Impact of self-steepening on incoherent dispersive spectral shocks and collapselike spectral singularities

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Incoherent dispersive shock waves and collapselike singularities have been recently predicted to occur in the spectral evolution of an incoherent optical wave that propagates in a noninstantaneous nonlinear medium. Here we extend this work by considering the generalized nonlinear Schrödinger equation. We show that self-steepening significantly affects these incoherent spectral singularities: (i) It leads to a delay in the development of incoherent dispersive shocks, and (ii) it arrests the incoherent collapse singularity. Furthermore, we show that the spectral collapselike behavior can be exploited to achieve a significant enhancement (by two orders of magnitudes) of the degree of coherence of the optical wave.

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I. INTRODUCTION

Shock waves have been thoroughly investigated during the last century in many different branches of physics [1]. The well-known phenomenon of viscous shock waves in a dissipative compressible fluid (gas) is characterized by a steep jump in gas velocity, density, and temperature across which dissipation of energy due to particle collisions regularizes the shock singularity. On the other hand, in conservative systems a different regularization occurs that entails the formation, owing to dispersion, of rapidly oscillating nonstationary structures, so-called undular bores or dispersive shock waves (DSWs). Since their mathematical construction in 1974 [2], DSWs have been recognized as a fundamental example of singular nonlinear wave behavior which has generated much interest among diverse areas of physics. Originally observed in plasmas [3] and water waves [4], DSWs are now the subject of intense theoretical and experimental studies in Bose-Einstein condensates [5], unitary Fermi gases [6], nonlinear optics [7,8], oceanography [4], quantum liquids [9], nonlinear chains or granular materials [10], and electrons [11]. Also notice that the role of structural disorder of the medium on the properties of DSWs has been investigated in the context of optical waves [12].

These previous studies on DSWs have been reported for coherent, i.e., deterministic, amplitudes of the waves. From a different perspective, we predicted in a recent work the existence of incoherent DSWs which manifest themselves as a wave breaking process (“gradient catastrophe”) in the spectral dynamics of the incoherent wave that evolves in a noninstantaneous nonlinear environment [13]. These incoherent shocks are unexpected because, in contrast to conventional shocks which are known to require a strong nonlinear regime, incoherent DSWs develop into the highly incoherent regime of propagation, in which linear dispersive effects dominate nonlinear effects. On the basis of a wave turbulence (WT) approach [14,15], the theory revealed that these incoherent singular objects are described, as a rule, by singular integro-differential kinetic equations (SIDKEs). This theoretical approach revealed interesting links with the three-dimensional vorticity equation in incompressible fluids [16], as well as with the integrable Benjamin-Ono (BO) equation [17] originally derived in hydrodynamics for stratified fluids and recently investigated in the semiclassical limit to study coherent wave breaking processes [18]. This kinetic theory provided a detailed description of the mechanism underlying the formation, as well as the inhibition, of spectral incoherent shocks: While incoherent DSWs are generated for a damped harmonic oscillator response function, for a purely exponential response the incoherent wave exhibits a different form of incoherent singular behavior, namely, a collapse singularity in the spectral evolution of the random wave [13].

Our aim in this article is to make a step toward the experimental observation of these novel singular incoherent entities. In this respect, nonlinear optical fibers constitute ideal test beds for their experimental study, owing to the easily tailorable noninstantaneous nonlinear response function, e.g., through the Raman effect, or other mechanisms involving liquid- and gas-filled photonic crystal fibers (PCFs) or surface plasmon polaritons [19]. The analysis reveals that the developments of incoherent DSWs and incoherent spectral collapses (ISCs) require a broad spectral bandwidth of the incoherent wave, whose description calls for a generalized nonlinear Schrödinger equation (GNLSE). In this article we go beyond the basic NLSE considered in Ref. [13] by studying the influence of higher-order effects, such as self-steepening, on the spectral dynamics of the incoherent optical wave. On the basis of the GNLSE, we extend the theory developed in Ref. [13], and show that self-steepening can deeply affect the dynamics of the incoherent wave: (i) It leads to a delay in the development of the incoherent DSW in the presence of a damped harmonic oscillator response, (ii) It inhibits the ISC singularity in the presence of an exponential response function. Furthermore, we show that the ISC-like behavior can be exploited to achieve a significant enhancement (by two orders of magnitudes) of the degree of coherence of the random optical wave.

II. GNLSE SIMULATIONS

The starting point is the usual GNLSE describing the propagation of broadband optical waves in highly nonlinear optical fiber systems [20,21],

$$-i\frac{\partial\psi(z,t)}{\partial z} = \sum_{j=2}^{m}\frac{i^j\beta_j}{j!}\frac{\partial^j\psi(z,t)}{\partial t^j} + \gamma\left(1+i\tau_s\frac{\partial}{\partial t}\right)\psi(z,t) \times \int_{-\infty}^{+\infty}\mathcal{R}(t')\,|\psi(z,t-t')|^2\,dt', \quad (1)$$

where γ refers to the nonlinear coefficient and $\mathcal{R}(t) = (1 - f_R)\delta(t) + f_R R(t)$ to the usual response function accounting for the instantaneous Kerr effect and the noninstantaneous Raman response $R(t)$, which is constrained by the causality condition $R(t) = 0$ for $t < 0$. Its typical width denotes the nonlinear response time τ_R, which needs to be compared with the "healing time" $\tau_0 = \sqrt{\beta_2 L_{nl}/2}$, where $L_{nl} = 1/(\gamma P)$ is the nonlinear length and P the averaged power. The healing time denotes the time scale for which linear and nonlinear effects are of the same order, e.g., the typical time period of modulational instability [15]. DSWs and ISCs develop in the highly noninstantaneous nonlinear ("long-range") regime $\tau_R \gg \tau_0$ [13,22]. Higher-order time derivatives in the GNLSE (1) originate in a Taylor series expansion of the dispersion curve of the fiber, $k(\omega_0 + \omega) = \sum_j \beta_j\omega^j/j!$, around the carrier angular frequency $\omega_0 = 2\pi c/\lambda_0$, where λ_0 is the central wavelength of the partially coherent source and c the speed of light. In this expansion, β_0 is the carrier wave number, β_1 the inverse group velocity, β_2 the (second-order) group velocity dispersion, and β_j, $j \geqslant 3$, the higher-order dispersion terms. Equation (1) also describes self-steepening through the term proportional to $\tau_s\partial/\partial t$, which accounts for the dispersion of the nonlinearity [20]. We remind that the GNLSE (1) conserves the "number of photons" $N = \int |\tilde{\psi}(\omega,z)|^2/(1+\tau_s\omega)\,d\omega$ [23]. We refer the reader to Refs. [20,21] for more details concerning the GNLSE model (1).

III. GENERALIZED KINETIC EQUATION

To describe the influence of self-steepening on the incoherent spectral dynamics, we generalize the previous theory [15,24] based on a WT statistical description of the random wave [14,25]. A key assumption of the theory is that the incoherent wave evolves in the weakly nonlinear regime, i.e., the highly incoherent regime in which linear dispersive effects dominate nonlinear effects, $t_c \ll \tau_0$, t_c being the time correlation [15,24]. Starting from the GNLSE (1), one obtains a closure of the hierarchy of moments equations, in which the averaged spectrum of the wave $\langle\tilde{\psi}_{\omega+\Omega/2}\tilde{\psi}^*_{\omega-\Omega/2}\rangle = n_\omega(z)\delta(\Omega)$ is governed by the generalized kinetic equation (GKE)

$$\partial_z n_\omega = \frac{\gamma}{\pi}n_\omega\int G(\omega,\omega')n_{\omega'}\,d\omega', \quad (2)$$

where $G(\omega,\omega') = (1+\tau_s\omega)g(\omega-\omega')$, and $g(\omega) = \mathrm{Im}[\tilde{R}(\omega)]$ is the imaginary part of the Fourier transform of $R(t)$. Note that in the limit $\tau_s = 0$, Eq. (2) recovers the kinetic equation previously derived in [15,24]. It is interesting to remark that the GKE (2) accounts for nonlinear dispersive effects (self-steepening), but not for linear dispersion effects—although linear dispersion plays a key role in the establishment of the weakly nonlinear regime. Simulations of the GKE (2) are found to be in quantitative agreement with GNLSE simulations, without adjustable parameters (see Fig. 1), an important fact that was already pointed out in previous works without self-steepening ($\tau_s = 0$) [24]. However, the validity of the GKE becomes questionable when the optical spectrum experiences the presence of a zero-dispersion frequency (ZDF) of the fiber. Near a ZDF, linear dispersive effects become perturbative: The dynamics turns out to be dominated by nonlinear effects, which invalidates the weakly nonlinear assumption underlying the derivation of the GKE (2). In this case one needs to include higher-order contributions in the closure of the hierarchy of the moments equation: To next order, the instantaneous Kerr nonlinearity coupled to higher-order dispersion leads to a collision term that describes the well-known process of supercontinuum (SC) generation through optical wave thermalization [26] (also see [15,27]). Here, in order to avoid SC spectral broadening which inevitably alters the dynamics of DSWs and ISCs, we assume that the spectrum evolves far from a ZDF. We finally note that the GKE can easily be generalized by including a frequency dependence of the nonlinear Kerr coefficient, so that γ becomes a function $\gamma(\omega)$ in Eq. (2).

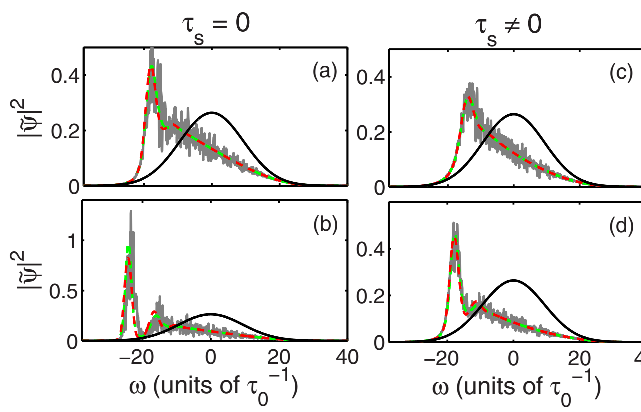

FIG. 1. (Color online) Incoherent DSWs (without spectral background) in the absence (a),(b), and in the presence (c),(d) of self-steepening, $\tau_s = 1/\omega_0$. Numerical simulations of the GNLSE (1) with a damped harmonic oscillator response (gray), of the GKE (2) (green), of the SIDKE (4) (dashed red): $z = 140L_{nl}$ for (a) and (c); $z = 240L_{nl}$ for (b) and (d). The initial condition is in solid black. Parameters are $\tau_R = 1.5\tau_0$, $\tau_s = \tau_0/37.8$, and $\sigma/(2\pi) = 1.5\tau_0^{-1}$, with $\tau_0 = \sqrt{\beta_2 L_{nl}/2}$, $\tau_R = 32$ fs, $\eta = \tau_R/\tau_1 = 2.6$, and $f_R = 0.18$.

IV. SINGULARITIES OF THE CONVOLUTION OPERATOR

Incoherent DSWs and ISC singularities have been shown to develop in the regime where the spectral bandwidth of the optical wave, $\Delta\omega$, is much larger than the spectral gain bandwidth $\Delta\omega_g \sim 1/\tau_R$ of $g(\omega)$ ($\Delta\omega \gg \Delta\omega_g$) [13]. In this long-range regime, the tail behavior of the gain spectrum $g(\omega)$ plays a key role in the spectral dynamics. Because of the causality condition of $R(t)$, the function $g(\omega)$ always decays algebraically at infinity, e.g., $\sim 1/\omega^3$ for a damped harmonic oscillator and $\sim 1/\omega$ for an exponential response. This slow decay introduces singularities into the convolution operator of the GKE (2), $M_\omega = \int g(\omega-u)n_u(z)\,du$, which in turn leads to divergent integrals. These singularities can be addressed accurately by introducing the Hilbert operator $\mathcal{H}f(\omega) = \pi^{-1}\mathrm{P}\int_{-\infty}^{+\infty} \frac{f(\omega-u)}{u}du$, where P denotes the Cauchy principal value. Considering the long-range regime $\tau_R \gg \tau_0$, it was shown in the Supplemental Material of Ref. [13] that the convolution operator takes the form

$$M_\omega = -\frac{\pi \bar{R}(0)}{\tau_R}\mathcal{H}n_\omega + \frac{\pi \bar{R}^{(1)}(0)}{\tau_R^2}\partial_\omega n_\omega + \frac{\pi \bar{R}^{(2)}(0)}{2\tau_R^3}\mathcal{H}\partial_\omega^2 n_\omega - \frac{\pi \bar{R}^{(3)}(0)}{6\tau_R^4}\partial_\omega^3 n_\omega + o\left(\frac{1}{\tau_R^4}\right), \tag{3}$$

where $\bar{R}(t)$ is a smooth function defined by $R(t) = \tau_R^{-1}\bar{R}(t/\tau_R)H(t)$, $\bar{R}^{(n)}(0)$ denoting the nth derivative at $t=0$ and $H(t)$ the Heaviside function. The derivation of Eq. (3) is tricky. It was reported in detail in the Supplemental Material of Ref. [13], although Eq. (3) was not reported in the main Letter. The writing of Eq. (3) is instructive, in that it makes explicitly apparent that the dominant terms in the convolution operator M_ω are determined by the behavior of the response function near the origin $t=0$. In the following we consider different representative examples of response functions to illustrate different qualitative spectral dynamics of the incoherent wave.

V. DAMPED HARMONIC OSCILLATOR RESPONSE

A. Without background

We first consider the example of the damped harmonic oscillator, i.e., the usual Raman-like response function $R(t) = H(t)\frac{1+\eta^2}{\eta\tau_R}\sin(\eta t/\tau_R)\exp(-t/\tau_R)$ [20]. This function is *continuous* at $t=0$, so that the first term in the expansion of the convolution operator (3) vanishes, $\bar{R}^{(0)}(0) = 0$. Then plugging Eq. (3) into the GKE (2), one obtains the SIDKE

$$\partial_z n_\omega = \frac{\gamma(1+\eta^2)}{\tau_R^2}(1+\tau_s\omega)\left(n_\omega\partial_\omega n_\omega - \frac{1}{\tau_R}n_\omega\mathcal{H}\partial_\omega^2 n_\omega\right). \tag{4}$$

Despite the self-steepening term, the leading-order Burgers term in (4) predicts that the spectral dynamics should develop a DSW, whose singularity is regularized by the nonlinear dispersive term involving the Hilbert operator. The self-steepening effect should delay the formation of the shock, since the front of the shock develops in the low-frequency part of the spectrum ($\omega < 0$).

We have confirmed these predictions by performing simulations of the GNLSE (1). The initial condition is a partially coherent wave with Gaussian spectrum ($\sim\exp[-\omega^2/(2\sigma^2)]$) and random spectral phases, i.e., the incoherent wave exhibits fluctuations that are statistically stationary in time. The simulations of the GNLSE (1) are reported in Fig. 1, in the absence ($\tau_s = 0$, left column) and in the presence ($\tau_s \simeq 1/\omega_0$, right column) of self-steepening. As expected, self-steepening leads to a delay in the development of the incoherent DSW. Let us stress that a quantitative agreement has been obtained between the simulations of the GNLSE (1), the GKE (2), and the SIDKE (4), without adjustable parameters. We also verified by GNLSE simulations that perturbative higher-order dispersion effects do not affect the spectral dynamics of the incoherent wave, in agreement with the theory.

B. With background

Let us now study the impact of self-steepening in the regime in which the incoherent wave evolves in the presence of a significant background spectral noise $n_\omega(z) = n_0 + \tilde{n}_\omega(z)$. This regime is interesting because the corresponding SIDKE was shown to recover the integrable BO equation in the absence of self-steepening [13]. Here, starting from Eq. (4) and considering a multiscale expansion with $\tilde{n}_\omega(z) \sim n_0/\tau_R$, we obtain the SIDKE

$$\partial_z\tilde{n}_\omega - \gamma(1+\eta^2)\frac{1+\tau_s\omega}{\tau_R^2}n_0\partial_\omega\tilde{n}_\omega = \gamma(1+\eta^2)\frac{1+\tau_s\omega}{\tau_R^2}\left(\tilde{n}_\omega\partial_\omega\tilde{n}_\omega - \frac{1}{\tau_R}n_0\mathcal{H}\partial_\omega^2\tilde{n}_\omega\right). \tag{5}$$

It becomes apparent that in the absence of self-steepening ($\tau_s = 0$), the second term in the left-hand side can be removed by means of a change of the Galilean reference frame in frequency space, so that Eq. (5) recovers the BO equation. The evolution of the spectrum has been reported in Fig. 2, in which an initial dark perturbation (spectral hole) has been superimposed on the spectral background. The spectral evolution is characterized by an expansion (rarefaction) wave on the leading edge and a gradient catastrophe on the trailing edge, which is regularized by an expanding dispersive wave train, both features being described in detail by the SIDKE (5). As illustrated by Fig. 2, self-steepening ($\tau_s \neq 0$) slows down the spectral redshift of the whole expansion wave, a feature that can be interpreted in a way analogous to the case without

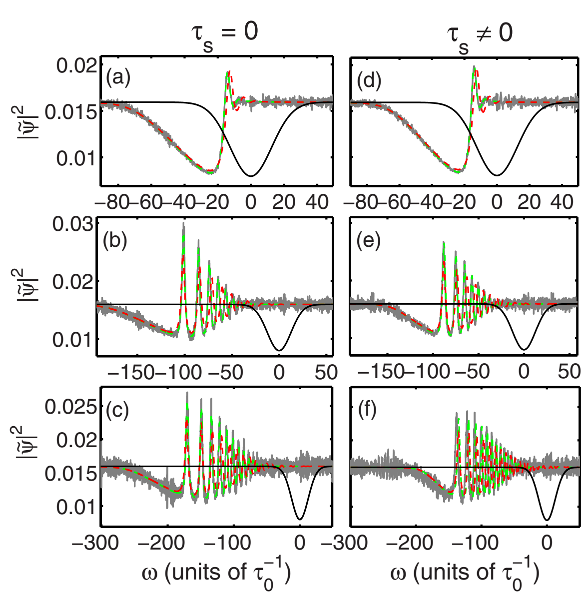

FIG. 2. (Color online) Incoherent DSWs from a darklike input spectrum with background noise [$n_0 \gg \tilde{n}_\omega(z)$] in the absence (a)–(c) and in the presence (d)–(f) of self-steepening, $\tau_s = 1/\omega_0$. Numerical simulations of the GNLSE (1) with a damped harmonic oscillator response (tiny gray), of the GKE (2) (green), of the SIDKE (5) (dashed red): $z = 6\times10^3$ (a),(d), $z = 20\times10^3$ (b),(e), and $z = 30\times10^3$ (c),(f), in units of L_{nl}. The initial condition is in solid black. Parameters are $\tau_R = 2\tau_0$, $\tau_s = \tau_0/400$, $\sigma/(2\pi) = 2\tau_0^{-1}$, $\tau_R = 32$ fs, $\eta = \tau_R/\tau_1 = 1$, and $f_R = 1$. Note that the ω scale is different for each row.

background discussed through Fig. 1 and the SIDKE (4). Note that DSWs can also be generated starting from a bright initial condition, as was discussed in Ref. [13].

The derivation of the SIDKE (5) may be exploited to analyze the impact of self-steepening ($\tau_s \neq 0$) by means of a perturbation theory of the integrable BO equation, in relation to previous studies dealing with the perturbed Korteweg–de Vries (KdV) equation [28,29]. In the previous works [28] the propagation of shallow-water solitary waves and nonlinear periodic waves (undular bores) over a gradual slope with bottom friction were considered in the framework of the KdV equation with slowly variable coefficients. Note, however, that the nature of the perturbation introduced by self-steepening in the BO Eq. (5) is quite complex, since it is both inhomogeneous and nonlocal, and more importantly, it cannot be treated as a weak perturbation. Indeed, the corrective terms proportional to $\tau_s\omega$ become of order 1 when the front of the spectral amplitude $\tilde{n}_\omega(z)$ approaches $\omega \sim -\omega_0$, a feature that will be shown to be responsible for the inhibition of the ISC singularity in the next section. Therefore caution should be exercised when applying to this problem the Whitham modulation theory [1], or a standard multiple-scale perturbation method [30] whose validity should be restricted to frequency intervals far from $-\omega_0$.

VI. EXPONENTIAL RESPONSE FUNCTION

A. Without background

We now consider a purely exponential response function $R(t) = H(t)\exp(-t/\tau_R)/\tau_R$, whose discontinuity at $t = 0$ completely changes the dynamics since it is now the first term in (3) which is dominant. One may question the physical relevance of the discontinuity of the response function at $t = 0$. Such discontinuity arises whenever the "excitation time," say τ_e, of a system is much smaller than the smaller relevant time scale of the problem—here the time correlation of the incoherent wave, t_c. Let us illustrate this aspect by considering the overdamped harmonic oscillator, whose normalized response function reads $R(t) = H(t)w_0^2(\nu^2 - w_0^2)^{-1/2}\exp(-\nu t)\sinh[(\nu^2 - w_0^2)^{-1/2}t]$. In the regime $\nu \gg w_0$, $R(t) \sim 2w_0^2 H(t)(\nu^2 - w_0^2)^{-1/2}[\exp(-t/\tau_R) - \exp(-t/\tau_e)]$, where $\tau_e = 1/(2\nu)$ and $\tau_R = 2\nu/w_0^2$. Although $R(t)$ is continuous at $t = 0$, in the overdamped regime ($\nu \gg w_0$) the excitation time can be neglected whenever $\tau_e \ll t_c$, so that the response function behaves as an exponential discontinuous function, $R(t) = H(t)\exp(-t/\tau_R)/\tau_R$. The validity of the criterion $\tau_e \ll t_c$ has been checked by direct numerical simulations of the GNLSE (1).

In the limit of a purely exponential response function, we have $\bar{R}^{(0)}(0) \neq 0$ in Eq. (3), which gives the SIDKE

$$\partial_z n_\omega = \frac{\gamma(1+\tau_s\omega)}{\tau_R}\left(-n_\omega \mathcal{H} n_\omega - \frac{1}{\tau_R} n_\omega \partial_\omega n_\omega + \frac{1}{2\tau_R^2} n_\omega \mathcal{H}\partial_\omega^2 n_\omega\right). \tag{6}$$

In the absence of self-steepening, $\tau_s = 0$, the leading-order term in (6) was discussed in some details in Ref. [13], in relation to an analytical solution originally derived in [16]. It

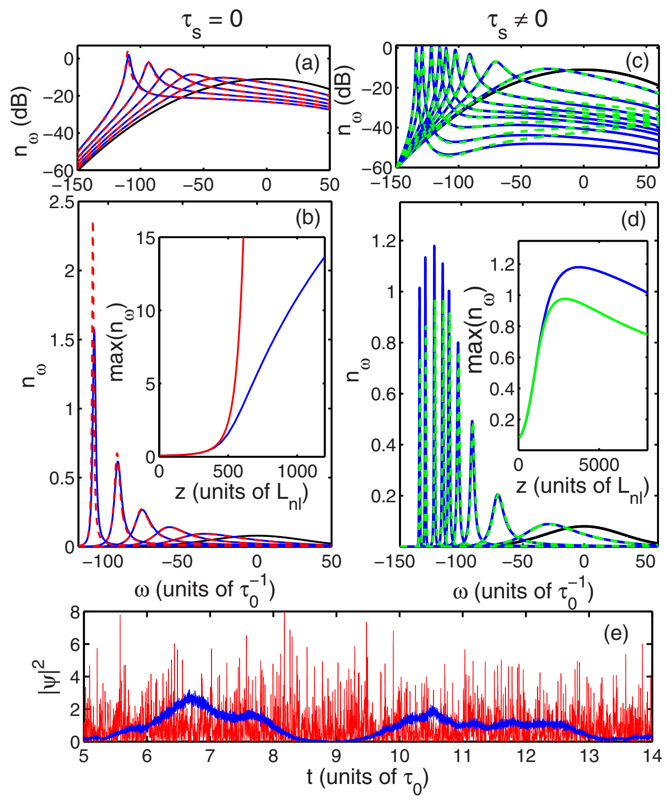

FIG. 3. (Color online) Exponential response function: Inhibition of ISC singularity by self-steepening ($\tau_s = 1/\omega_0$). Simulations of the GKE (2) in the absence [(a),(b), left column], and in the presence [(c),(d), right column] of self-steepening, in $\log_{10}$ scale (a),(c) and normal scale (b),(d). The dashed red line in (a) and (b) denotes the analytical solution of Eq. (7) with $\tau_s = 0$ describing the ISC singularity. The dashed green line in (c) and (d) denotes the simulations of the SIDKE (6) with $\tau_s \neq 0$ which leads to the inhibition of the ISC singularity. The insets show the corresponding evolutions of the maximum of the spectral peak, $\max(n_\omega)$, vs z: the GKE (2) blue, Eq. (7) red, and the SIDKE (6) green. From right to left: in (a),(b) $z = 0,100,200,300,400,500$; in (c),(d) $z = 0,100,500,1000,1500,2000,2500,3500,5500,8000$ in units of L_{nl}, $\tau_s = \tau_0/150$, $\tau_R = 5\tau_0$, $\sigma/(2\pi) = 5\tau_0^{-1}$, and $f_R = 1$. (e) Simulations of the GNLSE (1) showing $|\psi|^2(t)$ at $z = 0$ (red) and $z = 20\,000 L_{nl}$ (blue): collapselike behavior leads to an increase of the time correlation of the random wave by approximately two orders of magnitude [$\sigma/(2\pi) = 50\tau_0^{-1}, \tau_s = \tau_0/4 \times 10^3$, and $\tau_R = 5\tau_0$].

was shown that the spectrum exhibits a collapselike behavior with $\tau_s = 0$, while the spectral peak is shifted toward the low-frequency components ($\omega < 0$) with a constant velocity. The presence of self-steepening changes this behavior in a substantial way. This is illustrated in Fig. 3, which reports numerical simulations of the GKE (2) in the absence (left column) and in the presence (right column) of self-steepening. The main result is that self-steepening arrests the ISC, as illustrated by the evolutions of the maxima of the spectral amplitudes and the decreasing of the moments of the spectrum reported in Fig. 4.

The impact of self-steepening on the dynamics is mainly captured by the leading-order term in the SIDKE (6)—note in this respect that the second Burgers term would generate a

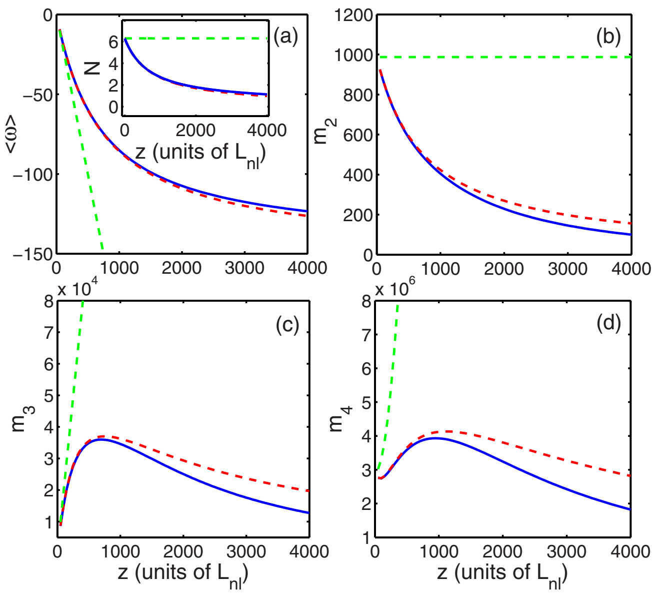

FIG. 4. (Color online) Evolutions of the first- (a), second- (b), third- (c), and fourth- (d) order moments obtained by plotting Eqs. (8)–(11) (dashed red), and by numerical simulation of the GKE (2) (blue) ($m_j \equiv \langle \bar{\omega^j} \rangle$, $j \geqslant 2$). The inset in (a) shows N vs z obtained by simulation of the GKE (2) (blue) and analytically from Eq. (7) (dashed red). The dashed green lines show the cases $\tau_s = 0$. Parameters are the same as in Figs. 3(c) and 3(d).

shock toward the high-frequency components ($\omega > 0$). In the following we thus analyze the reduced equation

$$\partial_{\tilde{z}} n_\omega = -(1 + \tau_s \omega) n_\omega \mathcal{H} n_\omega, \tag{7}$$

where $\tilde{z} = \gamma z/\tau_R$. Because of self-steepening, Eq. (7) does not conserve the power, $N(z) = \int n_\omega(z)\, d\omega$, which decreases according to $N(z) = N_0/(1 + N_0 \tau_s \tilde{z}/2)$, where $N_0 = N(\tilde{z} = 0)$ [see the inset in Fig. 4(a)]. The dynamics can be characterized through the evolutions of the moments of the spectrum, which can be calculated in explicit form. We denote $\langle \omega^p \rangle(z) = \frac{1}{N(z)} \int \omega^p n_\omega(z)\, d\omega$, $\langle \omega_0^p \rangle = \frac{1}{N_0} \int \omega^p n_\omega^0\, d\omega$, $\langle \bar{\omega^p} \rangle = \langle (\omega - \langle \omega \rangle)^p \rangle$, and assume that the initial spectrum $n_\omega(z = 0) = n_\omega^0$ is even. We obtain from Eq. (7)

$$\langle \omega \rangle = -\frac{N_0 \tilde{z}}{2 + \tau_s N_0 \tilde{z}}, \tag{8}$$

$$\langle \bar{\omega^2} \rangle = \frac{\langle \omega_0^2 \rangle}{1 + \tau_s N_0 \tilde{z}/2}, \tag{9}$$

$$\langle \bar{\omega^3} \rangle = \frac{2 N_0 \tilde{z}}{(2 + \tau_s N_0 \tilde{z})^2} \langle \omega_0^2 \rangle, \tag{10}$$

$$\langle \bar{\omega^4} \rangle = \frac{2\langle \omega_0^4 \rangle}{2 + \tau_s N_0 \tilde{z}} + \frac{2\langle \omega_0^2 \rangle N_0^2 \tilde{z}^2}{(2 + \tau_s N_0 \tilde{z})^3} - \frac{2\langle \omega_0^2 \rangle^2 \tau_s N_0 \tilde{z}}{(2 + \tau_s N_0 \tilde{z})^2}. \tag{11}$$

The moment $\langle \omega \rangle$ gives the mean or central frequency of the spectrum, while the moments $\langle \bar{\omega^p} \rangle$ describe the dispersion of the spectrum around its mean. Without self-steepening, $\tau_s = 0$, Eq. (8) shows that the spectral peak is redshifted with a constant velocity and thus virtually crosses the zero-frequency component of the optical field, $\omega = -\omega_0$. Self-steepening regularizes this nonphysical negative frequency propagation through a significant deceleration of the spectral peak in the frequency domain, a noteworthy property confirmed by the simulations of the GKE (2) [see Fig. 4(a)].

The evolutions of higher-order moments Eqs. (9)–(11) are deeply affected by self-steepening, as illustrated by Figs. 4(b)–4(d). Note in Fig. 4(b) that the second-order moment is constant for $\tau_s = 0$, in spite of the ISC singularity. This is due to the contribution of the long spectral tail which leaves the spectrum behind itself while it is redshifted; see Fig. 3(a). Conversely, the second-order moment decreases with $\tau_s \neq 0$, because self-steepening effectively reduces the weight of the long spectral tail (see Fig. 3). This is confirmed by the analysis of higher-order moments—especially the fourth-order moment which is more sensitive to the spectral long tail [Fig. 4(d)]. In the absence of self-steepening, the third- and fourth-order moments exhibit a monotonic growth, which is saturated and then inverted by self-steepening, so that all moments decrease with propagation, consistently with the inhibition of the ISC singularity. Also, the deceleration of the spectral peak effectively reduces the asymmetry of the whole spectrum n_ω, as revealed by the impact of self-steepening on the third-order moment in Fig. 4(c). We finally point out that, although self-steepening arrests the ISC, the dramatic spectral narrowing of the field shown in Fig. 3 can be exploited to significantly increase the degree of coherence of the incoherent wave, as remarkably illustrated in Fig. 3(e).

B. With background

We conclude this section devoted to the exponential response function by remarking that this study can be extended to the regime in which the random wave evolves in the presence of a spectral background, in a way analogous to what has been done above for the damped harmonic oscillator response. By splitting $n_\omega(z) = n_0 + \tilde{n}_\omega(z)$, with $\tilde{n}_\omega(z) \sim n_0/\tau_R$, it was shown in Ref. [13] that the spectral background dramatically changes the behavior of the system: Instead of the ISC singularity, the incoherent spectrum was shown to exhibit a regular periodic evolution. We have extended this previous study by considering the influence of self-steepening on such periodic behavior. The analysis reveals a rather intuitive result: Self-steepening breaks the periodic evolution of the spectrum, a feature that can easily be interpreted since the perturbative term $1 + \tau_s \omega$ in front of the corresponding SIDKE slows down (accelerates) the dynamics in the low- (high-) frequency part of the spectrum $\omega < 0$ ($\omega > 0$).

VII. GENERALITY OF THE THEORY

We finally note that the theory discussed here can be applied to any form of the response function. For instance, if $R(t)$ has a continuous derivative at $t = 0$, e.g., $R(t) = H(t) t^2 \exp(-t/\tau_R)/(2\tau_R^3)$, the first two terms in the expansion of M_ω in (3) vanish identically, so that the corresponding SIDKE is dominated by nonlinear dispersive

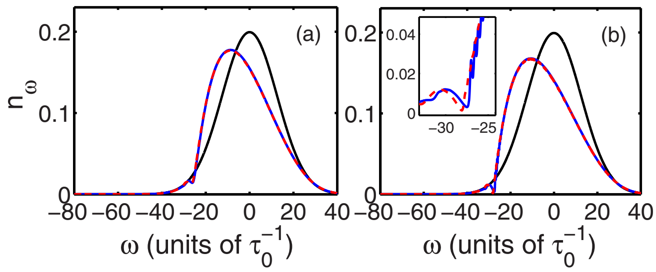

FIG. 5. (Color online) Response function $R(t)$ with a continuous derivative at $t = 0$: Simulations of the GKE (2) (blue) and the SIDKE (12) (dashed red) at $z = 7 \times 10^4 L_{nl}$ (a) and $z = 10^5 L_{nl}$ (b) (initial condition shown dark): Self-steepening and spectral oscillations occur simultaneously ($\tau_R = 4\tau_0$, $\tau_s = \tau_0/80$). The inset shows a zoom.

terms

$$\partial_z n_\omega = \frac{\gamma(1+\tau_s\omega)}{2\tau_R^3}\left(n_\omega \mathcal{H}\partial_\omega^2 n_\omega + \frac{1}{\tau_R} n_\omega \partial_\omega^3 n_\omega\right). \qquad (12)$$

Accordingly, the dynamics of the spectral front exhibits self-steepening and spectral oscillations at the same time, as illustrated in Fig. 5.

VIII. CONCLUSION

In summary, we have studied the impact of self-steepening on different types of singularities that occur in the spectral evolution of a random wave in the presence of a highly noninstantaneous nonlinear response. We have shown that self-steepening leads to a delay in the development of incoherent DSWs in the presence of a damped harmonic oscillator response, while it inhibits the ISC singularity in the presence of an exponential response function. Furthermore, the spectral collapselike behavior can be exploited to enhance in a significant way the degree of coherence of the random wave. The influence of a spectral background on the evolution of the random wave has been also considered. The generality and the validity of our mathematical treatment have been confirmed by direct numerical simulations.

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