DIVERSIFICATION IN FINANCIAL NETWORKS MAY INCREASE SYSTEMIC RISK

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Abstract. It has been pointed out in the macroeconomics and financial risk literature that risk-sharing by diversification in a financial network may increase the systemic risk. This means roughly that while individual agents in the network, for example banks, perceive their risk of default or insolvency decrease as a result of cooperation, the overall risk, that is, the risk that several agents may default simultaneously, or nearly so, may in fact increase. We present the results of a recent mathematical study that addresses this issue, relying on a mean-field model of interacting diffusions and its large deviations behavior. We also review briefly some recent literature that addresses similar issues.

Key words. systemic risk, mean field, large deviations, dynamic phase transitions

AMS subject classifications. 60F10, 60K35, 91B30, 82C26

1. Introduction. Systemic risk is the risk that a large number of components of an interconnected financial system fail within a short time thus leading to the overall failure of the financial system. What is particularly interesting is that the onset of this overall failure can occur even when the individual agents in the system perceive that their own individual risk of failure is diminished by diversification through cooperation. This phenomenon is described and put into a broader perspective in Haldane's recent presentation [8], which was the starting point of our own work [7]. In this review we first introduce and describe our model for the role of cooperation in determining systemic risk. It is a system of bistable diffusion processes that interact through their mean field. Failure is formulated as dynamic phenomenon analogous to a phase transition and it is analyzed using the theory of large deviations. We then describe briefly some related recent literature on contagion and risk amplification that give an idea of what mathematical models are being currently used to address such systemic risk issues.

In section 2 we consider a simple model of interacting agents for which systemic risk can be assessed analytically in some interesting cases [7]. Each agent can be in one of two states, a normal and a failed one, and it can undergo transitions between them. We assume that the dynamic evolution of each agent has the following features. First, there is an intrinsic stabilization mechanism that tends to keep the agents near the normal state. Second, there are external destabilizing forces that tend to push away from the normal state and are modeled by Brownian motions. Third, there is cooperation among the agents that acts as a stabilizer, modeled by a mean field. In such a system there is a decrease in the risk of destabilization or "failure" for each agent because of the cooperation. However, the effect of cooperation on the overall or system's risk is to increase it. Systemic risk is defined here as the small probability of an overall transition out of the normal state. We describe how for the models under consideration, and in a certain regime of parameters, the systemic risk increases with increasing cooperation. Our aim is to elucidate mathematically the tradeoff between

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individual risk and systemic risk for a class interacting systems subject to failure.

In section 3 we describe three mathematical models that consider overall or systemic failure of financial systems. The first is the study of contagion in a random network of interacting financial agents [11]. The main result is that the degree of interconnectivity of the network determines if failure of one, or a few, components will spread out to the whole system, that is, become a contagion. This is a form of instability or dynamic phase transition that is different from what we consider and it provides some perspective on the kind of mathematical models that can be used. The second model [2] that we discuss briefly addresses roughly the same question that we address, that is, how reducing individual risk by diversification can increase systemic risk. A static model is used and systemic risk is defined through a cost function of the size of the default. For convex cost functions it is shown that diversification is not the best strategy for reducing systemic risk. Finally the third model [1] introduces a feedback mechanism for default acceleration in a system of interacting diffusions. In this model the effect of risk diversification by cooperation is counterbalanced by this default acceleration mechanism.

2. A Bistable Mean-Field Model for Systemic Risk.

2.1. The Model. We consider a mean-field model of an interacting system of diffusions in order to explain why the risk diversification may increase the systemic risk. Consider a banking system with N banks. For $j = 1, \ldots, N$, we let $x_j(t)$ denote a risk index for bank j. We say that at time t, the bank j is in the safe or normal state if $x_j(t) \approx -1$ and in the failed state if $x_j(t) \approx 1$. We model the evolution of the $x_j(t)$'s as continuous-time processes satisfying the system of Itô stochastic differential equations:

$$dx_j(t) = -h(x_j^3(t) - x_j(t))dt + \theta(\bar{x}(t) - x_j(t))dt + \sigma dw_j(t). \tag{2.1}$$

Here $U(y) := y^3 - y$ is the gradient of the two-well potential function $V(y) := y^4/4 - y^2/2$. Without the last two terms in (2.1), $x_j = \pm 1$ are the stable states of (2.1), and therefore we may call -1 the safe or normal state and $+1$ the failed state. The positive constant h is a measure of intrinsic stability, indicating how difficult it is for a bank to transit from one state to the other. In the second term, $\bar{x}(t) := \sum_j x_j(t)/N$ is the empirical mean of the risks and is taken as the risk index of the whole system. The parameter $\theta > 0$ quantifies the degree of cooperation between banks since when this parameter is large the individual bank j tends to behave like the overall system. More precisely, if the whole banking system is in the good state, any bank can share its risk with the other banks so that it can reduce its own risk. On the other hand, a well-behaved bank might also default because it shares risk with many ill-behaved banks. The size of θ quantifies the strength of risk diversification. Finally $w_j(t)$ is the external shock to the bank j and σ is its amplitude. We model $\{w_j(t)\}_{j=1}^N$ as independent Brownian motions.

2.2. The Mean-Field Limit. In order to analyze the systemic behavior we need to consider the dynamical behavior of the empirical mean $\bar{x}(t)$. However, the system is nonlinear so there is no closed equation for $\bar{x}(t)$. We therefore need to consider the empirical density of the diffusions, not only the empirical mean, which evolves in an infinitely dimensional space. Let $M_1(\mathbb{R})$ be the space of probability measures and $C([0,T], M_1(\mathbb{R}))$ be the space of continuous $M_1(\mathbb{R})$-valued processes on $[0,T]$; both spaces are endowed with the standard topologies. We define $X_N(t, dy) := \frac{1}{N}\sum_{j=1}^N \delta_{x_j(t)}(dy)$, and it is not difficult to see that $X_N \in C([0,T], M_1(\mathbb{R}))$ and $\bar{x}(t) =$

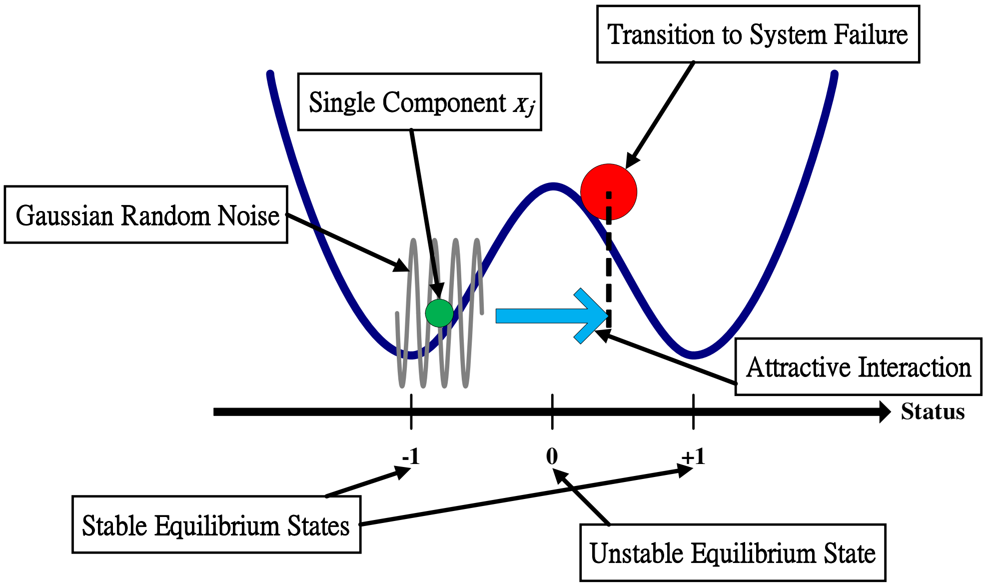

FIG. 2.1. *Schematic of the bistable interacting diffusions system.*

$\int yX_N(t,dy)$. We cite the well-known mean-field limit theorem (proved by Dawson [3]) where as $N \to \infty$, X_N converges in law to a $M_1(\mathbb{R})$-valued deterministic process satisfying the Fokker-Planck equation

$$\frac{\partial}{\partial t}u = h\frac{\partial}{\partial y}[(y^3-y)u] - \theta\frac{\partial}{\partial y}\left\{\left[\int yu(t,dy)-y\right]u\right\} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}u. \tag{2.2}$$

One can easily see the connection between (2.2) and (2.1), and indeed the proof is based on Itô's formula and the martingale formulation of diffusions.

Because of the mean-field limit theorem, we can analyze u in (2.2) instead of the empirical density X_N, which we can view as a perturbation of u for N large. Although the full explicit solution of (2.2) is not known, equilibrium solutions can be found. Assume that the first order moment of the equilibrium is ξ, and then the equilibrium $u^e_\xi(y)$ satisfies

$$h\frac{d}{dy}[(y^3-y)u^e_\xi] - \theta\frac{d}{dy}[(\xi-y)u^e_\xi] + \frac{1}{2}\sigma^2\frac{d^2}{dy^2}u^e_\xi = 0,$$

which implies that

$$u^e_\xi(y) = \frac{1}{Z_\xi\sqrt{2\pi\frac{\sigma^2}{2\theta}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2}(\frac{1}{4}y^4-\frac{1}{2}y^2)\right\}, \tag{2.3}$$

where Z_ξ is the normalization constant. Since the first order moment is ξ, we have the compatibility condition:

$$\xi = m(\xi) := \int yu^e_\xi(y)dy. \tag{2.4}$$

Clearly $\xi = 0$ satisfies (2.4), but there may be other solutions as well. We refer to [3] for the following general result: given h and θ, there exists a critical $\sigma_c > 0$

such that (2.4) has two additional non-zero solutions $\pm\xi_b$ if and only if $\sigma < \sigma_c$. This has a straightforward interpretation. When $\sigma \geq \sigma_c$, the randomness dominates the interaction among the components, that is, $\theta(\bar{x}(t) - x_j(t))dt$ is negligible. In this case, the model behaves like a set of N independent diffusions and then roughly one half of them stay around -1 and the rest stay around $+1$ so the average is 0. On the other hand, when $\sigma < \sigma_c$ then the interactive and stabilizing forces are significantly larger than the random disturbances. Therefore most particles stay around the same place ($-\xi_b$ or $+\xi_b$).

We can get a more explicit expression for σ_c when h is small. In that case, u_ξ^e can be viewed as a pertubation of the Gaussian density function and we can solve (2.4) explicitly. It turns out that

$$\xi_b = \sqrt{1 - 3\frac{\sigma^2}{2\theta}} \left(1 + h\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1 - 2(\sigma^2/2\theta)}{1 - 3(\sigma^2/2\theta)}\right) + O(h^2), \tag{2.5}$$

and ξ_b exists if and only if $3\sigma^2 < 2\theta$.

2.3. Large Deviations. For modeling systemic risk, we need to specify the system's safe (normal) and failed states. We assume that $\sigma < \sigma_c$ throughout so that there are two stable equilibria: $u_{\pm\xi_b}^e$. Let $u_{-\xi_b}^e$ denote the system's safe state and $u_{+\xi_b}^e$ denote the system's failed state. If all agents are in the safe state initially, $x_j(0) = -1$, then we expect that for large N and large t, $X_N(t) \approx u_{\xi_b}^e$ and $\bar{x}(t) \approx -\xi_b$. We assume that $X_N(0) \to u_{-\xi_b}^e$ and $\bar{x}(0) \to -\xi_b$ as $N \to \infty$. If so, one would think that for large N, $X_N(t) \approx u_{-\xi_b}^e$ and $\bar{x}(t) \approx -\xi_b$ for all t.

However, as long as N is large but finite, randomness still matters and given a finite time horizon $[0, T]$ the following event A happens with *small but nonzero* probability:

$$A = \{X_N(t), 0 \leq t \leq T : X_N(0) \approx u_{-\xi_b}^e, X_N(T) \approx u_{+\xi_b}^e\}. \tag{2.6}$$

The *rare event* A captures a *phase transition*, in the sense that the system moves from the safe state to the failed state. Now the systemic risk in this framework is naturally defined as $\mathbf{P}(A)$, the probability that the system transitions from the system's safe state $u_{-\xi_b}^e$ at time $t = 0$ to the system's failed state $u_{+\xi_b}^e$ at time $t = T$.

We need, therefore, to compute $\mathbf{P}(A)$. Because A is a rare event we use large deviations to compute it. The general large deviations principle and related studies for similar interacting particle system has been carried out by Dawson and Gärtner [4, 5]. Although everything here can be proven rigorously, we ignore the technicalities so as to give a simplified description of the results. The Dawson-Gärtner large deviations theory says that for N large,

$$\mathbf{P}(A) \approx \exp\left(-N \inf_{\phi \in A} I_h(A)\right),$$

where $I_h(\phi)$ is the *rate function*:

$$I_h(\phi) = \frac{1}{2\sigma^2}\int_0^T \sup_{f:\langle \phi, f_y^2\rangle \neq 0} \langle \phi_t - \mathcal{L}_\phi^*\phi - h\mathcal{M}^*\phi, f\rangle^2 / \langle \phi, f_y^2\rangle dt, \quad \langle \mu, f\rangle = \int_{-\infty}^{\infty} \mu(dy) f(y),$$

$$\mathcal{L}_\psi^*\phi = \frac{1}{2}\sigma^2\phi_{yy} + \theta\frac{\partial}{\partial y}\left\{\left[y - \int y\psi(t, dy)\right]\phi\right\}, \quad \mathcal{M}^*\phi = \frac{\partial}{\partial y}\left[(y^3 - y)\phi\right].$$

This result is an infinite dimensional analog of the Freidlin-Wentzell theorem for finite dimensional SDEs with small noise (see [6]). To get a feeling for the Dawson-Gärtner theory we note that $\phi_t - \mathcal{L}^*_\phi \phi - h\mathcal{M}^*\phi$ is just the same differential operator in (2.2) so $I_h(u) = 0$ if u solves (2.2) and $\mathbf{P}(A) = 1$, provided we have $u \in A$. Moreover, if a path ϕ is far from u in the sense that $\phi_t - \mathcal{L}^*_\phi \phi - h\mathcal{M}^*\phi$ is significantly different from zero, $I_h(\phi)$ is large, meaning that it is less likely to have $X_N \approx \phi$. In other words, $I_h(\cdot)$ is a measure for how difficult it is for X_N to deviate from u. The higher value $I_h(\phi)$ is, the less likely $X_N \approx \phi$.

Now the problem is that it is very hard to compute $\inf_{\phi\in A} I_h(A)$ with the given A in (2.6). It is an infinitely-dimensional, nonlinear variational problem. However, for small h, we are can find a very good approximation for it [7]. More precisely, for any $\epsilon > 0$, there are sufficiently small $h > 0$ such that

$$\frac{2\xi_b^2}{\sigma^2 T} - \epsilon < \inf_{\phi\in A} I_h(\phi) < \frac{2\xi_b^2}{\sigma^2 T} + \epsilon,$$

where ξ_b is given in (2.5).

One might ask, why this small-h assumption: is it realistic to take h small? In fact, the small-h case is the only interesting one. Virtually all of the model-based studies of systemic risk that we have found in the literature show that individual stability has the monotone effect to the systemic stability, and systemic collapse happens extremely rarely if there is strong or moderate individual stability. Our numerical simulations also show that it is very unlikely to see a phase transition, within a reasonable time horizon, even for moderate h. Thus we may conclude that indeed $\mathbf{P}(A)$ is nonzero for h that are not small, but the expected time of the system failure might be hundreds or thousands of years. Another rationale is that the only incentive for a bank to share risk is that its capital reserves, roughly modeled by the parameter h, are relatively lower than the uncertainty, σ that it faces. In this case, the bank is concerned about the risk it is taking and wants to diversify it. If its capital reserves are relatively high, the bank has no incentive to share the risk.

2.4. Fluctuation Theory for Individuals. Before we go to the main result that risk diversification may increase the systemic risk, we need to review briefly the classical fluctuation theory for individual agents. Assume that all x_j are in the vicinity of -1 so that we can linearize them:

$$x_j(t) = -1 + \delta x_j(t), \quad \bar{x}(t) = -1 + \delta\bar{x}(t), \quad \delta\bar{x}(t) = \frac{1}{N}\sum_{j=1}^{N} \delta x_j(t).$$

Then $\delta x_j(t)$ and $\delta\bar{x}(t)$ satisfy the linear SDEs:

$$d\delta x_j = -(\theta + 2h)\delta x_j dt + \theta\delta\bar{x}dt + \sigma dw_j, \quad d\delta\bar{x} = -2h\delta\bar{x}dt + \frac{\sigma}{N}\sum_{j=1}^{N} dw_j,$$

with $\delta x_j(0) = \delta\bar{x}(0) = 0$. The processes $\delta x_j(t)$ and $\delta\bar{x}(t)$ are Gaussian and their mean and variance are easily calculated. For all $t \geq 0$, $\mathbf{E}\delta x_j(t) = \mathbf{E}\delta\bar{x}(t) = 0$ and $\mathbf{Var}\delta\bar{x}(t) = \frac{\sigma^2}{N}(1 - e^{-4ht})$. In addition, $\mathbf{Var}\delta x_j(t) \to \frac{\sigma^2}{2(\theta+2h)}(1 - e^{-2(\theta+2h)t})$ as $N \to \infty$, uniformly in $t \geq 0$. Thus σ^2/N and $\sigma^2/2(\theta + 2h)$ should be small so that the linearizations are legitimate.

2.5. Why the Risk-Sharing May Increase the Systemic Risk. Now we are ready to present the main result. Assume that σ^2/N and $\sigma^2/2(\theta+2h)$ are sufficiently small and N is large. From the above fluctuation theory, the risk index $x_j(t)$ of the bank j is approximately a Gaussian process with the stationary distribution $\mathcal{N}(-1, \sigma^2/2(\theta+2h))$. Suppose σ^2 is high. Such a σ^2 may mean that the economy is more uncertain or the bank is more risk-prone. In this case, in order to keep the risk index $x_j(t)$ at a safe level, the bank may increase its intrinsic stability (capital reserve), h, or share risk with the other banks, that is, increase θ. Because increasing h is generally much more costly (affects profits) than increasing θ, and at the individual agent level there is no real difference to risk between increasing h or θ, the banks are likely to increase θ to diversify risk. Note that $\sigma^2/2(\theta+2h) \lesssim \sigma^2/2\theta$ when σ^2 and θ are significantly larger than h. Thus, the bank can maintain low individual risk even with high external uncertainty by fixing a low ratio $\sigma^2/2\theta$.

How does this risk-sharing (increase σ but keep $\sigma^2/2\theta$ low) impact the systemic risk? From the large deviation section above we know that the systemic risk is

$$\mathbf{P}(A) \approx \exp\left(-N\frac{2\xi_b^2}{\sigma^2 T}\right),$$

$$\xi_b = \sqrt{1-3\frac{\sigma^2}{2\theta}}\left(1 + h\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1-2(\sigma^2/2\theta)}{1-3(\sigma^2/2\theta)}\right) + O(h^2).$$

We see that there are additional systemic-level σ^2's in the exponent and ξ_b, which can not be observed by the individual agents, increasing the systemic risk, even if the individual risk $\sigma^2/2\theta$ is fixed and low. In other words, the individuals may believe that they are able to take more external risk by diversifying it, but an increase of external uncertainty still destabilizes the system.

3. Review of Some Models for Systemic Risk. We will review briefly three models of systemic risk that we have selected from the recent literature on the subject. We do this in order to give an idea of the type of mathematical models that are being used and the results obtained.

3.1. The Bank of England Model. We review the systemic risk analysis considered in [11, 10, 9]. A banking system is modeled as a random network where the nodes represent the banks and the edges represent the interbank relations (loans and borrowing). Using the notation in [10, 9], each bank i is characterized by four activities quantified by the external assets e_i, the interbank loans l_i, the deposits d_i and the interbank borrowing b_i. The e_i and l_i are considered as assets for the bank i, whereas b_i and d_i are its liabilities. Its net worth γ_i is $\gamma_i = (e_i + l_i) - (d_i + b_i)$. The bank i is solvent if $\gamma_i \geq 0$ and otherwise the bank i defaults. The network is a random Erdös-Rényi graph. This means that the directed edge from i to j (that is, when the bank i lends to the bank j) exists with probability p independently of the other edges.

Focus is on the default contagion due to the failure of a *single* bank. Assume that there is an external shock to the bank i so that it loses a fraction of its external assets e_i and the net worth γ_i becomes negative. Therefore, by definition, the bank i defaults (Phase I failure) and its creditors lose their loans because of the default of i. If the loss of the creditor j is higher than its net worth γ_j, then the bank j also defaults (Phase II failure) and j's creditors lose their assets accordingly and so on (Phase III failure and more).

The main interest in this problem is in calculating how the parameters (e_i, l_i, d_i, b_i, p and the number of banks N) affect the extent or spreading of defaults. For simplicity the parameters are assumed to be uniform among all banks. This uniformity also allows the asymptotic analysis by the mean-field approximation as $N \to \infty$ and the approximation is in excellent agreements with numerical simulation (see [10]).

We briefly explain the mean-field approximation in [10], which is different from the mean-field model considered in the previous section. Since the edges in the Erdös-Rényi network are independent identically distributed $0-1$ random variables, in the mean-field approximation we let the in/out degrees of each node be the deterministic average number $z = p(N-1)$. Phase I failure is straightforward: a randomly chosen bank fails when its loss is larger than the net worth γ. Note that the size of Phase I failure is always one. Phase II failure is also clear: because the failed bank has z creditors, there are z failed banks in Phase II if each creditor's net worth γ is lower than the loss divided by z, or no banks fail in Phase II otherwise. We see here a phase transition: as z increases, the size of Phase II failure also increases because the failed bank propagates the shock to more neighbors. However, once z is higher than a critical value, the neighbors can absorb the shock without defaulting. Phase III failure is more complicated to describe and we refer to [10]. But we note that, as in Phase II, Phase III failure occurs only for z smaller than a critical value.

In [11, 10] extensive numerical tests are carried out to assess the effect of the various parameters. Although the extent of the defaults is not linear in the parameters, some of qualitative behavior agrees with what may be expected: the number of defaults monotonically decreases with the net worth γ and monotonically increases with the fraction of interbank assets. However, the most interesting effect is the impact of the Erdös-Rényi probability $p = (N-1)/z$, which can be viewed as the interconnectivity of the network. Numerical studies show that the number of defaults is an M-shaped curve of p. This can be explained by the mean-field approximation where the number of the Phase II defaults is a linear function of p for p less than the critical value p_{II} and vanishes for $p \geq p_{\text{II}}$. The number of the Phase III defaults is nonzero only if p is smaller than the critical value $p_{\text{III}} < p_{\text{II}}$. If additional failures (Phase IV, V, ...) are negligible, the number of defaults is the sum of those in the first three Phase failures, which gives an M-shape function of p (note that the Phase I failure is always one).

3.2. Individual versus Systemic Risk. We explain here the general framework, rather than a particular mathematical model, considered in [2] so as to argue why the individual's optimum policy could be very different from the system's optimum policy to minimize risk. Suppose that this system has N banks and M assets, and at time $t = 0$ each bank can invest these M assets. Let X_{ij} denote the fixed allocation of bank i to asset j at $t = 0$. If V_j is the loss of the asset j between $t = 0$ and $t = 1$, then the total loss of the bank i at $t = 1$ is $\sum_{j=1}^{M} X_{ij} V_j$, and i defaults when $\sum_{j=1}^{M} X_{ij} V_j > \gamma_i$, the capital reserve of i.

For simplicity, we assume that for all i, $\gamma_i = \gamma$, that $\sum_{j=1}^{M} X_{ij} = 1$ and that V_j, $j = 1, \dots, M$ are independent identically distributed random variables. Then clearly the optimal strategy of each bank i is to take $X_{ij} = 1/M$ for $j = 1, \dots, M$. That is, each individual should invest equally in all assets to diversify risk.

What is the optimal strategy for the system? This is a mainly question about the way to define systemic risk. In [2], systemic risk is defined as a cost function of the number of defaults. If the cost function is a linear function of the number of defaults, then the system's optimal is equal to the individual's optimal because we believe that the cost of two defaults at the same time equals twice the cost of a single default. However, if the cost function is super-linear, more precisely convex, then the system's optimum is different because now we believe that simultaneous defaults are much more expensive. In this situation, the best strategy for the system is that each bank invests in a *single but distinct* asset, so that when some assets lose their values, only a fraction of the banks default and the systemic risk is still low.

In this framework the result is based entirely on the form of the cost function. In [2] it is argued that the cost function is indeed super-linear at the level of the economy and the society in general. In this case the regulators face a dilemma since the individual and system's optima are very different. Reconciling these differences becomes a challenge.

3.3. The Model of Financial Accelerators. Battiston et al. use the concept of the *financial accelerator* to explain why risk diversification in fact increases the individual as well as the systemic risk. The system is a k-regular graph and the set k_i are the edges of the bank i. Let ρ_i indicate the robustness of the bank i assumed to satisfy the SDEs

$$d\rho_i = \sum_{j\in k_i} W_{ij}(\rho_j(t)-\rho_i(t))dt + \sigma \sum_{j\in k_i} W_{ij} d\xi_j(t) + h(\rho_i(t), \rho_i(t-dt))dt. \tag{3.1}$$

The bank i defaults if $\rho_i < 0$. The first term in (3.1) is the interbank cooperation and W_{ij} are non-negative weights with $\sum_j W_{ij} = 1$. Here $\xi_j(t)$ are independent Brownian motions representing the random influences, and $h(\rho_i(t), \rho_i(t-dt))$ is the financial accelerator defined as

$$h(\rho_i(t), \rho_i(t-dt)) = \begin{cases} -\alpha, & \rho_i(t)-\rho_i(t-dt) < -\epsilon\sigma/\sqrt{k}dt, \\ 0, & \text{otherwise.} \end{cases}$$

The financial accelerator models a positive feedback to speed of the decline when it is past a threshold. We also note that when the diversify k increases, it is easier to trigger the financial accelerator, and therefore the diversification may destabilize the individual risk. In order to obtain an analytical result, they replace $h(\rho_i(t), \rho_i(t-dt))$ by its expected value $-\alpha q$ (a constant), where $q(k, \sigma, \alpha, \epsilon)$ is the probability that $h(\rho_i(t), \rho_i(t-dt)) = -\alpha$:

$$q = \frac{\Phi(-\epsilon)}{1-\Phi(\alpha\sqrt{k}/\sigma-\epsilon)+\Phi(-\epsilon)},$$

where Φ is the Gaussian cumulative distribution. Then they study the linearized SDE

$$d\rho_i = \sum_{j\in k_i} W_{ij}(\rho_j(t)-\rho_i(t))dt + \sigma \sum_{j\in k_i} W_{ij} d\xi_j(t) - \alpha q(k, \sigma, \alpha, \epsilon)dt. \tag{3.2}$$

Let P_f be the individual risk defined as $P_f = 1/T_f$, where T_f is the mean first passage time that a single bank defaults. Without the financial accelerator, $P_f(k)$ is a monotonically decreasing function of the diversity k. However, with the financial accelerator, $P_f(k)$ reaches a minimum at certain value k^*. This is interpreted as an indication that diversification is not always a good thing.

4. Summary and Conclusion. We have described through a mathematical model why risk-sharing may increase the systemic risk. We have used a classical model of bistable diffusions with mean-field interactions for the analysis of dynamic phase transitions that is widely studied in physics and other fields. The interpretation of our analytical result in the systemic risk context is that an increase in the systemic risk can come from a qualitative inconsistency between central and tail probabilities. This means that the individual agents (central) and the overall system (large deviations) perceive risk in different ways. On the one hand, the central probability analysis shows that the individual agents can diversify their risks as long as the system is stable. On the other hand, the system is not always stable: it may fail with a small probability that in fact increases with σ. From a physical point of view, $\{\sigma w_j(t)\}_{j=1}^N$ quantifies the energy injected into the system. Because this system is closed, the overall energy will not diminish by diversification. Eventually the high energy makes the system more volatile and increases the probability of a systemic failure.

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