SIAM J. APPL. MATH.
Vol. 67, No. 6, pp. 1718–1739

PULSE PROPAGATION AND TIME REVERSAL IN RANDOM WAVEGUIDES*

JOSSELIN GARNIER† AND GEORGE PAPANICOLAOU‡

Abstract. Mode coupling in a random waveguide can be analyzed with asymptotic analysis based on separation of scales when the propagation distance is large compared to the size of the random inhomogeneities, which have small variance, and when the wavelength is comparable to the scale of the inhomogeneities. In this paper we study the asymptotic form of the joint distribution of the mode amplitudes at different frequencies. We derive a deterministic system of transport equations that describe the evolution of mode powers. This result is applied to the computations of pulse spreading in a random waveguide. It is also applied to the analysis of time reversal in a random waveguide. We show that randomness enhances spatial refocusing and that diffraction-limited focal spots can be obtained even with small-size time-reversal mirrors. The refocused field is statistically stable for broadband pulses in general. We show here that it is also stable for narrowband pulses, provided that the time-reversal mirror is large enough.

Key words. acoustic waveguides, random media, asymptotic analysis

AMS subject classifications. 76B15, 35Q99, 60F05

DOI. 10.1137/060659235

1. Introduction.

In a perfect waveguide, energy propagates through its guided wave modes, which do not interact with each other. Imperfections in the material or in the geometrical properties of the waveguide induce mode coupling. These imperfections are usually small, but their effects accumulate over large propagation distances and can be significant. In this paper we consider wave propagation in an acoustic waveguide whose bulk modulus is a three-dimensional random function. Using the propagating modes of the unperturbed waveguide, we can reduce the three-dimensional wave propagation problem to the analysis of a system of coupled ordinary differential equations with random coefficients. It is in the frequency domain that the mode amplitudes satisfy these differential equations. We can analyze them as a system of random differential equations in a diffusion approximation, in which the mode amplitudes are a multidimensional diffusion process. This coupled mode asymptotic analysis was considered previously for applications in underwater acoustics [14, 5], fiber optics [17], and quantum mechanics [11].

The main purpose of this paper is to analyze the asymptotic behavior of the coupled mode amplitudes in the time domain. This requires the analysis of the joint distribution of the mode amplitudes at two nearby frequencies, which results in a system of transport equations for the mode powers. This is done in section 6. In section 7 we apply this result to the analysis of pulse spreading in a random waveguide.

In section 8 we consider time reversal in a random waveguide and present the first analysis of the phenomenon of side-lobe suppression, which has been observed in experiments [15] and in numerical simulations [2]. Side-lobe suppression is the main

*Received by the editors May 8, 2006; accepted for publication (in revised form) May 18, 2007; published electronically October 5, 2007.
http://www.siam.org/journals/siap/67-6/65923.html

†Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions, Université Paris VII, 2 Place Jussieu, 75251 Paris Cedex 5, France (garnier@math.jussieu.fr).
‡Mathematics Department, Stanford University, Stanford, CA 94305 (papanico@math.stanford.edu). The work of this author was supported by grants ONR N00014-02-1-0088 and NSF DMS-0354674-001.

application of our asymptotic analysis. Time-reversal refocusing has been studied extensively both experimentally and theoretically, in various contexts such as in ultrasound and underwater acoustics, as reviewed in [7, 8]. A time-reversal mirror is an active array of transducers that records a signal, time reverses it, and re-emits it into the medium. The waves generated at the time-reversal mirror propagate back to their source and focus near it, as if wave propagation was run in reverse time. Surprisingly, random inhomogeneities enhance the refocusing of the time-reversed waves near the original source location [4, 8]. Ultrasonic wave propagation and time reversal in homogeneous waveguides is studied experimentally and theoretically in [22]. Time-reversal refocusing was experimentally investigated in underwater acoustics in [15, 23], where the random inhomogeneities in the environment reduce the side-lobes that are seen in refocusing in a homogeneous waveguide.

In addition to enhanced refocusing and side-lobe suppression, statistical stability of the refocused field is critical for applications in communications and detection. Statistical stability means that the refocused field does not depend on the particular realization of the random medium. It has been studied in *broadband* regimes of propagation in one-dimensional random media [3, 10], in three-dimensional randomly layered media [9, 10], and in three-dimensional wave propagation in the paraxial approximation [2, 21]. Stabilization of the refocused field for broadband pulses results from the superposition of its many approximately uncorrelated frequency components. We show in section 8 that in random waveguides we have statistical stability of the refocused field even for *narrowband* pulses, provided that the number of propagating modes and the size of the time-reversal mirror are large enough.

2. Propagation in homogeneous waveguides.

In this section we study wave propagation in an acoustic waveguide that supports a finite number of propagating modes. In an ideal waveguide the geometric structure and the medium parameters can have a general form in the transverse directions but must be homogeneous along the waveguide axis. There are two general types of ideal waveguides: those that surround a homogeneous region with a confining boundary, and those in which the confinement is achieved with a transversely varying index of refraction. We will present the analysis of the effects of random perturbations on waveguides of the first type and will illustrate specific results with a planar waveguide. The main difference in working with waveguides of the second type is that the transverse wave mode profiles depend on the frequency, but this does not affect the theory we present here.

2.1. Modeling of the waveguide.

We consider linear acoustic waves propagating in three space dimensions modeled by the system of wave equations

$$\rho(\mathbf{r})\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F}\,, \qquad \frac{1}{K(\mathbf{r})}\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} = 0\,, \tag{2.1}$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ is the density of the medium, and K is the bulk modulus. The source is modeled by the forcing term $\mathbf{F}(t,\mathbf{r})$. We assume that the transverse profile of the waveguide is a simply connected region $\mathcal{D}$ in two dimensions. The direction of propagation along the waveguide axis is z and the transverse coordinates are denoted by $\mathbf{x} \in \mathcal{D}$. In the interior of the waveguide the medium parameters are homogeneous:

$$\rho(\mathbf{r}) \equiv \bar{\rho}, \qquad K(\mathbf{r}) = \bar{K} \qquad \text{for } \mathbf{x} \in \mathcal{D} \text{ and } z \in \mathbb{R}\,.$$

By differentiating the second equation of (2.1) with respect to time and substituting the first equation into it, we get the standard wave equation for the pressure field,

$$\Delta p - \frac{1}{\bar{c}^2}\frac{\partial^2 p}{\partial t^2} = \nabla \cdot \mathbf{F}\,, \tag{2.2}$$

where $\Delta = \Delta_\perp + \partial_z^2$ and $\Delta_\perp$ is the transverse Laplacian. The sound speed is $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$. We must now prescribe boundary conditions on the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$. In underwater acoustics, or in seismic wave propagation, the density is much smaller outside than inside the waveguide. This means that we must use a pressure release boundary condition since the pressure is very weak outside, and therefore, by continuity, the pressure is zero just inside the waveguide. Motivated by such examples, we will use Dirichlet boundary conditions

$$p(t,\mathbf{x},z) = 0 \qquad \text{for } \mathbf{x} \in \partial\mathcal{D} \text{ and } z \in \mathbb{R}\,. \tag{2.3}$$

We could also consider other types of boundary conditions if, for example, the boundary of the waveguide is a rigid wall, in which case the normal velocity vanishes. By (2.1) we obtain Neumann boundary conditions for the pressure.

2.2. The propagating and evanescent modes. A waveguide mode is a monochromatic wave $p(t,\mathbf{x},z) = \hat{p}(\omega,\mathbf{x},z)e^{-i\omega t}$ with frequency ω, where $\hat{p}(\omega,\mathbf{x},z)$ satisfies the time harmonic form of the wave equation (2.2) without a source term:

$$\partial_z^2 \hat{p}(\omega,\mathbf{x},z) + \Delta_\perp \hat{p}(\omega,\mathbf{x},z) + k^2(\omega)\hat{p}(\omega,\mathbf{x},z) = 0\,. \tag{2.4}$$

Here $k = \omega/\bar{c}$ is the wavenumber and we have Dirichlet boundary conditions on $\partial\mathcal{D}$. The transverse Laplacian in $\mathcal{D}$ with Dirichlet boundary conditions on $\partial\mathcal{D}$ is self-adjoint in $L^2(\mathcal{D})$. Its spectrum is an infinite number of discrete eigenvalues

$$-\Delta_\perp \phi_j(\mathbf{x}) = \lambda_j \phi_j(\mathbf{x}),\ \mathbf{x} \in \mathcal{D}, \qquad \phi_j(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\mathcal{D}$$

for $j = 1, 2, \ldots$. The eigenvalues are positive and nondecreasing, and we assume for simplicity that they are simple, so we have $0 < \lambda_1 < \lambda_2 < \cdots$. The eigenmodes are real and form an orthonormal set

$$\int_{\mathcal{D}} \phi_j(\mathbf{x})\phi_l(\mathbf{x})d\mathbf{x} = \delta_{jl}\,,$$

with $\delta_{jl} = 1$ if $j = l$ and 0 otherwise. For a given frequency ω, there exists a unique integer $N(\omega)$ such that $\lambda_{N(\omega)} \le k^2(\omega) < \lambda_{N(\omega)+1}$, with the convention that $N(\omega) = 0$ if $\lambda_1 > k(\omega)$. The modal wavenumbers $\beta_j(\omega) \ge 0$ for $j \le N(\omega)$ are defined by

$$\beta_j^2(\omega) = k^2(\omega) - \lambda_j\,. \tag{2.5}$$

The solutions $\hat{p}_j(\omega,\mathbf{x},z) = \phi_j(\mathbf{x})e^{\pm i\beta_j(\omega)z}$, $j = 1,\ldots,N(\omega)$, of the wave equation (2.4) are the propagating waveguide modes. For $j > N(\omega)$ we define the modal wavenumbers $\beta_j(\omega) > 0$ by $\beta_j^2(\omega) = \lambda_j - k^2(\omega)$, and the corresponding solutions $\hat{q}_j(\omega,\mathbf{x},z) = \phi_j(\mathbf{x})e^{\pm\beta_j(\omega)z}$ of the wave equation (2.4) are the evanescent modes.

In this paper we shall illustrate some results for the planar waveguide. This is the case where $\mathcal{D}$ is $(0,d)\times\mathbb{R}$, and we consider only solutions that depend on $x \in (0,d)$. In this case we have $\lambda_j = \pi^2 j^2/d^2$, $\phi_j(x) = \sqrt{2/d}\sin(\pi j x/d)$, $j \ge 1$, and the number of propagating modes is $N(\omega) = [(\omega d)/(\pi\bar{c})]$, where $[x]$ is the integer part of x.

2.3. Excitation conditions for a source. We consider a point-like source located at $(\mathbf{x}_0, z=0)$ that emits a signal with orientation in the z-direction:

$$\mathbf{F}(t,\mathbf{x},z) = f(t)\delta(\mathbf{x}-\mathbf{x}_0)\delta(z)\mathbf{e}_z\,.$$

Here $\mathbf{e}_z$ is the unit vector pointing in the z-direction. By the first equation of (2.1), this source term implies that the pressure satisfies the following jump conditions across the plane $z=0$:

$$\hat{p}(\omega,\mathbf{x},z=0^+) - \hat{p}(\omega,\mathbf{x},z=0^-) = \hat{f}(\omega)\delta(\mathbf{x}-\mathbf{x}_0)\,,$$

while the second equation of (2.1) implies that there is no jump in the longitudinal velocity so that the pressure field also satisfies $\partial_z\hat{p}(\omega,\mathbf{x},z=0^+) = \partial_z\hat{p}(\omega,\mathbf{x},z=0^-)$. Here $\hat{f}$ is the Fourier transform of f with respect to time:

$$\hat{f}(\omega) = \int f(t)e^{i\omega t}dt, \qquad f(t) = \frac{1}{2\pi}\int \hat{f}(\omega)e^{-i\omega t}d\omega\,.$$

The pressure field can be written as a superposition of the complete set of modes,

$$\begin{aligned}\hat{p}(\omega,\mathbf{x},z) = &\left[\sum_{j=1}^{N}\frac{\hat{a}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{i\beta_j z}\phi_j(\mathbf{x}) + \sum_{j=N+1}^{\infty}\frac{\hat{c}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{-\beta_j z}\phi_j(\mathbf{x})\right]\mathbf{1}_{(0,\infty)}(z)\\ &+\left[\sum_{j=1}^{N}\frac{\hat{b}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{-i\beta_j z}\phi_j(\mathbf{x}) + \sum_{j=N+1}^{\infty}\frac{\hat{d}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{\beta_j z}\phi_j(\mathbf{x})\right]\mathbf{1}_{(-\infty,0)}(z)\,,\end{aligned}$$

where $\hat{a}_j$ is the amplitude of the jth right-going mode propagating in the right half-space $z>0$, $\hat{b}_j$ is the amplitude of the jth left-going mode propagating in the left half-space $z<0$, and $\hat{c}_j$ (resp., $\hat{d}_j$) is the amplitude of the jth right-going (resp., left-going) evanescent mode. Substituting this expansion into the jump conditions, multiplying by $\phi_j(\mathbf{x})$, integrating with respect to $\mathbf{x}$ over $\mathcal{D}$, and using the orthogonality of the modes, we express the mode amplitudes in terms of the source:

$$\hat{a}_j(\omega) = -\hat{b}_j(\omega) = \frac{\sqrt{\beta_j(\omega)}}{2}\hat{f}(\omega)\phi_j(\mathbf{x}_0)\,, \qquad \hat{c}_j(\omega) = -\hat{d}_j(\omega) = -\frac{\sqrt{\beta_j(\omega)}}{2}\hat{f}(\omega)\phi_j(\mathbf{x}_0)\,.$$

3. Mode coupling in random waveguides. We consider a randomly perturbed waveguide section occupying the region $z\in[0,L/\varepsilon^2]$, with two homogeneous waveguides occupying the two half-spaces $z<0$ and $z>L/\varepsilon^2$. The bulk modulus and the density have the form

$$\begin{aligned}\frac{1}{K(\mathbf{x},z)} &= \begin{cases}\frac{1}{\bar{K}}\left(1+\varepsilon\nu(\mathbf{x},z)\right) & \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in[0,L/\varepsilon^2],\\ \frac{1}{\bar{K}} & \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in(-\infty,0)\cup(L/\varepsilon^2,\infty),\end{cases}\\ \rho(\mathbf{x},z) &= \bar{\rho} \quad \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in(-\infty,\infty)\,,\end{aligned}$$

where ν is a zero-mean, stationary, and ergodic random process with respect to the axis coordinate z. Moreover, it is assumed to possess enough decorrelation; more precisely, it fulfills the condition that "ν is ϕ-mixing, with $\phi\in L^{1/2}(\mathbb{R}^+)$" [16, section 4.6.2].

The perturbed wave equation satisfied by the pressure field is

$$\Delta p - \frac{1+\varepsilon\nu(\mathbf{x},z)}{\bar{c}^2}\frac{\partial^2 p}{\partial t^2} = \nabla\cdot\mathbf{F}\,, \tag{3.1}$$

where the average sound speed is $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$. The pressure field also satisfies the boundary conditions (2.3). We consider that a point-like source located at $(\mathbf{x}_0, 0)$ emits a pulse $f(t)$ and we denote by $\hat{a}_{j,0}(\omega)$ the initial mode amplitudes as described in the previous section. The weak fluctuations of the medium parameters induce a coupling between the propagating modes, as well as between propagating and evanescent modes, which build up and become of order one after a propagation distance of order ε^{-2}, as expected from the diffusion approximation theory.

3.1. Coupled amplitude equations. We fix the frequency ω and expand the field $\hat{p}$ inside the randomly perturbed waveguide in terms of the transverse eigenmodes,

$$\hat{p}(\omega,\mathbf{x},z) = \sum_{j=1}^{N(\omega)} \phi_j(\mathbf{x})\hat{p}_j(\omega,z) + \sum_{j=N(\omega)+1}^{\infty} \phi_j(\mathbf{x})\hat{q}_j(\omega,z)\,, \tag{3.2}$$

where $\hat{p}_j$ is the amplitude of the jth propagating mode and $\hat{q}_j$ is the amplitude of the jth evanescent mode. We introduce the right-going and left-going mode amplitudes $\hat{a}_j(\omega,z)$ and $\hat{b}_j(\omega,z)$, defined by

$$\hat{p}_j = \frac{1}{\sqrt{\beta_j}}\left(\hat{a}_j e^{i\beta_j z} + \hat{b}_j e^{-i\beta_j z}\right)\,, \qquad \frac{d\hat{p}_j}{dz} = i\sqrt{\beta_j}\left(\hat{a}_j e^{i\beta_j z} - \hat{b}_j e^{-i\beta_j z}\right)$$

for $j \le N(\omega)$. The total field $\hat{p}$ satisfies the time harmonic wave equation

$$\Delta\hat{p}(\omega,\mathbf{x},z) + k^2(\omega)(1+\varepsilon\nu(\mathbf{x},z))\hat{p}(\omega,\mathbf{x},z) = 0\,. \tag{3.3}$$

Using (3.2) in this equation, multiplying it by $\phi_l(\mathbf{x})$, and integrating over $\mathbf{x} \in \mathcal{D}$, we deduce from the orthogonality of the eigenmodes $(\phi_j)_{j\ge 1}$ the following system of coupled differential equations for the mode amplitudes:

$$\frac{d\hat{a}_j}{dz} = \frac{i\varepsilon k^2}{2}\sum_{1\le l\le N}\frac{C_{jl}(z)}{\sqrt{\beta_j\beta_l}}\left(\hat{a}_l e^{i(\beta_l-\beta_j)z} + \hat{b}_l e^{-i(\beta_l+\beta_j)z}\right)\,, \tag{3.4}$$

$$\frac{d\hat{b}_j}{dz} = -\frac{i\varepsilon k^2}{2}\sum_{1\le l\le N}\frac{C_{jl}(z)}{\sqrt{\beta_j\beta_l}}\left(\hat{a}_l e^{i(\beta_l+\beta_j)z} + \hat{b}_l e^{i(\beta_j-\beta_l)z}\right)\,, \tag{3.5}$$

where

$$C_{jl}(z) = \int_{\mathcal{D}} \phi_j(\mathbf{x})\phi_l(\mathbf{x})\nu(\mathbf{x},z)d\mathbf{x}\,, \tag{3.6}$$

and we have neglected the evanescent modes. The system (3.4)–(3.5) is complemented with the boundary conditions

$$\hat{a}_j(\omega,0) = \hat{a}_{j,0}(\omega), \qquad \hat{b}_j\left(\omega,\frac{L}{\varepsilon^2}\right) = 0 \tag{3.7}$$

for the propagating modes. The second condition indicates that no wave is incoming from the right.

It is possible to carry out a complete asymptotic analysis, including the coupling with the evanescent modes, in a general context of random ordinary differential equations [20], and specifically for random waveguide problems [13, 12] and shallow water waves [18]. The evanescent modes do affect the propagation statistics in the asymptotic regime considered here. A detailed asymptotic analysis shows, however, that the coupling with evanescent modes does not remove energy from the propagating modes. This coupling changes only the frequency-dependent phases of the propagating mode amplitudes. The overall effect for these modes is an additional dispersive term, which is given in terms of the two-point statistics of the random process ν, in the same asymptotic regime as the one considered here. This effective dispersion affects the operator $\mathcal{L}$ for the statistics of the complex mode amplitudes in (4.1) but not the operator $\mathcal{L}_P$ for their square modulus in (4.10). Therefore, all results that involve the propagation of energy, which includes nearly all results presented here, are not affected by the evanescent modes. We indicate in the following where the results are affected and refer to [12] for their form when full evanescent coupling is included.

3.2. Propagator matrices. We introduce the rescaled propagating mode amplitudes $\hat{a}_j^\varepsilon$, $\hat{b}_j^\varepsilon$, $j = 1, \dots, N(\omega)$, given by

$$\hat{a}_j^\varepsilon(\omega, z) = \hat{a}_j\left(\omega, \frac{z}{\varepsilon^2}\right), \qquad \hat{b}_j^\varepsilon(\omega, z) = \hat{b}_j\left(\omega, \frac{z}{\varepsilon^2}\right). \tag{3.8}$$

The two-point linear boundary value problem (3.4), (3.5), (3.7) for $(\hat{a}^\varepsilon, \hat{b}^\varepsilon)$ can be solved using propagator matrices. We first put the problem in vector-matrix form:

$$\frac{dX_\omega^\varepsilon}{dz} = \frac{1}{\varepsilon}\mathbf{H}_\omega\left(\frac{z}{\varepsilon^2}\right) X_\omega^\varepsilon. \tag{3.9}$$

Here the $2N(\omega)$-vector X_ω^ε, obtained by concatenating the $N(\omega)$-vectors $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$ and the $2N(\omega) \times 2N(\omega)$ matrix $\mathbf{H}_\omega$, is defined by

$$X_\omega^\varepsilon(z) = \begin{bmatrix} \hat{a}^\varepsilon(\omega, z) \\ \hat{b}^\varepsilon(\omega, z) \end{bmatrix}, \qquad \mathbf{H}_\omega(z) = \begin{bmatrix} \mathbf{H}_\omega^{(a)}(z) & \mathbf{H}_\omega^{(b)}(z) \\ \overline{\mathbf{H}_\omega^{(b)}(z)} & \overline{\mathbf{H}_\omega^{(a)}(z)} \end{bmatrix}, \tag{3.10}$$

where the entries of the $N(\omega) \times N(\omega)$ matrices $\mathbf{H}_\omega^{(a)}(z)$ and $\mathbf{H}_\omega^{(b)}(z)$ are given by

$$H_{\omega,jl}^{(a)}(z) = \frac{ik^2}{2}\frac{C_{jl}(z)}{\sqrt{\beta_j\beta_l}}e^{i(\beta_l-\beta_j)z}, \qquad H_{\omega,jl}^{(b)}(z) = \frac{ik^2}{2}\frac{C_{jl}(z)}{\sqrt{\beta_j\beta_l}}e^{-i(\beta_l+\beta_j)z}. \tag{3.11}$$

The propagator matrices $\mathbf{P}_\omega^\varepsilon(z)$ are the $2N(\omega) \times 2N(\omega)$ random matrices solution of the initial value problem

$$\frac{d\mathbf{P}_\omega^\varepsilon}{dz} = \frac{1}{\varepsilon}\mathbf{H}_\omega\left(\frac{z}{\varepsilon^2}\right)\mathbf{P}_\omega^\varepsilon, \tag{3.12}$$

with the initial condition $\mathbf{P}_\omega^\varepsilon(z=0) = \mathbf{I}$. The solution of (3.4), (3.5), (3.7) satisfies

$$\begin{bmatrix} \hat{a}^\varepsilon(\omega, L) \\ 0 \end{bmatrix} = \mathbf{P}_\omega^\varepsilon(L)\begin{bmatrix} \hat{a}_0(\omega) \\ \hat{b}^\varepsilon(\omega, 0) \end{bmatrix}, \tag{3.13}$$

so that $\hat{a}^\varepsilon(\omega, L)$ can be expressed in terms of the entries of the propagator matrix $\mathbf{P}_\omega^\varepsilon(L)$. The symmetry relation (3.10) satisfied by the matrix $\mathbf{H}_\omega$ imposes the condition that the propagator has the form

$$\mathbf{P}_\omega^\varepsilon(z) = \begin{bmatrix} \mathbf{P}_\omega^{\varepsilon,a}(z) & \mathbf{P}_\omega^{\varepsilon,b}(z) \\ \overline{\mathbf{P}_\omega^{\varepsilon,b}(z)} & \overline{\mathbf{P}_\omega^{\varepsilon,a}(z)} \end{bmatrix}, \tag{3.14}$$

where $\mathbf{P}_\omega^{\varepsilon,a}(z)$ and $\mathbf{P}_\omega^{\varepsilon,b}(z)$ are $N(\omega) \times N(\omega)$ matrices. Note that the matrix $\mathbf{P}_\omega^{\varepsilon,a}$ describes the coupling between different right-going modes, while $\mathbf{P}_\omega^{\varepsilon,b}$ describes the coupling between right-going and left-going modes.

3.3. The forward scattering approximation. The limit as $\varepsilon \to 0$ of $\mathbf{P}_\omega^{\varepsilon}$ can be obtained and identified as a multidimensional diffusion process, meaning that the entries of the limit matrix satisfy a system of linear stochastic differential equations. This follows from the application of the diffusion-approximation theorem proved in [19]. The stochastic differential equations for the limit entries of $\mathbf{P}_\omega^{\varepsilon,b}(z)$ are coupled to the limit entries of $\mathbf{P}_\omega^{\varepsilon,a}(z)$ through the coefficients

$$\int_0^\infty \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos((\beta_j(\omega) + \beta_l(\omega))z)\, dz, \qquad j,l = 1,\dots,N(\omega)\,.$$

This is because the phase factors present in the matrix $\mathbf{H}_\omega^{(b)}(z)$ are $\pm(\beta_j + \beta_l)z$. On the other hand, the stochastic differential equations for the limit entries of $\mathbf{P}_\omega^{\varepsilon,a}(z)$ are coupled to each other through the coefficients

$$\int_0^\infty \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos((\beta_j(\omega) - \beta_l(\omega))z)\, dz, \qquad j,l = 1,\dots,N(\omega)\,.$$

This is because the phase factors present in the matrix $\mathbf{H}_\omega^{(a)}(z)$ are $\pm(\beta_j - \beta_l)z$. If we assume that the power spectral density of the process ν (i.e., the Fourier transform of its z-autocorrelation function) possesses a cut-off frequency, then it is natural to consider the case where

$$\int_0^\infty \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos((\beta_j(\omega) + \beta_l(\omega))z)\, dz = 0, \qquad j,l = 1,\dots,N(\omega), \tag{3.15}$$

while (at least) some of the intracoupling coefficients (those with $|j-l| = 1$) are not zero. As a result of this assumption, the asymptotic coupling between $\mathbf{P}_\omega^{\varepsilon,a}(z)$ and $\mathbf{P}_\omega^{\varepsilon,b}(z)$ becomes zero. If we also take into account the initial condition $\mathbf{P}_\omega^{\varepsilon,b}(z=0) = \mathbf{0}$, then the limit of $\mathbf{P}_\omega^{\varepsilon,b}(z)$ is $\mathbf{0}$.

In the forward scattering approximation, we neglect the left-going (backward) propagating modes. As we have just seen, it is valid in the limit $\varepsilon \to 0$ when the condition (3.15) holds. In this case we can consider the simplified coupled mode equation given by

$$\frac{d\hat{a}^\varepsilon}{dz} = \frac{1}{\varepsilon}\mathbf{H}_\omega^{(a)}\left(\frac{z}{\varepsilon^2}\right)\hat{a}^\varepsilon\,, \tag{3.16}$$

where $\mathbf{H}_\omega^{(a)}$ is the $N(\omega) \times N(\omega)$ complex matrix given by (3.11). The system (3.16) is provided with the initial condition $\hat{a}_j^\varepsilon(\omega, z=0) = \hat{a}_{j,0}(\omega)$. Note that the matrix $\mathbf{H}_\omega^{(a)}$ is skew Hermitian, which implies the conservation relation $\sum_{j=1}^N |\hat{a}_j^\varepsilon(L)|^2 = \sum_{j=1}^N |\hat{a}_{j,0}|^2$. We finally introduce the *transfer*, or propagator matrix $\mathbf{T}^\varepsilon(\omega, z)$, which is the fundamental solution of (3.16). It is the $N(\omega) \times N(\omega)$ matrix solution of

$$\frac{d}{dz}\mathbf{T}^\varepsilon(\omega, z) = \frac{1}{\varepsilon}\mathbf{H}_\omega^{(a)}\left(\frac{z}{\varepsilon^2}\right)\mathbf{T}^\varepsilon(\omega, z)\,, \tag{3.17}$$

starting from $\mathbf{T}^\varepsilon(\omega, 0) = \mathbf{I}$. The (j,l)-entry of the transfer matrix is the transmission coefficient $T_{jl}^\varepsilon(\omega, L)$, i.e., the output amplitude of the mode j when the input wave is a pure l mode with amplitude one. The transfer matrix $\mathbf{T}^\varepsilon(\omega, L)$ is unitary because $\mathbf{H}_\omega^{(a)}$ is skew Hermitian.

4. The time harmonic problem. In this section we consider the system of random differential equations (3.16) for a single frequency ω. Most of the results presented in this section can be found in [14] and are known collectively as the "coupled mode theory." We reproduce this theory because the original two-frequency analysis presented in the next section will give a new point of view on it.

4.1. The coupled mode diffusion process. We now apply the diffusion approximation theorem [19] to the system (3.16). The limit distribution of $\hat{a}^\varepsilon$ as $\varepsilon \to 0$ is a diffusion on $\mathbb{C}^{N(\omega)}$. We will assume that the longitudinal wavenumbers β_j, along with their sums and differences, are distinct. In this case the infinitesimal generator of the limit $\hat{a}$ has a simple form, provided we write it in terms of $\hat{a}$ and $\overline{\hat{a}}$ rather than in terms of the real and imaginary parts of $\hat{a}$. We get the following result.

PROPOSITION 4.1. *The mode amplitudes* $(\hat{a}_j^\varepsilon(\omega, z))_{j=1,\dots,N}$ *converge in distribution as* $\varepsilon \to 0$ *to the diffusion process* $(\hat{a}_j(\omega, z))_{j=1,\dots,N}$*, whose infinitesimal generator is*

$$\mathcal{L} = \frac{1}{4}\sum_{j\neq l}\Gamma_{jl}^{(c)}(\omega)\left(A_{jl}\overline{A_{jl}} + \overline{A_{jl}}A_{jl}\right) + \frac{1}{2}\sum_{j,l}\Gamma_{jl}^{(1)}(\omega)A_{jj}\overline{A_{ll}}$$

$$+\frac{i}{4}\sum_{j\neq l}\Gamma_{jl}^{(s)}(\omega)(A_{ll} - A_{jj})\,, \tag{4.1}$$

$$A_{jl} = \hat{a}_j\frac{\partial}{\partial \hat{a}_l} - \overline{\hat{a}_l}\frac{\partial}{\partial \overline{\hat{a}_j}} = -\overline{A_{lj}}\,. \tag{4.2}$$

Here we have defined the complex derivatives in the standard way: if $z = x+iy$*, then* $\partial_z = (1/2)(\partial_x - i\partial_y)$ *and* $\partial_{\overline{z}} = (1/2)(\partial_x + i\partial_y)$*. The coefficients* $\Gamma^{(c)}$*,* $\Gamma^{(s)}$*, and* $\Gamma^{(1)}$ *are given by*

$$\Gamma_{jl}^{(c)}(\omega) = \frac{\omega^4\gamma_{jl}^{(c)}(\omega)}{4\bar{c}^4\beta_j(\omega)\beta_l(\omega)} \quad \textit{if } j\neq l\,, \qquad \Gamma_{jj}^{(c)}(\omega) = -\sum_{n\neq j}\Gamma_{jn}^{(c)}(\omega)\,, \tag{4.3}$$

$$\gamma_{jl}^{(c)}(\omega) = 2\int_0^\infty \cos\left((\beta_j(\omega) - \beta_l(\omega))z\right)\mathbb{E}[C_{jl}(0)C_{jl}(z)]dz\,, \tag{4.4}$$

$$\Gamma_{jl}^{(s)}(\omega) = \frac{\omega^4\gamma_{jl}^{(s)}(\omega)}{4\bar{c}^4\beta_j(\omega)\beta_l(\omega)} \quad \textit{if } j\neq l\,, \qquad \Gamma_{jj}^{(s)}(\omega) = -\sum_{n\neq j}\Gamma_{jn}^{(s)}(\omega)\,, \tag{4.5}$$

$$\gamma_{jl}^{(s)}(\omega) = 2\int_0^\infty \sin\left((\beta_j(\omega) - \beta_l(\omega))z\right)\mathbb{E}[C_{jl}(0)C_{jl}(z)]dz\,, \tag{4.6}$$

$$\Gamma_{jl}^{(1)}(\omega) = \frac{\omega^4\gamma_{jl}^{(1)}}{4\bar{c}^4\beta_j(\omega)\beta_l(\omega)} \quad \textit{for all } j,l\,, \tag{4.7}$$

$$\gamma_{jl}^{(1)} = 2\int_0^\infty \mathbb{E}[C_{jj}(0)C_{ll}(z)]dz\,. \tag{4.8}$$

Let us discuss some qualitative properties of the diffusion process $\hat{a}$.

(1) The coefficients of the second derivatives of the generator $\mathcal{L}$ are homogeneous of degree two, while the coefficients of the first derivatives are homogeneous of degree one. As a consequence we can write closed differential equations for moments of any order, as we shall see in the next sections.

(2) The coefficients $\gamma_{jl}^{(c)}$, and thus $\Gamma_{jl}^{(c)}$, are proportional to the power spectral densities of the stationary process $C_{jl}(z)$ for $j\neq l$. They are, therefore, nonnegative. In this paper we assume that the off-diagonal entries of the matrix $\Gamma^{(c)}$ are positive.

(3) We have $A_{jn}\big(\sum_{l=1}^N |\hat{a}_l|^2\big) = 0$ for any j,n, so that the infinitesimal generator satisfies $\mathcal{L}\big(\sum_{l=1}^N |\hat{a}_l|^2\big) = 0$. This implies that the diffusion process is supported on a sphere of $\mathbb{C}^N$, whose radius R_0 is determined by the initial condition $R_0^2 = \sum_{l=1}^N |\hat{a}_{l,0}(\omega)|^2$. The operator $\mathcal{L}$ is not self-adjoint on the sphere because of the term $\Gamma^{(s)}$ in (4.1). This means that the process is not reversible. However, the uniform measure on the sphere is invariant, and the generator is strongly elliptic. From the theory of irreducible Markov processes with compact state space, we know that the process is ergodic, which means in particular that for large z, the limit process $\hat{a}(z)$ converges to the uniform distribution over the sphere of radius R_0. This fact can be used to compute the limit distribution of the mode powers $(|\hat{a}_j|^2)_{j=1,\dots,N}$ for large z, which is the uniform distribution over $\mathcal{H}_N$,

$$\mathcal{H}_N = \left\{ (P_j)_{j=1,\dots,N},\, P_j \geq 0,\, \sum_{j=1}^N P_j = R_0^2 \right\} . \tag{4.9}$$

In the next section we carry out a more detailed analysis that is valid for any z.

(4) As noted at the end of section 3.1, coupling to the evanescent modes does affect $\mathcal{L}$ in (4.1). All the coefficients (4.3)–(4.8) remain the same except for $\Gamma_{jl}^{(s)}(\omega)$ in (4.5) which has an additional term [12]. This modification affects (6.1) and (6.7) in the following sections.

4.2. Coupled power equations. The generator of the limit process $\hat{a}$ possesses an important symmetry, which follows from noting that, when applying the generator to a function of $(|\hat{a}_1|^2, \dots, |\hat{a}_N|^2)$, we obtain another function of $(|\hat{a}_1|^2, \dots, |\hat{a}_N|^2)$. This implies that the limit process $(|\hat{a}_1(z)|^2)_{j=1,\dots,N}$ is itself a Markov process.

PROPOSITION 4.2. *The mode powers $(|\hat{a}_j^\varepsilon(\omega,z)|^2)_{j=1,\dots,N}$ converge in distribution as $\varepsilon \to$ to the diffusion process $(P_j(\omega,z))_{j=1,\dots,N}$, whose infinitesimal generator is*

$$\mathcal{L}_P = \sum_{j\neq l} \Gamma_{jl}^{(c)}(\omega) \left[P_l P_j \left(\frac{\partial}{\partial P_j} - \frac{\partial}{\partial P_l} \right) \frac{\partial}{\partial P_j} + (P_l - P_j) \frac{\partial}{\partial P_j} \right] . \tag{4.10}$$

As pointed out above, the diffusion process $(P_j(\omega,z))_{j=1,\dots,N}$ is supported in $\mathcal{H}_N$. As a first application of this result, we compute the mean mode powers:

$$P_j^{(1)}(\omega,z) = \mathbb{E}[P_j(\omega,z)] = \lim_{\varepsilon\to 0} \mathbb{E}[|\hat{a}_j^\varepsilon(\omega,z)|^2] .$$

Using the generator $\mathcal{L}_P$ we get the following proposition.

PROPOSITION 4.3. *The mean mode powers $\mathbb{E}[|\hat{a}_j^\varepsilon(\omega,z)|^2]$ converge to $P_j^{(1)}(\omega,z)$, which is the solution of the linear system*

$$\frac{dP_j^{(1)}}{dz} = \sum_{n\neq j} \Gamma_{jn}^{(c)}(\omega) \left(P_n^{(1)} - P_j^{(1)} \right) , \tag{4.11}$$

starting from $P_j^{(1)}(\omega, z=0) = |\hat{a}_{j,0}(\omega)|^2$, $j = 1,\dots,N$.

The solution of this system can be written in terms of the exponential of the matrix $\Gamma^{(c)}(\omega)$. We note that the vector $P^{(1)}(\omega,z)$ has a probabilistic interpretation, which we consider in some detail in section 6.3. We give here some basic properties. First, the matrix $\Gamma^{(c)}(\omega)$ is symmetric and real, its off-diagonal terms are positive, and its diagonal terms are negative. The sums over the rows and columns are all zero. As a consequence of the Perron–Frobenius theorem, $\Gamma^{(c)}(\omega)$ has zero as a simple eigenvalue, and all other eigenvalues are negative. The eigenvector associated with the zero eigenvalue is the uniform vector $(1, \dots, 1)^T$. This shows that

$$\sup_{j=1,\dots,N} \left| P_j^{(1)}(\omega, z) - \frac{1}{N} R_0^2 \right| \le C e^{-z/L_{\text{equip}}(\omega)},$$

where $R_0^2 = \sum_{j=1}^N |\hat{a}_{j,0}(\omega)|^2$ and $L_{\text{equip}}(\omega)$ is the absolute value of the reciprocal of the second eigenvalue of $\Gamma^{(c)}(\omega)$. In other words, the mean mode powers converge exponentially fast to the uniform distribution, which means that we have asymptotic *equipartition* of mode energy. The length $L_{\text{equip}}(\omega)$ is the equipartition distance for the mean mode powers.

4.3. Fluctuations theory. Proposition 4.2 also allows us to study the fluctuations of the mode powers by looking at the fourth-order moments of the mode amplitudes:

$$P_{jl}^{(2)}(\omega, z) = \lim_{\varepsilon \to 0} \mathbb{E}\left[|\hat{a}_j^\varepsilon(\omega, z)|^2 |\hat{a}_l^\varepsilon(\omega, z)|^2 \right] = \mathbb{E}[P_j(\omega, z) P_l(\omega, z)] \, .$$

Using the generator $\mathcal{L}_P$ we get a system of ordinary differential equations for limit fourth-order moments $(P_{jl}^{(2)})_{j,l=1,\dots,N}$ of the form

$$\begin{aligned} \frac{dP_{jj}^{(2)}}{dz} &= \sum_{n \neq j} \Gamma_{jn}^{(c)} \left(4P_{jn}^{(2)} - 2P_{jj}^{(2)} \right) , \\ \frac{dP_{jl}^{(2)}}{dz} &= -2\Gamma_{jl}^{(c)} P_{jl}^{(2)} + \sum_n \Gamma_{ln}^{(c)} \left(P_{jn}^{(2)} - P_{jl}^{(2)} \right) + \sum_n \Gamma_{jn}^{(c)} \left(P_{ln}^{(2)} - P_{jl}^{(2)} \right), \qquad j \neq l \, . \end{aligned}$$

The initial conditions are $P_{jl}^{(2)}(z=0) = |\hat{a}_{j,0}|^2 |\hat{a}_{l,0}|^2$. This is a system of linear ordinary differential equations with constant coefficients that can be solved by computing the exponent of the evolution matrix.

It is straightforward to check that the function $P_{jl}^{(2)} \equiv 1 + \delta_{jl}$ is a stationary solution of the fourth-order moment system. Using the positivity of $\Gamma_{jl}^{(c)}$, $j \neq l$, we conclude that this stationary solution is asymptotically stable, which means that the solution $P_{jl}^{(2)}(z)$ starting from $P_{jl}^{(2)}(z=0) = |\hat{a}_{j,0}|^2 |\hat{a}_{l,0}|^2$ converges as $z \to \infty$ to

$$P_{jl}^{(2)}(z) \stackrel{z \to \infty}{\longrightarrow} \begin{cases} \dfrac{1}{N(N+1)} R_0^4 & \text{if } j \neq l \, , \\ \dfrac{2}{N(N+1)} R_0^4 & \text{if } j = l \, , \end{cases}$$

where $R_0^2 = \sum_{j=1}^N |\hat{a}_{j,0}|^2$. This implies that the correlation of $P_j(z)$ and $P_l(z)$ converges to $-1/(N-1)$ if $j \neq l$ and to $(N-1)/(N+1)$ if $j = l$ as $z \to \infty$. We see from the $j \neq l$ result that if, in addition, the number of modes N becomes large, then the mode powers become uncorrelated. The $j = l$ result shows that, whatever the number of modes N, the mode powers P_j are not statistically stable quantities.

5. Pulse propagation in waveguides. Bandwidth plays a basic role in the propagation of pulses in a waveguide because of dispersion. We assume that a point source located inside the waveguide at $(z = 0, \mathbf{x} = \mathbf{x}_0)$ emits a pulse with carrier frequency ω_0 and bandwidth of order ε^2:

$$\mathbf{F}^\varepsilon(t,\mathbf{x},z) = f^\varepsilon(t)\delta(\mathbf{x}-\mathbf{x}_0)\delta(z)\mathbf{e}_z\,, \qquad f^\varepsilon(t) = f(\varepsilon^2 t)e^{i\omega_0 t}\,. \tag{5.1}$$

We have assumed in this model a narrowband source whose duration is of order ε^{-2}, that is, of the same order as the travel time through the waveguide whose length is L/ε^2. This is not the typical situation encountered in ultrasonic and underwater sound experiments in connection with time reversal [22, 15], where broadband pulses are used and, in the notation of this paper, $f^\varepsilon(t) = f(\varepsilon^p t)e^{i\omega_0 t}$ with $0 \le p < 2$. The analysis of these broadband regimes is carried out in [10]. The main results are similar in the two regimes, except for those regarding statistical stability in time reversal. We discuss this issue in sections 8.3 and 8.4, where we show that statistical stability in the narrowband case (5.1) can be achieved by mode diversity rather than by frequency diversity. The latter is typical for time-reversal refocusing of broadband pulses [10]. This is the reason that we consider the narrowband case in this paper.

The point source (5.1) generates left-going propagating modes that we do not need to consider, as they propagate in a homogeneous half-space, and right-going modes that we do analyze. As shown in section 2.3, the interface conditions at $z = 0$, which are initial conditions in the forward scattering approximation, have the form

$$\hat{a}_j^\varepsilon(\omega,0) = \frac{1}{2}\sqrt{\beta_j(\omega)}\hat{f}^\varepsilon(\omega)\phi_j(\mathbf{x}_0)\,, \qquad \hat{f}^\varepsilon(\omega) = \frac{1}{\varepsilon^2}\hat{f}\left(\frac{\omega-\omega_0}{\varepsilon^2}\right)$$

for $j \le N(\omega)$. The transmitted field observed in the plane $z = L/\varepsilon^2$ and at time t/ε^2 has the form

$$p_{tr}^\varepsilon(t,\mathbf{x},L) := p\left(\frac{t}{\varepsilon^2},\mathbf{x},\frac{L}{\varepsilon^2}\right)\,,$$

$$p_{tr}^\varepsilon(t,\mathbf{x},L) \;=\; \frac{1}{4\pi\varepsilon^2}\int \sum_{j,l=1}^{N(\omega)} \frac{\sqrt{\beta_l(\omega)}}{\sqrt{\beta_j(\omega)}}\phi_j(\mathbf{x})\phi_l(\mathbf{x}_0)\hat{f}\left(\frac{\omega-\omega_0}{\varepsilon^2}\right)T_{jl}^\varepsilon(\omega)e^{i\frac{\beta_j(\omega)L-\omega t}{\varepsilon^2}}d\omega\,.$$

We change variables $\omega = \omega_0 + \varepsilon^2 h$ and expand $\beta_j(\omega_0+\varepsilon^2 h)$ with respect to ε:

$$\begin{aligned} p_{tr}^\varepsilon(t,\mathbf{x},L) = \frac{1}{4\pi}\int \sum_{j,l=1}^{N} \frac{\sqrt{\beta_l}}{\sqrt{\beta_j}}\phi_j(\mathbf{x})\phi_l(\mathbf{x}_0) \\ \times \hat{f}(h)T_{jl}^\varepsilon(\omega_0+\varepsilon^2 h)e^{i\frac{\beta_j(\omega_0)L-\omega_0 t}{\varepsilon^2}}e^{i[\beta_j'(\omega_0)L-t]h}dh\,. \end{aligned} \tag{5.2}$$

Here $\beta_j'(\omega)$ is the ω-derivative of $\beta_j(\omega)$. We do not show the dependence of N on ω_0 after we approximate $N(\omega_0+\varepsilon^2 h)$ by $N(\omega_0)$.

In a homogeneous waveguide we have that $T_{jl}^\varepsilon = \delta_{jl}$ and

$$p_{tr}^\varepsilon(t,\mathbf{x},L) = \frac{1}{2}\sum_{j=1}^{N}\phi_j(\mathbf{x})\phi_j(\mathbf{x}_0)e^{i\frac{\beta_j(\omega_0)L-\omega_0 t}{\varepsilon^2}}f\left(t-\beta_j'(\omega_0)L\right)\,. \tag{5.3}$$

The transmitted field is therefore a superposition of modes, each of which is centered around its travel time $\beta_j'(\omega_0)L$. The modal dispersion makes the overall spreading of the transmitted field linearly increasing with L.

In a random waveguide, the integral representation (5.2) shows that the moments of the transmitted field depend on the joint statistics of the entries of the transfer matrix at different frequencies. The next section will give the needed results.

6. Statistics of the transfer matrix.

6.1. Single frequency statistics of the transfer matrix. Using Proposition 4.1 with the special initial conditions $\hat{a}_j(0) = \delta_{jl}$, we obtain the first moments of the transmission coefficients.

PROPOSITION 6.1. *The expectations of the transmission coefficients* $\mathbb{E}[T^{\varepsilon}_{jl}(\omega, L)]$ *converge to zero as* $\varepsilon \to 0$ *if* $j \neq l$ *and to* $\bar{T}_j(\omega, L)$ *if* $j = l$*, where*

$$\bar{T}_j(\omega, L) = \exp\left(\frac{\Gamma^{(c)}_{jj}(\omega)L}{2} - \frac{\Gamma^{(1)}_{jj}(\omega)L}{2} + \frac{i\Gamma^{(s)}_{jj}(\omega)L}{2}\right). \tag{6.1}$$

The real part of the exponential factor is $[\Gamma^{(c)}_{jj}(\omega) - \Gamma^{(1)}_{jj}(\omega)]L/2$. The coefficient $\Gamma^{(c)}_{jj}(\omega)$ is negative. The coefficient $\Gamma^{(1)}_{jj}(\omega)$ is nonnegative because it is proportional to the power spectral density of C_{jj} at zero-frequency. As a result, the damping coefficient has a negative real part, and therefore the mean transmission coefficients decay exponentially with propagation distance.

Using Proposition 4.3, we immediately get the following result.

PROPOSITION 6.2. *The mean square moduli of the transmission coefficients have limits as* $\varepsilon \to 0$*,* $\lim_{\varepsilon\to 0}\mathbb{E}[|T^{\varepsilon}_{jl}(\omega, L)|^2] = \mathcal{T}^{(l)}_j(\omega, L)$*, which are the solutions of the system of linear equations*

$$\frac{d\mathcal{T}^{(l)}_j}{dz} = \sum_{n\neq j}\Gamma^{(c)}_{jn}(\omega)\left(\mathcal{T}^{(l)}_n - \mathcal{T}^{(l)}_j\right), \qquad \mathcal{T}^{(l)}_j(\omega, z = 0) = \delta_{jl}\,. \tag{6.2}$$

The coefficients $\Gamma^{(c)}_{jl}$ *are given by* (4.3).

From the analysis of section 4.2, we know that the mean square moduli of the entries of the transfer matrix converge exponentially fast to the constant $1/N$.

6.2. Transport equations for the autocorrelation of the transfer matrix. In many physically interesting contexts, such as in calculating the mean transmitted intensity or the mean refocused field amplitude, we need two-frequency statistical information. We now introduce a proposition that describes the two-frequency statistical properties that we will need in the applications discussed in this paper.

PROPOSITION 6.3. *The autocorrelation function of the transmission coefficients at two nearby frequencies admits a limit as* $\varepsilon \to 0$*:*

$$\mathbb{E}[T^{\varepsilon}_{jj}(\omega, L)\overline{T^{\varepsilon}_{ll}(\omega - \varepsilon^2 h, L)}] \stackrel{\varepsilon\to 0}{\longrightarrow} e^{Q_{jl}(\omega)L} \text{ if } j \neq l\,, \tag{6.3}$$

$$\mathbb{E}[T^{\varepsilon}_{jl}(\omega, L)\overline{T^{\varepsilon}_{jl}(\omega - \varepsilon^2 h, L)}] \stackrel{\varepsilon\to 0}{\longrightarrow} e^{-i\beta_j'(\omega)hL}\int \mathcal{W}^{(l)}_j(\omega, \tau, L)e^{ih\tau}d\tau\,, \tag{6.4}$$

$$\mathbb{E}[T^{\varepsilon}_{jm}(\omega, L)\overline{T^{\varepsilon}_{ln}(\omega - \varepsilon^2 h, L)}] \stackrel{\varepsilon\to 0}{\longrightarrow} 0 \text{ in the other cases,} \tag{6.5}$$

where $(\mathcal{W}^{(l)}_j(\omega, \tau, z))_{j=1,\ldots,N(\omega)}$ *is the solution of the system of transport equations*

$$\frac{\partial\mathcal{W}^{(l)}_j}{\partial z} + \beta_j'(\omega)\frac{\partial\mathcal{W}^{(l)}_j}{\partial\tau} = \sum_{n\neq j}\Gamma^{(c)}_{jn}(\omega)\left(\mathcal{W}^{(l)}_n - \mathcal{W}^{(l)}_j\right), \tag{6.6}$$

starting from $\mathcal{W}_j^{(l)}(\omega,\tau,z=0)=\delta(\tau)\delta_{jl}$. *The coefficients* $\Gamma_{jl}^{(c)}$ *are given by* (4.3). *The damping factors* Q_{jl} *are*

$$Q_{jl}(\omega) = \frac{\Gamma_{jj}^{(c)}(\omega)+\Gamma_{ll}^{(c)}(\omega)}{2} - \frac{\Gamma_{jj}^{(1)}(\omega)+\Gamma_{ll}^{(1)}(\omega)-2\Gamma_{jl}^{(1)}(\omega)}{2} + i\frac{\Gamma_{jj}^{(s)}(\omega)-\Gamma_{ll}^{(s)}(\omega)}{2}\,. \tag{6.7}$$

We note that the real parts of the damping factors Q_{jl} are negative and that the solutions of the transport equations are measures. If $j\neq l$, then $\mathcal{W}_j^{(l)}$ has a continuous density, but $\mathcal{W}_l^{(l)}$ has a Dirac mass at $\tau=\beta_l'(\omega)z$ with weight $\exp(\Gamma_{ll}^{(c)}z)$, where $\Gamma_{ll}^{(c)}$ is given by (4.3), and a continuous density denoted by $\mathcal{W}_{l,c}^{(l)}(\omega,\tau,z)$:

$$\mathcal{W}_l^{(l)}(\omega,\tau,z)d\tau = e^{\Gamma_{ll}^{(c)}(\omega)z}\delta(\tau-\beta_l'(\omega)z)d\tau + \mathcal{W}_{l,c}^{(l)}(\omega,\tau,z)d\tau\,.$$

Note also that by integrating the system of transport equations with respect to τ, we recover the result of Proposition 6.2.

The system of transport equations describes the coupling between the N right-going modes. It is the main theoretical result of this paper. It describes the evolution of the coupled powers of the modes in frequency and time, with transport velocities equal to the group velocities of the modes $1/\beta_j'(\omega)$. Therefore, the transport equations (6.6) could have been written as the natural space-time generalization of the coupled power equations (6.2). The mathematical content of Proposition 6.3 gives the precise connection between the quantities that satisfy this simple and intuitive space-time extension of (6.2) and the moments of the random transfer matrix. The two-frequency nature of the statistical quantities that satisfy the transport equations is clear in Proposition 6.3.

Proof. For fixed indices m and n we consider the product of two transfer matrices,

$$U_{jl}^{\varepsilon}(\omega,h,z) = T_{jm}^{\varepsilon}(\omega,z)\overline{T_{ln}^{\varepsilon}(\omega-\varepsilon^2 h,z)}\,,$$

and note that it is the solution of

$$\frac{dU_{jl}^{\varepsilon}}{dz} = \sum_{j_1=1}^{N}\frac{1}{\varepsilon}H_{\omega,jj_1}^{(a)}\left(\frac{z}{\varepsilon^2}\right)U_{j_1l}^{\varepsilon} + \sum_{l_1=1}^{N}\frac{1}{\varepsilon}\overline{H_{\omega-\varepsilon^2h,ll_1}^{(a)}}\left(\frac{z}{\varepsilon^2}\right)U_{jl_1}^{\varepsilon}\,,$$

with the initial conditions $U_{jl}^{\varepsilon}(\omega,h,z=0)=\delta_{mj}\delta_{nl}$. We expand $\beta_.(\omega-\varepsilon^2h)$ with respect to ε, and we introduce the Fourier transform

$$V_{jl}^{\varepsilon}(\omega,\tau,z) = \frac{1}{2\pi}\int e^{-ih(\tau-\beta_l'(\omega)z)}U_{jl}^{\varepsilon}(\omega,h,z)dh\,,$$

which is the solution of

$$\frac{\partial V_{jl}^{\varepsilon}}{\partial z} + \beta_l'(\omega)\frac{\partial V_{jl}^{\varepsilon}}{\partial \tau} = \sum_{j_1=1}^{N}\frac{1}{\varepsilon}H_{\omega,jj_1}^{(a)}\left(\frac{z}{\varepsilon^2}\right)V_{j_1l}^{\varepsilon} + \sum_{l_1=1}^{N}\frac{1}{\varepsilon}\overline{H_{\omega,ll_1}^{(a)}}\left(\frac{z}{\varepsilon^2}\right)V_{jl_1}^{\varepsilon}\,,$$

with the initial conditions $V_{jl}^{\varepsilon}(\omega,\tau,z=0)=\delta_{mj}\delta_{nl}\delta(\tau)$. We can now apply the diffusion approximation theorem [19] and get the result stated in the proposition. ∎

6.3. Probabilistic interpretation of the transport equations. The transport equations (6.6) have a probabilistic representation that can be used for Monte Carlo simulations as well as for getting a diffusion approximation result. It is primarily this diffusion approximation that we want to derive in this section. We will use it in the applications that follow. We introduce the jump Markov process J_z, whose state space is $\{1,\dots,N(\omega)\}$ and whose infinitesimal generator is

$$\mathcal{L}\phi(j) = \sum_{l\neq j} \Gamma^{(c)}_{jl}(\omega)\left(\phi(l)-\phi(j)\right) . \tag{6.8}$$

We also define the process $\mathcal{B}_z$ by

$$\mathcal{B}_z = \int_0^z \beta'_{J_s} ds\,, \quad z\geq 0\,, \tag{6.9}$$

which is well defined because J_z is piecewise constant. In a manner similar to that in [1], we get the probabilistic representation of the solutions to system (6.2) and the solutions to the transport equations (6.6) in terms of the jump Markov process J_z:

$$\mathcal{T}^{(n)}_j(\omega,L) = \mathbb{P}\left(J_L = j \mid J_0 = n\right) , \tag{6.10}$$

$$\int_{\tau_0}^{\tau_1} \mathcal{W}^{(n)}_j(\omega,\tau,L)d\tau = \mathbb{P}\left(J_L = j\ ,\ \mathcal{B}_L\in[\tau_0,\tau_1] \mid J_0 = n\right) . \tag{6.11}$$

The process J_z is an irreducible, reversible, and ergodic Markov process. Its distribution converges as $z\to\infty$ to the uniform distribution over $\{1,\dots,N\}$. The convergence is exponential with a rate that is equal to the second eigenvalue of the matrix $\Gamma^{(c)} = (\Gamma^{(c)}_{jl})_{j,l=1,\dots,N}$. The first eigenvalue of this matrix is zero and the associated eigenvector is the uniform distribution over $\{1,\dots,N\}$. The second eigenvalue can be written in the form $-1/L_{\text{equip}}$, which defines the equipartition distance L_{equip}.

We next determine the asymptotic distribution of the process $\mathcal{B}_z$. From the ergodic theorem we have that with probability one,

$$\frac{\mathcal{B}_z}{z} \stackrel{z\to\infty}{\longrightarrow} \overline{\beta'(\omega)}\,,\ \text{where } \overline{\beta'(\omega)} = \frac{1}{N(\omega)}\sum_{j=1}^{N(\omega)} \beta'_j(\omega)\,. \tag{6.12}$$

We can interpret the z large limit to mean that z is considerably larger than L_{equip}.

For a planar waveguide we have that $\beta_j = \sqrt{\omega^2/c^2 - \pi^2 j^2/d^2}$ and $N(\omega) = [(\omega d)/(\pi\bar{c})]$. In the continuum limit $N(\omega)\gg 1$, we obtain the expression $\overline{\beta'(\omega)} = \pi/(2\bar{c})$, which is independent of ω. This ω independence property is likely to hold for a broad class of waveguides.

By applying a central limit theorem for functionals of ergodic Markov processes, we find that, in distribution,

$$\frac{\mathcal{B}_z - \overline{\beta'(\omega)}z}{\sqrt{z}} \stackrel{z\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2_{\beta'(\omega)})\,. \tag{6.13}$$

Here $\mathcal{N}(0,\sigma^2_{\beta'(\omega)})$ is a zero-mean Gaussian random variable with variance

$$\sigma^2_{\beta'(\omega)} = 2\int_0^\infty \mathbb{E}_e\left[(\beta'_{J_0}(\omega)-\overline{\beta'(\omega)})(\beta'_{J_s}(\omega)-\overline{\beta'(\omega)})\right] ds\,, \tag{6.14}$$

where $\mathbb{E}_e$ stands for expectation with respect to the stationary process J_z. These limit theorems imply that when $L \gg L_{\rm equip}$, we have

$$\mathcal{T}_j^{(n)}(\omega, L) \overset{L\gg L_{\rm equip}}{\simeq} \frac{1}{N(\omega)}\,, \tag{6.15}$$

$$\mathcal{W}_j^{(n)}(\omega, \tau, L) \overset{L\gg L_{\rm equip}}{\simeq} \frac{1}{N(\omega)} \frac{1}{\sqrt{2\pi\sigma^2_{\beta'(\omega)}L}} \exp\left(-\frac{(\tau - \overline{\beta'(\omega)}L)^2}{2\sigma^2_{\beta'(\omega)}L}\right)\,. \tag{6.16}$$

The asymptotic result (6.15) shows that $\mathcal{T}_j^{(n)}$ becomes independent of n, the initial mode index, and uniform over $j \in \{1, \ldots, N(\omega)\}$. This is the regime of energy equipartition among all propagating modes. The asymptotic result (6.16) is equivalent to the diffusion approximation for the system of transport equations (6.6).

7. Pulse propagation in random waveguides. We consider the transmitted field (5.2) obtained in the setup described in section 5, and we now address the case of a random waveguide. By Proposition 6.1, the mean transmitted field in the asymptotic $\varepsilon \to 0$ is given by

$$\mathbb{E}[p^\varepsilon_{tr}(t, \mathbf{x}, L)] = \frac{1}{2}\sum_{j=1}^N \phi_j(\mathbf{x})\phi_j(\mathbf{x}_0)\bar{T}_j(\omega_0, L)e^{i\frac{\beta_j(\omega_0)L-\omega_0 t}{\varepsilon^2}} f\left(t - \beta_j'(\omega_0)L\right)\,.$$

As in the homogeneous case, the mean field is a superposition of modes in the random case, but the mean transmission coefficients are exponentially damped and vanish for L large, $L > L_{\rm equip}(\omega_0)$. Therefore, the mean field vanishes for large L. We now turn our attention to the mean intensity, which accounts for the conversion of the coherent field into incoherent wave fluctuations. We express the transmitted intensity as the expectation of a double integral

$$\begin{aligned}\mathbb{E}\left[|p^\varepsilon_{tr}(t, \mathbf{x}, L)|^2\right] &= \frac{1}{16\pi^2}\sum_{j,l=1}^N \sum_{m,n=1}^N \frac{\sqrt{\beta_l\beta_n}}{\sqrt{\beta_j\beta_m}}\phi_j(\mathbf{x})\phi_l(\mathbf{x}_0)\phi_m(\mathbf{x})\phi_n(\mathbf{x}_0)\\ &\times e^{i\frac{[\beta_j(\omega_0)-\beta_m(\omega_0)]L}{\varepsilon^2}}\int\int \hat{f}(h)\overline{\hat{f}(h')}\mathbb{E}[T^\varepsilon_{jl}(\omega_0+\varepsilon^2 h)\overline{T^\varepsilon_{mn}(\omega_0+\varepsilon^2 h')}]\\ &\times e^{i[\beta_j'(\omega_0)L-t]h-[\beta_m'(\omega_0)L-t]h'}dhdh'\,.\end{aligned}$$

Using Proposition 6.3 we see that there are two contributions to this integral:

$$\mathbb{E}\left[|p^\varepsilon_{tr}(t, \mathbf{x}, L)|^2\right] = I_1^\varepsilon(t, \mathbf{x}, L) + I_2^\varepsilon(t, \mathbf{x}, L)\,. \tag{7.1}$$

The limit of the first contribution is

$$\begin{aligned}I_1^\varepsilon(t, \mathbf{x}, L) &\overset{\varepsilon\to 0}{\simeq} \frac{1}{4}\sum_{j\neq m=1}^N \phi_j(\mathbf{x})\phi_j(\mathbf{x}_0)\phi_m(\mathbf{x})\phi_m(\mathbf{x}_0)e^{i\frac{[\beta_j(\omega_0)-\beta_m(\omega_0)]L}{\varepsilon^2}}\\ &\times e^{Q_{jm}(\omega_0)L}f(t-\beta_j'(\omega_0)L)f(t-\beta_m'(\omega_0)L)\,.\end{aligned} \tag{7.2}$$

We see that it decays exponentially with the propagation distance because of the damping factors $\exp(Q_{jm}(\omega_0)L)$. We can therefore neglect this contribution for $L \gg L_{\rm equip}(\omega_0)$. The limit of the second contribution is

$$I_2^\varepsilon(t, \mathbf{x}, L) \overset{\varepsilon\to 0}{\simeq} \frac{1}{4}\sum_{j,l=1}^N \frac{\beta_l}{\beta_j}\phi_j^2(\mathbf{x})\phi_l^2(\mathbf{x}_0)\int \mathcal{W}_j^{(l)}(\omega_0, \tau, L)f(t-\tau)^2 d\tau\,. \tag{7.3}$$

In the asymptotic equipartition regime $L \gg L_{\rm equip}(\omega_0)$ we use the diffusion approximation (6.16). We conclude that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[|p_{tr}^\varepsilon(t,\mathbf{x},L)|^2 \right] \overset{L \gg L_{\rm equip}}{\simeq} H_{\omega_0,\mathbf{x}_0}(\mathbf{x}) \times [K_{\omega_0,L} * (f^2)](t)\,, \tag{7.4}$$

where the spatial profile $H_{\omega_0,\mathbf{x}_0}$ and the time convolution kernel $K_{\omega_0,L}$ are given by

$$H_{\omega_0,\mathbf{x}_0}(\mathbf{x}) = \frac{1}{4N(\omega_0)} \sum_{j=1}^{N(\omega_0)} \frac{\phi_j^2(\mathbf{x})}{\beta_j(\omega_0)} \times \sum_{l=1}^{N(\omega_0)} \phi_l^2(\mathbf{x}_0)\beta_l(\omega_0)\,, \tag{7.5}$$

$$K_{\omega_0,L}(t) = \frac{1}{\sqrt{2\pi\sigma^2_{\beta'(\omega_0)}L}} \exp\left(-\frac{(t-\overline{\beta'(\omega_0)}L)^2}{2\sigma^2_{\beta'(\omega_0)}L} \right). \tag{7.6}$$

To sum up, the main results of this section are that

- the mean field decays exponentially with propagation distance,
- the mean transmitted intensity converges to the transverse spatial profile $H_{\omega_0,\mathbf{x}_0}$, and
- the mean transmitted intensity is concentrated around the time $\overline{\beta'(\omega_0)}L$ with a spread that is of the order of $\sigma_{\beta'(\omega_0)}\sqrt{L} \sim \sqrt{LL_{\rm equip}(\omega_0)}/\bar{c}$ for a pulse with carrier frequency ω_0.

Note that $\sigma_{\beta'(\omega_0)}\sqrt{L} \ll L/\bar{c}$, which means that pulse spreading increases as $\sqrt{L}$ in a random waveguide while it increases linearly in a homogeneous one. This is because the modes are strongly coupled together and propagate with the same "average" group velocity $1/\overline{\beta'(\omega_0)}$ in the random waveguide. The "average" group velocity is the harmonic average of the group velocities of the modes $1/\beta_j'(\omega_0)$. These results are intuitively clear, but they were not discussed in detail in the early literature [14, 5]. We note that in (7.4) it is the mean of the pulse intensity that has the asymptotic form that we have derived. The pulse intensity fluctuations can also be computed and are not small.

8. Time reversal in a waveguide.

8.1. Time reversal setup. We now consider time reversal in a waveguide. A point source located in the plane $z=0$ at the lateral position $\mathbf{x}_0$ emits a pulse $f^\varepsilon(t)$ of the form (5.1). A time-reversal mirror is located in the plane $z = L/\varepsilon^2$ and occupies the subdomain $\mathcal{D}_M \subset \mathcal{D}$. The transmitted wave observed in the plane $z = L/\varepsilon^2$ at time t/ε^2 is (5.2). The time-reversal mirror records the field from time t_0/ε^2 up to time t_1/ε^2, time reverses it, and sends it back into the waveguide. The new source at the time-reversal mirror that generates the back propagating waves is

$$\mathbf{F}^\varepsilon_{\rm TR}(t,\mathbf{x},z) = f^\varepsilon_{\rm TR}(t,\mathbf{x})\delta\left(z - \frac{L}{\varepsilon^2}\right)\mathbf{e}_z\,, \tag{8.1}$$
$$f^\varepsilon_{\rm TR}(t,\mathbf{x}) = p^\varepsilon_{tr}\left(t_1 - \varepsilon^2 t, \mathbf{x}, L\right) G_1(t_1 - \varepsilon^2 t) G_2(\mathbf{x})\,,$$

where p^ε_{tr} is given by (5.2), G_1 is the time-window function of the form $G_1(t) = \mathbf{1}_{[t_0,t_1]}(t)$, and G_2 is the spatial-window function $G_2(\mathbf{x}) = \mathbf{1}_{\mathcal{D}_M}(\mathbf{x})$. We have seen that the power delay spread of the transmitted signal is not very long in the forward scattering approximation. This is because there is no backscattering to produce long codas (i.e., long incoherent wave fluctuations). Moreover, we focus our attention more on spatial effects in this paper, so it is reasonable to assume that we record the field for all time at the time-reversal mirror. This means that we have $G_1(t) = 1$ and $\hat{G}_1(h) = 2\pi\delta(h)$. Therefore, the left-going propagating modes generated by this source have amplitudes

$$\hat{b}_m(\omega) = -\frac{\sqrt{\beta_m(\omega)}}{2} \int_{\mathcal{D}} \hat{f}^{\varepsilon}_{\mathrm{TR}}(\omega, \mathbf{x})\phi_m(\mathbf{x}) d\mathbf{x} e^{i\beta_m(\omega)\frac{L}{\varepsilon^2}}$$

$$= \frac{1}{4\varepsilon^2} \sum_{j,l=1}^{N} \frac{\sqrt{\beta_l\beta_m}}{\sqrt{\beta_j}} M_{mj}\phi_l(\mathbf{x}_0) \overline{\hat{f}\left(\frac{\omega-\omega_0}{\varepsilon^2}\right)} \overline{T^{\varepsilon}_{jl}(\omega)} e^{i[\beta_m(\omega)-\beta_j(\omega)]\frac{L}{\varepsilon^2} + i\omega\frac{t_1}{\varepsilon^2}}, \tag{8.2}$$

where the coupling coefficients M_{jl} are given by

$$M_{jl} = \int_{\mathcal{D}} \phi_j(\mathbf{x}) G_2(\mathbf{x}) \phi_l(\mathbf{x}) d\mathbf{x}. \tag{8.3}$$

We have explicit formulas for the coupling coefficients M_{jl} in two cases as follows:

- If the mirror spans the complete cross section $\mathcal{D}$ of the waveguide, then we have $G_2(\mathbf{x}) = 1$ and $M_{jl} = \delta_{jl}$.
- If the mirror is point-like at $\mathbf{x} = \mathbf{x}_1$, meaning $G_2(\mathbf{x}) = |\mathcal{D}|\delta(\mathbf{x}-\mathbf{x}_1)$, with the factor $|\mathcal{D}|$ added for dimensional consistency, then $M_{jl} = |\mathcal{D}|\phi_j(\mathbf{x}_1)\phi_l(\mathbf{x}_1)$.

The refocused field observed in the plane $z = 0$ in the Fourier domain is given by

$$\hat{p}_{\mathrm{TR}}(\omega, \mathbf{x}, 0) = \sum_{m,n=1}^{N} \frac{(\mathbf{T}^{\varepsilon})^T_{nm}(\omega)\hat{b}_m(\omega)}{\sqrt{\beta_n}} \phi_n(\mathbf{x}). \tag{8.4}$$

Here $(\mathbf{T}^{\varepsilon})^T(\omega)$ is the transfer matrix for the left-going modes propagating from L/ε^2 to 0, and it is the transpose of $\mathbf{T}^{\varepsilon}(\omega)$. This follows from the unitarity of the transfer matrix $\mathbf{T}^{\varepsilon}(\omega)$. In the time domain, the refocused field observed at time $t_{\mathrm{obs}}/\varepsilon^2$ is

$$p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right) = \frac{1}{4\pi} \sum_{j,l,m,n=1}^{N} \frac{\sqrt{\beta_l\beta_m}}{\sqrt{\beta_j\beta_n}} M_{mj}\phi_n(\mathbf{x})\phi_l(\mathbf{x}_0) e^{i[\beta_m-\beta_j](\omega_0)\frac{L}{\varepsilon^2} + i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}}$$

$$\times \int \overline{\hat{f}(h)} \overline{T^{\varepsilon}_{jl}(\omega_0+\varepsilon^2 h)} T^{\varepsilon}_{mn}(\omega_0+\varepsilon^2 h) e^{i\{[\beta'_m-\beta'_j](\omega_0)L + (t_1-t_{\mathrm{obs}})\}h} dh. \tag{8.5}$$

8.2. Refocusing in a homogeneous waveguide. In this case, $T^{\varepsilon}_{jl} = \delta_{jl}$ and the refocused field is

$$p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right) = \frac{1}{2} e^{i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}} \sum_{j,m=1}^{N} e^{i[\beta_m-\beta_j](\omega_0)\frac{L}{\varepsilon^2}}$$

$$\times M_{mj}\phi_m(\mathbf{x})\phi_j(\mathbf{x}_0) f\left([\beta'_m-\beta'_j](\omega_0)L + t_1 - t_{\mathrm{obs}}\right). \tag{8.6}$$

The refocused field is a weighted sum of modes, whose weights depend on the mirror through the coefficients M_{mj}. The oscillatory terms in (8.6) produce transverse sidelobes in the refocused field, as seen in Figure 8.1.

8.3. The mean refocused field in a random waveguide. In the analysis of time reversal considered up to now [3, 9, 2, 21], statistical stability is shown by a frequency decoherence argument. More precisely, it is shown that the refocused field is the superposition of many approximately uncorrelated frequency components, which ensures self-averaging in the time domain. This argument cannot be used in the narrowband case (5.1) that we consider here. This is because the decoherence frequency, which is of order ε^2, is comparable to the bandwidth, which is also of order ε^2. In broadband cases with a bandwidth of order ε^p, $p \in [0, 2)$, statistical stability can be obtained by the usual frequency decoherence argument [10].

First we compute the mean refocused field and then consider the statistical stability. By taking the expectation of (8.5), we find that the mean refocused field involves the second-order moments of the transfer matrix. From Proposition 6.3 we have the limit values of these second-order moments, so we can write

$$\mathbb{E}\left[p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right)\right] = p_1^\varepsilon + p_2^\varepsilon\,, \tag{8.7}$$

$$p_1^\varepsilon \overset{\varepsilon\to 0}{\simeq} \frac{1}{2}\sum_{j\neq m=1}^{N} M_{mj}\phi_m(\mathbf{x})\phi_j(\mathbf{x}_0)e^{i[\beta_m-\beta_j](\omega_0)\frac{L}{\varepsilon^2}+i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}} \times e^{Q_{jm}(\omega_0)L} f\left([\beta_m'-\beta_j'](\omega_0)L + t_1 - t_{\mathrm{obs}}\right)\,,$$

$$p_2^\varepsilon \overset{\varepsilon\to 0}{\simeq} \frac{1}{2}e^{i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}} f\left(t_1 - t_{\mathrm{obs}}\right)\sum_{j,l=1}^{N} M_{jj}\phi_l(\mathbf{x})\phi_l(\mathbf{x}_0)\mathcal{T}_j^{(l)}(\omega_0, L)\,.$$

The term p_1^ε decays exponentially with propagation distance because of the damping factors coming from Q_{jm}. We can therefore neglect this term in the asymptotic equipartition regime. The term p_2^ε does contribute and it refocuses around the time $t_{\mathrm{obs}} = t_1$ with the original pulse shape time reversed. The spatial focusing profile is a weighted sum of modes, with weights that depend on the time-reversal mirror through the coefficients M_{jl} and on the mean square transmission coefficients $\mathcal{T}_j^{(l)}$.

In the asymptotic equipartition regime $L \gg L_{\mathrm{equip}}$, the coefficients $\mathcal{T}_j^{(l)}(\omega_0, L)$ converge to $1/N$ for all j and l, which for the mean refocused field gives

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right)\right] \overset{L\gg L_{\mathrm{equip}}}{\simeq} e^{i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}} f\left(t_1 - t_{\mathrm{obs}}\right) \times\frac{1}{N(\omega_0)}\sum_{j}^{N(\omega_0)} M_{jj} \times \frac{1}{2}\sum_{l=1}^{N(\omega_0)}\phi_l(\mathbf{x})\phi_l(\mathbf{x}_0)\,. \tag{8.8}$$

We note the difference between this expression and (8.6) in a homogeneous waveguide. The oscillatory terms in (8.6) which generate the side-lobes are suppressed in (8.8). The analytical understanding of side-lobe suppression in time reversal in random waveguides is a new result in this paper.

The spatial refocusing profile can then be computed explicitly because it does not depend on the mirror shape or size. In the case of a planar waveguide, we have $\phi_j(x) = \sqrt{2/d}\sin(\pi j x/d)$, and in the continuum limit $N \gg 1$, we have

$$\frac{1}{2}\sum_{l=1}^{N}\phi_l(x)\phi_l(x_0) \overset{N\gg 1}{\simeq} \frac{1}{\lambda_0}\mathrm{sinc}\left(2\pi\frac{x-x_0}{\lambda_0}\right)\,.$$

The mean refocused field is therefore concentrated around the original source location x_0 with a resolution of half of a wavelength, which is the diffraction limit.

8.4. Statistical stability of the refocused field. As we noted already, we cannot claim that the refocused field is statistically stable by using the same argument as in the broadband case, because here we have a narrowband pulse. However, we can achieve statistical stability through the summation over the modes. We will show this in the quasi-monochromatic case, where the pulse envelope $f(t) = 1$ and $\hat{f}(h) = 2\pi\delta(h)$. In this case, it is clear that statistical stability cannot arise from time averaging, and the refocused field is

$$p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right) = \frac{1}{2}\sum_{j,l,m,n=1}^{N} \frac{\sqrt{\beta_l\beta_m}}{\sqrt{\beta_j\beta_n}} M_{mj}\phi_n(\mathbf{x})\phi_l(\mathbf{x}_0) \times e^{i[\beta_m-\beta_j](\omega_0)\frac{L}{\varepsilon^2}+i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}}\overline{T^{\varepsilon}_{jl}(\omega_0)}T^{\varepsilon}_{mn}(\omega_0)\,. \tag{8.9}$$

From (8.8), the mean refocused field at $\mathbf{x} = \mathbf{x}_0$ is in the asymptotic equipartition regime

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right] \overset{L\gg L_{\mathrm{equip}}}{\simeq} e^{i\omega_0\frac{t_1-t_{\mathrm{obs}}}{\varepsilon^2}}\frac{R_0^2}{2N}\sum_{j=1}^{N} M_{jj}\,, \tag{8.10}$$

where $R_0^2 = \sum_{l=1}^N \phi_l^2(\mathbf{x}_0)$. We now compute the second moment of the refocused field observed at $\mathbf{x} = \mathbf{x}_0$. By taking the expectation of the square of (8.9), we find that this moment involves fourth-order moments of the transfer matrix. From the results of section 4.3, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}[\overline{T^{\varepsilon}_{jl}}T^{\varepsilon}_{mn}\overline{T^{\varepsilon}_{j'l'}}T^{\varepsilon}_{m'n'}] \overset{L\gg L_{\mathrm{equip}}}{\simeq} \begin{cases} \frac{2}{N(N+1)} & \text{if } (j,l)=(m,n)=(j',l')=(m',n')\,, \\ \frac{1}{N(N+1)} & \text{if } (j,l)=(m,n)\neq(j',l')=(m',n')\,, \\ \frac{1}{N(N+1)} & \text{if } (j,l)=(m',n')\neq(j',l')=(m,n)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Using these fourth-order moment results in the expression for the second moment of the refocused field we see that, in the limit $\varepsilon \to 0$ and in the asymptotic equipartition regime $L \gg L_{\mathrm{equip}}$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right|^2\right] \overset{L\gg L_{\mathrm{equip}}}{\simeq} \frac{R_0^4}{4N(N+1)}\left[\left(\sum_j M_{jj}\right)^2 + \sum_{j,j'} M_{jj'}^2\right].$$

Let us introduce the relative standard deviation S of the refocused field amplitude,

$$S^2 := \lim_{\varepsilon\to 0}\frac{\mathbb{E}\left[\left|p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right|^2\right] - \left|\mathbb{E}\left[p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right]\right|^2}{\left|\mathbb{E}\left[p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right]\right|^2}\,. \tag{8.11}$$

We have statistical stability when S is small. In the asymptotic equipartition regime, S^2 is given by

$$S^2 \overset{L\gg L_{\mathrm{equip}}}{\simeq} -\frac{1}{N+1} + \frac{N}{N+1}\frac{1}{Q_{\mathrm{mirror}}}\,, \qquad Q_{\mathrm{mirror}} = \frac{\sum_{j,l=1}^N M_{jj}M_{ll}}{\sum_{j,l=1}^N M_{jl}^2}\,. \tag{8.12}$$

The quality factor Q_{mirror} depends only on the time-reversal mirror. We will have statistical stability when the number of modes N is large and when the quality factor Q_{mirror} is large. This analytical criterion for statistical stability in narrowband time reversal is a new result in this paper.

We can consider two extreme cases:

- If the time-reversal mirror spans the waveguide cross section, then $M_{jl} = \delta_{jl}$ and the quality factor is equal to N, which is optimal since the relative standard deviation is then zero for any N. This result is not surprising since the time-reversal mirror records the transmitted signal fully, in both time and space, which implies optimal refocusing.
- If the time-reversal mirror is point-like at $\mathbf{x}_1$, then $M_{jl} = \phi_j(\mathbf{x}_1)\phi_l(\mathbf{x}_1)$ and the quality factor is 1, which is bad, because the relative standard deviation S is asymptotically equal to $\sqrt{N-1}/\sqrt{N+1}$. The fluctuations of the refocused field are, therefore, of the same order as the mean field, which means that there is no statistical stability.

In the next section, we address a particular case which allows explicit calculations.

8.5. Numerical illustration. In this section we illustrate the time reversal results in the quasi-monochromatic case for a particular random waveguide. We consider a random planar waveguide with diameter d. The random process ν is stationary and mixing in the z-direction. Its autocorrelation function is

$$\mathbb{E}[\nu(x,z)\nu(x',z')] = \sigma^2 \exp\left(-\frac{|z-z'|}{l_c}\right) R(x,x')\,,$$

where the support of R is contained in $[0,d]^2$, σ is the standard deviation of the medium fluctuations, and l_c is the axial correlation length. This decomposition of the autocorrelation function makes it easy to compute the effective coefficients $\Gamma^{(c)}_{jl}$, $\Gamma^{(s)}_{jl}$, and $\Gamma^{(1)}_{jl}$. We obtain

$$\gamma^{(c)}_{jl} = \frac{2\sigma^2 l_c G_{l,j}}{1+(\beta_j-\beta_l)^2 l_c^2}\,, \qquad \gamma^{(s)}_{jl} = \frac{2\sigma^2(\beta_j-\beta_l) l_c^2 G_{l,j}}{1+(\beta_j-\beta_l)^2 l_c^2}\,, \qquad \gamma^{(1)}_{jl} = 2 l_c G_{l,j}\,,$$
$$G_{l,j} = S_{j-l,j-l} + S_{j+l,j+l} - S_{j-l,j+l} - S_{j+l,j-l}\,,$$

where

$$S_{j,l} = \frac{1}{d^2}\int_0^d\int_0^d \cos\left(\frac{j\pi x}{d}\right)\cos\left(\frac{l\pi x'}{d}\right) R(x,x')dxdx'\,.$$

For simplicity, we shall make two hypotheses. First, we introduce a band-limiting idealization; i.e., we assume that the support of S lies with a finite square so that $S(j,l) \neq 0$ only if $|j| \leq 1$ and $|l| \leq 1$. Second, we assume that l_c is smaller than $\beta_j - \beta_{j-1}$ for all j. The expressions of the effective coefficients can then be simplified. The matrix $\Gamma^{(s)}_{jl}$ is essentially zero, while the matrices $\Gamma^{(c)}_{jl}$ and $\Gamma^{(s)}_{jl}$ are tridiagonal,

$$\Gamma^{(c)}_{jl} \simeq \Gamma^{(1)}_{jl} \simeq \frac{\omega^4\sigma^2 l_c d S_{1,1}}{4\bar{c}^4\beta_j(\omega)\beta_l(\omega)} \text{ if } |j-l| = 1\,, \qquad \Gamma^{(1)}_{jj} \simeq \frac{\omega^4\sigma^2 l_c d S_{0,0}}{4\bar{c}^4\beta_j^2(\omega)}\,,$$

and $\Gamma^{(c)}_{jj}$ is chosen so that the lines of the matrix are zero.

Homogeneous waveguide. In the homogeneous case the spatial profile of the refocused field is (8.6). Let us consider a time-reversal mirror of size a located in $x \in [d/2-a/2, d/2+a/2]$: $G_2(x) = \mathbf{1}_{[d/2-a/2,d/2+a/2]}(x)$. We then have

$$M_{jl} = \frac{a}{d}\left[\cos\left(\frac{(j-l)\pi}{2}\right)\operatorname{sinc}\left(\frac{(j-l)\pi a}{2d}\right) - \cos\left(\frac{(j+l)\pi}{2}\right)\operatorname{sinc}\left(\frac{(j+l)\pi a}{2d}\right)\right].$$

Using these formulas, we plot in Figure 8.1 the spatial profile of the refocused field for different sizes a of the time-reversal mirror. The peak at the original source location is there in all cases, but for small time-reversal mirrors, large side-lobes appear.

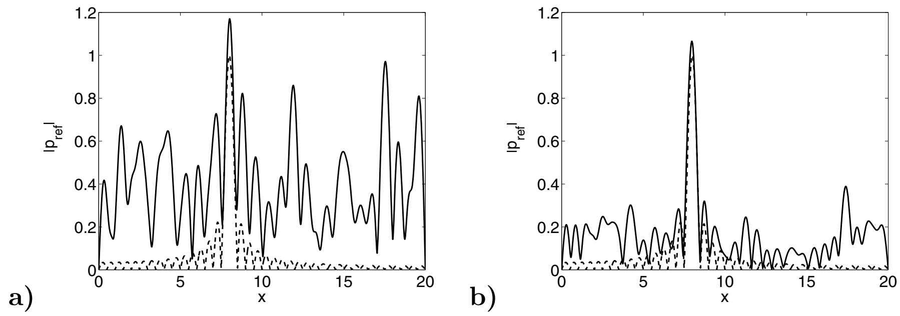

FIG. 8.1. *Transverse profile of the refocused field in a homogeneous waveguide with diameter d and length L. Here $d = 20$, $L = 200$, and $\lambda_0 = 1$, so there are* 40 *modes. The original source location is $x_0 = 8$. The dashed curve is the sinc profile, which is the focusing profile of a full-size time-reversal mirror. The solid curves are refocusing profiles for time-reversal mirrors of size* $a = 2.5$ (a) *and* $a = 10$ (b).

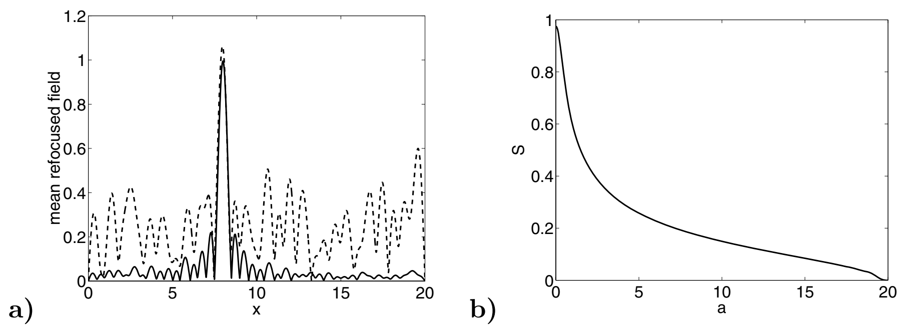

FIG. 8.2. (a) *Transverse profile of the mean refocused field in a random waveguide with diameter d and length L. Here $d = 20$, $L = 200$, $\lambda_0 = 1$, the time-reversal mirror has size $a = 5$, and the original source location is $x_0 = 8$. The axial correlation length of the random medium is $l_c = 0.25$ and the random fluctuations have standard deviation σ. The dashed curve is the spatial profile obtained in homogeneous medium $\sigma = 0$. The solid lines stand for the mean profiles obtained in random media $\sigma = 0.015$. The random case is very close to the equipartition regime, for which the focusing profile is a sinc.* (b) *The relative standard deviation S, from* (8.12)*, of the refocused field in the equipartition regime as a function of the mirror size a. Here $d = 20$ and $\lambda_0 = 1$.*

Random waveguide. The mean spatial profile of the refocused field is (8.7). In the absence of randomness, this formula reduces to (8.6). The mean spatial profile is plotted in Figure 8.2(a), which illustrates the transition from the poor spatial refocusing in a homogeneous medium to the diffraction-limited refocusing obtained in the equipartition regime. In the equipartition regime $L \gg L_{\text{equip}}$, the mean spatial profile is given by (8.8) and becomes independent of the mirror size, up to an amplitude factor. However, the statistical stability of the refocused field depends on the size of the time-reversal mirror, as shown in Figure 8.2(b).

9. Summary and conclusions. The main result of this paper is the derivation from first principles of the system of transport equations (6.6) for the coupled mode powers, in the asymptotic limit of section 3. It is easy to write such equations by simply adding a time-dispersive term in (4.11). We identify here the field quantity that has the mean which satisfies this space-time transport equation.

We apply the transport equation (6.6) to pulse spreading in order to get (7.4), which shows how randomness reduces time dispersion at the expense of introducing random fluctuations. We also apply it to time reversal in a random waveguide, in a narrowband regime, and show in (8.8) how side-lobes are suppressed in refocusing. We also show in (8.12) how statistical stability depends on the quality factor Q_{mirror}. This quality factor does not depend on the random medium in the energy equipartition regime, but it does depend on the size of the time-reversal mirror relative to the waveguide cross section.

REFERENCES

[1] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. White, *Frequency content of randomly scattered signals*, SIAM Rev., 33 (1991), pp. 519–625.

[2] P. Blomgren, G. Papanicolaou, and H. Zhao, *Super-resolution in time-reversal acoustics*, J. Acoust. Soc. Amer., 111 (2002), pp. 230–248.

[3] J.-F. Clouet and J.-P. Fouque, *A time reversal-method for an acoustical pulse propagating in randomly layered media*, Wave Motion, 25 (1997), pp. 361–368.

[4] A. Derode, P. Roux, and M. Fink, *Robust acoustic time reversal with high-order multiple scattering*, Phys. Rev. Lett., 75 (1995), pp. 4206–4209.

[5] L. B. Dozier and F. D. Tappert, *Statistics of normal mode amplitudes in a random ocean.* I. *Theory*, J. Acoust. Soc. Amer., 63 (1978), pp. 353–365.

[6] L. B. Dozier and F. D. Tappert, *Statistics of normal mode amplitudes in a random ocean.* II. *Computations*, J. Acoust. Soc. Amer., 64 (1978), pp. 533–547.

[7] M. Fink, *Time reversed acoustics*, Physics Today, 20 (1997), pp. 34–40.

[8] M. Fink, D. Cassereau, A. Derode, C. Prada, P. Roux, M. Tanter, J.-L. Thomas, and F. Wu, *Time-reversed acoustics*, Reports on Progress in Physics, 63 (2000), pp. 1933–1995.

[9] J.-P. Fouque, J. Garnier, A. Nachbin, and K. Sølna, *Time reversal refocusing for point source in randomly layered media*, Wave Motion, 42 (2005), pp. 238–260.

[10] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer, New York, 2007.

[11] J. Garnier, *Energy distribution of the quantum harmonic oscillator under random time-dependent perturbations*, Phys. Rev. E, 60 (1999), pp. 3676–3687.

[12] J. Garnier, *The role of evanescent modes in randomly perturbed single-mode waveguides*, Discrete Contin. Dyn. Syst. Ser. B, 8 (2007), pp. 455–472.

[13] W. Kohler, *Power reflection at the input of a randomly perturbed rectangular waveguide*, SIAM J. Appl. Math., 32 (1977), pp. 521–533.

[14] W. Kohler and G. Papanicolaou, *Wave propagation in a randomly inhomogeneous ocean*, in Wave Propagation and Underwater Acoustics, Lecture Notes in Phys. 70, J. B. Keller and J. S. Papadakis, eds., Springer, Berlin, 1977, pp. 153–223.

[15] W. A. Kuperman, W. S. Hodgkiss, H. C. Song, T. Akal, C. Ferla, and D. R. Jackson, *Phase conjugation in the ocean: Experimental demonstration of an acoustic time-reversal mirror*, J. Acoust. Soc. Amer., 103 (1998), pp. 25–40.

[16] H. J. Kushner, *Approximation and Weak Convergence Methods for Random Processes*, MIT Press, Cambridge, MA, 1984.

[17] D. Marcuse, *Theory of Dielectric Optical Waveguides*, Academic Press, New York, 1974.

[18] A. Nachbin and G. Papanicolaou, *Water waves in shallow channels of rapidly varying depth*, J. Fluid Mech., 241 (1992), pp. 311–332.

[19] G. Papanicolaou and W. Kohler, *Asymptotic theory of mixing stochastic ordinary differential equations*, Comm. Pure Appl. Math., 27 (1974), pp. 641–668.

[20] G. Papanicolaou and W. Kohler, *Asymptotic analysis of deterministic and stochastic equations with rapidly varying components*, Comm. Math. Phys., 45 (1975), pp. 217–232.

[21] G. Papanicolaou, L. Ryzhik, and K. Sølna, *Statistical stability in time reversal*, SIAM J. Appl. Math., 64 (2004), pp. 1133–1155.

[22] P. Roux and M. Fink, *Time reversal in a waveguide: Study of the temporal and spatial focusing*, J. Acoust. Soc. Amer., 107 (2000), pp. 2418–2429.

[23] H. C. Song, W. A. Kuperman, and W. S. Hodgkiss, *Iterative time reversal in the ocean*, J. Acoust. Soc. Amer., 105 (1999), pp. 3176–3184.