Length-scale competition for the sine-Gordon kink in a random environment

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This paper deals with the transmission of a kink in a random medium described by a randomly perturbed sine-Gordon equation. Different kinds of perturbations are addressed, with time or spatial random fluctuations, with or without damping. We derive effective evolution equations for the kink velocity and width by applying a perturbation theory of the inverse scattering transform and limit theorems of stochastic calculus. Results are very different compared to a randomly perturbed nonlinear Schrödinger equation. The effect of a random perturbation is shown to depend strongly on the interplay of the correlation length of the perturbation and the kink width.

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I. INTRODUCTION

The sine-Gordon (SG) equation models many applications in solid-state physics.[1] As an important example we may think at long Josephson junctions (LJJ's) which received new interest with the appearance of high-temperature superconductors.[2] The SG equation is also a paradigm for the study of topological kinks. A realistic SG model should also include small random fluctuations of the parameters of the equation. The study of the soliton dynamics driven by small perturbations has attracted much attention. Kink dynamics along nonlinear random systems can be studied by using a collective-coordinate approach.[3–8] The kink is treated as a quasiparticle and the solution is sought in the form of a kink with slowly varying parameters. Substituting this ansatz into a conservation law establishes a set of ordinary differential equations for these parameters. This approach can be generalized to nonintegrable systems[9] and is valid while radiative losses are negligible. A two-stage scheme was proposed by McLaughlin and Scott[10] where the authors first compute slow modulations of the kink parameters and then derive the first-order correction by a constructed radiative Green's function. This work was further extended to give analytical representation of the complicated Green's function.[11] Based on the method of separation of variables, a direct approach in the study of soliton perturbations has been developed by Yan *et al.*[12] and Tang and Wang,[13] where the solution is expanded in power series with the leading-order term being the original solution, and the higher-order terms being derived through a set of linearized equations.

A powerful approach to the study of soliton dynamics under random perturbations is perturbation theory based on the inverse scattering transform (IST).[14–16] It is well known[17] that the solution of the direct-scattering transform for a linear operator connected with the Lax pair of a nonlinear evolution equation yields a set of scattering data. The discrete spectrum describes localized states while the continuous spectrum describes dispersive waves. The spectrum is conserved in time in case of an exact integrable system, which implies in particular that the solitons and the dispersive waves do not interact. In the presence of perturbations the evolutions of the scattering data are coupled and obey a closed form but complicated system of equations. Using these equations an iterative scheme was proposed to calculate the solution by successive approximations.[14,15] This method is efficient when dealing with short-time dynamics or very small perturbations, so that the expansion of the solution at first or second order is sufficient to describe the soliton dynamics.[18] For long-time dynamics, when the corrective terms corresponding to the radiation emitted by the soliton become of the same order of magnitude as the soliton term, few results are available.

In this paper we discuss general types of perturbations and proceed under a different asymptotic framework. Our main contribution is that we use the inverse scattering transform so as to take into account both the variations of the soliton part and the radiative part of the wave. Both effects and their interplay are important, especially when the correlation length of the perturbation is of the same order as the soliton width. The interaction of different length scales is an important issue in localization theory for wave propagation in linear media,[19,20] so the relationship of the soliton width and the correlation length of the perturbation is expected to have a fundamental effect on the questions we are trying to answer. Our approach is based on a separation of scales technique (between the fast oscillations of the kink parameters and the slow emission of radiation) and on the existence of conservation laws. We consider the influence of small random perturbations and aim at reporting the possible asymptotic behaviors when the amplitudes of the random fluctuations are small while the size of the system is large enough so that an evolution of the order of 1 of the soliton and radiation is observed. The effective dynamics for the soliton width and velocity can then be computed.

II. THE HOMOGENEOUS SINE-GORDON EQUATION

The equation we consider in this paper is the SG equation

$$u_{tt} - u_{xx} + \sin(u) = 0, \tag{1}$$

which governs the evolution of a real field u. The subscripts x and t stand for partial derivatives with respect to position and time, respectively. This integrable equation supports moving nonlinear excitations in the form of kinks, so we can study the effects of various perturbations with the known analytic behavior of the unperturbed dynamics. We begin by a short review of the inverse scattering transform applied to the SG equation. Details of the theory can be found for instance in the review by Manakov *et al.*[17]

A. Direct transform: The scattering problem

The Lax commutative representation of the SG equation is $[L,A]=0$ with

$$L=\frac{\partial}{\partial x}-\frac{i}{2}\left[\left(\lambda-\frac{\cos(u)}{4\lambda}\right)\sigma_3-\frac{\sin(u)}{4\lambda}\sigma_2+\frac{u_x-u_t}{2}\sigma_1\right],$$

$$A=\frac{\partial}{\partial t}+\frac{i}{2}\left[\left(\lambda+\frac{\cos(u)}{4\lambda}\right)\sigma_3+\frac{\sin(u)}{4\lambda}\sigma_2+\frac{u_x-u_t}{2}\sigma_1\right],$$

where the σ_i, $i=1,2,3$, are the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and $\lambda\in\mathbb{R}^+$ is the real spectral parameter. The scattering problem for the operator L under the conditions that $u\to 0 \bmod 2\pi$ and $u_t\to 0$ as $|x|\to\infty$ puts into evidence the existence of two fundamental Jost functions that are defined as the solutions of $Lf=0$ that satisfy the following boundary conditions at $-$ and $+\infty$:

$$\psi(x,\lambda)\overset{x\to+\infty}{\simeq}\begin{pmatrix}1\\0\end{pmatrix}e^{ik(\lambda)x/2},\quad \phi(x,\lambda)\overset{x\to-\infty}{\simeq}\begin{pmatrix}0\\1\end{pmatrix}e^{-ik(\lambda)x/2},$$

where the wave number k is given by

$$k(\lambda)=\lambda-\frac{1}{4\lambda}.$$

If we denote by $\overline{\cdot}$ the involution operator that turns a row vector f into the row vector $\overline{f}:=(-f_2^*,f_1^*)^T$, then it turns out that $\overline{\psi}$ is also a solution of $L\overline{\psi}=0$. Furthermore ψ and $\overline{\psi}$ are linearly independent and form a base of the space of the solutions of $Lf=0$. It can then be proved that the Jost functions are related by

$$\phi(x,\lambda)=a(\lambda)\overline{\psi}(x,\lambda)+b(\lambda)\psi(x,\lambda), \tag{2}$$

$$\psi(x,\lambda)=-a(\lambda)\overline{\phi}(x,\lambda)+b^*(\lambda)\phi(x,\lambda). \tag{3}$$

Injecting the second equality into the first one, we also exhibit the following conservation relation:

$$|a(\lambda)|^2+|b(\lambda)|^2=1. \tag{4}$$

Using $L\psi=0$ we get two more conservation relations that concern the Euclidian norms of the Jost functions ψ and ϕ:

$$|\psi_1(x,\lambda)|^2+|\psi_2(x,\lambda)|^2=1,\quad |\phi_1(x,\lambda)|^2+|\phi_2(x,\lambda)|^2=1.$$

Multiplying Eq. (2) by the vector $\overline{\psi}^*$, we get an explicit representation of the coefficient a as the Wronskian of ψ and ϕ:

$$a(\lambda)=W(\psi,\phi)=\psi_1(x,\lambda)\phi_2(x,\lambda)-\psi_2(x,\lambda)\phi_1(x,\lambda). \tag{5}$$

Additional symmetry identities are

$$\psi_1(x,\lambda)=\psi_1^*(x,-\lambda),\quad \psi_2(x,\lambda)=-\psi_2^*(x,-\lambda),$$

$$\phi_1(x,\lambda)=-\phi_1^*(x,-\lambda),\quad \phi_2(x,\lambda)=\phi_2^*(x,-\lambda),$$

$$a(\lambda)=a^*(-\lambda),\quad b(\lambda)=-b^*(-\lambda).$$

Besides $\psi\exp[-ik(\lambda)x/2]$ and $\phi\exp[ik(\lambda)x/2]$ can be analytically continued into the upper complex half plane $\mathrm{Im}(\lambda)\geqslant 0$ where they have no singularity. From Eq. (5) we can define an analytic continuation of $a(\lambda)$ into the upper complex half plane. A noticeable feature then appears. If λ_j is a zero of $a(\lambda)$, then f and g are linearly dependent, so there exists a coefficient ρ_r such that $\phi(x,\lambda_j)=\rho_j\psi(x,\lambda_j)$. The corresponding eigenfunction is bounded and decays exponentially as $x\to+\infty$ (because $|\psi|\sim e^{-\mathrm{Im}\,\lambda_j x}$) and as $x\to-\infty$ (because $|\phi|\sim e^{+\mathrm{Im}\,\lambda_j x}$). Thus λ_j is an element of the point spectrum of L. It can then be proved that the set $(a(\lambda),b(\lambda),\lambda_j,\rho_j,a'(\lambda_j))$ characterizes the Jost functions ϕ and ψ and the solution u.

B. Time evolutions of the scattering data

The time equations for the scattering data are

$$a(t,\lambda)=a(t_0,\lambda),\quad \lambda\in\mathbb{R}^+, \tag{6}$$

$$b(t,\lambda)=b(t_0,\lambda)\exp[-i\omega(\lambda)(t-t_0)],\quad \lambda\in\mathbb{R}^+, \tag{7}$$

$$\rho_j(t)=\rho_j(t_0)\exp[-i\omega(\lambda_j)(t-t_0)],\quad j=1,\ldots,N, \tag{8}$$

where $\omega(\lambda)=\lambda+1/(4\lambda)$. Note that $(k(\lambda),\omega(\lambda))$ obeys the dispersion relation $-\omega^2+k^2+1=0$ of the linearized SG equation.

C. The inverse scattering transform

The inverse transform is essentially based on the resolution of the linear integrodifferential Gelfand-Levitan-Marchenko equation. The closed set of equations for the function ψ reads

$$\begin{aligned}\overline{\psi}(x,\lambda)\exp[ik(\lambda)x/2]&=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{j=1}^{N}\frac{\rho_j}{a'(\lambda_j)}\frac{\psi(x,\lambda_j)}{\lambda-\lambda_j}\exp[ik(\lambda_j)x/2]\\&\quad+\frac{1}{2i\pi}\int_{-\infty}^{\infty}d\lambda'\frac{b}{a}(\lambda')\frac{\psi(x,\lambda')}{\lambda'-\lambda+i0}\exp[ik(\lambda')x/2],\end{aligned}$$

$$\begin{aligned}\overline{\psi}(x,\lambda_l)\exp[ik(\lambda_l)x/2]&=\begin{pmatrix}0\\1\end{pmatrix}+\sum_{j=1}^{N}\frac{\rho_j}{a'(\lambda_j)}\frac{\psi(x,\lambda_j)}{\lambda_l^*-\lambda_j}\exp[ik(\lambda_j)x/2]\\&\quad+\frac{1}{2i\pi}\int_{-\infty}^{\infty}d\lambda'\frac{b}{a}(\lambda')\frac{\psi(x,\lambda')}{\lambda'-\lambda_l^*}\exp[ik(\lambda')x/2].\end{aligned}$$

Given the set of scattering data,

$$\cos(u/2)=(-1)^{u(+\infty)/(2\pi)}\Bigg(1-\sum_{j=1}^{N}\frac{\rho_j}{\lambda_j a'(\lambda_j)}\psi_2(x,\lambda_j)\times\exp[ik(\lambda_j)x/2]+\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\frac{b(\lambda)}{a(\lambda)}\psi_2(x,\lambda)\exp[ik(\lambda)x/2]\Bigg).$$

D. Kink

The so-called kinks are solutions of the SG equation that propagate at constant velocities without deformation:

$$u_s(x,t)=4\arctan\left[\exp\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right]. \tag{9}$$

Note that the soliton velocity $v\in(0,1)$ also completely characterizes the soliton width $\sqrt{1-v^2}$. Accordingly the soliton is all the narrower as its velocity is larger. In terms of IST the kink corresponds to the discrete eigenvalue $\lambda_s=i\nu$, where

$$\nu=\frac{1}{2}\sqrt{\frac{1+v}{1-v}}.$$

The kink has the corresponding Jost functions

$$\psi_s(x,\lambda)=\frac{\exp[ik(\lambda)x/2]}{\lambda+i\nu}\begin{pmatrix}\lambda+i\nu\tanh(z)\\ \nu\operatorname{sech}(z)\end{pmatrix},$$

$$\phi_s(x,\lambda)=\frac{\exp[-ik(\lambda)x/2]}{\lambda+i\nu}\begin{pmatrix}-\nu\operatorname{sech}(z)\\ \lambda-i\nu\tanh(z)\end{pmatrix},$$

and scattering data

$$a_s(\lambda)=\frac{\lambda-i\nu}{\lambda+i\nu},\quad b_s(\lambda)=0,\quad \rho_s=i\exp\left(\frac{vt}{\sqrt{1-v^2}}\right).$$

In the LJJ framework the kink represents a fluxon, i.e., a magnetic flux quantum.

E. Conserved quantities

Conserved quantities can be worked out as in any integrable system. The momentum

$$\mathcal{P}=-\int_{-\infty}^{\infty}u_t u_x dx \tag{10}$$

and the energy (Hamiltonian)

$$\mathcal{E}=\int_{-\infty}^{\infty}\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+[1-\cos(u)]dx \tag{11}$$

are two of the infinite number of conserved quantities for the sine-Gordon equation. They can also be expressed in terms of the scattering data as

$$\mathcal{P}=\sum_{j=1}^{N}\frac{8v_i}{\sqrt{1-v_i^2}}-\frac{1}{\pi}\int_0^{\infty}(4-\lambda^{-2})\ln(|a|^2)(\lambda)d\lambda, \tag{12}$$

$$\mathcal{E}=\sum_{j=1}^{N}\frac{8}{\sqrt{1-v_i^2}}-\frac{1}{\pi}\int_0^{\infty}(4+\lambda^{-2})\ln(|a|^2)(\lambda)d\lambda. \tag{13}$$

III. PERTURBATION

We consider a perturbed sine-Gordon equation with a nonzero right-hand side:

$$u_{tt}+u_{xx}-\sin(u)=\varepsilon R(u)(t,x). \tag{14}$$

The small parameter $\varepsilon\in(0,1)$ characterizes the amplitude of the perturbation which has the general form

$$R(u)=f(u,m),$$

where f is a local function of u [i.e., $f(u,m)(t,x)$ depends only on u and its partial derivatives evaluated at point (t,x)] that depends also on a driving random process m. We consider either time random perturbations or spatially random perturbations in the sense that either m is a time-dependent random process or a space-dependent random process. An example of the first case is $R(u)=m(t)u_{xx}$, while an example of a spatially random perturbation is $R(u)=m(x)u_{xx}$ or else $R(u)=(m(x)u_x)_x$.

In case of time random perturbations the driving process $(m(t))_{t\in\mathbb{R}}$ is assumed to be a zero-mean, stationary and ergodic process. The exact technical condition is that m should be "ϕ mixing" (in the sense discussed by Kushner,[21] in Sec. 4-6-2) with $\phi\in L^{1/2}$. The autocorrelation function of m is denoted by

$$\gamma_m(t):=\langle m(0)m(t)\rangle=\langle m(t')m(t'+t)\rangle, \tag{15}$$

where the brackets stand for the statistical average with respect to the stationary distribution of m. The ϕ-mixing condition with $\phi\in L^{1/2}$ means in particular that the perturbation has enough decorrelation properties so that the integral $\int|\gamma_m(t)|^{1/2}dt$ is finite. We can then introduce the Fourier transform of the autocorrelation function of the perturbation m,

$$\hat{\gamma}_m(\omega):=\int_{-\infty}^{\infty}\gamma_m(t)\cos(\omega t)dt, \tag{16}$$

which is non-negative real valued since it is proportional to the power spectral density by the Wiener-Khintchine theorem.[22] For instance, if the perturbation m is a white noise $\gamma_m(t)=\sigma^2\delta(t)$, then the spectrum of the perturbation is flat and given by $\hat{\gamma}_m(\omega)=\sigma^2$ for any ω. In case of spatially random perturbations similar assumptions are made on the random process $(m(x))_{x\in\mathbb{R}}$.

We investigate the propagation of a kink driven by the perturbation εR. Our method is based on the inverse scattering transform. The random perturbation induces variations of the spectral data. Calculating these changes we are able to find the effective evolution of the field and calculate the characteristic parameters of the wave. We are interested in the effective dynamics of the kink propagating over long times T/ε^2. The total energy or the total momentum are conserved by the addressed perturbations, but the discrete and continuous components evolve during the propagation. The evolution of the continuous component corresponding to radiation can be found from the evolution equations of the Jost coefficients. Note that the spectral density of the radiative power has already been obtained for various types of regular perturbations[23,24] and then extended to random perturbations.[25] The evolution of the kink velocity and width can then be derived from the conservation laws.

We now describe the evolutions of the Jost coefficients a and b during the propagation. They satisfy the following exact equations:[26]

$$\frac{\partial a}{\partial t}=-\frac{i\varepsilon}{4}\int_{-\infty}^{\infty}dxR(u)(t,x)(\psi_1\phi_1-\psi_2\phi_2), \tag{17}$$

$$\frac{\partial b}{\partial t}=-i\omega(\lambda)b-\frac{i\varepsilon}{4}\int_{-\infty}^{\infty}dxR(u)(t,x)(\psi_1^*\phi_2+\psi_2^*\phi_1). \tag{18}$$

The momentum and energy obey the equations

$$\frac{d\mathcal{P}}{dt}=-\varepsilon\int_{-\infty}^{\infty}R(u)(t,x)u_x(t,x)dx, \tag{19}$$

$$\frac{d\mathcal{E}}{dt}=\varepsilon\int_{-\infty}^{\infty}R(u)(t,x)u_t(t,x)dx. \tag{20}$$

IV. SPATIALLY RANDOM NONLINEARITY

A. The perturbed model

We consider in this section that the nonlinear term of the SG equation is spatially randomly perturbed:

$$R(u)=m(x)\sin(u). \tag{21}$$

In the framework of the LJJ, this corresponds to a random variation of the maximum Josephson current density.[27] The perturbed Hamiltonian of the system is $\mathcal{E}_0+\varepsilon\mathcal{E}_1$, where

$$\mathcal{E}_1=-\int_{-\infty}^{\infty}[1-\cos(u)]m(x)dx.$$

We then get that, with probability that goes to 1 as $\varepsilon\to 0$, the scattered wave at time t/ε^2 consists of one kink with velocity $v^\varepsilon(t)$ [and width $\sqrt{1-v^\varepsilon(t)^2}$] and radiation. The process $(v^\varepsilon(t))_{t\in[0,T]}$ converges in probability as $\varepsilon\to 0$ to the deterministic function $(v_l(t))_{t\in[0,T]}$ which satisfies the ordinary differential equation

$$\frac{dv_l}{dt}=F(v_l), \tag{22}$$

where

$$F(v)=-\frac{(1-v^2)^{3/2}}{8v}\int_{-\infty}^{\infty}dkE(k) \tag{23}$$

and $E(k)$ is the scattered energy density per unit time,

$$E(k)=\hat{\gamma}_m\left(\frac{\sqrt{k^2+1}}{v}-k\right)\frac{\pi(1-v^2)^2}{4v^3}\times\frac{\left(\frac{\sqrt{k^2+1}}{v}-k\right)^2}{\cosh\left(\sqrt{k^2+1}\frac{\sqrt{1-v^2}}{v}\frac{\pi}{2}\right)^2}. \tag{24}$$

Note that $F(v)$ is always nonpositive which means that the kink is slowing down. Note that the soliton width is given asymptotically by $\sqrt{1-v_l^2(t)}$, which means that the soliton also broadens.

B. Derivation of the effective evolution equation

The proofs of the results stated in the above section as well as in the forthcoming sections are based on a perturbed IST. We outline the main steps of the analysis that follows the strategy developed by Garnier[28] for the nonlinear Schrödinger equation and we underline the key points.

1. A priori estimates

The total energy $\mathcal{E}=\mathcal{E}_0+\varepsilon\mathcal{E}_1$ is preserved by the perturbed SG equation. First assume that m is a bounded process. Sobolev inequalities then prove that the H^1 norm and thus the L^∞ norm of $[1-\cos(u)](t,.)$ are uniformly bounded with respect to $t\in\mathbb{R}$ and $\varepsilon\in(0,1)$. This in turn implies that $\varepsilon\mathcal{E}_1$ can be bounded uniformly with respect to $t\in\mathbb{R}$ by $2\mathcal{E}_0\|m\|_\infty\varepsilon$. Let us now assume that m is not bounded, but is a Gaussian process. Then $\sup_{x\in[0,L/\varepsilon^2]}|m(x)|\leq K\varepsilon|\ln(\varepsilon)|^{1/2}$ and $\varepsilon\mathcal{E}_1$ can be bounded uniformly with respect to $t\in\mathbb{R}$ by $2\mathcal{E}_0K\varepsilon|\ln(\varepsilon)|^{1/2}$.

2. Prove the stability of the zero of the Jost coefficient a

The zero corresponds to the kink. This part strongly relies on the analytical properties of a in the upper complex half plane. We apply Rouché's theorem so as to prove that the number of zeros is constant. This method is efficient to prove that the zero is preserved, but it does not bring control on its precise location in the upper half plane. This step is not sufficient to compute the variations of the kink parameters.

3. Compute the radiation

The Jost coefficients a and b satisfy the coupled equations (17) and (18). In a first approximation we can substitute for ϕ, ψ, and u in the right-hand sides of Eqs. (17) and (18) the corresponding expressions ϕ_s, ψ_s, and u_s for the unperturbed kink:

$$\frac{\partial b}{\partial t}=-i\omega(\lambda)b-\frac{i\varepsilon}{4(\lambda^2+\nu^2)}\int_{-\infty}^{\infty}dxR(u_s)\times\left[\lambda^2-\nu^2-2i\nu\lambda\tanh\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right]\exp[-ik(\lambda)x].$$

From these equations we can estimate the amount of radiation which is emitted during some time interval in terms of energy. We are then able to deduce the evolution equation of the kink velocity by using the conservation of the total energy. For times of the order of $O(1)$, since $\mathcal{E}_{tot}$ is conserved, the variations $\Delta(\cdots)$ of the relevant quantities are linked together by the relations

$$0=\Delta\left(\frac{8}{\sqrt{1-v^2}}\right)+\int_0^\infty \Delta E(\lambda)d\lambda+\varepsilon\Delta(\mathcal{E}_1), \tag{25}$$

where $E(\lambda)$ is the scattered energy density per unit time:

$$E(\lambda)=-\pi^{-1}(4+\lambda^{-2})\ln(|a|^2)(\lambda)$$
$$=\pi^{-1}(4+\lambda^{-2})\ln\left(1+\frac{|b|^2}{|a|^2}\right)(\lambda).$$

$\Delta E(\lambda)$ is of the order of ε^2, but the last term in the expression of the total energy as well as in the balance identity (25) is of the order of ε. Thus our strategy is not efficient for estimating the variations of the kink parameters for times of the order of 1. Actually the collective-coordinate approach or the variational approach aims at dealing with this regime by taking care of $\mathcal{E}_1$. Let us now consider times of the order of ε^{-2}. $\Delta E(\lambda)$ is now of the order of 1, while the last term in the expression of the total energy is of the order of ε by the *a priori* estimates. Thus we can efficiently compute the long-time behavior of the kink parameters in the asymptotic framework $\varepsilon\to 0$, when the last term in the expression of the total energy is uniformly negligible. Applying probabilistic limit theorems (approximation-diffusion) establishes that the kink parameters converge in probability to nonrandom functions which satisfy

$$F(v)=-\frac{(1-v^2)^{3/2}}{8v}\int_0^\infty d\lambda E(\lambda),$$

where $E(\lambda)$ is the mean scattered energy density per unit time:

$$E(\lambda)=\hat{\gamma}_m\left(\lambda\frac{1-v}{v}+\frac{1}{4\lambda}\frac{1+v}{v}\right)\frac{\pi}{16}$$
$$\times\frac{(4+\lambda^{-2})\left(\lambda^2+\frac{1}{4}\frac{1+v}{1-v}\right)^2}{\lambda^2\cosh\left[\left(\lambda+\frac{1}{4\lambda}\right)\frac{\sqrt{1-v^2}}{v}\frac{\pi}{2}\right]^2}$$
$$\times\frac{(1-v^2)^2(1-v)^2}{v^5}.$$

Performing the change of variables $\lambda\mapsto k:=\lambda-1/(4\lambda)$ establishes the result.

4. Compute the form of the scattered wave

By applying the inverse scattering transform we find that the total wave is given by the sum of a kink and of radiation.

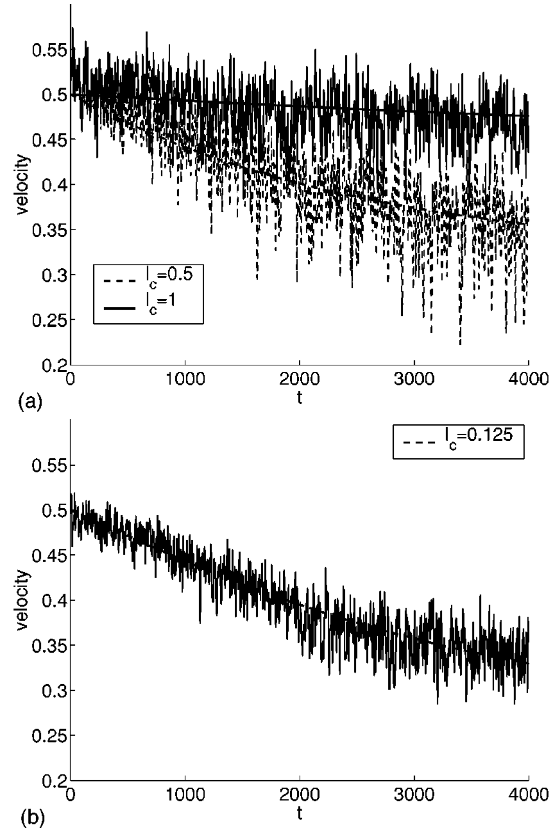

FIG. 1. Kink velocity as a function of time. The perturbation model is $R(u)=m(x)\sin(u)$ with m a Gaussian process with zero mean and autocorrelation function $\gamma_m(x)=\sigma_0^2\exp[-x^2/(4l_c^2)]$ and power spectral density $\hat{\gamma}(k)=\exp(-k^2l_c^2)2\sqrt{\pi}l_c\sigma_0^2$ with $\sigma_0=0.05$. We start from the initial condition $v(t=0)=v_0$ and consider different correlation lengths l_c. The thick lines represent the theoretical solutions v_l, while the thin lines are the numerical solutions.

Radiation looks like any linear dispersive wave far from the kink, but it has complex structure in the neighborhood of the kink. Roughly speaking, the support of the radiation lies in an interval with length of the order of ε^{-2}. Since the L^2 norm of $1-\cos(u)$ is bounded, we can expect that the amplitude of the radiation is of the order of ε around 0 behind the kink and around 2π in front of the kink. It can be rigorously proved that the amplitude of the radiation can be bounded above by $K\varepsilon|\ln\varepsilon|$.[28]

C. Numerical simulations

To check our theoretical predictions we perform full numerical simulations of Eqs. (14) and (21). For this purpose Eqs. (14) and (21) are transformed into a set of difference equations in the spatial domain (with second-order accuracy) and we apply an explicit sixth-order Runge-Kutta scheme. We use a shifting computational domain which is always centered at the center of mass of the solution. Moreover we impose boundaries of this domain which absorb outgoing waves. The reader can observe in Fig. 1 a good agreement between the numerical simulations and the theoretical predictions. We would like to comment on the oscillations of the velocity that can be observed. Indeed, in the configuration at hand, the Hamiltonian is preserved, which also reads as

$$\mathcal{E}_0=\frac{8}{\sqrt{1-v(t)^2}}-\frac{1}{\pi}\int_0^\infty(4+\lambda^{-2})\ln(|a|^2)(t,\lambda)d\lambda$$
$$+\varepsilon\int_{-\infty}^{\infty}m(x)\{\cos[u_0(x)]-\cos[u(t,x)]\}\,dx,$$

where u_0 is the initial condition. The last term of the right-hand side is negligible in the asymptotic framework $\varepsilon\to 0$, but when $\varepsilon>0$ it gives rise to local fluctuations of the unperturbed Hamiltonian as defined by Eq. (11), hence local fluctuations of the velocity whose amplitudes are of the order of ε over times of the order of 1 (the important point for the asymptotic analysis is that they are still of the order of ε for times of the order of ε^{-2}). The fluctuations are all the more important as the correlation length of the medium is of the same order or even larger than the kink width [see Fig. 1(a)]. They are smoothed when the correlation length is much smaller than the kink width [see Fig. 1(b)].

V. SPATIALLY RANDOM DISPERSION

A. First model: A simple Hamiltonian form

In the first part of this section devoted to spatially random dispersion we analyze the case where the perturbation reads

$$R(u)=(m(x)u_x)_x. \tag{26}$$

The perturbed Hamiltonian is $\mathcal{E}_0+\varepsilon\mathcal{E}_1$, where

$$\mathcal{E}_1=\frac{1}{2}\int_{-\infty}^{\infty}m(x)u_x^2dx.$$

We get the same conclusion as in the case of the nonlinear perturbation, by taking care to define the scattered energy density per unit time as

$$E(k)=\hat{\gamma}_m\left(\frac{\sqrt{k^2+1}}{v}-k\right)\frac{\pi}{4v^3}\frac{\left(\frac{\sqrt{k^2+1}}{v}-k\right)^2}{\cosh\left(\sqrt{k^2+1}\frac{\sqrt{1-v^2}}{v}\frac{\pi}{2}\right)^2}. \tag{27}$$

Note that this is the same expression as for the spatially random nonlinearity, up to a factor $(1-v^2)^2$.

B. A model without radiation

Let us now consider the following perturbation model:

$$R(u)=(m(x)u_x)_x+m(x)u_{xx}=m_xu_x+2mu_{xx}. \tag{28}$$

This model preserves the total momentum. Calculations show that this perturbation gives rise to no radiation: $E(k)\equiv 0$. This is not at all surprising. Indeed the change of variable

$$x\mapsto X\quad\text{such that}\quad\frac{dX}{dx}=\sqrt{1+2\varepsilon m(x)} \tag{29}$$

is well defined as soon as $\sup_{x\in[0,L/\varepsilon^2]}|2m(x)|<1/\varepsilon$ which is always the case in our framework. In this new frame the new function $\tilde{u}(t,x):=u(t,X(x))$ satisfies

$$\tilde{u}_x=\sqrt{1+2\varepsilon m(x)}u_x,$$

$$\tilde{u}_{xx}=[1+2\varepsilon m(x)]u_{xx}+\frac{\varepsilon m_x(x)}{\sqrt{1+2\varepsilon m(x)}}u_x.$$

Then the perturbed SG equation reads as

$$\tilde{u}_{tt}-\tilde{u}_{xx}+\sin(\tilde{u})=\varepsilon m_x(x)\times\left(\frac{1}{\sqrt{1+2\varepsilon m(x)}}-\frac{1}{1+2\varepsilon m(x)}\right)\tilde{u}_x.$$

The perturbation can be expanded as $\varepsilon^2m(x)m_x(x)u_x+O(\varepsilon^3)$. For propagation times of the order of ε^{-2} this $O(\varepsilon^2)$ zero-mean perturbation does not involve any macroscopic (i.e., of the order of 1) modification of the kink velocity and width.

C. A natural model of spatially random dispersion

Let us finally address the following case:

$$R(u)=m(x)u_{xx}. \tag{30}$$

This model is the most natural model of spatially random dispersion. In the LJJ framework it corresponds to fluctuations of the inductance of the junction.[29] In terms of perturbation analysis this model presents the *a priori* drawback that it neither preserves the momentum nor the energy of the solution. Thus it seems not possible to apply our method to this perturbation. However if we perform the change of space variable (29), then we get that the perturbed SG equation reads in this new frame as

$$\tilde{u}_{tt}-\tilde{u}_{xx}+\sin(\tilde{u})=\varepsilon(m(x)\tilde{u}_x)_x+\text{negligible terms}.$$

This is our first model (26) up to terms which have negligible influence for propagation times of the order of ε^{-2}. The decay of the kink velocity thus obeys the effective equations (23) and (27) as $\varepsilon\to 0$.

Note that the fact that perturbations (30) and (26) are equivalent follows from the second-order dispersion. A third-order dispersion (such as the one of the Korteweg-de Vries equation) would have led us to a different conclusion.

D. Numerical simulations

To check our theoretical predictions we have performed full numerical simulations of Eqs. (14) and (26), Eqs. (14) and (28), and Eqs. (14) and (30). Note that the dispersive perturbation was discretized as follows for the three models:

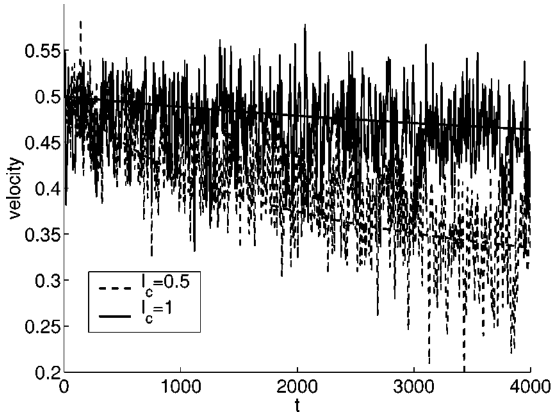

FIG. 2. Kink velocity as a function of time. The perturbation model is $R(u)=m(x)u_{xx}$ with m a Gaussian process with zero mean and autocorrelation function $\gamma_m(x)=\sigma_0^2\exp[-x^2/(4l_c^2)]$ and power spectral density $\hat{\gamma}(k)=\exp(-k^2l_c^2)2\sqrt{\pi}l_c\sigma_0^2$ with $\sigma_0=0.05$. We start from the initial condition $v(t=0)=v_0$. The solid lines stand for $l_c=1$ and the dashes lines for $l_c=0.5$. The thick lines represent the theoretical solutions v_l, while the thin lines are the numerical solutions.

$$\text{model 1: } (m_x u)_x=\frac{m_{j+1}+m_j}{2}u_{j+1}^n+\frac{m_{j-1}+m_j}{2}u_{j-1}^n - \frac{m_{j-1}+2m_j+m_{j+1}}{2}u_j^n,$$

$$\text{model 3: } m u_{xx}=m_j(u_{j+1}^n+u_{j-1}^n-2u_j^n),$$

and model 2=model 1+model 3. Full numerical simulations for these models show complete agreement with the theoretical predictions (see Fig. 2).

VI. TIME RANDOM NONLINEARITY

The perturbation that is addressed in this section is the following:

$$R(u)=m(t)\sin(u). \tag{31}$$

This perturbation conserves the total momentum. By applying the same strategy as in Sec. IV B but using the momentum conservation law, we get that, with probability that goes to 1 as $\varepsilon\to 0$, the scattered wave at time t/ε^2 consists of one kink with velocity $v^\varepsilon(t)$ [and width $\sqrt{1-v^\varepsilon(t)^2}$] and radiation. The process $(v^\varepsilon(t))_{t\in[0,T]}$ converges in probability as $\varepsilon\to 0$ to the deterministic function $(v_l(t))_{t\in[0,T]}$ which satisfies the ordinary differential equation (ODE)

$$\frac{dv_l}{dt}=F(v_l), \tag{32}$$

where

$$F(v)=-\frac{(1-v^2)^{3/2}}{8}\int_{-\infty}^{\infty}P(k)dk, \tag{33}$$

and $P(k)$ is the scattered momentum density per unit time:

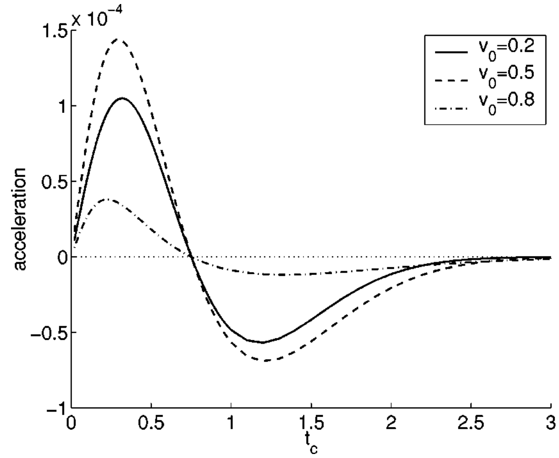

FIG. 3. Kink acceleration. The perturbation model is $R(u)=m(t)\sin(u)$ with m a Gaussian process with zero mean and autocorrelation function $\gamma_m(t)=\sigma_0^2\exp[-t^2/(4t_c^2)]$ and power spectral density $\hat{\gamma}=\exp(-\omega^2t_c^2)2\sqrt{\pi}t_c\sigma_0^2$ with $\sigma_0=0.1$. The lines represent the functions $F(v_0)$ as a function of the coherence time.

$$P(k)=\hat{\gamma}_m(\sqrt{1+k^2}-kv)\times\frac{(\sqrt{k^2+1}-kv)^2}{\sqrt{1+k^2}}\frac{\pi k(1-v^2)^2}{4\cosh(k\sqrt{1-v^2}\pi/2)^2}. \tag{34}$$

It appears that the acceleration function $F(v)$ does not possess a definite sign, and can be either positive or negative as shown in Fig. 3. Actually this phenomenon can be studied analytically and we now show that the sign of the acceleration function essentially depends on the coherence time of the perturbation. For short-correlation time (in the sense that $\sqrt{1-v^2}t_c\ll 1$), we have $\hat{\gamma}(\omega)\simeq\sigma^2$ for $\omega t_c\ll 1$, so

$$P(k)+P(-k)\simeq-\sigma^2\frac{\pi(1-v)^2vk^2}{\cosh\left(k\sqrt{1-v^2}\,\frac{\pi}{2}\right)^2},$$

which in turn implies that

$$F(v)\simeq\frac{(1-v^2)^{7/2}v}{8}\sigma^2\int_0^\infty dk\frac{\pi k^2}{\cosh(k\sqrt{1-v^2}\pi/2)^2} =\frac{\sigma^2}{12}v(1-v^2)^2$$

which is positive. Thus the kink is speeding up. Its velocity increases to 1. The kink also becomes narrower. Its width $\sqrt{1-v_l^2(t)}$ decays to 0.

For long-correlation time (in the sense that $\sqrt{1-v^2}t_c\gg 1$), the integral in Eq. (33) concentrates around the wave number $k_c=v/\sqrt{1-v^2}$ which is such that $\sqrt{1+k^2}-kv$ is minimal. And then

$$F(v)\simeq-\frac{\pi v(1-v^2)^{3/2}}{32\cosh(v\pi/2)^2}\int_{-\infty}^{\infty}\hat{\gamma}_m(\sqrt{1+k^2}-kv)dk$$

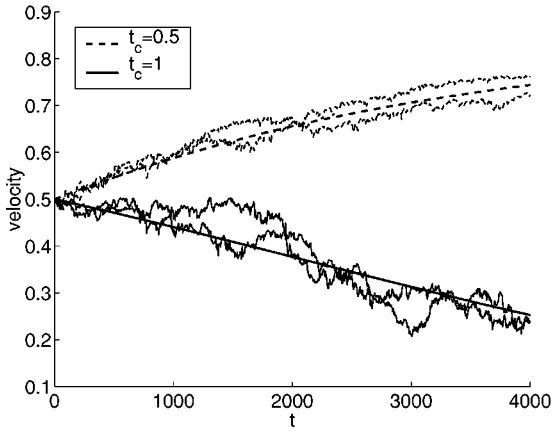

FIG. 4. Kink velocity as a function of time. The perturbation model is $R(u)=m(t)\sin(u)$ with m a Gaussian process with zero mean and autocorrelation function $\gamma_m(t)=\sigma_0^2\exp[-t^2/(4t_c^2)]$ and power spectral density $\hat{\gamma}=\exp(-\omega^2 t_c^2)2\sqrt{\pi}t_c\sigma_0^2$ with $\sigma_0=0.1$. We start from the initial condition $v(t=0)=v_0$. The solid lines stand for $t_c=1$ and the dashes lines for $t_c=0.5$. The thick lines represent the theoretical solutions v_l, while the thin lines are the numerical solutions.

which is negative. Thus the kink is slowing down and becomes wider. It eventually stops at some finite distance x_d. This distance can be computed numerically from the ODE (32), and in the case when the initial kink velocity is small, we can compute the explicit expression

$$x_d=\frac{16v_0}{\pi\overline{\sigma}^2},\quad \overline{\sigma}^2=\int_0^\infty \hat{\gamma}_m(\sqrt{1+k^2})dk.$$

It can be shown that these two behaviors are the only ones that can be observed. More exactly, for a given power spectrum $\hat{\gamma}$, there exists a critical value v_c of the initial kink velocity v_0 such that, if $v_0>v_c$, then the velocity goes to 1, while if $v_0<v_c$, then the velocity decays to 0 and the kink stops.

These theoretical predictions are confirmed by the numerical simulations that we carried out (see Fig. 4).

VII. TIME RANDOM DAMPING

We now address the case of a random damping. We consider that the damping has zero mean, and we analyze the effective influence of this damping onto the kink. The question whether the kink takes benefit or not of the damping fluctuations is not obvious. We are going to show that the answer is not unique and strongly depends on the spectral property of the damping process. Let us consider

$$R(u)=m(t)u_t. \tag{35}$$

Note that this perturbation gives rise to a computable change of the total momentum

$$\mathcal{P}(t)=\mathcal{P}(0)\exp[-M_d(t)],$$

where

$$M_d(t):=\varepsilon\int_0^t m(s)ds$$

is the accumulated dissipation. Once this identity is known, we can apply the very same approach as in the previous sections. There then appears a dichotomy. Indeed the application of a standard diffusion-approximation theorem to the accumulated dissipation shows that in the asymptotic $\varepsilon\to 0$, the total momentum in the scale t/ε^2 evolves as

$$\mathcal{P}(t/\varepsilon^2)\overset{\varepsilon\to 0}{\longrightarrow}\overline{\mathcal{P}}(t):=\mathcal{P}_0\exp(-\sigma W_t),$$

where W_t is a standard one-dimensional Brownian motion and $\sigma:=\sqrt{\hat{\gamma}_m(0)}$. By the Wiener-Khintchine theorem it is known that $\hat{\gamma}_m(0)$ is non-negative. We should then distinguish between the cases $\hat{\gamma}_m(0)=0$ (absence of an effective damping) and $\hat{\gamma}_m(0)>0$ (presence of an effective damping).

A. Dissipation without effective damping

We assume in this section that $\hat{\gamma}_m(0)=0$. We may think at the case where m is the time derivative w' of some stationary process w with continuously differentiable realizations, autocorrelation function γ_w, and power spectral density $\hat{\gamma}_w$. The autocorrelation functions of w and m are then related through the identity

$$\gamma_m(t)=\langle w'(s)w'(s+t)\rangle=-\langle w(s)w''(s+t)\rangle=-\gamma_w''(t),$$

so that their respective power spectral densities satisfy

$$\hat{\gamma}_m(\omega)=\omega^2\hat{\gamma}_w(\omega).$$

We then get that, with probability that goes to 1 as $\varepsilon\to 0$, the scattered wave at time t/ε^2 consists of one kink with velocity $v^\varepsilon(t)$ [and width $\sqrt{1-v^\varepsilon(t)^2}$] and radiation. The process $(v^\varepsilon(t))_{t\in[0,T]}$ converges in probability to the deterministic function $(v_l(t))_{t\in[0,T]}$ which satisfies the ordinary differential equation

$$\frac{dv_l}{dt}=F(v_l), \tag{36}$$

where

$$F(v)=-\frac{(1-v^2)^{3/2}}{8}\int_{-\infty}^\infty P(k)dk, \tag{37}$$

and $P(k)$ is the scattered momentum density per unit time:

$$P(k)=\hat{\gamma}_m(\sqrt{1+k^2}-kv)\frac{k}{\sqrt{1+k^2}}\frac{\pi v^4}{\cosh(k\sqrt{1-v^2}\pi/2)^2}. \tag{38}$$

As pointed out above, a very natural question from which we could expect a simple answer is the following: what is the total effect of this zero-mean dissipation? Surprisingly, the answer depends on the coherence time of the process m, and more generally on its power spectral density.

For short coherence time, we can expand $\hat{\gamma}$ as $\hat{\gamma}(\omega) \simeq \alpha\omega^2$ while $|\omega|t_c<1$, where $\alpha=\int t^2\gamma_m(t)dt/2=\int\gamma_w(t)dt>0$. Then for any wave number $k\in(0,[(1+v)t_c]^{-1})$

$$P(k)+P(-k)\simeq-\alpha\frac{4\pi k^2v^5}{\cosh(k\sqrt{1-v^2}\pi/2)^2},$$

so that, if $v<1-8t_c^2/\pi^2$

$$F(v)\simeq\frac{(1-v^2)^{3/2}}{2}\alpha\int_0^\infty dk\frac{\pi k^2v^5}{\cosh(k\sqrt{1-v^2}\pi/2)^2}=\frac{\alpha v^5}{3}$$

which is positive. Thus the kink is speeding up and becomes narrower. Note that, if $v\in(1-8t_c^2/\pi^2,1)$, then the above expansion is not valid anymore and one should take into account the complete expressions (37) and (38).

For long coherence time, the integral in Eq. (37) concentrates around the wave number $k_c=v/\sqrt{1-v^2}$ which is such that $\sqrt{1+k^2}-kv$ is minimal. Accordingly

$$F(v)\simeq-\frac{(1-v^2)^{3/2}}{8}\frac{\pi v^5}{\cosh(v\pi/2)^2}\int_{-\infty}^{\infty}\hat{\gamma}_m(\sqrt{1+k^2}-kv)dk$$

which is negative. Thus the kink is slowing down and becomes wider.

In Fig. 4 we compare numerical simulations with the theoretical predictions in the case $m=w'$ where w is a stationary random process with Gaussian statistics, Gaussian autocorrelation function $\gamma_w(t)=\exp[-t^2/(4t_c^2)]$ with coherence time t_c. The curve that stands for the kink velocity is randomly modulated at time scale 1. Indeed the total momentum is preserved in the asymptotic framework $\varepsilon\to 0$, but for $\varepsilon>0$

$$\mathcal{P}(t)=\mathcal{P}_0\exp\{-\varepsilon[w(t)-w(0)]\},$$

which in turn involve local fluctuations of the kink velocity

$$v_{local}(t_0+\Delta t)=v(t_0)\{v(t_0)^2+[1-v(t_0)^2]\times\exp[2\varepsilon w(t_0+\Delta t)-2\varepsilon w(t_0)]\}^{-1/2}.$$

These local fluctuations (of amplitude $\sim\varepsilon$) involve the observed short-scale modulations observed in Fig. 5(a). However these fluctuations have no importance when we consider the long-time behavior (at scale ε^{-2}) which is imposed by Eqs. (36)–(38). Furthermore, if we record numerically the values of the accumulated dissipation w_{num} and plot the numerical velocity v_{num} divided by the local fluctuations,

$$v_{comp}(t):=v_{num}(t)\{v_0^2+(1-v_0^2)\times\exp[2\varepsilon w_{num}(t)-2\varepsilon w_{num}(0)]\}^{-1/2},$$

we get a smooth curve that follows the asymptotic formulas (36)–(38) as shown in Fig. 5(b).

B. Dissipation with effective damping

We assume in this section that $\hat{\gamma}_m(0)>0$ and set $\sigma=\sqrt{\hat{\gamma}_m(0)}$. We then get that, with probability that goes to 1 as $\varepsilon\to 0$, the scattered wave at time t/ε^2 consists of one kink with velocity $v^\varepsilon(t)$ [and width $\sqrt{1-v^\varepsilon(t)^2}$] and radiation.

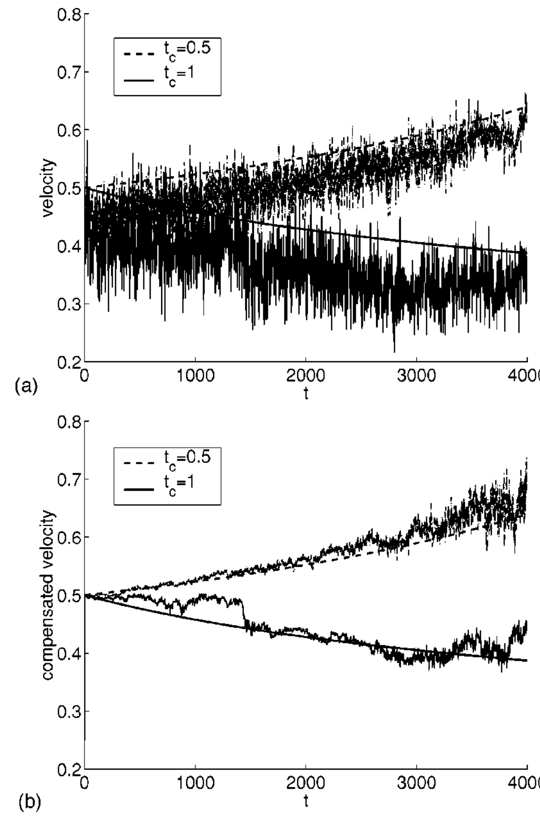

FIG. 5. Kink velocity as a function of time. The perturbation model is $R(u)=m(t)u_t$ with m a Gaussian process with zero mean and autocorrelation function $\gamma_m(t)=\sigma_0^2[1-t^2/(2t_c^2)]\exp[-t^2/(4t_c^2)]$ and power spectral density $\hat{\gamma}=\omega^2\exp(-\omega^2t_c^2)4\sqrt{\pi}t_c^3\sigma_0^2$ with $\sigma_0=0.1$. We start from the initial condition $v(t=0)=v_0$. Picture a plot compare the theoretical velocities v_l (thick lines) with the numerical velocities v_{num} (thin lines). In (b) we compare the theoretical velocities with the compensated numerical velocities defined by v_{comp}. This allows us to get rid of the fast oscillations while v_l is close to v_0.

The process $(v^\varepsilon(t))_{t\in[0,T]}$ converges in probability to the random function $(v_l(t))_{t\in[0,T]}$ which satisfies the *stochastic* differential equation

$$dv_l=F(v_l)dt-\sigma\frac{(1-v_l^2)^{3/2}}{(1-v_0^2)^{1/2}}v_0e^{-\sigma W_t}\circ dW_t,\quad v_l(0)=v_0, \tag{39}$$

where $F(v)$ is given by Eqs. (37) and (38) and $\circ$ stands for the Stratonovich integral. In case of a white noise $\gamma_m(t)=\sigma^2\delta(t)$ the scattered momentum density is odd, so we have $F(v)=0$. As a consequence the solution of the stochastic differential equation has a closed-form expression

$$v_l(t)=v_0[v_0^2+(1-v_0^2)\exp(2\sigma W_t)]^{-1/2}. \tag{40}$$

We could have obtained this expression by a more direct technique based on a collective-coordinate approach. However the application of this technique gives expression (40) as a result whatever the power spectral density of the damping fluctuations m, because this approach neglects the radiative effects. The above analysis shows that Eq. (40) holds true only in case of short-correlation noise, and that the radiative effects become truly important as the correlation time increases and the power spectral density $\hat{\gamma}_m$ cannot be assimilated to a flat spectrum.

Note that, in the white-noise case, a closed-form expression for the probability density function of v_l can be computed:

$$p_t(v) = \frac{1}{\sqrt{2\pi\sigma^2 t}\, v(1-v^2)} \times \exp\left[-\frac{1}{8\sigma^2 t} \ln^2\left(\frac{v_0^2}{1-v_0^2} \frac{1-v^2}{v^2} \right) \right] \mathbf{1}_{v\in[0,1]}. \tag{41}$$

As $\sigma^2 t \gg 1$, the probability density function becomes close to two Dirac masses with equal weights 1/2 at $v=0$ and $v=1$ (see Fig. 6).

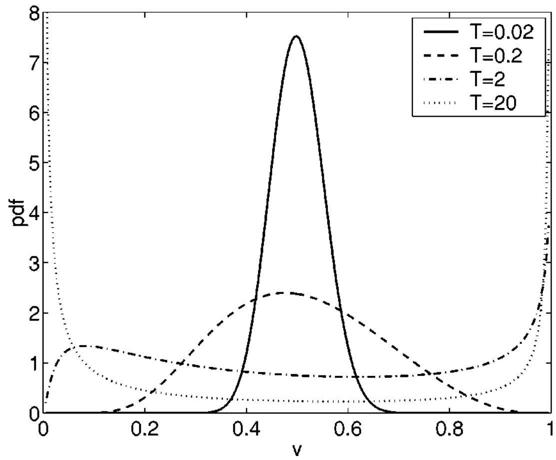

FIG. 6. Probability density function of the velocity for different normalized times $T=\sigma^2 t$. The perturbation model is $m(t)u_t$, where m is a white noise with covariance function $\sigma^2\delta(t)$. We start from the initial condition $v(T=0)=v_0$.

VIII. CONCLUSION

We have studied the propagation of a kink by a random sine-Gordon equation by applying the inverse scattering transform. Our method can be applied to a large class of random perturbations that are stationary and ergodic. The only but important condition is that some integral of motion should be preserved by the perturbation, or else that it could be *a priori* computed. Indeed, if the amount of scattered radiation can be estimated in very great generality, the feedback of the radiative scattering onto the kink evolution is imposed by the underlying conservation law and it strongly depends on the nature of the conserved quantity. If the energy or some modified form of the energy is preserved, then the kink velocity decays and its width increases. This is the case for many realistic perturbation models with position-dependent coefficients. However we have also exhibited some time-dependent models where our analysis shows that the kink is speeded up and becomes narrow.

We have focused our attention on the role of the correlation length/time of the noise. The qualitative conclusion of this study is that time random perturbations with short coherence time involve a speeding-up and a narrowing of the kink, while time random perturbations with long coherence time make the kink slow down and broaden. This is true in particular for a stochastic damping. The importance of the interplay of the correlation length of the perturbation and the characteristic lengths of kinks may be seen as the manifestation of the general phenomenon which has been termed length-scale competition.[30]

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