Daylight imaging for virtual reflection seismology

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Abstract. This paper considers daylight imaging: uncontrolled noise sources deep in the earth crust emit waves that are recorded at the surface. Using a one-dimensional earth model it is shown that the autocorrelation function of the recorded signal is directly related to the response function of the earth crust. The response function is usually acquired during a reflexion seismology experiment and it can be extracted from the wave reflected by the earth crust and measured at the surface when an impulsive source is used at the surface as well. The result is obtained whatever the complexity of the medium. It shows that controlled impulsive sources are not necessary to carry out a geophysics survey but it is possible to use ambient noise sources and passive receivers only, because the correlation-based imaging technique can transform the passive sensors into virtual sources.

1. Introduction

It is now well-known that the Green's function of the wave equation in an inhomogeneous medium can be estimated by cross correlating signals emitted by ambient noise sources and recorded by passive sensors [**3, 8, 31, 32**]. In a homogeneous medium and when the source of the waves is a space-time stationary random field that is also delta-correlated in space and time, it was demonstrated in [**28, 30**] that the derivative of the cross correlation of the recorded signals is proportional to the symmetrized Green's function between the sensors (ie the difference of the causal and anticausal Green's functions). In an inhomogeneous medium and when the sources completely surround the region of the sensors the identity is still valid and it can be shown using the Helmholtz-Kirchhoff theorem [**15, 31**]. This is true even with spatially localized noise source distributions provided the waves propagate within an ergodic cavity [**3**]. At the physical level this result can be obtained in both open and closed environments provided that the recorded signals are equipartitioned [**25, 32, 26**]. In an open environment this means that the recorded signals are an uncorrelated and isotropic superposition of plane waves of all directions. In a closed environment it means that the recorded signals are superpositions of normal modes with random amplitudes that are statistically uncorrelated and identically distributed.

From a historical point of view, the emergence of the Green's function from cross correlations of ambient noise signals in a geophysical context was first pointed out

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by Jon Claerbout [**6, 7, 27**]. His statement was that it is possible to carry out a reflexion seismology experiment - which consists in measuring the wave reflected by the earth crust when an impulsive source is used at the surface - by computing the correlation function of the signals recorded at the surface by passive sensors and emitted by uncontrolled noise sources deep in the earth crust, in the so-called daylight configuration. The physical explanation given by Jon Claerbout of why daylight imaging is equivalent to reflexion seismology is quite simple and based on flux conservation. We give in this paper a complete mathematical analysis of this equivalence, which shows that the passive sensors used in daylight imaging can indeed be transformed into virtual sources by correlation techniques.

The paper is organised as follows. In Section 2 we introduce a general approach to wave propagation in a heterogeneous one-dimensional half-space with Dirichlet boundary condition at the surface. In Section 3 we characterize the response function obtained during a reflexion seismology experiment, in which a source emits a short pulse from the surface and the reflected waves are recorded at the surface as well. In Section 4 we describe the daylight imaging configuration, in which a deep noise source emits a stationary random signal that is recorded at the surface, and we clarify the relation between the autocorrelation of this signal and the response function.

2. Wave propagation in a one-dimensional complex medium

We consider the one-dimensional wave equation

$$\frac{1}{c^2(z)}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial z^2} = n(t,z) \tag{2.1}$$

in the half-space $z \in (-\infty, 0)$. The speed of propagation $c(z)$ is bounded from below and above by two positive constants and it is constant and equal to c_0 for $|z|$ large enough. The source term $n(t,z)$ is spatially compactly supported in $(-\infty, 0)$. The field $p(t,z)$ also satisfies the Dirichlet boundary condition at the surface $z = 0$:

$$p(t, z = 0) = 0. \tag{2.2}$$

When p models the pressure field, this boundary condition corresponds to the pressure release boundary condition used in geophysics. This special boundary condition comes from the fact that the density of air is much smaller than the density of the material in the earth crust.

We will study two different source configurations which are all spatially compactly supported in $(-\infty, 0)$. We first give some elementary results on the one-dimensional wave equation. A complete presentation of wave propagation in randomly layered media can be found in [**13**].

2.1. Radiation condition.

The time-harmonic field

$$\hat{p}(\omega, z) = \int p(t,z) e^{i\omega t} \mathrm{d}t$$

is solution of the Helmholtz equation

$$\frac{\partial^2 \hat{p}}{\partial z^2} + \frac{\omega^2}{c^2(z)}\hat{p} = -\hat{n}(\omega, z). \tag{2.3}$$

For some $z_{\rm i} < 0$, the medium is homogeneous in the region $z \in (-\infty, z_{\rm i}]$ where the speed of propagation is c_0. The radiation condition at $z \to -\infty$ then reads

$$\lim_{z \to -\infty} \frac{\partial}{\partial z}\hat{p}(\omega, z) + i\frac{\omega}{c_0}\hat{p}(\omega, z) = 0. \tag{2.4}$$

We assume that there is no source below $z_{\rm b} < z_{\rm i}$ (i.e., $n(t, z) \equiv 0$ for $z \le z_{\rm b}$). Then the field in the region $z \in (-\infty, z_{\rm b}]$ satisfies the homogeneous Helmholtz equation

$$\frac{\partial^2 \hat{p}}{\partial z^2} + \frac{\omega^2}{c_0^2}\hat{p} = 0,$$

and it has therefore the form

$$\hat{p}(\omega, z) = \hat{a}_-(\omega)e^{i\frac{\omega}{c_0}z} + \hat{b}_-(\omega)e^{-i\frac{\omega}{c_0}z}.$$

From the radiation condition (2.4), we find that necessarily $\hat{a}_-(\omega) = 0$, and the field is a down-going wave in the region $(-\infty, z_{\rm b}]$:

$$\hat{p}(\omega, z) = \hat{b}_-(\omega)e^{-i\frac{\omega}{c_0}z}.$$

We call it a down-going wave since it has the following form in the time-domain:

$$p(t, z) = \frac{1}{2\pi}\int \hat{p}(\omega, z)e^{-i\omega t}{\rm d}t = b_-\Big(t + \frac{z}{c_0}\Big),$$

which is indeed a wave profile propagating with constant velocity c_0 towards negative z.

2.2. Wave decomposition. We introduce the up- and down-going wave mode amplitudes:

$$\hat{a}(\omega, z) = \frac{1}{2}\Big[\hat{p}(\omega, z) + \frac{c_0}{i\omega}\frac{\partial \hat{p}}{\partial z}(\omega, z)\Big]e^{-i\frac{\omega}{c_0}z}, \tag{2.5}$$

$$\hat{b}(\omega, z) = \frac{1}{2}\Big[\hat{p}(\omega, z) - \frac{c_0}{i\omega}\frac{\partial \hat{p}}{\partial z}(\omega, z)\Big]e^{i\frac{\omega}{c_0}z}. \tag{2.6}$$

The time-harmonic field $\hat{p}$ can then be written as

$$\hat{p}(\omega, z) = \hat{a}(\omega, z)e^{i\frac{\omega}{c_0}z} + \hat{b}(\omega, z)e^{-i\frac{\omega}{c_0}z},$$

its derivative as

$$\frac{\partial \hat{p}}{\partial z}(\omega, z) = \frac{i\omega}{c_0}\Big[\hat{a}(\omega, z)e^{i\frac{\omega}{c_0}z} - \hat{b}(\omega, z)e^{-i\frac{\omega}{c_0}z}\Big],$$

and the mode amplitudes also satisfy

$$\frac{\partial \hat{a}(\omega, z)}{\partial z}e^{i\frac{\omega}{c_0}z} + \frac{\partial \hat{b}(\omega, z)}{\partial z}e^{-i\frac{\omega}{c_0}z} = 0.$$

The terminology is clear from the previous paragraph: in a region where the medium is homogeneous with velocity c_0, the wave mode amplitudes $\hat{a}$ and $\hat{b}$ do not depend on z, the up-going mode corresponds to a wave field of the form $a(t - z/c_0)$, and the down-going mode corresponds to a wave field of the form $b(t + z/c_0)$.

Substituting the mode decomposition into the Helmholtz equation (2.3) with $\hat{n} = 0$, we find that, in the regions where $\hat{n} = 0$, the wave mode amplitudes satisfy the linear system

$$\frac{\partial}{\partial z}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} = \mathbf{H}_\omega(z)\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}, \tag{2.7}$$

where

$$\mathbf{H}_\omega(z) = \frac{i\omega}{2}\Big(\frac{c_0}{c(z)^2} - \frac{1}{c_0}\Big)\begin{pmatrix} 1 & e^{-2i\frac{\omega}{c_0}z} \\ -e^{2i\frac{\omega}{c_0}z} & -1 \end{pmatrix}. \tag{2.8}$$

From (2.2) the wave mode amplitudes also satisfy the boundary condition at the free surface $z = 0$:

$$\hat{a}(\omega, z = 0) + \hat{b}(\omega, z = 0) = 0.$$

From (2.4) they satisfy the radiation condition at the bottom surface $z = z_{\rm b}$ (below which the medium is homogeneous and there is no source):

$$\hat{a}(\omega, z = z_{\rm b}) = 0.$$

Finally they also satisfy jump conditions at the location(s) of the source(s) that we describe in the next paragraph.

2.3. Source conditions. We assume that the source is of the form

$$n(t, z) = -f(t)\delta'(z - z_{\rm s}),$$

for some $z_{\rm s} \in (z_{\rm b}, 0)$. This form of the source is standard for the pressure field in the acoustic wave equation [**13**, Section 2.1.2, Eq. (2.8)]. By integrating the Helmholtz equation (2.3) accross $z = z_{\rm s}$, we get that the field $\hat{p}$ satisfies the jump conditions

$$\big[\partial_z \hat{p}(\omega, z)\big]_{z_{\rm s}^-}^{z_{\rm s}^+} = 0,$$
$$\big[\hat{p}(\omega, z)\big]_{z_{\rm s}^-}^{z_{\rm s}^+} = \hat{f}(\omega),$$

and therefore the wave mode amplitudes satisfy

$$\big[\hat{a}(\omega, z)e^{i\frac{\omega}{c_0}z} - \hat{b}(\omega, z)e^{-i\frac{\omega}{c_0}z}\big]_{z_{\rm s}^-}^{z_{\rm s}^+} = 0,$$
$$\big[\hat{a}(\omega, z)e^{i\frac{\omega}{c_0}z} + \hat{b}(\omega, z)e^{-i\frac{\omega}{c_0}z}\big]_{z_{\rm s}^-}^{z_{\rm s}^+} = \hat{f}(\omega),$$

which gives

$$\big[\hat{a}(\omega, z)\big]_{z_{\rm s}^-}^{z_{\rm s}^+} = \frac{1}{2}\hat{f}(\omega)e^{-i\frac{\omega}{c_0}z_{\rm s}}, \qquad \big[\hat{b}(\omega, z)\big]_{z_{\rm s}^-}^{z_{\rm s}^+} = \frac{1}{2}\hat{f}(\omega)e^{i\frac{\omega}{c_0}z_{\rm s}}. \tag{2.9}$$

2.4. Propagator. Any solution of the linear system (2.7) satisfies for any z, z' (such that there is no source in between z and z')

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}(\omega, z) = \mathbf{P}_\omega(z, z')\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}(\omega, z')$$

in terms of the propagator matrix $\mathbf{P}(z, z')$ solution of

$$\frac{\partial}{\partial z}\mathbf{P}_\omega(z, z') = \mathbf{H}_\omega(z)\mathbf{P}_\omega(z, z'), \tag{2.10}$$

starting from $\mathbf{P}_\omega(z = z', z') = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.

Lemma 2.1. *The propagator matrix has the form*

$$\mathbf{P}_\omega(z, z') = \begin{pmatrix} \alpha_\omega(z, z') & \overline{\beta_\omega(z, z')} \\ \beta_\omega(z, z') & \overline{\alpha_\omega(z, z')} \end{pmatrix}, \tag{2.11}$$

where $(\alpha_\omega(z,z'),\beta_\omega(z,z'))$ *is the solution of*

$$\frac{\partial}{\partial z}\begin{pmatrix}\alpha_\omega\\ \beta_\omega\end{pmatrix}(z,z') = \mathbf{H}_\omega(z)\begin{pmatrix}\alpha_\omega\\ \beta_\omega\end{pmatrix}(z,z'),$$

starting from $(\alpha_\omega(z=z',z'),\beta_\omega(z=z',z')) = (1,0)$.
The coefficients $(\alpha_\omega(z,z'),\beta_\omega(z,z'))$ *satisfy the energy conservation relation:*

$$|\alpha_\omega(z,z')|^2 - |\beta_\omega(z,z')|^2 = 1\,. \tag{2.12}$$

Proof. We consider (2.10). Applying the Jacobi's formula for the derivative of a determinant,

$$\frac{\partial \det(\mathbf{P}_\omega)}{\partial z} = \mathrm{Tr}\left(\mathrm{Adj}(\mathbf{P}_\omega)\frac{\partial \mathbf{P}_\omega}{\partial z}\right),$$

where $\mathrm{Adj}(\mathbf{P}_\omega)$ is the adjugate of $\mathbf{P}_\omega$, which satisfies $\mathbf{P}_\omega\mathrm{Adj}(\mathbf{P}_\omega) = \det(\mathbf{P}_\omega)\mathbf{I}$, and using (2.10) we get

$$\frac{\partial \det(\mathbf{P}_\omega)}{\partial z} = \mathrm{Tr}\left(\mathrm{Adj}(\mathbf{P}_\omega)\mathbf{H}_\omega\mathbf{P}_\omega\right) = \mathrm{Tr}\left(\mathbf{H}_\omega\mathbf{P}_\omega\mathrm{Adj}(\mathbf{P}_\omega)\right),$$

where we use $\mathrm{Tr}(\mathbf{MN}) = \mathrm{Tr}(\mathbf{NM})$. Using the relation between $\mathbf{P}_\omega$ and $\mathrm{Adj}(\mathbf{P}_\omega)$ we have

$$\frac{\partial \det(\mathbf{P}_\omega)}{\partial z} = \mathrm{Tr}\left(\mathbf{H}_\omega\right)\det(\mathbf{P}_\omega)\,.$$

Observe that the trace of the matrix $\mathbf{H}_\omega$ is zero. Thus the determinant of $\mathbf{P}_\omega$ is constant in z. The initial condition being the identity then gives

$$\det\big(\mathbf{P}_\omega(z,z')\big) = 1\,. \tag{2.13}$$

If $(\alpha_\omega,\beta_\omega)^T$ satisfies (2.7) with initial condition $(1,0)^T$ (here the superscript T stands for transpose), then a simple computation shows that $(\overline{\beta_\omega},\overline{\alpha_\omega})^T$ satisfies the same equation with initial condition $(0,1)^T$. Since this gives two linearly independent solutions, we deduce that the propagator $\mathbf{P}_\omega$ has the representation (2.11) with the relation (2.12) that follows from (2.13). The relation (2.12) is a manifestation of energy conservation. Indeed the energy density and the energy flux at the position z can be defined by

$$e(t,z) = \frac{1}{2c^2(z)}\partial_t p(t,z)^2 + \frac{1}{2}\partial_z p(t,z)^2, \qquad \pi(t,z) = -\partial_z p(t,z)\partial_t p(t,z),$$

and they satisfy $\partial_t\int_a^b e(t,z)\mathrm{d}z + \pi(t,b) - \pi(t,a) = 0$ if there is no source in $[a,b]$. For a time-harmonic (periodic) field,

$$p(t,z) = \frac{1}{2}\hat{p}(\omega,z)e^{-i\omega t} + c.c.,$$

where *c.c.* means complex conjugate, the time average

$$\langle\pi(\cdot,z)\rangle = -\frac{1}{2}\mathrm{Re}\Big(i\omega\partial_z\hat{p}(\omega,z)\overline{\hat{p}(\omega,z)}\Big)$$

must be constant as a function of z. In terms of the wave mode amplitudes (2.5-2.6), this reads

$$\langle\pi(\cdot,z)\rangle = \frac{\omega^2}{2c_0}\big(|\hat{a}(\omega,z)|^2 - |\hat{b}(\omega,z)|^2\big),$$

which gives (2.12). □

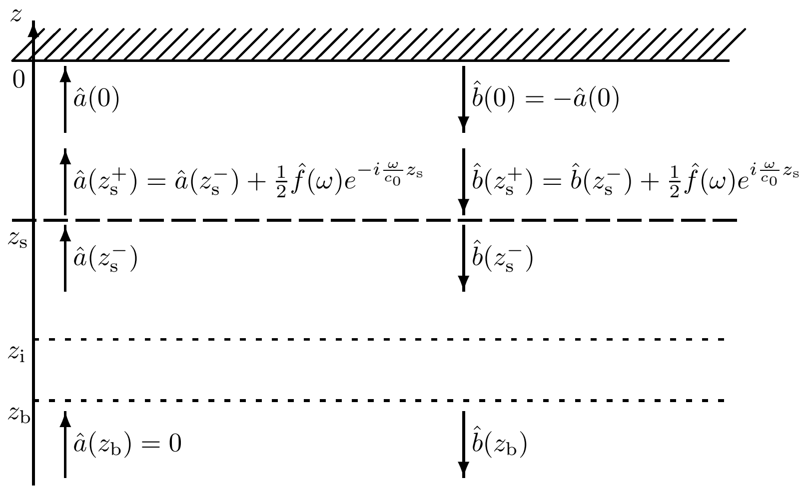

FIGURE 1. Boundary and jump conditions for the wave modes with a source $f(t)$ at $z = z_s$, free surface boundary condition at $z = 0$, and radiation condition at $z = z_b$. The medium is homogeneous below $z = z_i$.

3. Reflection seismology

We consider a situation corresponding to active reflection seismology [**4**]. A source located just below the free surface $z = 0$ emits a pulse $f(t)$ and the receiver located at the surface records the vectical velocity, which means that it records $\partial_z p_{rs}(t, z = 0)$ in our framework, where p_{rs} is the solution of (2.1) with the source $n(t,z) = -f(t)\delta'(z - z_s)$, with $z_s \simeq 0$. The purpose of the experiment is to measure the reflection operator $\mathcal{R}(t)$ or its Fourier transform $\hat{\mathcal{R}}(\omega)$ such that

$$\partial_z \hat p_{rs}(\omega, z = 0) = \frac{i\omega}{c_0} \hat{\mathcal{R}}(\omega) \hat f(\omega). \tag{3.1}$$

The following lemma expresses the reflection operator in terms of the propagator introduced above. It will be useful in the next subsection when we will show that the reflection operator can also be extracted from the correlation function of ambient noise signals.

LEMMA 3.1. *The signal recorded by the sensor at $z = 0$ is*

$$\partial_z \hat p_{rs}(\omega, z = 0) = \frac{i\omega}{c_0} \hat{\mathcal{R}}(\omega) \hat f(\omega), \tag{3.2}$$

with

$$\hat{\mathcal{R}}(\omega) = \frac{\alpha_\omega(z_i, 0) + \overline{\beta_\omega(z_i, 0)}}{\alpha_\omega(z_i, 0) - \overline{\beta_\omega(z_i, 0)}}. \tag{3.3}$$

Proof. The sensor records:

$$\partial_z \hat p_{rs}(\omega, z = 0) = \frac{i\omega}{c_0} \big[\hat a_{rs}(\omega, 0) - \hat b_{rs}(\omega, 0) \big].$$

The free surface condition reads:

$$\hat a_{rs}(\omega, 0) + \hat b_{rs}(\omega, 0) = 0,$$

the presence of the source just below the surface imposes:

$$\begin{pmatrix} \hat{a}_{\rm rs}(\omega,0) \\ \hat{b}_{\rm rs}(\omega,0) \end{pmatrix} = \begin{pmatrix} \hat{a}_{\rm rs}(\omega,0^-) \\ \hat{b}_{\rm rs}(\omega,0^-) \end{pmatrix} + \frac{1}{2}\hat{f}(\omega)\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

the propagator equation gives:

$$\begin{pmatrix} \hat{a}_{\rm rs}(\omega,z_{\rm i}) \\ \hat{b}_{\rm rs}(\omega,z_{\rm i}) \end{pmatrix} = \mathbf{P}_\omega(z_{\rm i},0)\begin{pmatrix} \hat{a}_{\rm rs}(\omega,0^-) \\ \hat{b}_{\rm rs}(\omega,0^-) \end{pmatrix},$$

and the radiation condition reads:

$$\hat{a}_{\rm rs}(\omega,z_{\rm i}) = \hat{a}_{\rm rs}(\omega,z_{\rm b}) = 0.$$

These relations form a linear system that we can solve and we find

$$\hat{a}_{\rm rs}(\omega,0) = -\hat{b}_{\rm rs}(\omega,0) = \frac{1}{2}\frac{\alpha_\omega(z_{\rm i},0)+\overline{\beta_\omega(z_{\rm i},0)}}{\alpha_\omega(z_{\rm i},0)-\overline{\beta_\omega(z_{\rm i},0)}}\hat{f}(\omega),$$

which gives the desired result. □

4. Daylight imaging

We consider a situation corresponding to daylight imaging as described by Jon Claerbout [**27**]. A source located at an unknown location $z_{\rm n}$ below the region of interest $[z_{\rm i},0]$ emits an unknown noise signal $g(t)$. The receiver located at the surface records the vectical velocity and it evaluates its autocorrelation function. This means that the receiver evaluates

$$C_{{\rm di},T}(\tau) = \frac{1}{T}\int_0^T \partial_z p_{\rm di}(t,z=0)\partial_z p_{\rm di}(t+\tau,z=0){\rm d}t, \tag{4.1}$$

where $p_{\rm di}$ is the solution of (2.1) with the source $n(t,z) = -g'(t)\delta(z-z_{\rm n})$. When the source emits a stationary noise signal $g(t)$ with mean zero and covariance function

$$F(t) = \langle g(t')g(t'+t)\rangle,$$

we will show that the autocorrelation function can be expressed in terms of the reflection operator $\hat{\mathcal{R}}(\omega)$ introduced in the previous section and the power spectral density $\hat{F}(\omega)$ of the source. We first express the autocorrelation function in terms of the propagator and power spectrum.

LEMMA 4.1. *When $T\to\infty$ the empirical autocorrelation function $C_{{\rm di},T}(\tau)$ of the noise signals recorded by the sensor at the surface (4.1) converges to the statistical cross correlation*

$$C^{(1)}_{\rm di}(\tau) = \langle \partial_z p_{\rm di}(0,z=0)\partial_z p_{\rm di}(\tau,z=0)\rangle, \tag{4.2}$$

where

$$C^{(1)}_{\rm di}(\tau) = \frac{1}{2\pi}\int \frac{\omega^2}{c_0^2}\hat{\mathcal{S}}(\omega)\hat{F}(\omega)e^{-i\omega\tau}{\rm d}\omega, \tag{4.3}$$

and

$$\hat{\mathcal{S}}(\omega) = \frac{1}{|\alpha_\omega(z_{\rm i},0)-\overline{\beta_\omega(z_{\rm i},0)}|^2}. \tag{4.4}$$

Proof. The sensor records:

$$\partial_z \hat{p}_{\rm di}(\omega, z=0) = \frac{i\omega}{c_0}\big[\hat{a}_{\rm di}(\omega,0) - \hat{b}_{\rm di}(\omega,0)\big],$$

the free surface condition reads:

$$\hat{a}_{\rm di}(\omega,0) + \hat{b}_{\rm di}(\omega,0) = 0,$$

the propagator equation gives:

$$\begin{pmatrix} \hat{a}_{\rm di}(\omega,z_{\rm i}) \\ \hat{b}_{\rm di}(\omega,z_{\rm i}) \end{pmatrix} = \mathbf{P}_\omega(z_{\rm i},0) \begin{pmatrix} \hat{a}_{\rm di}(\omega,0) \\ \hat{b}_{\rm di}(\omega,0) \end{pmatrix},$$

the homogeneous propagation from $z = z_{\rm n}$ to $z = z_{\rm i}$ gives

$$\hat{a}_{\rm di}(\omega, z_{\rm n}^+) = \hat{a}_{\rm di}(\omega, z_{\rm i}), \qquad \hat{b}_{\rm di}(\omega, z_{\rm n}^+) = \hat{b}_{\rm di}(\omega, z_{\rm i}),$$

the presence of the source at $z = z_{\rm n}$ imposes:

$$\begin{pmatrix} \hat{a}_{\rm di}(\omega,z_{\rm n}^+) \\ \hat{b}_{\rm di}(\omega,z_{\rm n}^+) \end{pmatrix} = \begin{pmatrix} \hat{a}_{\rm di}(\omega,z_{\rm n}^-) \\ \hat{b}_{\rm di}(\omega,z_{\rm n}^-) \end{pmatrix} + \frac{1}{2}\hat{g}(\omega) \begin{pmatrix} e^{-i\frac{\omega}{c_0}z_{\rm n}} \\ e^{i\frac{\omega}{c_0}z_{\rm n}} \end{pmatrix},$$

and the radiation condition reads:

$$\hat{a}_{\rm di}(\omega, z_{\rm n}^-) = \hat{a}_{\rm di}(\omega, z_{\rm b}) = 0.$$

These relations form a linear system that we can solve and we find

$$\hat{a}_{\rm di}(\omega,0) = -\hat{b}_{\rm di}(\omega,0) = \frac{1}{2}\frac{1}{\alpha_\omega(z_{\rm i},0) - \overline{\beta_\omega(z_{\rm i},0)}}\hat{g}(\omega)e^{-i\frac{\omega}{c_0}z_{\rm n}},$$

which gives

$$\partial_z \hat{p}_{\rm di}(\omega, z=0) = \frac{i\omega}{c_0}\frac{1}{\alpha_\omega(z_{\rm i},0) - \overline{\beta_\omega(z_{\rm i},0)}}\hat{g}(\omega)e^{-i\frac{\omega}{c_0}z_{\rm n}}.$$

Substituting into (4.2):

$$\begin{aligned}
C_{\rm di}^{(1)}(\tau) &= \frac{1}{4\pi^2}\iint \left\langle \overline{\partial_z \hat{p}_{\rm di}(\omega)} \partial_z \hat{p}_{\rm di}(\omega')\right\rangle e^{-i\omega\tau} {\rm d}\omega {\rm d}\omega' \\
&= \frac{1}{4\pi^2}\iint \frac{\omega\omega'}{c_0^2} \overline{\frac{1}{\alpha_\omega(z_{\rm i},0) - \overline{\beta_\omega(z_{\rm i},0)}}}\, \frac{1}{\alpha_{\omega'}(z_{\rm i},0) - \overline{\beta_{\omega'}(z_{\rm i},0)}} \\
&\quad \times e^{i\frac{\omega-\omega'}{c_0}z_{\rm n}} \left\langle \overline{\hat{g}(\omega)}\hat{g}(\omega')\right\rangle e^{-i\omega\tau}{\rm d}\omega{\rm d}\omega' \\
&= \frac{1}{2\pi}\int \frac{\omega^2}{c_0^2}\Big|\frac{1}{\alpha_\omega(z_{\rm i},0) - \overline{\beta_\omega(z_{\rm i},0)}}\Big|^2 \hat{F}(\omega) e^{-i\omega\tau}{\rm d}\omega,
\end{aligned}$$

since

$$\begin{aligned}
\left\langle \overline{\hat{g}(\omega)}\hat{g}(\omega')\right\rangle &= \iint e^{-i\omega t + i\omega' t'} \left\langle g(t) g(t')\right\rangle {\rm d}t{\rm d}t' \\
&= \iint e^{-i\frac{\omega-\omega'}{2}(t-t') - i(\omega-\omega')\frac{t+t'}{2}} F(t-t'){\rm d}t{\rm d}t' \\
&= \int e^{i\frac{\omega+\omega'}{2}\tau}F(\tau){\rm d}\tau \int e^{i(\omega-\omega')T}{\rm d}T \\
&= 2\pi\hat{F}(\omega)\delta(\omega-\omega').
\end{aligned}$$

This gives the expression (4.3) of the statistical autocorrelation function $C_{\rm di}^{(1)}$.

The statistical stability of the empirical autocorrelation follows from the same arguments as in [**15**]: by computing the variance of the autocorrelation function (see Appendix A) we can see that it converges to zero as $T \to \infty$, and therefore (by Markov inequality):

$$C_{\mathrm{di},T}(\tau) \stackrel{T\to\infty}{\longrightarrow} C_{\mathrm{di}}^{(1)}(\tau) \tag{4.5}$$

in probability. This completes the proof of the lemma. □

We can now state the main result of this section that relates the autocorrelation of the noise signal recorded by the sensor at the surface in a daylight configuration with the signal recorded during an active reflection seismometry experiment.

Proposition 4.2. *If $\hat{f}(\omega) = \omega^2 \hat{F}(\omega)$ then the autocorrelation function of the noise signal recorded by the sensor at the surface in the daylight imaging experiment (4.1) is related to the signal recorded in the reflection seismology experiment (3.1):*

$$\partial_\tau C_{\mathrm{di}}^{(1)}(\tau) = -\frac{1}{2c_0}\big[\partial_z p_{\mathrm{rs}}(\tau, z=0) - \partial_z p_{\mathrm{rs}}(-\tau, z=0)\big]. \tag{4.6}$$

Proof. The important remark is that

$$\begin{aligned}
\frac{\hat{\mathcal{R}}(\omega) + \overline{\hat{\mathcal{R}}(\omega)}}{2} &= \frac{1}{2}\frac{\alpha_\omega(z_{\mathrm{i}},0) + \overline{\beta_\omega(z_{\mathrm{i}},0)}}{\alpha_\omega(z_{\mathrm{i}},0) - \overline{\beta_\omega(z_{\mathrm{i}},0)}} + \frac{1}{2}\frac{\overline{\alpha_\omega(z_{\mathrm{i}},0)} + \beta_\omega(z_{\mathrm{i}},0)}{\overline{\alpha_\omega(z_{\mathrm{i}},0)} - \beta_\omega(z_{\mathrm{i}},0)} \\
&= \frac{|\alpha_\omega(z_{\mathrm{i}},0)|^2 - |\beta_\omega(z_{\mathrm{i}},0)|^2}{|\alpha_\omega(z_{\mathrm{i}},0) - \overline{\beta_\omega(z_{\mathrm{i}},0)}|^2} \\
&= \frac{1}{|\alpha_\omega(z_{\mathrm{i}},0) - \overline{\beta_\omega(z_{\mathrm{i}},0)}|^2} \\
&= \hat{\mathcal{S}}(\omega).
\end{aligned}$$

Therefore, provided $\hat{f}(\omega) = \omega^2 \hat{F}(\omega)$, we have

$$2i\omega c_0 \hat{C}_{\mathrm{di}}^{(1)}(\omega) = \partial_z \hat{p}_{\mathrm{rs}}(\omega, z=0) - \overline{\partial_z \hat{p}_{\mathrm{rs}}(\omega, z=0)},$$

which gives the desired result after inverse Fourier transform. □

This proposition shows that it is possible to extract the reflection operator (3.1) from the correlation function (4.1).

5. Conclusion

This paper gives a mathematical description of daylight imaging and shows how the data generated by deep seismic noise sources and recorded by a sensor at the surface of the medium can be processed to emulate a reflection seismology experiment, whatever the complexity of the medium. This result is the line of the recent work on correlation-based imaging [**1, 10, 14, 15, 16, 17, 20, 21, 22, 24, 29, 31**] and virtual source imaging [**2, 18, 19**] (and their promising applications [**5, 9, 11, 12, 23**]) and it gives a mathematical justification of the original proposition by Jon Claerbout [**6, 7, 27**].

Appendix A. The covariance of the empirical autocorrelation function

The principle of the computation is the following one. We first write the covariance function of $C_{\mathrm{di},T}$ as a multiple integral which involves the fourth-order moment of the random process g. Since g is Gaussian, this fourth-order moment can be written as the sum of products of second-order moments, which makes the computation tractable.

Using (4.1) we have

$$C_{\mathrm{di},T}(\tau) = \frac{1}{(2\pi)^2 T} \iint \overline{\partial_z \hat{p}(\omega)} \partial_z \hat{p}(\omega') e^{i(\omega'-\omega)\frac{T}{2}} \mathrm{sinc}\Big(\frac{(\omega'-\omega)T}{2}\Big) \times e^{-i\omega'\tau} \overline{\hat{g}(\omega)} \hat{g}(\omega') \mathrm{d}\omega \mathrm{d}\omega' ,$$

where $\hat{p}(\omega)$ stands for $\hat{p}_{\mathrm{di}}(\omega, z=0)$ and the covariance function can be written as

$$\begin{aligned}
&\mathrm{Cov}\big(C_{\mathrm{di},T}(\tau), C_{\mathrm{di},T}(\tau')\big) = \\
&\frac{1}{(2\pi)^4 T^2} \iiiint \mathrm{d}\omega_1 \mathrm{d}\omega_1' \mathrm{d}\omega_2 \mathrm{d}\omega_2' \overline{\partial_z \hat{p}(\omega_1)} \partial_z \hat{p}(\omega_1') \overline{\partial_z \hat{p}(\omega_2)} \partial_z \hat{p}(\omega_2') \\
&\times e^{i(\omega_1'-\omega_1+\omega_2'-\omega_2)\frac{T}{2}} \mathrm{sinc}\Big(\frac{(\omega_1'-\omega_1)T}{2}\Big) \mathrm{sinc}\Big(\frac{(\omega_2'-\omega_2)T}{2}\Big) e^{-i\omega_1'\tau - i\omega_2'\tau'} \\
&\times \Big(\Big\langle \overline{\hat{g}(\omega_1)} \hat{g}(\omega_1') \overline{\hat{g}(\omega_2)} \hat{g}(\omega_2') \Big\rangle - \Big\langle \overline{\hat{g}(\omega_1)} \hat{g}(\omega_1') \Big\rangle \Big\langle \overline{\hat{g}(\omega_2)} \hat{g}(\omega_2') \Big\rangle \Big) .
\end{aligned} \tag{A.1}$$

The fourth-order moment of the Gaussian random process g is

$$\begin{aligned}
\Big\langle \overline{\hat{g}(\omega_1)} \hat{g}(\omega_1') \overline{\hat{g}(\omega_2)} \hat{g}(\omega_2') \Big\rangle &= \Big\langle \overline{\hat{g}(\omega_1)} \hat{g}(\omega_1') \Big\rangle \Big\langle \overline{\hat{g}(\omega_2)} \hat{g}(\omega_2') \Big\rangle \\
&+ \Big\langle \overline{\hat{g}(\omega_1) \hat{g}(\omega_2)} \Big\rangle \langle \hat{g}(\omega_1') \hat{g}(\omega_2') \rangle \\
&+ \Big\langle \overline{\hat{g}(\omega_1)} \hat{g}(\omega_2') \Big\rangle \Big\langle \hat{g}(\omega_1') \overline{\hat{g}(\omega_2)} \Big\rangle ,
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathrm{Cov}\Big(\overline{\hat{g}(\omega_1)} \hat{g}(\omega_1'), \overline{\hat{g}(\omega_2)} \hat{g}(\omega_2') \Big) &= (2\pi)^2 \hat{F}(\omega_1) \delta(\omega_1 + \omega_2) \hat{F}(\omega_1') \delta(\omega_1' + \omega_2') \\
&+ (2\pi)^2 \hat{F}(\omega_1) \delta(\omega_1 - \omega_2') \hat{F}(\omega_1') \delta(\omega_2 - \omega_1') .
\end{aligned}$$

Substituting into (A.1), we obtain for all $T > 0$ the following expression for the covariance function:

$$\begin{aligned}
&\mathrm{Cov}\big(C_{\mathrm{di},T}(\tau), C_{\mathrm{di},T}(\tau')\big) = \\
&\frac{1}{(2\pi)^2 T^2} \iiiint \mathrm{d}\omega_1 \mathrm{d}\omega_1' \mathrm{d}\omega_2 \mathrm{d}\omega_2' \overline{\partial_z \hat{p}(\omega_1)} \partial_z \hat{p}(\omega_1') \overline{\partial_z \hat{p}(\omega_2)} \partial_z \hat{p}(\omega_2') \\
&\times e^{i(\omega_1'-\omega_1+\omega_2'-\omega_2)\frac{T}{2}} \mathrm{sinc}\Big(\frac{(\omega_1'-\omega_1)T}{2}\Big) \mathrm{sinc}\Big(\frac{(\omega_2'-\omega_2)T}{2}\Big) e^{-i\omega_1'\tau - i\omega_2'\tau'} \\
&\times \hat{F}(\omega_1) \hat{F}(\omega_1') \Big(\delta(\omega_1 + \omega_2) \delta(\omega_1' + \omega_2') + \delta(\omega_1 - \omega_2') \delta(\omega_2 - \omega_1') \Big) ,
\end{aligned} \tag{A.2}$$

which gives

$$\begin{aligned}
\mathrm{Cov}\big(C_{\mathrm{di},T}(\tau), C_{\mathrm{di},T}(\tau')\big) &= \frac{1}{4\pi^2 T^2} \iint \mathrm{d}\omega \mathrm{d}\omega' \hat{F}(\omega) \hat{F}(\omega') |\partial_z \hat{p}(\omega)|^2 |\partial_z \hat{p}(\omega')|^2 \\
&\times \mathrm{sinc}^2\Big(\frac{(\omega'-\omega)T}{2}\Big) \Big[e^{-i\omega'\tau + i\omega'\tau'} + e^{-i\omega'\tau - i\omega\tau'} \Big] .
\end{aligned} \tag{A.3}$$

Taking the limit $T \to \infty$, and using the fact that $\int \mathrm{sinc}^2 s ds = \pi$, we see that the variance is of order $1/T$:

$$2\pi T \mathrm{Var}\big(C_{\mathrm{di},T}(\tau)\big) \overset{T\to\infty}{\longrightarrow} \int \hat{F}(\omega)^2 |\partial_z \hat{p}(\omega)|^4 \Big[1 + e^{-2i\omega\tau}\Big] \mathrm{d}\omega \,, \tag{A.4}$$

which quantifies the convergence rate in (4.5). Note that the first term of the asymptotic variance does not depend on τ, which means that it corresponds to fluctuations for the cross correlation around its mean $C_{\mathrm{di}}^{(1)}$ that are stationary and extend over the whole time axis. The second term corresponds to local fluctuations, localized around $\tau = 0$. The time scale of the fluctuations of the cross correlation can be quantified from the asymptotic covariance function

$$2\pi T \mathrm{Cov}\big(C_{\mathrm{di},T}(\tau), C_{\mathrm{di},T}(\tau+\Delta\tau)\big) \overset{T\to\infty}{\longrightarrow} \int e^{-i\omega\Delta\tau} \hat{F}(\omega)^2 |\partial_z \hat{p}(\omega)|^4 \Big[1 + e^{-2i\omega\tau}\Big] \mathrm{d}\omega \,,$$

which shows that the decoherence time of the fluctuations of $C_{\mathrm{di},T}$ is proportional to the decoherence time of the source.

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