbf{B}\circ dW_z^2 \\ &+\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_3\mathbf{A}\circ dW_z^3 + \frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_1\mathbf{A}\circ dW_z^4 \\ &-i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_3\mathbf{A}\circ dW_z^5 - i\frac{\omega\kappa}{2c}\sqrt{\gamma_c(2\omega)}\boldsymbol{\sigma}_1\mathbf{A}\circ dW_z^6 \\ &+i\frac{\omega^2\kappa^2}{2c^2}\gamma_s(2\omega)\mathbf{B}dz, \end{aligned} \tag{A.5}$$

starting from $\mathbf{A}(z=0,\omega)=\mathbf{I}$ and $\mathbf{B}(z=0,\omega)=\mathbf{0}$. One can then check that

$$\mathbf{R}(z,\omega) = \overline{\mathbf{B}(z,\omega)}\,\overline{\mathbf{A}^{-1}(z,\omega)}, \qquad \mathbf{T}(z,\omega) = \overline{\mathbf{A}^{-1}(z,\omega)} \tag{A.6}$$

satisfy (5.1-5.2) and are therefore the limits in distribution of the reflection and transmission matrices in (4.15).

Appendix B. Energy Relations

The energy density of an elastic wave field $\boldsymbol{u}(t,\boldsymbol{x})$ in a isotropic medium is defined by [9]

$$e = \frac{1}{2}\sum_{k=1}^3 \rho\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \frac{1}{2}\sum_{i,j=1}^3 \sigma_{ij}\varepsilon_{ij}.$$

The energy flux $\boldsymbol{F}=(F_j)_{j=1}^3$ such that

$$\frac{\partial e}{\partial t} + \nabla\cdot\boldsymbol{F} = 0$$

is

$$F_j = -\sum_{i=1}^3 \sigma_{ji}\frac{\partial u_i}{\partial t}.$$

For a shear wave propagating along the z-axis, these equations read:

$$e = \frac{\rho}{2}\sum_{k=1}^2\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \frac{K}{2}\sum_{k=1}^2\Big(\frac{\partial u_k}{\partial z}\Big)^2,$$

and the associated flux F (in the z-direction) such that

$$\frac{\partial e}{\partial t} + \frac{\partial F}{\partial z} = 0$$

is

$$F = -K \sum_{k=1}^{2} \frac{\partial u_k}{\partial t} \frac{\partial u_k}{\partial z}.$$

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