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OPTICAL SOLITONS IN RANDOM MEDIA

BY

Fatkhulla Abdullaev
Physical-Technical Institute, Uzbek Academy of Sciences
700084, Tashkent-84, G.Mavlyanov str., 2-b, Uzbekistan

Josselin Garnier
Laboratoire de Statistique et Probabilite, Universite Paul Sabatier, 118 Route de Narbonne Toulouse Cedex, France

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§ 1. Introduction

Solitons are now considered as one of the most important objects in nonlinear physics. They can be found in many areas of physics including hydrodynamics, condensed matter, nonlinear optics, and atomic Bose-Einstein condensates. Optical solitons in fibers and spatial solitons (self-supported optical beams) in nonlinear media have attracted special attention. It is widely believed that optical solitons will be key elements for the next generation of high speed optical communication systems and ultrafast optical devices. Spatial solitons are considered as the base elements for all-optical logical and switching devices.

The existence of solitons is connected with the balance between the fiber dispersion or the linear diffraction and the medium nonlinearity. Optical solitons in fibers have been predicted by Hasegawa and Tappert [1973] and observed experimentally by Mollenauer, Stolen, and Gordon [1980]. Optical fibers are media with cubic nonlinearity. Mathematically the optical pulse propagation in a slowly varying amplitude approximation is described by the nonlinear Schrödinger equation (NLSE) for the electric field. This equation is integrable and can be solved by the use of the Inverse Scattering Transform (IST). Fortunately, many properties of real optical fiber systems such as dissipation, amplification, Raman processes, self-steepening, and higher-order dispersion can be described as *weak* perturbations of the NLSE. It is therefore possible to develop a perturbation theory in order to explain many phenomena connected with optical solitons in fibers and related devices, like soliton couplers, soliton laser etc. Finally, the propagation of short pulses in birefringent fibers was investigated. In these media cross phase modulation processes play important roles. Mathematically the pulse propagation can be described by the Manakov system and its nearly integrable and nonintegrable modifications, which supports vector optical solitons. Perturbations of the Manakov system can be dealt with similar approaches as for the scalar NLSE.

The efficiency of using solitons in optical communication systems is limited by several factors. The most important ones are: amplifier noise, inhomogeneities of the fiber material and the fiber transverse profile along the propagation, and random birefringence. The nonlinear interaction processes of trains of solitons are also factors of information loss in single-mode fibers. As was shown by Gordon and Haus [1986] and by Elgin [1985] the distributed noise leads to the soliton Gordon-Haus (GH) jitter and to fundamental limits on the bit error rate in soliton communication systems. Some estimates of the effects of random variations of fibers on soliton propagation were also obtained in this period, mainly using the adiabatic approximation of the perturbation theory for solitons (see the book by Abdullaev, Darmanyan, and Khabibullaev [1993]). More precise results were derived later by Baker and Elgin [1998], Falkovich, Kolokolov, Lebedev, and Turitsyn [2001], Biswas [2001]. Finally, optical soliton propagation in fibers with random birefringence was investigated. It was shown that the scale of variations of birefringence is much smaller that the characteristic scales of the problem like dispersion and nonlinear lengths. As was shown by Wai, Menyuk, and Chen [1991], Evangelides, Mollenauer, Gordon, and Bergano [1992] the averaging over the fast variations can be performed and the averaged dynamics is described by the Manakov system. Using collective variable methods Ueda and Kath [1992] derived the Fokker-Planck equation to find the probability densities for the polarization state of a propagating vector soliton. Higher-order corrections to the averaged Manakov system lead to random wave generation by the soliton that limits the error free bit rate in optical communication systems as shown by Lakoba and Kaup [1997], Chen and Haus [2000a], Chung, Chertkov, and Lebedev [2004]. The investigation of this Polarization Mode Dispersion (PMD) process requires to develop the perturbation theory for the Manakov system admitting the calculation of radiative effects. The analysis of the perturbed Manakov system performed by Midrio, Wabnitz, and Franco [1996], Shechnovich and Doktorov [1997] has deal with the adiabatic approximation. In the Appendix 2 we present our results for the calculation of the radiation effects in the perturbed Manakov system, based on the Gelfand-Levitan-Marchenko representation and the Hamiltonian formulation.

Many stochastic problems are connected with the long-scale propagation of optical solitons: generation of waves by picosecond and femtosecond solitons in random fibers, interaction of solitons, existence of dissipative solitons. Some of these problems were solved recently by Abdullaev, Caputo, and Flytzanis [1994], Abdullaev and Baizakov [2000], Chertkov, Chung, Dyachenko, Gabitov, Kolokolov, and Lebedev [2003], Abdullaev, Navotny, and Baizakov [2004], Chung, Chertkov, and Lebedev [2004] and some other remain open.

The discovery of dispersion-managed (DM) solitons by Smith, Knox, Doran, Blow, and Bennion [1996], Gabitov and Turitsyn [1996] has opened new possibilities for soliton optical communication. Periodic and strong modulations of the fiber dispersion are used to generate a DM soliton. The DM soliton is breathing during propagation and has enhanced energy in comparison with the standard optical soliton. This type of pulse is more robust against the action of the GH jitter. But since dispersion is modulated, new types of stochastic problems appear. It is important to understand the stability of the DM solitons against random variations of the chromatic and waveguide dispersions existing in fibers. As measurements show, the random variations of dispersion can be the order of the average dispersion value and so they play important role in the short pulse degradation. Mathematically this problem is more complicated since we have to deal with the NLSE with strongly periodically and randomly varying dispersion coefficient. For the analysis of this problem, the variational method (used by Abdullaev and Baizakov [2000], Malomed and Berntson [2001], Garnier [2002], Schafer, Moore, and Jones [2002]) and a frequency domain approach (used by Chertkov, Gabitov, Lushnikov, Moeser, and Toroczkai [2002]) are effective. DM soliton evolution under PMD was considered recently by Karlsson [1998], Chen and Haus [2000b], Abdullaev, Umarov, Wahiddin, and Navotny [2000] and experimentally by Xie, Sunnerud, Karlsson, and Andrekson [2001].

Other areas of optical solitons are connected with the self-supported optical beams (spatial solitons) and spatio-temporal pulses (optical bullets). In media supporting such structures the nonlinearity can be much larger than in optical fibers and so the effective scales are much shorter. Optical solitons become important as they are used as elements of all-optical logical devices and switchers.

Only cubic nonlinearity has been discussed in the previous paragraphs, but quadratic nonlinear media are also important for theoretical and practical purposes. The quadratic soliton formation in such media is possible due to the balance between the linear diffraction (dispersion) and nonlinear terms involving phase mismatchs. The soliton has a more complicated form than in the fiber case and it involves in general the interaction between three waves. An important partial case which is considered in this review is a fundamental wave (FW) and a second harmonic (SH) wave interaction. The system of equations describing this process is not integrable in general. One-dimensional (1D) solitons have been predicted by Karamzin and Sukhorukov [1974] and experimentally observed in a planar nonlinear waveguide by Baek, Schiek, Stegeman, and Sohler [1997] (see also the reviews by Buryak, Di Trapani, Skryabin, and Trillo [2002] and by Etrich, Lederer, Malomed, Peschel, and Peschel [2000]). The question of the stability of such solitons in random media is still uninvestigated. In distinction with the solitons in cubic nonlinear media, quadratic solitons are very sensitive to the phase matching condition. The randomness of the medium should lead to the distortion of the phase matching and to the quadratic soliton degradation. Since the equations are nonintegrable exact analytical methods are absent. Numerical simulations for the random mismatch and quadratic nonlinearity performed by Torner and Stegeman [1997], Clausen, Bang, Kivshar, and Christiansen [1997], Abdullaev, Darmanyan, Kobyakov, Schmidt, and Lederer [1999] show the pure dissipative nature of the soliton decay.

The problems mentioned here above involve the pulse dynamics driven by perturbations which are random in the evolutional variable. In the case of spatial solitons it is important to consider the propagation in a medium which is randomly varying with the spatial transverse variable. It corresponds to the random linear $V_1(x)$ and/or nonlinear potentials $V_2(x)|u|^2$ in the NLSE. This problem is also important for many areas of physics. It was investigated by many authors Gredeskul and Kivshar [1992], Bronski [1998], Knapp [1995], but recently it was solved using the IST approach for nonlinear and dispersive perturbations Garnier [1998], Garnier [2001].

Practically the evolution of optical bullets in random media remains uninvestigated. Some results were obtained recently by Gaididei and Christiansen [1998], Fibich and Papanicolaou [1999], Abdullaev, Bronski, and Galimzyanov [2003], Yannacopoulos, Frantzeskakis, Polymilis, and Hizanidis [2002]. The dynamics of discrete optical solitons in disordered 1D arrays of planar waveguides and 2D arrays of fibers has recently attracted attention - see the experiments performed by Pertsch, Peschel, Kobelke, Schuster, Bartelt, Nolte, Tunnermann, and Lederer [2004]. But the theory of wave phenomena in such structures is absent. Only some numerical simulations for 1D nonlinear disordered arrays have been performed by Kopidakis and Aubry [2000].

In this review we intend to give a description of the modern status of the theory and experiment about propagation and interaction of optical solitons in random media. We start from a short derivation of the main equations used for the description of optical soliton evolution in presence of cubic or quadratic nonlinearity (Section 2). The first part of the review is devoted to the dynamics of solitonic pulses in fibers with random parameters. The propagation of standard optical solitons (Section 3) as well as dispersion-managed solitons (Section 4) under random fluctuations of dispersion and nonlinearity (amplification) is considered. The inverse scattering transform, the variational approach, and the averaging of the NLSE in the frequency domain are applied for the investigation of the solitonic processes in fibers with such types of inhomogeneities. Other related issues are addressed: interaction of solitons in single-mode fibers with fluctuating parameters and propagation of femtosecond pulses using the randomly perturbed modified NLSE. In Section 5 we study the influence of random birefringence on the optical soliton propagation. PMD is a limiting factor for long distance optical fiber communication systems. It can be overcome by solitonic propagation. The propagation can be analyzed using the averaging of the original system of coupled NLSEs over the rapid variations of the birefringence. The resulting system is the randomly perturbed Manakov system. The propagation of the vector optical solitons, radiative effects, internal motion, etc, are described using the IST approach. The influence of PMD on dispersion-managed solitons is studied at the end of this section. In Section 6 we address the propagation of optical solitons in quadratic random media. Analytical and numerical results are described for two types of fluctuations along the propagation direction: random mismatch or fluctuating nonlinearity. Section 7 is devoted to the propagation of spatial solitons (beams) in media with spatially varying parameters. We first address the propagation of a soliton in a long slab with an index of refraction which is random in the transverse direction. The problem is treated analytically and numerically. The role of random dispersive perturbations in the propagation of solitons is studied through the example of an array of planar waveguides with random tunnel coupling. The results of recent experiments in 2D disordered fiber arrays are discussed. The case of an index of refraction randomly varying in the transverse and longitudinal directions is considered using the moments method. In the final section the evolution of 2D solitons under fluctuations of the medium parameters is investigated. The analysis is based on the modulation theory for the Townes soliton. In Appendices the IST approachs for the scalar and vector NLSEs are given.

Throughout the study we present different approaches (inverse scattering transform, moments approach, variational approach, averaging of partial differential equations). We discuss their respective domains of applicability from a theoretical point of view.

§ 2. The main equations

2.1. DERIVATION OF THE NONLINEAR SCHRÖDINGER EQUATION

In this section we reproduce standard arguments that can be found for instance in [Haus and Wong [1996]]. The transverse profile of the index of refraction of an optical fiber can be designed so that the fiber supports only one electromagnetic mode, with two possible polarizations. Field patterns of greater transverse variation are not guided but are lost to radiation. A single-mode fiber propagates modes of two polarizations. Polarization-maintaining fibers are sufficiently birefringent that a mode launched in one linear polarization along one of the axes of birefringence remains polarized along this axis. However, polarization-maintaining fibers are expensive and have higher losses than regular fibers, and hence are not used in long-distance fiber communications. Regular fibers still possess birefringence of the order of 10^{-7} . This means that within 10^7 wavelengths, i.e., about 10 m, the polarization changes uncontrollably. The depolarization length is much shorter than the dispersion length, and the length within which the optical Kerr effect produces self-phase modulation. On these larger distance scales, the mode can be treated in a first approximation as a mode of a single average polarization.

A pulse propagating along the fiber experiences dispersion and the optical Kerr effect. Dispersion is called normal, or positive, if the group velocity decreases with increasing frequency; it is anomalous, or negative, if the change is in the opposite direction. Dispersion acting alone is a linear effect that does not change the spectrum of the pulse. Different frequencies travel with different group velocities and thus a pulse spreads in time. The optical Kerr effect is an intensity-induced refractive index change. The Kerr coefficient is defined to be positive if the refractive index increases with increasing intensity. This index change leads to a time-dependent phase shift. Since the time derivative of phase is related to the frequency, the optical Kerr effect usually produces a change of the pulse spectrum. The combination of positive dispersion and a positive Kerr coefficient leads to temporal and spectral broadening of the pulse. A fiber with negative dispersion and a positive Kerr coefficient can propagate a pulse with no distortion. This may be surprising, at first, since dispersion affects the pulse in the time domain, the Kerr effect in the frequency domain. However, a small time-dependent phase shift added to a Fourier transform-limited pulse does not change the spectrum to first order. If this phase shift is canceled by dispersion in the same fiber, the pulse does not change its shape or its spectrum as it propagates. This propagation of an optical soliton is governed by the nonlinear Schrödinger equation (NLSE), which we now derive.

Consider the propagation of an electromagnetic mode of one polarization along a single-mode optical fiber. The fact that the polarization varies along the fiber due to the natural birefringence of the fiber will be taken up in Section 5. In the frequency domain, the amplitude $\hat{a}(z,\omega)$ of the mode obeys the differential equation $\hat{a}_z(z,\omega) = i\beta(\omega)\hat{a}(z,\omega)$ where $\beta(\omega)$ is the propagation constant. The spectrum of $\hat{a}(z,\omega)$ is assumed to be confined to a frequency regime around ω_0. A Taylor expansion of $\beta(\omega_0)$ in frequency around the carrier frequency ω_0 gives $\beta(\omega) \simeq \beta(\omega_0) + \beta'\Delta\omega + \beta''\Delta\omega^2/2$, where $\Delta\omega = \omega - \omega_0$. If we introduce the new envelope variable $\hat{a}(z,\omega) = \exp(i\beta(\omega_0)z - i\omega_0 t)\hat{v}(z,\Delta\omega)$, we obtain the equation for $\hat{v}$: $\hat{v}_z = i(\beta'\Delta\omega + \beta''\Delta\omega^2/2)\hat{v}$. In the time domain this equation reads

$$v_z = -\beta' v_t - \frac{i}{2}\beta'' v_{tt}$$

Transformation of the independent variables z and t into a frame represented by $z' = z$, $t' = t - \beta' z$ comoving with the group velocity $1/\beta'$, removes the first-order time derivative. To keep the notation simple, we shall drop the primes on z and t, getting the simple equation as the result

$$v_z = -\frac{i}{2}\beta'' v_{tt} \tag{2.1}$$

This is the linear Schrödinger equation for a free particle in one dimension (with z and t interchanged).

If the fiber is nonlinear, the propagation constant acquires an intensity-dependent contribution; and $\beta(\omega)$ has to be supplemented by the Kerr contribution, which is a change of the refractive index proportional to the optical intensity. The process is known as four-wave mixing, since Fourier components at ω_1 and ω_2 mix with a Fourier component at ω_3 to produce a phase shift at $\omega = \omega_3 - \omega_2 + \omega_1$. If a distribution of frequencies is involved, a convolution has to be carried out in the frequency domain. The nonlinear index n_2 produces a phase shift proportional to

$$n_2 \int d\omega_1 \int d\omega_2 v(z,\omega_1) v^*(z,\omega_2) v(z,\omega + \omega_2 - \omega_1)$$

When transformed back into the time domain, this integral becomes

$$n_2 \partial_t [|v(z,t)|^2] v(z,t)$$

Thus, the Kerr effect is expressed much more simply in the time domain, as long as the Kerr coefficient is frequency independent. The change of amplitude Δv of the pulse propagating through a differential length of fiber Δz can be derived from standard perturbation theory:

$$\Delta v = i\frac{\omega_0}{c}n_2\frac{|v(z,t)|^2}{A_{eff}}v(z,t)\Delta z$$

Here n_2 is the Kerr coefficient ($n_2 \simeq 3\ 10^{-20}$ m^2/W for silica fiber), A_{eff} is the effective mode area, obtained by averaging the phase shift using the mode profile $e(x,y)$ over the fiber, that is, by integrating over the transverse dimensions x and y:

$$\frac{1}{A_{eff}} = \frac{\int\int |e(x,y)|^4 dxdy}{\int\int |e(x,y)|^2 dxdy}$$

We have assumed that $|v(z,t)|^2$ is normalized to the power and the mode pattern $e(x,y)$ has been so normalized that $\int\int |e(x,y)|^2 dxdy$ is dimensionless. When Eq. (2.1) is supplemented by the Kerr effect, we obtain the NLSE

$$v_z = -\frac{i\beta''}{2}v_{tt} + i\kappa|v|^2 v \tag{2.2}$$

where $\kappa = \omega_0 n_2/(cA_{eff})$. The nonlinearity in this equation can compensate for the dispersion: a pulse need not disperse since it can dig its own potential well, which provides confinement. This happens when $\beta'' < 0$. Indeed, a solitary-wave solution of Eq. (2.2) is

$$v_S(z,t) = A_0 \mathrm{sech}\left(\frac{t-\beta''\omega_1 z}{\tau}\right)\exp\left(i\frac{\kappa|A_0|^2+\beta''\omega_1^2}{2}z\right)$$

where $1/\tau^2 = -\kappa|A_0|^2/\beta''$. This constraint shows that dispersion has to be indeed negative. We next introduce a reference time t_0 (of the order of the pulse width) and we define the dimensionless time $T = t/t_0$ and distance $Z = z/z_0$ where $z_0 = t_0^2/|\beta''|$ is the dispersion distance. The dimensionless field

$$u(T,Z) = \sqrt{\kappa z_0}u(Tt_0, Zz_0)$$

satisfies the normalized NLSE

$$iu_Z + \frac{1}{2}u_{TT} + |u|^2 u = 0 \tag{2.3}$$

The NLSE is integrable (see Appendix 1) and supports soliton solutions with a sech envelop.

2.2. DERIVATION OF $\chi^{(2)}$ SYSTEM

The induced polarization of the medium **P** can be expanded as powers of the electric field amplitude

$$\mathbf{P} = \epsilon_0[\chi^{(1)} \cdot \mathbf{E} + \chi^{(2)} : (\mathbf{E}, \mathbf{E}) + \cdots], \tag{2.4}$$

where ϵ_0 is the vacuum permittivity and $\chi^{(i)}$, $i = 1, 2, ...$ is the $i-$th order susceptibility. The second order susceptibility is responsible for the second-harmonic generation. Let us derive the equations for the fundamental waves (FW) and second harmonic (SH) fields. The light wave with the frequency ω can feel the nonlinear response of the χ^2 terms through the interaction of ω and 2ω components. In a film waveguide geometry we can look for fields of the form

$$E(X,Y,Z;t) = \sum_i \mathbf{e}_i \bar{E}_i + c.c.,$$
$$\bar{E}_1(X,Y,Z;t) = A_1(X,Z) f_A(Y) e^{i(k_1 Z - \omega_1 t)},$$
$$\bar{E}_2(X,Y,Z;t) = A_2(X,Z) f_B(Y) e^{i(k_2 Z - \omega_2 t)}. \tag{2.5}$$

where c.c. stands for "complex conjugate". Here $\mathbf{e_i}$ are unit polarization vectors, $k_{1,2} = \omega n_{1,2}/c$ are the propagation constants, $n_{1,2}$ are the effective mode indices, $f_A(Y)$, $f_B(Y)$ are the propagating modes in the transverse Y direction, and $A_{1,2}$ are the slowly varying envelopes for FW and SH fields, respectively. It is assumed that the frequencies of the interacting waves are exactly matched ($2\omega_1 = \omega_2$) and wavevectors are almost matched ($2k_1(\omega_1) - k_2(\omega_2) = \Delta k$, $\Delta k \ll k_i$). Substituting this expansion into the Maxwell equation

$$\Delta \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\chi * \mathbf{E}) - \nabla(\nabla \cdot \mathbf{E}) = 0, \tag{2.6}$$

and averaging over the transverse profile we obtain the system of equations for the FW (A_1) and SH (A_2) fields,

$$iA_{1\mathrm{Z}} + \frac{1}{2k_1} A_{1\mathrm{XX}} + \Gamma(X,Z) A_1^* A_2 e^{-i\Delta k Z} = 0, \tag{2.7}$$

$$iA_{2\mathrm{Z}} + \frac{1}{2k_2} A_{2\mathrm{XX}} + \Gamma(X,Z) A_1^2 e^{i\Delta k Z} = 0. \tag{2.8}$$

The effective nonlinear coefficient $\Gamma(X,Z)$ has the form

$$\Gamma(X,Z) = \sqrt{\frac{2}{\varepsilon_0 c^3 n_1^2 n_2}} \omega d_{eff} \frac{\int (f_A(Y))^2 f_B(Y) dY}{\int_{-\infty}^{\infty} |f_A(Y)|^2 dY \sqrt{\int_{-\infty}^{\infty} |f_B(Y)|^2 dY}}, \tag{2.9}$$

where d_{eff} is an effective component of the nonlinear susceptibility tensor $\chi^{(2)}$ and $f_{A,B}(Y)$ are dimensionless mode profiles. Eqs. (2.7-2.8) can be recast into the convenient dimensionless form with the following transformation

$$A_1 = \frac{P_0}{\sqrt{2}} E_1, \quad A_2 = P_0 E_2 e^{i\Delta k Z}, \quad Z = L_D z, \quad X = X_0 x,$$

where $P_0 = (\Gamma L_d)^{-1}$, $L_D = \omega n_\omega X_0^2/c$ is the diffraction length, and X_0 is the initial beam width. Then Eqs. (2.7-2.8) take the form

$$\begin{aligned} &iE_{1z} + \frac{1}{2}E_{1xx} + E_1^* E_2 = 0, \\ &iE_{2z} + \frac{1}{4}E_{2xx} + \frac{1}{2}E_1^2 - qE_2 = 0, \end{aligned} \tag{2.10}$$

where $q = \Delta k L_D$ is the dimensionless mismatch. The total energy

$$\mathcal{E} = \int_{-\infty}^{\infty} (|E_1|^2 + 2|E_2|^2) dx, \tag{2.11}$$

and the Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{\infty} \left[\frac{1}{2}|E_{1x}|^2 + \frac{1}{4}|E_{2x}|^2 - \frac{1}{2}(E_1^{*2}E_2 + E_1^2 E_2^*) + q|E_2|^2 \right] dx. \tag{2.12}$$

are two conserved quantities. The equations of motion can be written in the form

$$iE_{1z} = \frac{\delta H}{\delta E_1^*}, \quad iE_{2z} = \frac{\delta H}{\delta E_2^*}. \tag{2.13}$$

Although the system is not integrable, it has different types of solitonic solutions. A bright soliton solution known for the phase mismatch $q = -3$ has the form

$$E_1(z,x) = 3\text{sech}^2(x)e^{2iz}, \quad E_2(z,x) = 3\text{sech}^2(x)e^{4iz}. \tag{2.14}$$

The solution will be used in section 6 in the numerical simulations of the stochastic version of the $\chi^{(2)}$ system. This analytical solution holds only for *negative* mismatch $q < 0$, whereas the whole family of one-parametric solitons can be found by numerical means. For details we refer to the reviews by Buryak, Di Trapani, Skryabin, and Trillo [2002], Etrich, Lederer, Malomed, Peschel, and Peschel [2000]. We can obtain moving solutions

$$\begin{aligned} E_1 &= 3\text{sech}^2(x - vz)e^{2iz + ivx - i\frac{v^2 z}{2}}, \\ E_2 &= 3\text{sech}^2(x - vz)e^{4iz + 2ivx - iv^2 z}. \end{aligned} \tag{2.15}$$

This transformation to moving solutions is valid only for this system, describing spatial $\chi^{(2)}$ solitons.

When the mismatch is *positive* and $q \gg 1$, we obtain from the second equation in the system (2.10) that $E_2 \approx E_1^2/(2q)$. Substituting this result into the first equation we get the NLSE for E_1:

$$iE_{1z} + \frac{1}{2}E_{1xx} + \frac{1}{2q}|E_1|^2 E_1 = 0. \tag{2.16}$$

Thus the solutions in this limit are

$$E_1 = 2\sqrt{2q}\nu \operatorname{sech}(2\nu x)e^{i\phi_s}, \quad E_2 = 4\nu^2 \operatorname{sech}^2(2\nu x)e^{2i\phi_s}. \tag{2.17}$$

It is seen that the amplitude of the SH soliton is smaller than the amplitude of the FW soliton.

§ 3. Solitons in random single-mode fibers

Let us consider an optical pulse propagating in a single-mode optical fiber with randomly varying nonlinear and dispersion parameters. The dimensionless envelope of the electric field obeys the modified NLSE

$$iu_z + \frac{d(z)}{2}u_{tt} + c(z)|u|^2 u = 0, \tag{3.1}$$

where $d(z) = 1 + m(z)$ and $c(z) = 1 + n(z)$ model the fluctuations of the dispersion and nonlinearity, and m and n are assumed to be zero-mean random processes. We address the propagation of a soliton or a train of solitons governed by (3.1).

3.1. DIFFERENT APPROACHES

In the present time a universal approach to describe pulse propagation in random medium is absent. For randomly perturbed nonintegrable systems we can use the variational approach. For systems close to integrable a perturbation technique based on the IST can be applied. The expansion of the solution over a basis of eigenfunctions of the unperturbed system is also possible. We review these techniques in this section.

3.1.1. The variational approach

Let us consider the propagation of a single soliton in a random optical medium. The Lagrangian density for (3.1) is

$$\mathcal{L} = \frac{i}{2}(u^* u_z - u u_z^*) - \frac{d(z)}{2}|u_t|^2 + \frac{c(z)}{2}|u|^4 \tag{3.2}$$

The key to what follows is the choice of the trial function for u to substitute into the Lagrangian [Anderson [1983]]. As pointed out in different papers, the chirped soliton trial function turns out to give good predictions:

$$u(z,t) = A(z)\mathrm{sech}\left[\frac{t}{a(z)}\right]\exp[ib(z)t^2], \tag{3.3}$$

where $A(z)$, $a(z)$, $b(z)$ describe the complex amplitude, the width, and the chirp of the soliton, respectively. When substituting this ansatz into the Lagrangian density, the averaged is

$$\begin{aligned}\int_{-\infty}^{\infty} dt\mathcal{L}dt &= -\frac{\pi^2}{6}|A|^2a^3b_z - 2|A|^2a(\mathrm{Arg}(A))_z \\ &\quad -d(z)\left[\frac{|A|^2}{3a} + \frac{\pi^2}{3}|A|^2a^3b^2\right] + \frac{2}{3}c(z)|A|^4a\end{aligned}$$

When equating to zero the variations of $\int_0^L dz \int_{-\infty}^{\infty} dt\mathcal{L}$, the equations for the soliton parameters A, a, b are obtained

$$b(z) = \frac{a_z}{2ad(z)}, \tag{3.4}$$

$$(a|A|^2)_z = 0, \quad N^2 = a|A|^2, \tag{3.5}$$

$$a_{zz} = \frac{4d^2(z)}{\pi^2a^3} - \frac{4N^2d(z)c(z)}{\pi^2a^2} + \frac{a_zd_z(z)}{d(z)}, \tag{3.6}$$

$$(\arg(A))_z = -\frac{d(z)}{3a^2} + \frac{5N^2c(z)}{6a}. \tag{3.7}$$

As can be seen from (3.6), the evolution of the soliton width under random perturbations is described by the motion equation of a unit-mass particle in a nonstationary anharmonic potential. We shall see that this analogy opens ways to investigate the soliton dynamics.

It is well-known that the validity of the variational approach is limited. Its main drawback is the same as its main advantage: inserting a trial function into the Lagrangian dramatically reduces the number of degrees of freedom of the system. This simplifies the problem, but this may lead to wrong predictions as the trajectories of the system are forced to follow the constraints imposed by the choice of the trial function. In the case of the random NLSE the variational approach can be improved by taking a trial function which consists of a soliton-like pulse with variable parameters plus a linear dispersive term taking into account the generation of dispersive radiation upon pulse evolution [Kath and Smith [1995]]. In any case the predictions of the variational approach should be confirmed by full numerical simulations of the random NLSE.

3.1.2. Inverse Scattering Transform

We now study the propagation of a single soliton in a random medium by applying a perturbation theory of the IST (see Appendix 1). The propagating soliton emits radiation due to scattering with the random medium so the total field can be decomposed as the sum of a localized soliton part u_S (associated to the discrete eigenvalue $\lambda_S = \mu + i\nu$) and delocalized radiation (associated to the continuous spectrum). u_S is of the form

$$u_S(z,t) = 2\nu \mathrm{sech}[2\nu(t - t_S(z))] \exp[i\phi_S(z)],$$

where the amplitude 2ν is slowly varying. The total energy $\mathcal{E}$ is preserved by the perturbed equation (3.1) and it can be written as the sum of the soliton energy and the radiative energy:

$$\mathcal{E} := \int_{-\infty}^{\infty} |u|^2 dt = 4\nu - \frac{1}{\pi} \int_{-\infty}^{\infty} \ln[1 - |b(z,\lambda)|^2] d\lambda, \tag{3.8}$$

where λ is the spectral parameter and b is the Jost coefficient introduced in the IST theory (see Appendix 1). The space evolution of the Jost coefficient is $b_z = -2i\lambda^2 b$ for the unperturbed NLSE, which shows that the radiative energy, and consequently the soliton energy, are preserved. In the case of the random NLSE (3.1) the evolution of the Jost coefficient can be found from the perturbation theory based on the IST [see Eq. (10.10)]. Qualitatively, the energy of the radiation increases which involves a decay of the soliton energy. Quantitatively, in the first approximation the right-hand-side of Eq. (10.10) can be computed by substituting the corresponding expressions for the soliton part:

$$b_z = -2i\lambda^2 b - \exp[i\phi_S(z) - 2i\lambda t_S(z)] \frac{2\pi\nu[(\lambda-\mu)^2 + \nu^2]}{\cosh(\frac{\pi}{2}\frac{\lambda-\mu}{\nu})} [m(z) - n(z)],$$

From (3.8) the mean emittted spectral power is $P(\lambda) = 2\mathrm{Re}\langle b^* b_z\rangle/\pi$ and we find the evolution equation

$$\nu_z = -\frac{1}{4}\int_{-\infty}^{\infty} P(\lambda)d\lambda \tag{3.9}$$

In the white noise case $\langle (m-n)(z)(m-n)(z')\rangle = D\delta(z-z')$, the mean emitted spectral power is

$$P(\lambda) = \frac{D\pi[(\lambda-\mu)^2+\nu^2]^2}{\cosh^2(\frac{\pi}{2}\frac{\lambda-\mu}{\nu})} \tag{3.10}$$

Substituting into (3.9) and integrating with respect to λ gives a power law for the soliton decay. This decay will be analyzed in detail in the forthcoming sections.

3.1.3. Kaup perturbation technique

Motivated by the weakness of the disorder we can write the solution as $u = u_S + u_L$ where u_S is the soliton (or multi-soliton) part and u_L the radiative part. We can expand the radiative part u_L over the eigenfunctions of the operator describing the evolution of a linear perturbation about the single-soliton profile of the unperturbed NLSE. This is possible because the complete system of eigenfunctions was found by Kaup [1990]. In the presence of random perturbations the coefficients of the decomposition of u_L are slowly varying, and using the orthogonality properties of the eigenfunctions it is possible to write the evolution equations of these coefficients by a projection technique. Chertkov, Chung, Dyachenko, Gabitov, Kolokolov, and Lebedev [2003] apply this method for the study of the impact of random dispersion, and moreover extend the technique for addressing the evolution of a multi-soliton solution.

3.2. SINGLE-SOLITON DRIVEN BY RANDOM PERTURBATIONS

3.2.1. Random dispersion

Random fluctuations of dispersion may occur as shown by Kodama, Maruta, and Hasegawa [1994], Wabnitz, Kodama, and Aceves [1995]. A general theory is presented by Lin and Agrawal [2002] to describe the effects of dispersion fluctuations on optical pulses propagating inside single-mode fibers modeled as a linear dispersive medium. Random dispersion has been shown by Abdullaev, Darmanyan, Kobyakov, and Lederer [1996] to involve dramatic effects on the modulational instability of stationary waves because of a stochastic parametric resonance phenomenon. In this section we shall analyze the stability of the soliton with respect to random fluctuations of the dispersion.

Approximate scale characteristics of the dispersion noise present in real fibers can be extracted from experimental results [Mollenauer, Mamyshev, and Neubelt [1996], Nakajima, Ohashi, and Tateda [1997], Mollenauer and Gripp [1998]]. These results show that the typical distance of noticeable change in the dispersion value is shorter than 100 m (The resolution of the experimental methods vary from 100 m to 1 km, whereas one expects that the typical scale of the variations is actually 10-100 m, which is the size of the production facility). For constant-dispersion fibers, the amplifier spacing is of the order of 50 km, and for dispersion-managed fibers, the period of a typical dispersion map is also of the order of 50 km. These scales are much longer than that of the dispersion variation. Therefore, according to the approximation-diffusion theory [Papanicolaou and Kohler [1974]], the natural m at the larger scales can be treated as a homogeneous Gaussian random process with zero mean and delta-correlated correlation function $\langle m(z)m(z')\rangle = D\delta(z-z')$, where $D = \int \langle m(z)m(z+h)\rangle\, dh$. Measurements show that fluctuations of the dispersion coefficient in dispersion shifted fibers are of the order of its averaged value $\delta\beta_2 \sim 0.5$ ps^2/km. Therefore, for the pulse duration ~ 7 ps with the nonlinear length $z_{nl} = 1/\kappa P_0 \sim 250$ km, the noise intensity in dimensionless variables is estimated as $D \sim l_d \Delta d^2 \sim 10^{-2} - 10^{-3}$. The scale when the solitons interaction feel the noise effect is $\sim 1/\sqrt{D} \sim 10 z_{nl} \sim 2500$ km.

A soliton, propagating through a fiber, emits radiation due to disorder and, consequently, loses its energy. However, in the case of weak disorder (weakness of disorder is actually required for successful fiber performance) the destruction of the soliton is slow, thus making an adiabatic description of this problem possible. The adiabaticity implies separation of dynamical degrees of freedom into slow and fast modes.

Slow modes describe evolution of the soliton itself while the fast modes correspond to the radiation. The soliton keeps its shape so that, at each instant, the soliton is close to a stationary solution of the noiseless NLSE with the soliton parameters (position, width, phase, and phase velocity) evolving slowly. Waves shed by a soliton are moving away from it. One finds that at any z, however large, the radiation in an immediate vicinity of the soliton is much less intense than the soliton itself, i.e., the soliton

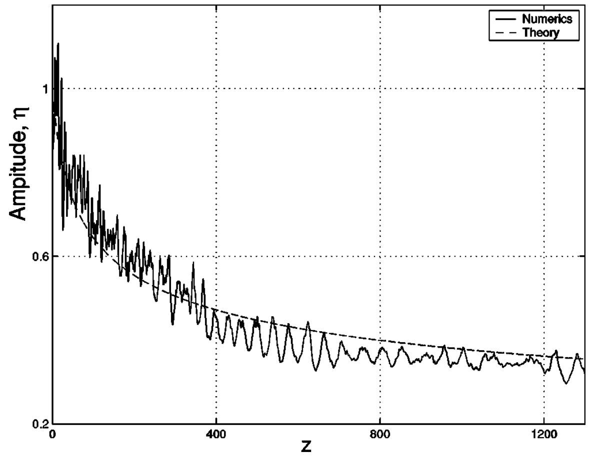

Figure 1: Dependence of the soliton amplitude $\eta = 2\nu$, measured in units of its initial value, on the dimensionless coordinate along the fiber z, is shown for disorder strength $D = 0.18$. Solid and dashed curves represent theory, resulted in Eq. (3.11), and numerics for a representative realization of the disorder, respectively [reprinted from Chertkov, Chung, Dyachenko, Gabitov, Kolokolov, and Lebedev (2003)].

is always distinguishable from the radiation. Since the soliton energy is converted into radiation, its amplitude 2ν decays with z. The degradation law is deterministic in spite of the original stochastic setting. This is due to the fact that the variation of ν is determined by an integral over z, which is a self-averaged quantity at large z. The soliton degradation law is

$$\nu(z) = \nu_0(1 + 128\nu_0^4 Dz/15)^{-1/4} \tag{3.11}$$

This formula was derived with the perturbed IST approach by Abdullaev, Caputo, and Flytzanis [1994], by the variational approach by Abdullaev, Bronski, and Papanicolaou [2000] and by the Kaup perturbation technique by Chertkov, Chung, Dyachenko, Gabitov, Kolokolov, and Lebedev [2003]. It was confirmed by numerical simulations carried out by Chertkov, Gabitov, Lushnikov, Moeser, and Toroczkai [2002] and by Abdullaev, Navotny, and Baizakov [2004]. Equation (3.11) shows that the soliton starts to degrade essentially at $z \sim 1/(D\nu_0^4)$(see Figure 1).

3.2.2. Random nonlinearity

Random fluctuations of the nonlinear coefficient can have several origins. The nonlinear index of refraction of the fiber may be varying, but this is not generally the main factor. Imperfections of the geometry of the fiber core may cause variations of the transverse profile of the mode of the fiber, which in turn induce fluctuations of the nonlinear coefficient of the NLSE because the effective core area plays a role in this coefficient [Bauer and Melnikov [1995]]. Finally, losses in the fiber are usually compensated by a series of amplifiers. As a result a z-varying damping term appears in the right-hand side of the NLSE, which is equivalent after a change of variable to a z-varying nonlinear coefficient [Gordon [1992]]. The purpose of this section is the study of the propagation of optical solitons in fibers with a randomly modulated nonlinear parameter.

In the framework of the variational approach this problem can be reduced to the dynamics of a unit-mass particle in a randomly perturbed central potential. In this case the governing equation for the soliton width coincides with the Kepler problem in a randomly perturbed potential. Using this analogy we obtain a system of equations in action-angle variables which describes the behavior of a chirped pulse in a randomly inhomogeneous fiber. In Abdullaev, Abdumalikov, and Baizakov [1997], Abdullaev, Bronski, and Papanicolaou [2000] the behavior of soliton-like pulses in long distance optical lines is studied on the basis of asymptotic properties of the soliton width evolution. Analytic estimations for the decay length of solitons propagating in randomly inhomogeneous fibers are derived in the case where n is assumed to be a zero mean Gaussian random function with correlation function $\langle n(z)n(z')\rangle = D\delta(z-z')$. The obtained formulas were confirmed first by numerical simulations, and also by the perturbed IST. Indeed, there exists a point at which the results obtained by the variational approach and the IST coincide. In the effective particle model this point corresponds to the minimum of the potential well, i.e. the single-soliton initial state. Then the estimation of the decay length can be deduced by the IST and it is found that the soliton amplitude decay has exactly the same rate (3.11) as the one triggered by random dispersion.

3.2.3. A random Kepler problem

The variational approach shows that the soliton width dynamics under random perturbations can be represented as a particle dynamics in a random anharmonic potential. This is useful to study the oscillations of the soliton width, but also the soliton disintegration distance that can be expressed in terms of the mean exit time of the equivalent particle from the Kepler potential [Abdullaev, Bronski, and Papanicolaou [2000]].

The resulting equation for the soliton width a is the randomly perturbed Kepler problem

$$a_{zz} = -U'(a) - \gamma(z)V'(a), \tag{3.12}$$

where $U(a)$ is the Kepler potential

$$U(a) = \frac{2}{\pi^2 a^2} - \frac{4N^2}{\pi^2 a}. \tag{3.13}$$

$\gamma(z)$ is assumed to be a Gaussian white noise with noise level D, and V is the perturbation potential which is equal to a^2 for fluctuating quadratic potential and $4N^2/(\pi^2 a)$ for a fluctuating nonlinearity. The Hamiltonian for the unperturbed Kepler problem is $H_0 = \frac{1}{2}(a_z)^2 + U(a)$. The Hamiltonian for the perturbed Kepler problem (3.12) is

$$H \equiv H_0 + \gamma(z)V \equiv \frac{(a_z)^2}{2} + U(a) + \gamma(z)V. \tag{3.14}$$

In order to continue the analysis of the random Kepler problem, we must first transform the unperturbed Kepler problem to action-angle variables. We summarize here the relevant facts that we need, with details given in the texts by Landau and Lifshitz [1974].

The minimum of the potential $U(a)$ occurs at $a_c = 1/N^2$ and it is equal to $U_0 = -2N^4/\pi^2$. The frequency ω_0 of small oscillations about the minimum is given by $\omega_0 = 2N^4/\pi$. This is the frequency of the width oscillations of an unperturbed soliton as it propagates down a homogeneous fiber.

For large oscillations the qualitative behavior of the orbits of the Kepler problem is determined by the energy

$$E_0 = 2a_0^2 b_0^2 + \frac{2}{\pi^2 a_0^2} - \frac{4N^2}{\pi^2 a_0},$$

where b_0 is the initial chirp and a_0 is the initial width. When $E_0 < 0$ (i.e $1 + \pi^2 a_0^4 b_0^2 < 2N^2 a_0$, corresponding to a sufficiently weak initial chirp) the orbits are closed, corresponding to oscillatory motion. In the the case $E_0 > 0$, corresponding to sufficiently large initial chirp, the orbits are unbounded, and the asymptotic motion is qualitatively like the motion of a free particle. In this regime the soliton does not persist, but instead spreads out and is lost ($a \to \infty$). The interesting questions in soliton propagation arise when the unperturbed motion is oscillatory, and so we consider the regime $E_0 < 0$.

We now make a change to action-angle variables. For the Kepler problem, where the phase space is two dimensional, there is one action variable J and the conjugate angle variable Θ. The action variable is given by

$$J = \frac{1}{2\pi}\oint a_z da = \frac{2\sqrt{2}N^2}{\pi^2\sqrt{-E}} - \frac{2}{\pi}.$$

Solving for the total energy E in terms of the action J gives the unperturbed Hamiltonian

$$H_0 = E = -\frac{8N^4}{\pi^2(\pi J + 2)^2}, \tag{3.15}$$

which is a function of J only. The change of variables to action-angle variables is canonical, so that Hamilton's equations retain their form

$$\begin{aligned} \frac{dJ}{dz} &= -\frac{\partial H_0(J)}{\partial \Theta} = 0, \\ \frac{d\Theta}{dz} &= \frac{\partial H_0(J)}{\partial J} = \omega(J), \end{aligned}$$

where

$$\omega(J) = \frac{dH_0}{dJ} = \frac{16N^4}{\pi(\pi J + 2)^3} = \frac{\pi^2}{\sqrt{2}N^2}(-E)^{3/2}. \tag{3.16}$$

The following implicit representation for the orbits is useful in the perturbation calculations. The position a and the "time" z can be expressed parametrically in terms of a variable ξ (called the Kepler parameter) by

$$a = b(1 - e_0\cos\xi), \qquad \Theta = \omega(J)z = \xi - e_0\sin\xi, \tag{3.17}$$

where the eccentricity $e_0 > 0$ and the semi-major axis b are given in terms of the constant E as follows

$$e_0^2 = 1 - \frac{\pi^2|E|}{2N^4} = 1 - \frac{4}{(\pi J + 2)^2}, \qquad b = \frac{2N^2}{\pi^2|E|} = \frac{(\pi J + 2)^2}{4N^2}$$

This parametric representation determines a implicitly as a function of z. Note that as Θ varies over $(0, 2\pi)$, ξ also varies over $(0, 2\pi)$. From equation (3.17) it follows that the soliton width oscillates between the minimum value $a_{min} = b(1-e_0)$ and the maximum value $a_{max} = b(1+e_0)$. The value of the action J varies from 0, for oscillations near the bottom of potential well, to ∞ for oscillations near the separatrix $E = 0$. The frequency $\omega(J)$ varies from $2N^4/\pi$ for oscillations near the bottom of the potential well to 0 for oscillations near the separatrix.

It is also useful to relate the initial action J_0 to the initial chirp b_0 and initial width a_0:

$$\pi J_0 + 2 = \frac{2N^2}{\sqrt{\frac{2N^2}{a_0} - \frac{1}{a_0^2} - \pi^2 b_0^2 a_0^2}}. \tag{3.18}$$

Let us address the case where the coefficients of the NLSE vary randomly. In this case the soliton parameters evolve according to a set of random ODE's, and we can derive a Fokker-Planck equation for the evolution of the probability distribution for the soliton parameters. The fluctuating quadratic potential perturbation and the fluctuating nonlinear term perturbation can be analyzed with the white noise model directly, using the Stratonovich interpretation. In the case of fluctuating dispersion the treatment is slightly more subtle (see for detail Abdullaev, Bronski, and Papanicolaou [2000]). In action-angle variables, the perturbed Hamiltonian has the form

$$H = H_0(J) + \gamma(z)V(J,\Theta), \tag{3.19}$$

where $V(J,\Theta)$ is obtained from (3.14). In general V, while simple in the original variables, is quite complicated in the action-angle variables. However for weak disorder we do not need the explicit form of V, but only averages over Θ. These averages are simple to calculate using the implicit representation given in (3.17).

In the presence of weak disorder the action-angle variables for the unperturbed Kepler problem are a convenient framework for the analysis of the perturbed problem because the change in the action is proportional to the small parameter. Since the change to action-angle variables is canonical the Hamiltonian structure is preserved, and the perturbed equations become

$$\begin{aligned} \frac{dJ}{dz} &= -\gamma(z)\frac{\partial V}{\partial \Theta}, \\ \frac{d\Theta}{dz} &= \omega(J) + \gamma(z)\frac{\partial V}{\partial J}. \end{aligned} \tag{3.20}$$

The Fokker-Planck equation for the evolution of the probability distribution function $P(z,J)$ of $J(z)$, after averaging over Θ, is given by

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial J}\left(A(J)\frac{\partial P}{\partial J}\right), \qquad A(J) = \frac{1}{2\pi}\int_0^{2\pi} (V_\Theta(J,\Theta))^2 d\Theta, \tag{3.21}$$

with $P(0,J) = \delta(J - J_0)$. We note that this averaging calculation breaks down very near the separatrix, since $\omega(J)$ vanishes there, and the perturbations can no longer be considered small.

The mean exit time (distance) to reach ∞ starting from J_0 is

$$\tau = \int_{J_0}^{\infty} \frac{J dJ}{A(J)},$$

For the fluctuating nonlinearity, the perturbation of the Hamiltonian is $V = (4N^2)/[\pi^2 a(J,\theta)]$ and, starting from a single-state soliton state $J = 0$, the mean disintegration distance is proportional to $a_0^4/D \sim 1/(D\nu_0^4)$. The same result holds true for the fluctuating dispersion, and different quantitative estimates of the soliton disintegration distance can be found in [Abdullaev, Bronski, and Papanicolaou [2000]].

3.3. INTERACTION OF SOLITONS IN RANDOM MEDIUM

3.3.1. Interaction induced by radiation shedding

Using the Kaup perturbation technique Chertkov, Chung, Dyachenko, Gabitov, Kolokolov, and Lebedev [2003] study the interaction of solitons when successive solitons are far from each other so that the interaction through the tails can be considered as negligible. Radiation emitted by a soliton acts as a multiplicative noise on another soliton. The interaction is extremely long range, due to the one-dimensional nature of the system and also because of the reflectionless feature of the radiation. At any z all solitons separated from a given one by $|t| \leq z$ act on this soliton with a force, which is zero on average. Fluctuations of the force result in a Gaussian jitter of the soliton position. We find that in the two soliton case ,i.e. for the pattern consisting of two solitons only, so that no other solitons are present anywhere in the $|t| \leq z$ vicinity of the pair fluctuations in their relative position δy is a zero-mean Gaussian random variable with the variance

$$\langle \delta y^2 \rangle \simeq 1.5\nu_0^8[1 + \cos(2\Delta\phi)]D^2 z^3, \tag{3.22}$$

where $\Delta\phi$ is the intersoliton phase mismatch (see Figure 2). In the general multi-soliton case intersoliton interaction caused by radiation leads to an essential shift of the solitons at the distance $z \sim N^{-1/3}D^{-2/3}$, where N is the number of solitons in the channel. The interaction causes the soliton to jitter randomly. The soliton displacement δy is a zero-mean Gaussian random variable, with the variance $\langle \delta y^2 \rangle \sim D^2 z^3 N$. If N does not grow

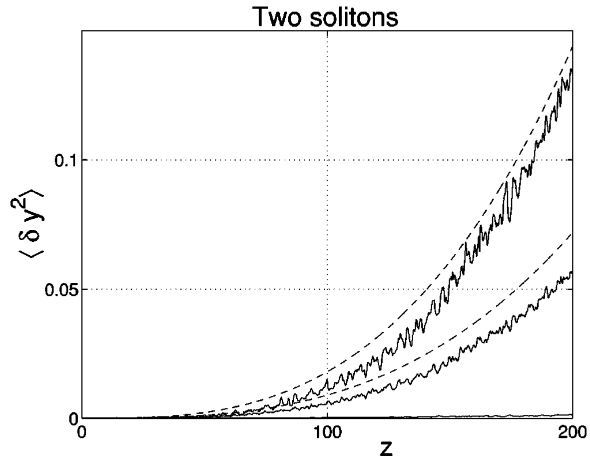

Figure 2: Dependence of the mean square value of the intersoliton separation $\langle \delta y^2 \rangle$ measured in units of the soliton width square, on the dimensionless position along the fiber z is shown. The disorder strength is $D = 0.00125$. Three different sets of curves for the three different values of the intersoliton phase mismatch $\Delta\phi = 0,\ \pi/4,\ \pi/2$ are presented. Dashed curves represent the analytical result given by Eq. (3.22). Solid curves represent results of numerics. Each curve is the result of averaging over 15 different realizations of disorder. [reprinted from Chertkov, Chung, Dyachenko, Gabitov, Kolokolov, and Lebedev (2003)].

with z (e.g., there are only finite number of solitons propagating in the channel) the z-dependence of the jitter is the same as the one given by the Elgin-Gordon-Haus jitter [Elgin [1985], Gordon and Haus [1986], Falkovich, Kolokolov, Lebedev, and Turitsyn [2001]] developed under the action of random additive noise (short-correlated both in t and z noise of amplifiers in the fiber system). However, if the flow of information is continuous, i.e., if the front of radiation shed by the given soliton sweeps more and more solitons with increasing z, $N \sim z$, the efficiency of the interaction grows with z in a faster way, $\delta y \sim z^2$, thus overwhelming the Elgin-Gordon-Haus jitter in long-haul transmission.

3.3.2. Interaction induced by tail overlap

Using the perturbation theory for soliton interaction Karpman and Solov'ev [1981] study the interaction of solitons when this interaction is induced by the overlap of soliton tails. The soliton separation should be sufficiently large so that the interaction is weak and can be treated by expansion methods, but it should not be so large otherwise the interaction with the induced radiation dominates the interaction through the tails. The analysis performed by Abdullaev, Hensen, Bischoff, Sorensen, and Smeltink [1998b] for a two-soliton configuration with a random nonlinearity exhibits complicated phenomena. When the solitons have the same phase and amplitude 2ν, the same result as in the unperturbed NLSE is found, that is to say solitons experience small oscillations with the period $\sim \exp(-2\nu\Delta t_S)$, where Δt_S is the initial soliton separation. If the solitons have not the same amplitude initially, the mean square of the amplitude difference $\Delta\nu$ grows diffusively during propagation $\Delta\nu - \Delta\nu_0 \sim \Delta\nu_0 \exp(-(\nu_1+\nu_2)\Delta t_S)\sqrt{Dz}$. The asymmetry in the two soliton shapes in media with random fluctuations has also been observed numerically.

3.4. BEYOND THE WHITE NOISE MODEL

3.4.1. Pinning schemes

Periodic pinning of disorder was suggested recently to compensate for the random dispersion by Chertkov, Gabitov, Lushnikov, Moeser, and Toroczkai [2002]. This method comes in two modifications of distributed and point pinning. Distributed pinning applies to new fiber lines (not yet installed in the ground). The method requires controlling the integral dispersion (its fluctuating part) of a fiber piece prior to its connection to the line. A profile of the integral of the fluctuating part of the dispersion coefficient should be found, first, and then the suggestion is to cut this fiber at a zero point for the fluctuating part of the integral dispersion (closest to the end of the fiber piece). The other type of pinning, point pinning, was suggested for implementation in already installed fiber optics lines. At the points of access to the fiber optics line (e.g. at amplifier stations placed periodically along the fiber) it is suggested to measure the integral of the fluctuating part of dispersion, and then to compensate it to zero by inserting a small peace of a fiber with a very well controlled integral dispersion. If the pinning period, l, is short (i.e., if it is the shortest scale in the problem), the noise term m can be replaced by $\tilde{m}$ described by $\langle \tilde{m}(z)\tilde{m}(z')\rangle = -(Dl^2/12)\delta''(z-z')$ ($\tilde{m}$ actually corresponds to the "distributed pinning" case, while in the case of the "point pinning" the replacement should be $\tilde{m} \mapsto 2\tilde{m}$). Recalculation of all the major results for the pinned noise is straightforward. First of all, one gets the decay rate of a single soliton:

$$\nu(z) = \nu_0(1 + 2^{14}\nu_0^8 Dl^2 z/315)^{-1/8} \tag{3.23}$$

This expression, contrasted against Eq. (3.11), shows an essential reduction in the soliton decay since the critical propagation distance becomes $z \sim 1/[(\nu_0^4 D)(\nu_0^2 l)^2] \gg 1/(\nu_0^4 D)$. Second, in the multi-soliton case, the radiation mediated jitter can be estimated by $\delta y \sim \sqrt{Nz}$, which becomes less important asymptotically than the Elgin-Gordon-Hauss jitter.

3.4.2. Colored noise

It may happen that the correlation length of the fluctuations of the medium parameters is not the smallest length scale present in the problem. A more careful analysis taking into account the noise spectrum is then necessary. Let us consider a random dispersion or nonlinearity with autocorrelation function

$$\langle m(z)m(z')\rangle = \frac{D}{2l_c}\exp\left[-\frac{|z-z'|}{l_c}\right]$$

The mean emitted spectral power is then [Abdullaev, Hensen, Bischoff, Sorensen, and Smeltink [1998b]]

$$P(\lambda) = \frac{D\pi[(\lambda-\mu)^2+\nu^2]^2}{\{1+16l_c^2[(\lambda-\mu)^2+\nu^2]^2\}\cosh^2(\frac{\pi}{2}\frac{\lambda-\mu}{\nu})} \tag{3.24}$$

When $4\nu^2 l_c \ll 1$ we recover the white noise model. When $4\nu^2 l_c \gg 1$ we get by integrating (3.9) that the decay rate of the soliton amplitude is

$$\nu(z) = \nu_0 \exp(-\frac{Dz}{16 l_c^2})$$

which shows that increasing the correlation length leads to less damping. The difference between radiative emission in media with white and colored noise fluctuations can be explained as follows. In a periodically perturbed fiber there exists a resonance between the characteristic frequencies of the soliton and the modulation frequency of the medium, leading to the emission of waves generated by the soliton [Hasegawa and Kodama [1991]]. The associated resonance condition is $2n\pi/L = 4\nu^2$, $n \in \mathbb{N}$, where L is the modulation period. When the soliton amplitude decays, the soliton parameters detune from resonance, leading to the stabilization of soliton. In the case of a white-noise perturbation, all frequencies are present in the spectrum. Therefore the soliton cannot detune from resonance and it is continuously damped. In the case of a colored-noise perturbation, the spectrum has a cut-off frequency corresponding to the correlation length l_c. If the soliton phase velocity $2\nu^2$ is smaller than l_c, the soliton is detuned from resonance.

3.4.3. Dissipation and filters

The influence of dissipation and amplification on soliton propagation under random perturbations can be analyzed. Malomed [1996] considers the effects of weak amplification and filtering. The unchirped autosoliton under noise has been studied by Hasegawa and Kodama [1995]. The governing equation in presence of weak amplification, filtering, and noise is

$$iu_z + \frac{d(z)}{2}u_{tt} + c(z)|u|^2u = i\Gamma_0(\frac{1}{3}u + u_{tt}). \tag{3.25}$$

When d is constant the chirped autosoliton solution exists. The evolution of the width of the chirped soliton is described by the random Kepler problem with a damping term proportional to a_z, which in turn implies the addition of a linear damping term in the right-hand side of Eq. (3.20) equal to $-\Gamma J$, $\Gamma \simeq 4.3\Gamma_0$ [Abdullaev, Bronski, and Papanicolaou [2000]]. As a result the following expression for the mean exit time (distance) is

$$\tau = \int_{J_0}^{\infty} \frac{1}{A(J)} \int_0^J e^{\int_y^J \frac{\Gamma z}{A(z)} dz} dy dJ$$

This expression shows that the exit time is increasing with Γ, which means that the interplay of noise and dissipation can lead to the stabilization of soliton. This is also a confirmation of the stabilizing role of filters.

The existence of autosoliton in random media can be shown by the IST approach. In a fiber with fluctuating dispersion the soliton radiative decay can be compensated by the linear/nonlinear amplifiers. The action of the distributed gain can be described by the terms $i\delta_l u + i\delta_{nl}|u|^2u$ in the right-hand side of Eq. (3.1). A distinctive property of created solitons is that they recover the stable waveform when deformed. Numerical simulations show that the linear amplification gives rise to more pronounced oscillations around the fixed point during the transient period compared to nonlinear amplification. At longer propagation distances the oscillations are damped out and a stable dissipative soliton is formed. The amplitude of the autosoliton at the fixed point ν_{st} can be found from the balance equation for the soliton energy

$$\frac{d\mathcal{E}_S}{dz} = -\int_{-\infty}^{\infty} P(\lambda)d\lambda + 8\delta_l\nu + \frac{64}{3}\delta_{nl}\nu^3, \tag{3.26}$$

Taking the right-hand side equal to zero we get

$$\nu_{1,2st}^2 = \frac{5\delta_{nl}}{4D}\left(1 \pm \sqrt{1 + \frac{3\delta_l D}{5\delta_{nl}^2}}\right). \tag{3.27}$$

Two types of fixed points exist. The first type ν_{1st} is defined by the competition between the dissipation (induced by the randomness of the fiber dispersion) and the linear/nonlinear amplifications. For $\delta_l = 0$, $\delta_{nl} > 0$ (the nonlinear amplification dominates) we find $\nu_{1st}^2 = (5\delta_{nl})/(2D)$, and similarly for $\delta_{nl} = 0$, $\delta_l > 0$ (linear amplification dominates) $\nu_{1st}^2 = \sqrt{(15\delta_l)/(8D)}$. The second type of fixed points is defined by the competition between the linear dissipation $\delta_l < 0$ and nonlinear amplification $\delta_{nl} > 0$. For small D we can expand $\nu_{2st}^2 \approx (3|\delta_l|)/(8\delta_{nl}) + D(9\delta_l^2)/(160\delta_{nl}^3)$, which shows that the fluctuations increase the value of autosoliton amplitude in this case.

Simulations of the stochastic NLSE with nonconservative terms have been performed by Abdullaev, Navotny, and Baizakov [2004] and confirm the theoretical predictions. However, it should be pointed out that for the linear amplification case the stable pulse propagation is limited by the growth of the zero mode under amplification. When the initial pulse amplitude is close to this solution the instability growth becomes noticeable at distances $z \sim 1/\delta_{nl}$ (see for instance Akhmediev and Ankiewicz [2000]).

3.5. FEMTOSECOND SOLITONS IN RANDOM FIBERS

The previous analysis based on the randomly perturbed NLSE shows that the influence of the random dispersion is very strong for very short pulses. For solitons with durations < 100 fs it is necessary to consider the modified NLSE, taking into account the finite time of the response of the Kerr nonlinearity, the Raman effect, and third and higher-order dispersion. Below we consider a simplified model taking into account only the dispersion, nonlinearity, and self-steepening. The governing equation is

$$iu_z + \frac{1}{2}u_{tt} + |u|^2u + i\alpha(|u|^2u)_t = \epsilon(z,t)R(u), \tag{3.28}$$

where $\epsilon(z,t)$ is a random process. When $\epsilon = 0$ this equation is integrable for any α by means of the IST and the spectral information can be processed to study the soliton dynamics. It is important to notice that the additional term in α can be large, which allows us to work beyond the perturbation theory. A nontrivial example of application is the soliton jitter driven by distributed amplifier noise [Doktorov and Kuten [2001]]. In distinction from the Elgin-Gordon-Haus (EGH) effect which is based on the standard NLSE for picosecond pulses, the jitter can be suppressed by a proper choice of the pulse and fiber parameters. The distributed amplifier noise is represented by the additive white noise source in the right-hand side of Eq. (3.28) $\epsilon R(u) = s(z,t)$, with the autocorrelation function $\langle s^*(z,t)s(z',t')\rangle = D\delta(z-z')\delta(t-t')$. The soliton solution of Eq. (3.28) with $s = 0$ is

$$u(z,t) = \frac{i}{w}\frac{ke^{-x} + k^*e^{x}}{(ke^{x} + k^*e^{-x})^2}e^{i\psi}, \tag{3.29}$$

where $x = q(z) - t/w$, $\psi = vt + \phi(z)$, $q(z) = (t_0 + vz)/w$, and $\phi(z) = \phi_0 - \frac{1}{2}(v^2 - \frac{1}{w^2})z$. Here t_0 and ϕ_0 determines the initial position and phase of the soliton, v and w are the soliton velocity and width

$$v = \frac{1}{\alpha} - \frac{2(k^2 + k^{*2})}{\alpha}, \qquad w = -\frac{i\alpha}{2(k^2 - k^{*2})}, \qquad k = \xi - i\eta. \tag{3.30}$$

The behaviors of integrals of motion are quite different from the standard NLSE case. For example the soliton power is

$$\mathcal{E} = \int |u|^2 dt = \frac{4\gamma}{\alpha}, \qquad \gamma = \mathrm{Arg}(k^*), \qquad 0 < \gamma < \frac{\pi}{2}. \tag{3.31}$$

We recover the standard IST for the NLS equation by taking the limit $\alpha \to 0$, $k \approx 1/2 - \alpha\lambda_{nlse}/2$, $\lambda_{nlse} = \mu_{nlse} + i\nu_{nlse}$, so $v \to 2\mu_{nlse}$, $w \to 1/(2\nu_{nlse})$.

Using the perturbation theory for the IST of the modified NLSE, Doktorov and Kuten [2001] find the expression of the soliton jitter

$$\langle q(z)^2\rangle = \frac{Dz^3}{9}F(\alpha, v, \gamma). \tag{3.32}$$

Other contributions exist but they are proportional to z so they can be neglected for large propagation distances. The EGH theory gives the same formula for the jitter with $F = 1$. If $v < 1/\alpha$, then γ can be chosen so that $F < 1$. So by the proper choice of parameters the EGH jitter can be essentially reduced. Many problems for ultrashort pulses on the base of Eq. (3.28) are still unsolved. One of them is the radiation of femtosecond soliton under fluctuations of dispersion. We should note that the interesting effects in this model are observed for non small values of α. But in this case the validity of the model is questionable, since it is necessary to take into account higher-order dispersion of nonlinearity as well.

§ 4. Dispersion-managed solitons under random perturbations

The modified NLSE is of the form

$$iu_z + \frac{d(z)}{2}u_{tt} + c(z)|u|^2u = 0 \tag{4.1}$$

where d and c possess periodic modulations d_p and c_p with the period L (the so-called dispersion map) as well as random fluctuations modeled by zero-mean random processes d_r and c_r:

$$d(z) = d_p(z) + d_r(z), \qquad c(z) = c_p(z) + c_r(z)$$

The amplitudes of the periodic modulations can be large, while we consider small-amplitude random fluctuations.

4.1. PERIODIC AND RANDOM DISPERSION MODULATIONS

In dimensional units, the optical pulse propagation in a system with varying dispersion is governed by the NLSE (2.2)

$$iv_Z - \frac{1}{2}\beta_2(Z)v_{TT} + \kappa|v|^2v = 0,$$

where Z is the propagation distance (in km), T is the time in the frame moving with the group velocity (in ps), $P = |v|^2$ is the optical power (in W), β_2 is the group velocity dispersion coefficient (in ps^2/km). The coefficient β_2 is related to the usual dispersion parameter D by $\beta_2 = -\lambda_0^2 D/(2\pi c)$ where $c =$ 0.3 mm/ps is the speed of light, λ_0 is the carrier wavelength (in μm), and D is measured in ps/nm/km. The pulse energy $\mathcal{E}_{\text{pulse}} = \int |v|^2(Z,T)DT$.

is independent of Z. The carrier wavelength is $\lambda_0 = 1.55\ \mu$m for telecommunication applications. The nonlinear coefficient $\kappa = 2\pi n_2/(\lambda_0 A_{\rm eff})$ (in $W^{-1}m^{-1}$) where n_2 is the nonlinear refractive index ($n_2 \simeq 3\ 10^{-2}$ nm^2/W in glass) and $A_{\rm eff}$ is the effective fiber area (in μm^2). Typically $A_{\rm eff} \simeq 60$ μm^2.

We then introduce the typical nonlinear length $Z_0 := 1/(\kappa P_0)$ where P_0 is the typical pulse power. In most practical applications P_0 is equal to a few milliwatts. Say $P_0 = 2\ 10^{-3}$W, so that $Z_0 = 250$ km. We normalize the coordinate along the fiber $z = Z/Z_0$ and the envelope of the electric field $u = E/\sqrt{P_0}$. We also normalize the time $t = T/T_0$ where T_0 is the typical pulse width T_0, say $T_0 = 5$ ps, so that the propagation is governed by the dimensionless NLSE (4.1) with $c = 1$ and

$$d(z) = -\beta_2(Z_0 z) Z_0/T_0^2 = \lambda_0^2 D(Z_0 z) Z_0/(2\pi c T_0^2).$$

Let us first describe the periodic dispersion management. We assume that the fiber consists in the periodic concatenation of segments of alternate normal and anomalous fibers, so that the physical dispersion management is of the type:

$$D(Z) = \begin{cases} D_+ \text{ if } Z \bmod L_{\rm map} \in [0, L_+/2), \\ D_- \text{ if } Z \bmod L_{\rm map} \in [L_+/2, L_{\rm map} - L_+/2), \\ D_+ \text{ if } Z \bmod L_{\rm map} \in [L_{\rm map} - L_+/2, L_{\rm map}), \end{cases}$$

The map period is $L_{\rm map} = L_- + L_+$ (see Figure 3). The multi-scale analysis that is used in the next sections is based on the separation of the scales $L_{\rm map} \ll Z_0$. In dimensionless units the GVD coefficient d_p is of the form $d_p(z) = \tilde{d}_p(z) + d_0$ where

$$\tilde{d}_p(z) = \begin{cases} d_+ \text{ if } z \bmod l_{\rm map} \in [0, l_+/2), \\ d_- \text{ if } z \bmod l_{\rm map} \in [l_+/2, l_{\rm map} - l_+/2), \\ d_+ \text{ if } z \bmod l_{\rm map} \in [l_{\rm map} - l_+/2, l_{\rm map}), \end{cases}$$

$l_{\rm map} = L_{\rm map}/Z_0$, $l_\pm = L_\pm/Z_0$, $d_\pm = \lambda_0^2(D_\pm - D_m)Z_0/(2\pi c T_0^2)$, and $d_0 = \lambda_0^2 D_m Z_0/(2\pi c T_0^2)$ is the residual dispersion. Typical values are $|D_\pm| = 2 - 20$ ps/nm/km, $L_\pm = 20 - 200$ km, and $D_0 = 0 - 0.1$ ps/nm/km. In dimensionless units, $|d_\pm| = 20 - 200$, $l_\pm = 0.1 - 1$, and $d_0 = 0 - 1$. The so-called dispersion management (DM) strength is $D_L := d_+ l_+ = -d_- l_-$.

In this chapter we address the influence of small random dispersive fluctuations on the pulse propagation in DM systems. Considering the same type of fluctuations as in Section 3.2.1, we have the white noise model $\langle d_r(z) d_r(z') \rangle = D\delta(z - z')$ with $D \sim 10^{-2} - 10^{-3}$.

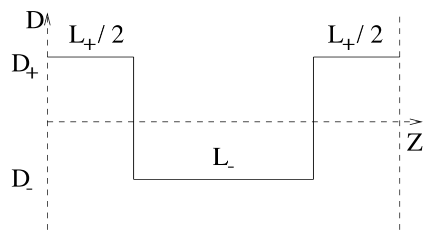

Figure 3: Two-step dispersion map.

4.2. DIFFERENT APPROACHES

A reduction of the NLSE to simpler equations is important for two reasons. First, to generate reliable statistics quantifying the effect of random coefficients on pulse evolution requires a large number of simulations. This is particularly true if the most relevant occurrences are exactly those that are least likely to occur in a straightforward Monte-Carlo approach, such as bit errors. It is therefore of obvious benefit to reduce the computational time for each simulation to produce meaningful results in a minimum of time. The second argument in favor of finding a reduction to the NLSE is that it is often difficult to obtain much insight from a partial differential equation. A reduction to a finite-dimensional system is generally more conducive to obtaining an analytical description of the pulse behavior. The resulting ordinary differential equations can be linearized about the dispersion-managed soliton for approximations valid over short distances, or they can be averaged over the dispersion map period to capture the behavior over longer scales [Turitsyn, Aceves, Jones, and Zharnitsky [1998]].

In this section we present a review of different methods used to reduce the NLSE. The first two approaches consist in reducing the partial differential equation to a finite-dimensional problem by deriving a set of ordinary differential equations for the pulse parameters. The last two approaches derive effective partial differential equations after an averaging of the NLSE over one map.

4.2.1. The variational approach

Employing the variational approach, the underlying NLSE can be reduced to a system of ordinary differential equations for the DM soliton parameters. The trial localized waveform is taken as

$$u(z,t) = A(z)Q\left[\frac{t}{a(z)}\right]\exp\left[i\frac{b(z)}{a(z)}\frac{t^2}{2}\right], \tag{4.2}$$

where A, a, and $b/(2a)$ are the complex amplitude, width, and chirp, respectively, and Q is the localized function specifying the pulse profile. Substituting the ansatz (4.2) into the average Lagrangian of Eq. (4.1) and equating to zero its variations, we arrive at the following variational equations:

$$a_z = d(z)b, \qquad b_z = \frac{C_1 d(z)}{a^3} - \frac{2C_2\mathcal{E}c(z)}{a^2}, \tag{4.3}$$

where the conserved quantity is the pulse energy $\mathcal{E} = \int |u|^2 dt = C_0|A|^2(z)a(z)$ and the constants C_j depend only on the shape function Q:

$$C_0 = \int |Q|^2 ds, \qquad C_1 = \frac{\int |Q_s|^2 ds}{\int s^2|Q|^2 ds}, \qquad C_2 = \frac{\int |Q|^4 ds}{4\left(\int s^2|Q|^2 ds\right).\left(\int |Q|^2 ds\right)}.$$

For the Gaussian approximation of the pulse function $Q(s) = \exp(-s^2)$ we have $C_0 = \sqrt{\pi/2}$, $C_1 = 4$, and $C_2 = 1/\sqrt{\pi}$.

4.2.2. Moment equations

Following Turitsyn, Aceves, Jones, and Zharnitsky [1998] we introduce the rms pulse width and the rms chirp of the pulse:

$$T_{rms} = \left(\frac{\int t^2|u|^2 dt}{\int |u|^2 dt}\right)^{1/2}, \qquad \frac{M_{rms}}{T_{rms}} = \frac{i}{4}\frac{\int t(uu_t^* - u^* u_t)dt}{\int t^2|u|^2 dt}.$$

It is easy to check that the evolution of T_{rms} and R_{rms} are

$$(T_{rms})_z = 2d(z)M_{rms}, \qquad (T_{rms}M_{rms})_z = \frac{d(z)}{2}\Omega_{rms}(z) - \frac{c(z)}{4}P_{rms}$$

where

$$\Omega_{rms}(z) = \frac{\int |u_t|^2 dt}{\int |u|^2 dt}, \qquad P_{rms}(z) = \frac{\int |u|^4 dt}{\int |u|^2 dt}$$

We thus get exact but not-closed equations. To close this system, we can 1) assume that the pulse has the self-similar structure (4.2). It is then possible to derive a closed-form system of ordinary differential equations for T_{rms} and M_{rms}, or alternatively a and b since both pairs are related to each other through the identities $a(z)/a(0) = T_{rms}(z)/T_{rms}(0)$ and $b(z)/a(z) = M_{rms}(z)/T_{rms}(z)$. The system that is derived is the same as the one obtained with the variational approach.

2) write the differential equations satisfied by Ω_{rms} and P_{rms} and make use of higher-order momentum equations. It is not possible to close the equations, but Turitsyn [1998] has succeeded in finding a five-dimensional system with five momenta (including T_{rms}, M_{rms}, Ω_{rms}, and P_{rms}) that can be closed by using only one assumption (a parabolic approximation of the phase near the pulse peak power location). After integration, it turns out that this system is exactly (4.3). This theoretical analysis and subsequent numerical simulations show that system (4.3) describes with good accuracy the DM soliton dynamics.

4.2.3. Path-averaged theory in the time domain

This method aims at estimating the deviations of a true DM soliton from the self-similar structure assumed in the variational or moment approaches. The large variation of the dispersion coefficient within one map does not allow a direct averaging of (4.1). The main idea is to use first a transformation that accounts for the fast pulse dynamics and to apply an averaging procedure to the transformed equation. Let us apply the following transformation

$$u(z,t) = \frac{N}{\sqrt{a_p(z)}} Q\left[z, \frac{t}{a_p(z)}\right] \exp\left[i\frac{b_p(z)}{a_p(z)}\frac{t^2}{2}\right]. \tag{4.4}$$

The rapid oscillations of pulse width and chirp are accounted for by the functions a_p and b_p satisfying

$$(a_p)_z = d_p(z) b_p, \qquad (b_p)_z = \frac{d_p(z)}{a_p^3} - \frac{2N^2 c_p(z)}{a_p^2}, \tag{4.5}$$

where N is a constant to be determined by the requirement that a_p and b_p are periodic functions. Note that we only take the periodic components of the dispersion and nonlinear coefficients in the system (4.5). Substituting (4.4) and (4.5) into (4.1) gives the modified NLSE

$$iQ_z + \frac{d_p}{2a_p^2}(Q_{xx} - x^2 Q) + \frac{c_p N^2}{a_p(z)}(|Q|^2 + x^2 Q)$$

$$= d_r\left[\frac{b_p^2}{2}x^2Q - \frac{1}{2a_p^2}Q_{xx} - \frac{ib_p}{2a_p}Q - \frac{ib_p}{a_p}xQ_x\right] - c_r\left[\frac{N^2}{a_p}|Q|^2Q\right] \quad (4.6)$$

We can expand the function $Q(z,x)$ over the system of Gauss-Hermite functions $Q(z,x) = \sum_n q_n(z)f_n(x)$. Multiplying (4.6) by f_n and integrating with respect to x, one obtains a system of coupled nonlinear ordinary differential equations for the coefficients q_n of the form

$$(q_n)_z + \frac{d_p}{a_p^2}(z)F_n(q) + \frac{c_p}{a_p}(z)G_n(q) = d_r(z)H_n(q,z) + c_r(z)I_n(q,z) \quad (4.7)$$

where $d_p/a_p^2(z)$, $c_p/a_p(z)$, $H_n(q,z)$, and $I_n(q,z)$ are z-periodic functions. In absence of random perturbations, it is possible to average this system over a map and to exhibit a stationary solution which is a pure DM soliton [Turitsyn, Schäfer, Spatschek, and Mezentzev [1999]]. In presence of perturbations, we can apply an adiabatic approach and write the slow evolutions of the coefficients q_n by averaging the right-hand side of (4.7) over a map. Note that, in practice, the system (4.7) is simplified by neglecting modes $n \geq n_0$ for a given n_0. If only the highest mode of this Gauss-Hermite expansion is considered, the resulting ordinary differential equations are the same as those found previously by the variational or moment approach [Schafer, Mezentsev, Spatschek, and Turitsyn [2001]].

4.2.4. Path-averaged theory in the frequency domain

If we neglect the nonlinear term in Eq. (4.1) we get that, in the spectral domain, the pulse phase experiences rapid oscillations during a map period. Nonlinear effects can be accounted for as a modification of this quasi-linear dynamics, and the parameters of the quasi-linear solution play the role of adiabatic invariants of the system. For simplicity we consider a version of (4.1) where the nonlinear coefficient is constant $c(z) \equiv c_0$. We expand the periodic component of the dispersion coefficient as $d_p(z) = d_0 + \tilde{d}_p(z)$, where $\tilde{d}_p$ has zero-mean. We introduce the accumulated dispersion function associated to the zero-mean periodic component and the (statistically) zero-mean random component d_r:

$$D(z) = \int_0^z \tilde{d}_p(\zeta) + d_r(\zeta)d\zeta$$

Chertkov, Gabitov, Lushnikov, Moeser, and Toroczkai [2002] apply a Fourier-like transform taking into account the zero-mean dispersion fluctuations

$$u(z,t) = \frac{1}{2\pi}\int \hat{u}(z,\omega)\exp\left[-i\omega t - i\omega^2 D(z)\right] d\omega$$

The resulting equation can be averaged by splitting $\hat{u}$ into a small-amplitude fast-varying part and a large-amplitude slowly varying component $\hat{v}$ which satisfies an integro-differential equation:

$$i\hat{v}_z - \omega^2 d_0 \hat{v} + c_0 \mathcal{L}\hat{v} = 0 \tag{4.8}$$

$$\mathcal{L}\hat{v} = \int \delta(\omega_1 + \omega_2 - \omega - \omega_3)\exp\left[-i\Delta D(z)\right]\hat{v}(\omega_1)\hat{v}(\omega_2)\hat{v}(\omega_3)^* d\omega_1 d\omega_2 d\omega_3$$

where $\Delta = \omega_1^2 + \omega_2^2 - \omega^2 - \omega_3^2$. In the absence of random fluctuations this path-averaged equation was first obtained by Gabitov and Turitsyn [1996], Gabitov, Shapiro, and Turitsyn [1996].

4.3. RANDOM DISPERSION

4.3.1. Pulse Broadening Induced by Random Dispersion

Particular configurations have first been addressed leading to simplified problems. Abdullaev and Baizakov [2000] assume that the DM strength is weak so that the averaged dynamics approach developed by Kutz, Holmes, Evangelidis, and Gordon [1998] is valid. In this approach a real map with fiber segments of alternating anomalous and normal dispersion is replaced with a uniform fiber with path-averaged dispersion d_{av}. The problem can be reduced to a randomly perturbed NLSE and using the same approach as in Section 3.2.3 it can be considered as a particular case of the random Kepler problem in the context of optical solitons. An explicit analytical expression is derived for the expected distance that a soliton propagates along a fiber with randomly varying dispersion before it disintegrates $L_{dis} = a_0^4/D$ where a_0 is the initial pulse width. Numerical simulations also show that the fluctuations of dispersion magnitudes of spans have more dramatic effects on DM solitons than the fluctuations of spans lengths (see figure 4). This observation has been confirmed by Xie, Mollenauer, and Mamysheva [2003].

Abdullaev and Navotny [2002] develop a mean field theory which is valid for very weak nonlinearity, as the averaging $\langle |u|^2 u\rangle = |\langle u\rangle|^2\langle u\rangle$ is performed, so that the interplay between random dispersion and nonlinearity

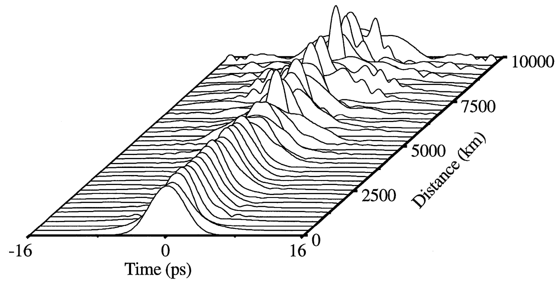

Figure 4: Disintegration of a soliton propagating in a DM line with randomly varying dispersion magnitudes of spans. The dispersion map corresponds to the concatenation of two segments of different fibers with dispersion $D_{\pm}$ and lengths $L_{\pm}$. The map parameters are $D_{+} = 2.446$ ps/nm/km, $D_{-} = 2.258$ ps/nm/km, and $L_{+} = L_{-} = 50$ km. The initial pulse width is 10 ps. In dimensionless units the parameters are $d_{+} = 26$, $d_{-} = -24$, $l_{+} = l_{-} = 0.14$ and the initial pulse is $u(0,t) = 2.45 \exp(-0.427t^2)$. The magnitudes of the spans are randomly varying with $\langle \delta d^2 / d_{\pm}^2 \rangle = 0.085^2$.

is not perfectly captured. In the case of a white noise model for the dispersion fluctuations the effect of the dispersion fluctuations can be described within the framework of a modified nonlinear Schrödinger equation with a frequency-dependent damping term ($\sim iu_{tttt}$). The presence of randomly modulated dispersion leads to the damping of optical pulses.

A second generation of work consists in reducing the problem using one of the approaches developed in Section 4.2, and then to carry out numerical simulations of the reduced problems. Malomed and Berntson [2001] consider a model of a long optical communication line consisting of alternating segments with anomalous and normal dispersion, whose lengths are picked randomly from a certain interval. As the first stage of the analysis, small changes in parameters of a quasi-Gaussian pulse passing a double-segment cell are calculated by means of the variational approximation. The evolution of the pulse passing many cells is approximated by smoothed ordinary differential equations with random coefficients, which are solved numerically. Simulations reveal slow long-scale dynamics of the pulse, frequently in the form of long-period oscillations of its width. It is found that the soliton is most stable in the case of zero path-average dispersion (PAD), less stable in the case of anomalous PAD, and least stable in the case of normal PAD. The soliton's stability also strongly depends on its energy, the soliton with low energy being much more robust than its high energy counterpart. This result has been confirmed by numerical simulations performed by Poutrina and Agrawal [2003]. Chertkov, Gabitov, Lushnikov, Moeser, and Toroczkai [2002] use the approach developed in Section 4.2.4 in the white noise case $\langle d_r(z)d_r(z')\rangle = D\delta(z-z')$. They show that the solution is not a self-averaging quantity so that a deterministic path-averaged approach is not applicable. They also predict that the length of the pulse destruction is $L_{dis} \sim a_0^4/D$ and they perform full numerical simulations of the path-averaged equation (4.8) to confirm the theoretical predictions.

These results based on numerical simulations have been confirmed by further work addressing the problem from a full theoretical point of view [Garnier [2002], Schafer, Moore, and Jones [2002]]. For short distances, a quasi-linear approach in the form of a simple perturbation expansion is sufficient to capture the broadening for each realization of the random dispersion. For intermediate distances, over which the noise and nonlinearity interact, the partial differential equation can be reduced to a relatively simple system of nonlinear stochastic ordinary differential equations. Finally, over long distances, the slow evolution of the pulse width can be obtained by applying an appropriate multiple-scales averaging procedure to yield a new, scaled noise process effecting pulse broadening. The main features exhibited by this analysis are the following ones. A low-energy soliton is more robust than a high-energy soliton with equivalent characteristics. The soliton robustness is also enhanced by strong dispersion management. More quantitatively, the impact of the random dispersion is proportional to the strength of disorder D and inversely proportional to the fourth power of the pulse width a_0. The mean pulse broadening grows like Dz/a_0^4, while the standard deviation of the pulse broadening grows like $\sqrt{Dz/a_0^4}$. Therefore the noise in dispersion presents a potential source of serious limitation for the next generation of high-speed communications.

4.3.2. Separation of Scales Technique

We present in more detail the results obtained by the application of the variational or moment approach and a technique of separation of scales which is valid for strong DM. We denote $\tilde{D}_p(z) = \int_0^z \tilde{d}_p(s)ds$ (which is a periodic function with period l_{map}) and we introduce the periodic orbits

$$\mathcal{A}(a_0, b_0, z) = \sqrt{a_0^2 + 2a_0 b_0 \tilde{D}_p(z) + \left(b_0^2 + 4a_0^{-2}\right)\tilde{D}_p(z)^2},$$

$$\mathcal{B}(a_0, b_0, z) = \frac{a_0 b_0 + \left(b_0^2 + 4a_0^{-2}\right) \tilde{D}_p(z)}{\sqrt{a_0^2 + 2a_0 b_0 \tilde{D}_p(z) + \left(b_0^2 + 4a_0^{-2}\right) \tilde{D}_p(z)^2}}.$$

If $c = 0$, $d_0 = 0$, and $d_r \equiv 0$, then $a(z) = \mathcal{A}(z)$ and $b(z) = \mathcal{B}(z)$ are the solutions of system 4.3 starting from (a_0, b_0). If the variations of the coefficients a and b over a period $l_{\rm map}$ involved by the nonlinearity c, the residual dispersion d_0 and the random dispersion d_r are small, then 1) the short scale dynamics of the parameters (a, b) is driven by $\tilde{d}_p$ and follows the periodic orbits $(\mathcal{A}, \mathcal{B})$. 2) the long-scale dynamics of the parameters (a, b) are driven by an effective system where the fast periodic oscillations have been averaged. The derivation of this effective system is performed by Garnier [2002]:

$$a(z) = \mathcal{A}(\bar{a}(z), \bar{b}(z), z), \qquad b(z) = \mathcal{B}(\bar{a}(z), \bar{b}(z), z), \tag{4.9}$$

where $(\bar{a}, \bar{b})$ obeys the effective system:

$$\frac{d\bar{a}}{dz} = f(\bar{a}, \bar{b}) + [d_0 + d_r(z)]\bar{b}, \qquad \frac{d\bar{b}}{dz} = g(\bar{a}, \bar{b}) + 4[d_0 + d_r(z)]\bar{a}^{-3}, \tag{4.10}$$

and the explicit expressions of the functions f and g are

$$f(a,b) = \frac{2C_E a^3 b}{D_L (4 + a^2 b^2)^{3/2}} \ln(\psi(a,b))$$
$$+ \frac{4C_E a^3}{D_L (4 + a^2 b^2)} \left(\frac{a - bD_L}{\sqrt{(2a - bD_L)^2 a^2 + 4D_L^2}} - \frac{a + bD_L}{\sqrt{(2a + bD_L)^2 a^2 + 4D_L^2}} \right)$$
$$g(a,b) = \frac{8C_E}{D_L (4 + a^2 b^2)^{3/2}} \ln(\psi(a,b))$$
$$+ \frac{4C_E}{D_L (4 + a^2 b^2)} \left(\frac{ba^3 - 4D_L}{\sqrt{(2a + bD_L)^2 a^2 + 4D_L^2}} - \frac{ba^3 + 4D_L}{\sqrt{(2a - bD_L)^2 a^2 + 4D_L^2}} \right)$$
$$\psi(a,b) = \frac{2a^3 b + \sqrt{4 + a^2 b^2}\sqrt{(2a + bD_L)^2 a^2 + 4D_L^2} + D_L (4 + a^2 b^2)}{2a^3 b + \sqrt{4 + a^2 b^2}\sqrt{(2a - bD_L)^2 a^2 + 4D_L^2} - D_L (4 + a^2 b^2)}.$$

Here $C_E = \mathcal{E}/\sqrt{\pi}$ and $D_L = d_+ l_+ = -d_- l_-$ is the DM strength. The pair $(\bar{a}, \bar{b})$ is a diffusion Markov process whose probability density function satisfies a Fokker-Planck equation. These equations can only be integrated numerically due to the complexity of the functions f and g, but expansions for small or large DM strengths can be performed. Also, a linear stability

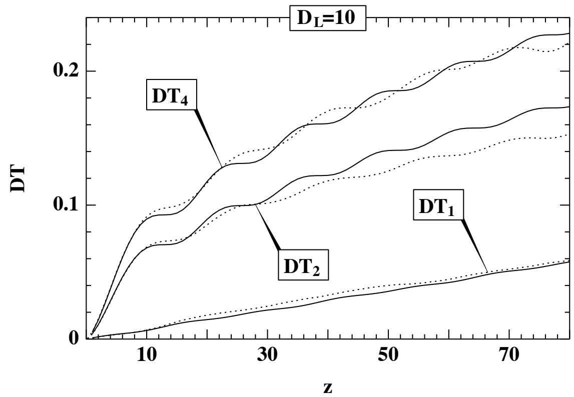

Figure 5: Pulse broadening in dimensionless units. The normalized first, second and fourth moments of the pulse width increments are plotted $DT_j = \left\langle (T_{rms}(z) - T_{rms0})^j \right\rangle^{1/j}$, $j = 1, 2, 4$. The solid lines stand for the theoretical values, the dotted lines represent the numerical values averaged over 10^4 realizations. Here $d_+ = -d_- = 100$, $d_0 = 1$, $l_+ = l_- = 0.1$, $D = 0.033, T_{rms0} = 1.68$.

analysis $a = a_s + a_1$ can be applied around the stable point $(a = a_s, b = 0)$ that corresponds to a DM soliton solution of (4.10) with $d_r = 0$. All moments have be computed by Garnier [2002] and we report the expression of the second moment of the pulse broadening

$$\langle a_1^2 \rangle = 8 \frac{D\partial_b f}{a_s^6 |\partial_a g|} \left(z - \frac{\sin(2\sqrt{\gamma} z)}{2\sqrt{\gamma}} \right), \tag{4.11}$$

where $\gamma = -\partial_b f \partial_a g > 0$ and the partial derivatives are computed at the stable point $(a = a_s, b = 0)$ (see Garnier [2002] for closed-form formulas). Comparisons of these theoretical predictions with full numerical simulations of the random NLSE are shown in Figure 5.

4.4. PINNING SCHEMES

The pinning method for a DM system consists of periodic compensation of accumulated fiber dispersion by insertion of an additional piece of fiber with a well controlled length and dispersion value. All components required for implementation of this method, including measurements of accumulated dispersion of a fiber span, are standard and well established in optical fiber communications. The pinning method can be implemented both for the upgrading of existing links and for the production of new optical cables. The pinning method is found to prevent from pulse deterioration and is capable of improving performance of high-speed optical fiber links [Chertkov, Gabitov, Lushnikov, Moeser, and Toroczkai [2002]]. The effect is even more dramatic than in the case of the standard soliton. Indeed, for a pinning period below some critical value, one observes a tendency toward statistically steady behavior: the average pulse width does not decay and the probability density function of the pulse width and amplitude does not change shape with z. This critical period is increasing with the noise level D. These theoretical findings are verified by direct numerical simulation, where no emission of radiation by the DM soliton can be observed. An independent set of numerical simulations performed by Xie, Mollenauer, and Mamysheva [2003] has recently confirmed these results. Thus, the pinning method is effective in optical fibers with and without dispersion management.

§ 5. Randomly birefringent fibers

Experiments have shown that Polarization Mode Dispersion (PMD) is one of the main limitations on fiber transmission links [Gisin, Pellaux, and Von der Weid [1991]]. PMD has its origin in the birefringence, i.e. the fact that the electric field is a vector field and the index of refraction of the medium depends on the polarization state (i.e. the unit vector pointing in the direction of the electric vector field). For a fixed position in the fiber, there are two orthogonal polarization eigenstates which correspond to the maximum and the minimum of the index of refraction. These two polarization states are parameterized by an angle with respect to a fixed pair of axes that is called the birefringence angle. The difference between the maximum and the minimum of the index of refraction is the birefringence strength. If the birefringence angle and strength were constant along the fiber, then a pulse polarized along one of the eigenstates would travel at constant velocity. However the birefringence angle is randomly varying which involves coupling between the two polarized modes. The modes travel with different velocities, which involves pulse spreading. Random birefringence results from variations of the fiber parameters such as the core radius or geometry. There exist various physical reasons for the fluctuations of the fiber parameters. They may be induced by mechanical distortions on fibers in practical use, such as point-like pressures or twists [Rasleigh [1983]]. They may also result from variations of ambient temperature or other environmental parameters [Biondini, Kath, and Menyuk [2002]]. In linear media PMD causes a separation between the two components of the pulse into the two principal polarizations. In nonlinear media a soliton may counteract this separation through the effective potential well produced by the local Kerr index. As a consequence it is expected that solitons could withstand PMD [Menyuk [1988], Mollenauer, Smith, Gordon, and Menyuk [1989]].

5.1. DERIVATION OF THE PERTURBED MANAKOV SYSTEM

The evolution of polarized fields in randomly birefringent fibers is governed by the coupled nonlinear Schrödinger equations with random PMD between two modes (polarizations) [Menyuk [1989]]:

$$i\mathbf{A}_z + K_0\mathbf{A} + iK_1\mathbf{A}_t - \frac{\beta''}{2}\mathbf{A}_{tt} + \kappa\mathbf{N} = 0 \tag{5.1}$$

where $\mathbf{A}$ is the column vector $(A_x, A_y)^T$ that denotes the envelopes of the electric field in the two eigenmodes. The z-dependent 2×2 matrices K_0 and K_1 describe random fiber birefringence. The dispersion coefficient β'' is the second derivative of the propagation constant with respect to frequency, which is negative (resp. positive) for anomalous (resp. normal) dispersion. In this section we assume that $\beta'' < 0$. Finally κ is the Kerr coefficient. The $\mathbf{N}$ term stands for the nonlinear terms:

$$\mathbf{N} = \begin{pmatrix} (|A_x|^2 + \alpha|A_y|^2)A_x + \frac{\alpha}{2}A_y^2A_x^* \\ (|A_y|^2 + \alpha|A_x|^2)A_y + \frac{\alpha}{2}A_x^2A_y^* \end{pmatrix}, \tag{5.2}$$

where the cross-phase modulation is $\alpha = 2/3$ for linearly birefringent fiber. As shown by Menyuk and Wai [1994], Wai and Menyuk [1996], one can eliminate the fast random birefringence variations that appear in Eq. (5.1) by means of a change of variables, that leads to the new vector equation:

$$i\mathbf{A_1}_z - \frac{\beta''}{2}\mathbf{A_1}_{tt} + \frac{8}{9}\kappa\mathbf{N_1} = iR_l\mathbf{A_1}_t + R_{nl}(\mathbf{A}_1) \tag{5.3}$$

where $\mathbf{A_1} \equiv M^{-1}\mathbf{A}$, $\mathbf{A_1} = (A_{1x}, A_{1y})^T$ represents the slow evolution of the field envelopes in the reference frame of the local polarization eigenmodes, and the matrix M obeys the equation $iM_z + K_0 M = 0$. The nonlinear term $\mathbf{N_1}$ is

$$\mathbf{N_1} = \begin{pmatrix} (|A_{1x}|^2 + |A_{1y}|^2)A_{1x} \\ (|A_{1x}|^2 + |A_{1y}|^2)A_{1y} \end{pmatrix}. \tag{5.4}$$

The left-hand side of (5.3) is known as the Manakov system which is an integrable system supporting soliton solutions (see Appendix 2). The first (resp. second) term of the right-hand side is the linear (resp. nonlinear) PMD. They are corrections to the Manakov system involving high order PMD and they have mean zero. In absence of losses M is unitary and R_l is a combination of three Pauli matrices

$$R_l(z) = m_1(z)\Sigma_1 + m_2(z)\Sigma_2 + m_3(z)\Sigma_3, \tag{5.5}$$

where

$$\Sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and m_j are real-valued random processes. R_l is associated with the linear coupling between the modes, as well as an accumulation of a phase mismatch. R_{nl} is the nonlinear PMD exhibited by Wai and Menyuk [1996]. As shown in Lakoba and Kaup [1997] it is usually smaller than the linear PMD.

The white noise model. Different PMD models have been developed by Menyuk and Wai [1994], Wai and Menyuk [1996], Lakoba and Kaup [1997]. These models become equivalent as soon as the correlation length of the random fluctuations of birefringence parameters is much smaller than the other characteristic lengths of the problem. The processes m_j can then be described by independent white random processes with zero-mean and autocorrelation functions $\langle m_i(z)m_j(z')\rangle = \sigma^2\delta_{ij}\delta(z-z')$.

The retarded-plate model. For numerical simulations, we may think at the commonly used retarded-plate model Marcuse, Menyuk, and Wai [1997]. The birefringence strength $\Delta\beta$ and its derivative $\Delta\beta'$ are constant; the birefringence angle is constant over elementary intervals with length Δz; at junctions between the fiber pieces with length Δz, random axial rotations are incorporated as well as additions of random phase differences between the two field components, so that the Stokes vector obeys a random walk over the Poincaré sphere. If Δz is small enough we can model this configuration by the white noise model with $\sigma^2 = \Delta\beta'^2\Delta z/12$. In this model the so-called polarization mode dispersion parameter is given by

$D_p = \sqrt{8/(3\pi)}\Delta\beta'\sqrt{\Delta z}$ [Poole and Wagner [1986]]. The parameter σ^2 is simply related to D_p through: $\sigma^2 = \pi D_p^2/32$. Chen and Haus [2000a] study the system (5.3) with a white noise model for PMD excitation m_j in order to obtain the analytical expression for the jitter due to the interaction with continuum component. Analogies with the noise driven harmonic oscillator was observed and confirmed by the numerical simulations of (5.3) with white noise perturbations. We shall focus our attention in the next section to the radiative soliton decay.

Dimensionless form. We introduce a reference time t_0 (of the order of the pulse width) and we define the dimensionless time $T = t/t_0$ and distance $Z = z/z_0$ where $z_0 = t_0^2/|\beta''|$ is the dispersion distance. The dimensionless field

$$\mathbf{U}(T, Z) = \sqrt{\frac{8\kappa z_0}{9}}\mathbf{A_1}(Tt_0, Zz_0)$$

satisfies the normalized equation:

$$i\mathbf{U}_Z + \frac{1}{2}\mathbf{U}_{TT} + |\mathbf{U}|^2\mathbf{U} = i\varepsilon R(Z)\mathbf{U}_T \tag{5.6}$$

where $\varepsilon = \sigma/\sqrt{|\beta''|}$,

$$R(Z) = \eta_1(Z)\Sigma_1 + \eta_2(Z)\Sigma_2 + \eta_3(Z)\Sigma_3,$$

and the normalized processes $\eta_j(Z) = \sqrt{z_0/\sigma^2}m_j(z_0 Z)$ are white noises with autocorrelation function $\langle m_j(Z)m_j(Z')\rangle = \delta_{ij}\delta(Z - Z')$. For consistency, note that the usual GVD parameter is $D = 2\pi c|\beta''|/\lambda^2$ where λ is the carrier wavelength of the pulse (1.55 μm for standard optical fiber applications). Thus $\varepsilon = \pi\sqrt{c}D_p/(4\lambda\sqrt{D})$. If D_p is expressed in ps/$\sqrt{\text{km}}$ and D is in ps/nm/km then $\varepsilon = 0.28D_p/\sqrt{D}$ for $\lambda = 1.55$ μm. The GVD parameter D is usually between 1–20 ps/nm/km. Typical values of the PMD parameter D_p have been measured in the range 0.1–1 ps/$\sqrt{\text{km}}$ [de Lignie, Nagel, and van Deventer [1994]]. Dispersion shifted fibers (which are of particular interest for soliton transmission) have been found to have particular high values Galtarossa, Gianello, Someda, and Schiano [1996]. The correlation length Δz of PMD varies around 0.1–1 km, which shows that the white noise model is valid.

5.2. RADIATIVE DAMPING OF SOLITONS

A perturbation theory of the IST for the Manakov system is presented in Appendix 2. This theory is valid for a large class of perturbations as soon as the total energy is preserved. It is formally equivalent to the method presented in Section 3.1.2 for the perturbed NLSE. It can applied to the analysis of soliton propagation driven by random PMD. It is found that the mean emitted spectral power is PMD is

$$P(\lambda) = \frac{2\varepsilon^2\pi\left(\nu^2 + (\mu-\lambda)^2\right)}{\cosh^2\left(\frac{\pi}{2}\frac{\mu-\lambda}{\nu}\right)}.$$

The conservation of the total energy allows us to find the soliton decay

$$\nu(Z) = \nu_0\left(1 + 16\varepsilon^2\nu_0^2 Z/3\right)^{-1/2}, \qquad (5.7)$$

which shows that the amplitude of the soliton decays as $z^{-1/2}$. Besides the momentum of the soliton is not affected at the leading order. This formula can also be obtained by means of the perturbation theory of the IST for the scalar NLSE [Lakoba and Kaup [1997]]. In the dimensional variables, we can consider an optical soliton with rms pulse width T_0. The time jitter is the prevailing phenomenon if the propagation distance is short $z \ll T_0^2/D_p^2$. For long propagation distances $z \geq T_0^2/D_p^2$ the radiative soliton decay is of order 1. As a consequence the inverse of the rms pulse width (proportional to the soliton energy) for long propagation distances is given by

$$T^2(z) = T_0^2 + \pi^3 D_p^2 z/288 = T_0^2 + \pi^2\sigma^2 z/9 \qquad (5.8)$$

This formula confirms the results obtained by a direct expansion technique by Matsumoto, Akagi, and Hasegawa [1997]. If we compare this expression with the one that describes the growth of the width of a linear pulse [Karlsson and Brentel [1999]]

$$\langle T^2\rangle = T_0^2 + 3\sigma^2 z - T_0^2\left(\sqrt{4 + \sigma^2 z/T_0^2} - 2\right) \overset{\sigma^2 z \ll T_0^2}{\simeq} T_0^2 + 2\sigma^2 z$$

we get that the soliton is more stable with respect to PMD fluctuations. Besides, the variance of the soliton width is also smaller than the variance of the linear pulse width (see Figure 6).

5.3. DISPERSION MANAGED SOLITONS AND PMD EFFECTS

DM solitons propagate along segments of fiber that are alternately positively and negatively dispersive. The scale of the dispersion management

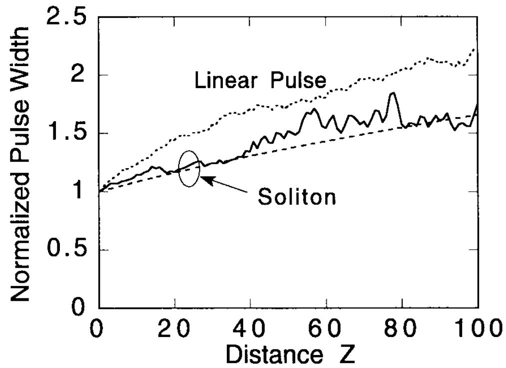

Figure 6: Normalized pulse width versus distance for a soliton and a linear pulse (i.e. a sech pulse in absence of dispersion and nonlinearity). Here $D = 0.25$ ps/nm/km and $D_p = 0.2062$ ps/$\sqrt{\text{km}}$. The distance is normalized by the dispersion distance [reprinted from Matsumoto, Akagi, and Hasegawa (1997)].

is usually much shorter than the soliton period. The scale of PMD is much shorter than that. PMD induced broadening and outage probability have been investigated by means of intensive numerical simulations. The qualitative picture is that the DM soliton system can achieve better performance than both the conventional soliton system and the linear system with respect to PMD [Xie, Sunnerud, Karlsson, and Andrekson [2001], Kylemark, Sunnerud, Karlsson, and Andrekson [2003]]. The quantitative picture is complicated as, apart from the PMD itself and the pulse energy and width, the performance of the conventional system depends on the average dispersion while the performance of the DM system also depends on the dispersion map strength [Sunnerud, Li, Xie, and Andrekson [2001]].

A complete analytical picture of the DM soliton dynamics under PMD is still lacking. Partial theoretical results are available. The DM soliton is treated as an average Manakov soliton by Chen and Haus [2000b]. Abdullaev, Umarov, Wahiddin, and Navotny [2000] apply a variational approach to describe the dynamics of vector solitons in dispersion-managed optical fiber with periodically and randomly inhomogeneous birefringence. With the help of a Lagrangian formalism an approximate system of equations is derived for the soliton parameters. The predictions for the evolution of the vector soliton parameters from variational equations have been compared with the results of numerical modeling of the governing coupled nonlinear Schrödinger equations, and the variational approach has been shown to give reliable results. Inhomogeneous birefringence affects primarily the relative distance and frequency of solitons, whereas chirp and intensity are slightly affected. Randomly inhomogeneous birefringence leads to diffusion growth of the mean square of the relative distance and may split vector solitons into their constitutional components. The dependence of the mean decay length L_d of the vector soliton on the strength of random birefringence and on the energy of the initial pulse is obtained numerically. There exists a threshold value of the soliton energy below which the dependence of L_d on the energy is weak and above which L_d decreases quickly.

§ 6. Solitons in random quadratic media

In the previous sections we considered the propagation of solitons in Kerr nonlinear media with random parameters. From the theoretical and experimental point of view it is important to study the dynamics of optical solitons in quadratic media with fluctuating parameters. Spatial solitons in quadratic (or $\chi^{(2)}$) media were studied for the first time two decades ago by Karamzin and Sukhorukov [1974]. They came again under intensive investigations recently [Kanashov and Rubenchik [1984], Torner [1995]]. The first observations of $\chi^{(2)}$ solitons in bulk media [Torruelas, Wang, Hagan, Van Stryland, Stegeman, Torner, and Menyuk [1995]] and in film waveguides [Schiek, Baek, and Stegeman [1996]] have triggered further experimental efforts. Exciting effects such as various scenarios of soliton interaction and switching operations have been observed by Constantini, De Angelis, Barthelemy, Bourliaguet, and Kermene [1998], Fuerst, Lawrence, Torruelas, and Stegeman [1997], Baek, Schiek, Stegeman, and Sohler [1997]. It should be mentioned, however, that these experiments have been performed in samples where the wave vector mismatch as well as the effective nonlinear coefficient can vary due to fluctuations of the impurity density of the material, the temperature, the effective mode area, etc. Although spatial $\chi^{(2)}$ solitons have been proven to be stable both in bulk medium and in waveguides, they may be unstable under some perturbations, especially phase sensitive ones. This feature distinguishes $\chi^{(2)}$ solitons from their Kerr-like counterparts that are not sensitive to random variations of the phase. Thus the dynamics of $\chi^{(2)}$ solitons under random, in particular phase sensitive, perturbations needs to be investigated.

In general, random variations of the linear dielectric constant lead to fluctuations of the wave vector mismatch. In addition, if the orientation of the crystallographic axes in a bulk medium or the effective mode area in planar waveguides vary with propagation distance, then the effective quadratic nonlinear coefficient is randomly modulated. Stochastic effects become even more pronounced when quasi-phase-matched (QPM) geometries are used. In QPM geometries a periodic modulation of the quadratic nonlinearity is used to compensate for the large mismatch. Randomly varying domain lengths contribute to the deviations of the phase mismatch and/or the effective quadratic nonlinearity from their mean values [Torner and Stegeman [1997]]. Thus, it is important to understand the effect of such fluctuations on nonlinear waves and, in particular, on quadratic solitons.

The $\chi^{(2)}$ soliton dynamics in media with fluctuating parameters is very complicated and it has been mainly numerically studied. Numerical simulations of soliton propagation in quadratic media with a fluctuating phase mismatch were carried out by by Torner and Stegeman [1997], Clausen, Bang, Kivshar, and Christiansen [1997], Abdullaev, Darmanyan, Kobyakov, Schmidt, and Lederer [1999]. It was observed that such fluctuations lead to a soliton decay which has a dissipative nature. Qualitatively, the effect of fluctuating parameters on the soliton propagation can be intuitively understood. Indeed, an important prerequisite for soliton formation in $\chi^{(2)}$ media is a balanced phase between the fundamental wave (FW) and the the second harmonic (SH). The medium fluctuations distort the phase balance and cause a net flow of energy between harmonics. As a result the soliton emits radiation and decays upon propagation. These quantitative arguments, however, have to be supported by theoretical estimations.

6.1. MEAN FIELD METHOD

In what follows we deal with one-dimensional spatial solitons thus assuming a film waveguide geometry. Beam propagation in quadratic nonlinear waveguides with variable parameters is described by two coupled equations for the FW (E_1) and SH (E_2) fields and in dimensionless variables has the form

$$iE_{1z} + \frac{1}{2}E_{1xx} + d(x,z)E_1^*E_2 + \beta_1(x,z)E_1 = 0, \tag{6.1}$$

$$iE_{2z} + \frac{1}{4}E_{2xx} + \frac{d(x,z)}{2}E_1^2 + (\beta_2(x,z) - q)E_2 = 0, \tag{6.2}$$

where q and $\beta_{1,2}$ are the deterministic and stochastic parts of the mismatch, respectively, and $d = 1 + \varepsilon(x,z)$ denotes the randomly varying nonlinearity. We assume that $\varepsilon(x,z)$ and $\beta_j(x,z)$, $j=1,2$ are independent white noise random processes. The correlation function for $\beta_j(z)$ is assumed to be $\langle\beta_j(z)\beta_j(z')\rangle=\sigma_j^2\delta(z-z')$, where $\sigma_j^2=l_{j,c}\left\langle\delta n_j^2\right\rangle/n_j^2$ and $l_{j,c}$, $j=1,2$, are the correlation lengths of the linear refraction indices. The correlation function for $\varepsilon(z)$ is assumed to be $\langle\varepsilon(z)\varepsilon(z')\rangle=\sigma^2\delta(x)$, with $\sigma^2=l_c\left\langle\delta d_{eff}^2\right\rangle/d_{eff}^2$, l_c is the correlation length of fluctuations of the quadratic nonlinearity.

The $\chi^{(2)}$ system is not integrable even without fluctuating parameters. Therefore some special, approximate methods, as e.g. the mean field method (MFM) has to be applied. Although the applicability of MFM is only justified for linear wave propagating in random media [Klyatzkin [1980]], recent studies have evidenced that this method can be successfully extended toward nonlinear problems as e.g. the dynamics of excitations in the disordered condensed matter [Zaiman [1980]] and of nonlinear waves in random weakly dispersive media [Abdullaev, Abdumalikov, and Baizakov [1997]]. Because under some circumstances the MFM results can be incorrect, they have to be double-checked by numerical means (see Figure 7). Typically, MFM provides correct results with regard to damping and its accuracy compares to that of the Born approximation as shown by Konotop and Vasquez [1994] and Abdullaev, Darmanyan, and Khabibullaev [1993].

Averaging the system Eqs. (6.1-6.2) over all realizations of the stochastic process and using MFM we obtain a system of equations for the mean fields $\langle E_1\rangle$ and $\langle E_2\rangle$

$$\begin{aligned}
i\langle E_1\rangle_z+\frac{1}{2}\langle E_1\rangle_{xx}+\langle E_1\rangle^*\langle E_2\rangle+\frac{i}{2}\sigma_1^2\langle E_1\rangle+&\\
\frac{i}{2}\sigma^2\left(\frac{1}{2}\left|\langle E_1\rangle\right|^2-\left|\langle E_2\rangle\right|^2\right)\langle E_1\rangle=0,&\\
i\langle E_2\rangle_z-q\langle E_2\rangle+\frac{1}{4}\langle E_2\rangle_{xx}+\frac{1}{2}\langle E_1\rangle^2+\frac{i}{2}\sigma_2^2\langle E_2\rangle+&\\
\frac{i}{2}\sigma^2\left|\langle E_1\rangle\right|^2\langle E_2\rangle=0.&
\end{aligned}\tag{6.3}$$

This shows that, in the MFM approximation, the influence of fluctuations is described by the effective *linear* and *nonlinear* losses acting on propagating solitons. Using these results it is possible to develop an analytical model describing the soliton evolution in quadratic media with fluctuating parameters based on the MFM equations (6.3) [Torner and Stegeman [1997]]. Numerical observations performed by Abdullaev, Darmanyan, Kobyakov,

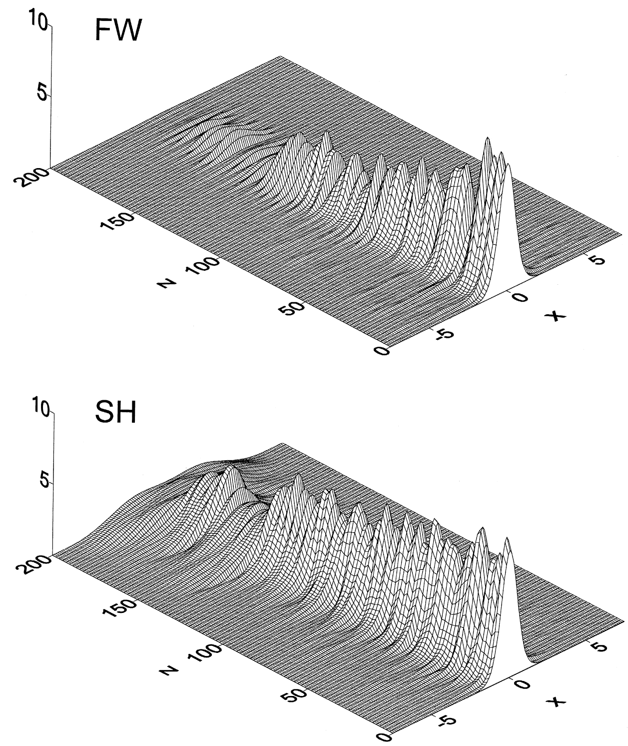

Figure 7: Typical evolution of the initial profile $E_1(0,x) = E_2(0,x) = 3\text{sech}^2(x)$ in a quadratic medium with fluctuating nonlinearity. Intensities of the FW and SH components are shown. Here $\sigma = 0.042$, $\sigma_{1,2} = 0$.

Schmidt, and Lederer [1999] indicate that the mean amplitudes decrease significantly but the mean widths of solitons change only slightly. Motivated by these observations we introduce the following ansatz to be substituted into (6.3)

$$\langle E_1\rangle = a(z)A_s(z,x), \qquad \langle E_2\rangle = b(z)B_s(z,x), \tag{6.4}$$

where $A_s(z,x)$ and $B_s(z,x)$ are the soliton solutions of the unperturbed system. The complex functions $a(z) = a_0(z)e^{i\phi(z)}$ and $b(z) = b_0(z)e^{i\psi(z)}$ describe the changes of the soliton's amplitudes and phases caused by the perturbation.

6.2. NONLINEAR DAMPING

Substituting Eqs. (6.4) into Eqs. (6.3) and performing an averaging over the transverse variable we get the following equations for the complex amplitudes

$$\begin{aligned} ia_\xi - a + a^* b + i\gamma\left(|a|^2 - 2\,|b|^2\right) a &= 0, \\ ib_\xi - \frac{1}{2}b + \frac{1}{2}a^2 + 2i\gamma\,|a|^2\, b &= 0, \end{aligned} \tag{6.5}$$

where $\xi = 4A_0z/5$, $\gamma = 3A_0\sigma^2/14$. The corresponding initial conditions (at $z = \xi = 0$) for the system (6.5) read as $a_0 = b_0 = 1$, $\varphi = \psi = 0$. An approximate solution to the system (6.5) for $\gamma \ll 1$ can be written in the form [Abdullaev, Darmanyan, Kobyakov, Schmidt, and Lederer [1999]]

$$\begin{aligned} \theta &= 2\phi - \psi \approx -\frac{8\gamma}{3}\left(1 - \cos\frac{3\xi}{2}\right), \\ b_0 &\approx \frac{1}{3}(2 + e^{-2\gamma\xi}) - \frac{8\gamma}{9}\sin\frac{3\xi}{2}, \\ a_0 &\approx \left(4b_0^2 - 3b_0 + 8\gamma b_0 \sin\frac{3\xi}{2}\right)^{1/2}. \end{aligned} \tag{6.6}$$

The comparison of the solution (6.6) with numerical results shows good agreement.

6.3. LINEAR DAMPING

Additionally we briefly discuss the behavior of $\chi^{(2)}$ solitons for fluctuations of the phase mismatch. As shown in the previous section in the framework of MFM this influence is described by *the linear* effective damping acting on both components. Applying the analytical approach described above, we obtain the system of ODE's for the functions a and b

$$ia_\xi - a + a^* b = -i\gamma_1 a,$$
$$ib_\xi - \frac{b}{2} + \frac{a^2}{2} = -i\gamma_2 b, \qquad (6.7)$$

where $\xi = 4A_0 z/5$, $\gamma_{1,2} = 5\sigma_{1,2}^2/(8A_0)$. The approximate solutions to this system are

$$\theta \approx \frac{2}{9}(2\gamma_1 - 5\gamma_2)\left(1 - \cos\frac{3\xi}{2}\right),$$
$$b_0 \approx \exp\left(-\frac{2}{9}(\gamma_1 + 2\gamma_2) + \frac{2(2\gamma_1 - 5\gamma_2)}{27}\sin\frac{3\xi}{2}\right),$$
$$a_0 \approx \left(4b_0^2 - 3b_0 - \frac{2}{3}b_0(2\gamma_1 - 5\gamma_2)\sin\frac{2\xi}{3}\right)^{1/2}. \qquad (6.8)$$

Note that $\theta = 0$ is not a solution to the system (6.7), therefore for $2\gamma_1 = 5\gamma_2$ higher order terms should be taken into account. The details of the theory of quadratic soliton propagation in lossy media can be found in [Darmanyan, Kobyakov, and Lederer [1999]]. It should be noted that for the mean fields the total energy decreases, while the original system (6.1-6.2) conserves the total energy. This discrepancy can be explained by the fact that the conserved total energy is the sum of the solitonic (the mean field) and random linear radiation parts. The decreasing soliton energy in MFM reflects a decreasing soliton content of the field. Numerical simulations for the fluctuating phase mismatch [Clausen, Bang, Kivshar, and Christiansen [1997]] and nonlinearity fluctuations [Abdullaev [1999]] confirm qualitatively this picture. However, in the case of fluctuating mismatch, MFM and numerics agree only qualitatively in the sense that MFM overestimates damping. The reason for this is that the stochastically varying mismatch enters into Eqs. (6.1-6.2) via the Brownian motion $W(z) = \int_0^z (2\beta_1(z') - \beta_2(z'))dz'$ in the exponent. Thus the effective nonlinearity is $d = \exp(-i(W(z) + q)$. A characteristic feature of the Brownian motion is that the deviations grow with distance. For the randomly modulated nonlinearity considered above this deviation is constant. Thus the fluctuations of the phase mismatch can be predicted by the MFM method less accurately compared to the case of randomly varying nonlinearity.

§ 7. Spatial solitons in random waveguides

In this section we consider the propagation of stable self-focused beams (called spatial solitons) in media with random parameters. We mainly deal with planar beams propagation in nonlinear waveguides with random parameters, since the governing equation is equivalent to the perturbed NLSE. As is well known the spatial soliton is unstable in free space with respect to perturbations in the transverse direction (see the discussion in the book Hasegawa and Kodama [1995] and in the review Kuznetzov, Rubenchik, and Zakharov [1986]). The case of propagation in planar waveguides is free from these difficulties. The beam is constrained in the transverse (say y-)direction by the profile of the linear refraction index, while the nonlinear refraction index defines a field distribution in the (say x-)direction.

7.1. SPATIAL SOLITONS IN PLANAR WAVEGUIDES WITH RANDOM PARAMETERS

The electromagnetic field in a planar waveguide can be represented as

$$E(x,z,t) = F(x,z)A(y)e^{i(k_0 z-\omega_0 t)}, \tag{7.1}$$

where A is the transverse profile of the propagating mode. In thin film this mode is described by the linear equation

$$A_{yy} + [k^2 f^2(y) - \beta_0^2]A = 0, \tag{7.2}$$

where $f^2(y)$ describes the profile of refraction index in the y-direction, β_0 is the propagation constant that corresponds to the ground state A. Here it is assumed that the waveguide supports only one propagating mode. Substituting this expression into the equation for the electric field

$$\Delta E + \frac{\omega^2}{c^2}(n_0 \pm n_2|E|^2)E = 0, \tag{7.3}$$

and averaging the wave equation over the transverse distribution we get the nonlinear Schrödinger equation for $F(x,z)$ of the form

$$2ik_0\tilde{F}_z + \tilde{F}_{xx} + 2k_0^2\sigma S|\tilde{F}|^2\tilde{F} = 0, \tag{7.4}$$

where $\tilde{F} = F\exp(i\beta_0^2 z/(2k_0))$, $S = \int A(y)^4 dy / \int A^2 dy$ and $\sigma = +1$, resp. -1, corresponds to a focusing and defocusing Kerr nonlinearity, respectively. Here the second derivative corresponds to spatial diffraction.

Introducing the dimensionless variables $z = z/z_d$, $x = x/x_0$, where $z_d = k_0 x_0^2$ is the diffraction length and x_0 is the characteristic size of the beam, and introducing the normalized amplitude $u(z,x) = \sqrt{k_0 z_d n_2 S/n_0}\tilde{F}$ we obtain

$$iu_z + \frac{1}{2}u_{xx} + \sigma|u|^2 u = V_1(x)u + V_2(x)|u|^2 u. \tag{7.5}$$

The unperturbed equation (7.5) (with zero right-hand side) supports soliton solutions in the focusing case $\sigma = +1$. The single spatial soliton solution describing the self-focused beam in the waveguide is

$$u_s(z,x) = 2\nu \mathrm{sech}(2\nu(x - 2\mu z) e^{2i\mu(x-2\mu z)+2i(\nu^2+\mu^2)z}. \tag{7.6}$$

The velocity 2μ is related to the angle θ in the (z,x)-plane of the beam propagation in the waveguide by the identity $2\mu = \tan(\theta)$.

We now address the effects of random fluctuations of the waveguide parameters on the soliton dynamics. We add in the right-hand side of Eq. (7.5) the functions $V_1(x)$ and $V_2(x)$ which describe transverse random inhomogeneities in the linear n_0 and nonlinear n_2 refractive indices. It is assumed that $\langle V_i(x)\rangle = 0$ and $\langle V_i(x)V_i(y)\rangle = D_i\delta(x-y)$, $i = 1,2$. In general the spatial random fluctuations could be dependent on both the x and z variables. The analysis of such fluctuations on pulse propagation will be presented in Subsection 7.3. In this section we consider in detail the simplified model when the random processes V_i depend only on the transverse variable x, since it is a fundamental model for many branches of physics. In distinction from the soliton propagation problem in optical fibers, where fluctuations were dependent on the evolutional variable (the time in standard NLSE), fluctuations in the transverse direction x correspond to the spatial variable in the standard NLSE and so the soliton dynamics is very different.

For the case of wave propagation in disordered 1D linear media the solution of the problem for the white noise disorder model is well known: the Anderson localization occurs with the localization length of the order of $L_{loc} \sim 8k^2/D_1$ for a pulse with carrier wavenumber k . The application of the IST approach to the randomly perturbed NLSE (see Appendix 1) gives equations for the evolution of soliton amplitude and velocity

$$\frac{d\nu}{dz} = F(\nu,\mu), \qquad \frac{d\mu}{dz} = G(\nu,\mu), \tag{7.7}$$

with the initial conditions $\nu(z=0) = \nu_0$, $\mu(z=0) = \mu_0$ corresponding to the input soliton. These equations take into account the emission of radiative waves. The expressions for F, G are complicated and are not shown here. The analysis performed by Garnier [1998] shows that two regimes exist:

1. For small-amplitude solitons $\nu_0 \ll \mu_0$, the soliton velocity is almost constant and the soliton amplitude decays exponentially

$$\nu(L) \approx \nu_0 e^{-L/L_1}, \qquad L_1 = \frac{32\mu_0^2}{D_1}$$

in the case of random linear refraction index V_1, and algebraically

$$\nu(L) \approx \frac{\nu_0}{(1+L/L_2)^{1/4}}, \qquad L_2 = \frac{3\mu_0^2}{4D_2\nu_0^4}$$

in the case of random nonlinear refraction index V_2.

2. If the soliton amplitude is large enough $\nu_0 \gg \mu_0$, then the soliton amplitude is almost constant and the soliton velocity decays to zero. The decay rate of the velocity is almost logarithmic.

It can be noted, that in the limit $\nu_0/\mu_0 \to 0$, the incoming soliton can be approximated by a linear wavepacket

$$u_0(z,x) \approx \int_{-\infty}^{\infty} dk f(k) e^{ikx - ik^2 z/2}, \quad \text{with } f(k) = \frac{1}{2}\cosh^{-1}\left(\frac{\pi}{4}\frac{(k-2\mu_0)}{\nu_0}\right),$$

whose spectrum is narrow around the carrier wavenumber $\tilde{k} = 2\mu_0$. The result for the small-amplitude soliton decay is in agreement with the linear approximation, since the length L_1 corresponding to a perturbation of the linear potential can be written in terms of the carrier wavenumber $\tilde{k}$ as $L_1 = 8\tilde{k}^2/D_1$. It is equal to the localization length of a narrowband pulse with carrier wavenumber $\tilde{k}$ scattered by a random slab.

Garnier [1998] also compares the predictions of the effective system (7.7) with full numerical simulations of the random NLSE. As shown by Figures 8,9 the agreement is very good.

The existence of two regimes in the soliton propagation in disordered media can be understood by comparing the different length scales existing in this problem. The characteristic length scales are the localization length L_{loc} and the nonlinear length L_{nl}. If the pulse energy is small (weak nonlinear regime), then the nonlinear length is larger then the localization length $L_{nl} > L_{loc}$, so that the exponential decay of the amplitude dominates. If the pulse energy is large, then $L_{nl} \ll L_{loc}$, and the disorder intensity is not enough to produce the localization and the decay of pulse is much weaker

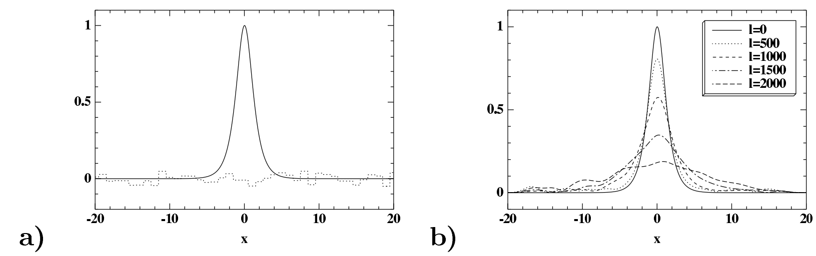

Figure 8: Picture a: Envelope of the initial soliton (solid line) whose initial coefficients are $\nu_0 = 0.5$, $\mu_0 = 0.4$. In dashed line is plotted a realization of the random potential V_1. Picture b: Soliton envelops when its center crosses different depth lines l for one of the realization of the random potential. The coordinate x is normalized around the depth line l.

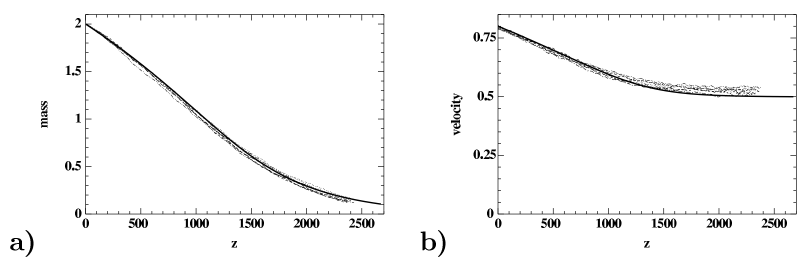

Figure 9: Evolution of the soliton whose initial coefficients are $\nu_0 = 0.5$, $\mu_0 = 0.4$. The mass $\mathcal{N} = 4\nu$ (resp. the velocity $v = 2\mu$) is plotted in picture a (resp. b). The thick solid lines represent the theoretical predictions. In thin dashed and dotted lines are plotted the simulated masses and velocities for 7 different realizations of the random potential.

then the exponential decay. If $L_{nl} \sim L_{loc}$ then for short propagation distances the soliton amplitude decay is moderate and for large propagation distances the exponential decay takes place.

7.2. SPATIAL SOLITONS UNDER RANDOM DISPERSIVE PERTURBATIONS

Spatial random dispersive perturbations naturally arise when the electromanetic beam propagation in planar waveguide arrays with randomly varying tunnel-coupling term is considered. The evolution of the amplitude of the complex electromagnetic field is described by the equation

$$iu_{n,z} + V_{n,n+1}u_{n+1} + V_{n,n-1}u_{n-1} + |u_n|^2 u_n = 0, \tag{7.8}$$

where $V_{n,n\pm1}$ are the linear tunnel coupling coefficients. Here two types of disorder are possible.

If we have disorder in the z-direction (i.e. $V_{n,n\pm1} = V_{n,n\pm1}(z)$) then we deal with the discrete analog of dispersion management in optical fibers - the so-called diffraction management for spatial solitons. It was studied recently by Ablowitz and Musslimani [2001].

If we have disorder in the x-direction ($x = nh$), then we deal with the spatial soliton evolution under spatial diffractive perturbations. If the evelope occupies a sufficiently large number of waveguides (typically ≥ 6), then we can apply a continuum limit and obtain the equation

$$iu_z + u_{xx} + 2|u|^2 u = \alpha V(x)u_{xx} + \beta V_x u_x + \gamma V_{xx} u, \tag{7.9}$$

with $\alpha = \beta = 1$, $\gamma = 2$ (see Mischall, Schmidt-Hattenberger, and Lederer [1994]). $V(x)$ is now assumed to be a coloured noise with the correlation function

$$\langle V(x)V(y)\rangle = R(x - y; l_c), \tag{7.10}$$

where l_c is the correlation length. To fix the ideas we can assume a noise with Gaussian autocorrelation function $R(x) = \sigma^2 \exp(-x^2/l_c^2)$, whose power spectral density is $\hat{R}(k) = \sqrt{\pi}(\sigma^2 l_c \exp(-k^2 l_c^2/4)$. For long propagation distances $z \sim 1/(\sigma^2 l_c)$ the IST analysis performed by Abdullaev and Garnier [1999] again predicts two regimes:

1. In the small soliton amplitude case $\nu_0 \ll \mu_0$, the soliton velocity is almost constant and the decay law for the soliton amplitude is exponential i.e.

$$\nu(L) \approx \nu_0 \exp(-L/L_1), \qquad L_1^{-1} = \mu_0^2 \hat{R}(4\mu_0)((\alpha + 4\gamma)^2 + 4\beta^2). \tag{7.11}$$

Two peaks exist in the spectrum of scattered radiation: a main peak at $k_{r,1} = -2\mu_0$ and a secondary peak at $k_{2,r} = 2\mu_0$ with the energy $\sim \nu^3 \hat{R}(0)$. If the carrier wavenumber of the soliton lies in the tail of the spectrum of perturbation then the contribution of the second peak dominates and the decay rate of the soliton amplitude is

$$\nu(L) \approx \frac{\nu_0}{(1 + L/L_2)^{1/2}}, \qquad L_2^{-1} = \frac{8}{27}\hat{R}(0)\nu_0^2\alpha^2. \tag{7.12}$$

The appearance of this law is connected with the random perturbation of the dispersion term $V(x)u_{xx}$. When ν becomes small enough the decay rate is again exponential.

2. In the large soliton amplitude case, $\nu_0 \gg \mu_0$, the soliton amplitude remains almost constant during the propagation, but the velocity decays according to the law

$$\mu(L) \approx \frac{\pi\nu_0}{2\ln(L)}. \tag{7.13}$$

If the spectrum of the noise decays faster than the Gaussian, then the velocity decays as

$$\mu(L) \approx \frac{\nu_0^2 l_c}{2\sqrt{\ln(L)}}. \tag{7.14}$$

Recently experimental beam propagation in a disordered 2D tunnel-coupled fiber array has been performed by Pertsch, Peschel, Kobelke, Schuster, Bartelt, Nolte, Tunnermann, and Lederer [2004]. The light dynamics in the hexagonal lattice is described by the coupled mode equations

$$\begin{aligned} &(i\partial_z + \beta_{mn} + \chi^{(3)}_{eff}|u_{mn}|^2)u_{mn} + V^{(1)}_{m,n}u_{m+1,n} \\ +&V^{(1)}u_{m-1,n}u_{m-1,n} + V^{(2)}_{m,n}u_{m,n+1} + V^{(2)}_{m,n-1}u_{m,n-1} \\ &+V^{(3)}_{m,n}u_{m+1,n+1} + V^{(3)}u_{m-1,n-1}u_{m-1,n-1} = 0. \end{aligned}$$

where $\chi^{(3)_{eff}} = 5.5$ km^{-1} W^{-1}. The coupling coefficient fluctuates around the mean value $V_0 = 46$ m^{-1} with $\sqrt{\langle(\delta V)^2\rangle} = 26.7$ m^{-1}. The stochastic variations of the core diameter across the fiber array leads to random variations of the propagation coefficients with $\sqrt{\langle(\delta\beta^2\rangle} = 140$ m$^{-1} \approx 3V_0$. Although these fluctuations are 5 orders of magnitude smaller than the mean value, they have a dramatic influence on the field evolution. Indeed the standard deviation is of the order of the width of the band of the homogeneous array $9V_0$. The mean value $\beta_0 = 11.4\ 10^6$ m^{-1} can be removed by a simple transformation of the field in the equations, but the fluctuations cannot.

It was observed that the linear modes become localized with a strong enough disorder - the Anderson localization phenomenon, predicted for random arrays by Abdullaev and Abdullaev [1980]. When the input power increases the linearly localized states evolves in two different ways. Some linear modes first expand and later contract when the power is increased. Other modes immediately contract. The behavior actually depends on the wavenumber of the localized state. If it in the upper part of the linear band, then the increasing power shifts the wavenumber out of the linear spectrum. If the initial wave number is at the lower edge of the linear band, then the nonlinearity shifts the wavenumber inside the the band. The influence of nonlinearity on the delocalized state leads initially to a slight expansion of the beam with the rapid collapsing to a single guide and as a result to the formation of a discrete optical soliton. The explanation of this experiment requires the theoretical investigation of the dynamics of discrete solitons in disordered lattices. Some results in this direction have been obtained by Garnier [2001], Kopidakis and Aubry [2000].

7.3. PULSE PROPAGATION IN NONLINEAR WAVEGUIDES WITH FLUCTUATING REFRACTION INDEX

In this section we consider the influence of the fluctuations of the medium parameters on the dynamics of intense optical pulses in nonlinear waveguides. We address the case of the spatio-temporal evolution of intense optical pulses in nonlinear waveguides with a randomly varying linear refraction index [Gaididei and Christiansen [1998]]. The pulse propagation is described by the 2D nonlinear Schrödinger equation

$$iu_z + \Delta u + |u|^2 u + \eta(x,z)u = 0, \qquad (7.15)$$

where $\Delta = \partial_x^2 + \partial_t^2$ and $\eta(x,z)$ models the fluctuations of the linear refraction index. It is well known that, when the fluctuations are absent, solutions of 2D NLSE (7.15) become singular if the initial energy excesses some critical value i.e. $\mathcal{E} > \mathcal{E}_c = 11.7$, where $\mathcal{E} = \int dx \int dt \rho$, $\rho = |u|^2$. The unperturbed Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt [|u_t|^2 + |u_x|^2 - \frac{1}{2}|u|^4]. \qquad (7.16)$$

It is interesting to investigate the influence of noise in $\eta(x,z)$ on the collapse of intense optical pulses. In the case where the random process is

δ-correlated with respect to z it is possible to obtain exact results by applying the virial theorem. We assume that η is a zero-mean Gaussian random process with autocorrelation function $\langle \eta(x,z), \eta(x',z') \rangle = C(x-x')\delta(z-z')$. The analysis is based on the equation satisfied by the second virial integral

$$\mathcal{I}(z) = \frac{1}{\mathcal{E}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt (x^2 + t^2)\rho \tag{7.17}$$

which is the pulse width. Together with $\mathcal{H}$ it obeys the differential equation

$$\begin{aligned} \frac{\mathcal{E}}{4} \frac{d^2 \mathcal{I}}{dz^2} &= 2\mathcal{H} + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \rho \eta_x, \\ \frac{d\mathcal{H}}{dz} &= \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \eta \rho_z . \end{aligned} \tag{7.18}$$

Applying the Furutsu-Novikov formulae to perform averaging over fluctuations [Klyatzkin [1980]]

$$\langle \eta(z) R(\eta) \rangle = \int_0^z dz' \, \langle \eta(z) \eta(z') \rangle \left\langle \frac{\delta R}{\delta \eta(z')} \right\rangle , \tag{7.19}$$

and using the principle of causality, we get the system of equations for the averaged values $\langle \mathcal{I} \rangle (z)$ and $\langle \mathcal{H} \rangle (z)$

$$\mathcal{E} \frac{d^2 \langle \mathcal{I} \rangle}{dz^2} = 8 \langle \mathcal{H} \rangle (z), \qquad \frac{d \langle \mathcal{H} \rangle}{dz} = \kappa \mathcal{E}, \tag{7.20}$$

where $\kappa = -C''(0) > 0$. By integrating equation (7.20) we get that the influence of fluctuations on the pulse width is described by

$$\langle \mathcal{I} \rangle (z) = I_0^2 + 4\epsilon z^2 + \frac{4}{3} \kappa z^3, \tag{7.21}$$

where $\mathcal{I}(0) = I_0$, $\mathcal{I}_z(0) = 0$, and $\mathcal{H}(0) = \epsilon \mathcal{E}$. If $\kappa = 0$ (no fluctuations) and $\epsilon < 0$, then $\mathcal{I}(z) = I_0(1 - z^2/z_c^2)$, which shows that collapse occurs before the critical propagation distance $z_c = \sqrt{I_0}/(2\sqrt{|\epsilon|})$. Fluctuations leads to the *increase* of this distance. In particular, if $\kappa > \kappa_{cr}$ with

$$\kappa_{cr} = \frac{4|\epsilon|^{3/2}}{\sqrt{3 I_0}}, \tag{7.22}$$

then $\langle \mathcal{I} \rangle (z) \geq 0$ for all $z > 0$, and the collapse on average is arrested. As a result the pulse will spread instead of collapsing.

§ 8. Two-dimensional solitons in random media

In this section we consider the influence of random fluctuations of the transverse profile of the waveguide on the dynamics of 2D solitons. The governing equation is the randomly perturbed 2D NLSE

$$iu_z + \Delta_\perp u + |u|^2 u + \epsilon_1 g(z)(x^2+y^2)u - \epsilon_2 |u|^4 u = 0, \tag{8.1}$$

where $\Delta_\perp = \partial_x^2 + \partial_y^2$ is the transverse Laplacian, $g(z) = g_0(1+\eta(z))$ is the amplitude of the quadratic potential imposed by the waveguide and $\eta(z)$ is a random process that models the fluctuations of the waveguide. The quintic nonlinear term can appear for example by expanding a saturable nonlinearity. A modulation theory has been proposed by Fibich and Papanicolaou [1999] for the analysis of the 2D soliton dynamics. The unperturbed 2D NLS equation ($\epsilon_1 = \epsilon_2 = 0$) has waveguide solutions of the form $u(r,z) = e^{iz} R_T(r)$, where the function $R_T(r)$ satisfies the boundary value problem

$$R_T'' + r^{-1} R_T' - R_T + R_T^3 = 0, \quad R_T'(0) = 0, \quad R_T(\infty) = 0. \tag{8.2}$$

The solution of this equation with the lowest energy (power) is called Townes soliton. It plays an important role in self-focusing theory and it has exactly the critical power for self-focusing:

$$\mathcal{E}_T \equiv \int_0^\infty R_T^2(r) r dr = \mathcal{E}_c \equiv 1.862, \tag{8.3}$$

while its Hamiltonian is equal to 0

$$\mathcal{H}_T = \int_0^\infty \left[(R_T')^2 - \frac{1}{2} R_T^4(r) \right] r dr = 0. \tag{8.4}$$

The solution of the perturbed 2D NLSE is searched for in the form of a modulated Townes soliton

$$u(r,z) \approx \frac{1}{a(z)} R_T\left(\frac{r}{a(z)}\right) e^{iS}, \qquad S = \sigma + \frac{a_z r^2}{4a}, \qquad \sigma_z = a^{-2}. \tag{8.5}$$

If the initial power is just above the critical value, then it is possible to derive an evolution equation for the function $a(z)$ starting from the approximation (8.5). The equation of modulation theory for the width $a(z)$ is

$$a^3 a_{zz} = -\beta_0 - \frac{1}{2M_0} f_1(z), \tag{8.6}$$

where $\beta_0 = \beta(0) - f_1(0)/(2M_0)$, $\beta(0) = (\mathcal{E} - \mathcal{E}_c)/M_0$, and $M_0 \equiv (1/4)\int_0^\infty r^3 dr R_T^2 \approx 0.55$. The auxiliary function is given by

$$f_1(z) = 2a(z)\mathrm{Re}\left[\frac{1}{2\pi}\int dxdy F(u)e^{-iS}(R_T + \rho\nabla R_T(\rho))\right], \qquad (8.7)$$

where $F(u) = -\epsilon_1 g(z)(x^2+y^2)u + \epsilon_2|u|^4 u$. In the lowest order approximation, the equation for the width takes the form

$$a_{zz} + 4\epsilon_1 g(z)a + U'(a) = 0, \qquad (8.8)$$

where the potential U is

$$U(a) = \frac{\epsilon_2 \mathcal{E}_c}{M_0 a^4} - \frac{\beta_0}{2a^2}. \qquad (8.9)$$

Similar equations appear during the study of 2D and 3D Bose-Einstein condensates (BECs) with time-varying trap potential [Abdullaev, Baizakov, and Konotop [2001], Abdullaev, Bronski, and Galimzyanov [2003], Garnier, Abdullaev and Baizakov [2004]]. Random modulations impose a distortion of the BEC due to the growths of focusing-defocusing oscillations. This destabilizing mechanism is similar to the random Kepler problem studied in the previous sections.

§ 9. Conclusion

In this review we have considered the propagation and interaction of optical solitons in random media. The dynamics of optical solitons in optical fibers with randomly varying dispersion, amplification and birefringence has been studied. The propagation of dispersion managed solitons in fiber links with random dispersion maps has been investigated. The influence of the fluctuations on the transverse variables has been considered for spatial solitons in waveguides and 2D optical bullets.

Some problems have been not addressed in the review. First, we have not considered the evolution of dark solitons in random fibers and bulk media. While recently some numerical results for dark solitons in the random dispersion-managed regime has been obtained by Hong and De-Xiu [2003], this class of problems is still mainly uninvestigated. Second, we have not considered the dynamics of partially coherent solitons, which has been intensively investigated recently, in particular in photorefractive crystals (see the recent review by Chen, Segev, and Christodoulides [2000b]).

However this problem involves the evolution of initial random conditions in nonlinear media and it is out of scope of this review.

It should be noted that we have mainly modeled the soliton propagation in random media by stochastic perturbations of integrable nonlinear wave equations such as the NLSE and the Manakov system. Fortunately many physical problems are described by these equations. The case of stochastic perturbations of nonintegrable wave equations, such as the system for FW and SH waves, random dispersion-managed solitons, and multidimensional solitons is still under investigation.

It should be of great interest to extend the approaches described in this review to the propagation of dissipative optical solitons in random media. Early work based on the adiabatic approximation show that for such solitons the noise effects are essentially suppressed (see the book by Hasegawa and Kodama [1995] and a recent work by Abdullaev, Navotny, and Baizakov [2004]). It should be also interesting to investigate the properties of discrete optical solitons in disordered arrays of planar waveguides and 2D fiber arrays. Such investigations are important for studying the interplay between nonlinearity, disorder, and discreteness. The theory of discrete optical solitons for the deterministic systems of fiber arrays has been developed by Aceves, Luther, De Angelis, Rubenchik, and Turitsyn [1995]. We should also note the recent experiments performed by Pertsch, Peschel, Kobelke, Schuster, Bartelt, Nolte, Tunnermann, and Lederer [2004] where properties of discrete solitons in disordered 2D fiber arrays have been studied.

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§ 10. Appendix 1. The Inverse Scattering Transform for the Nonlinear Schrödinger equation

10.1. THE UNPERTURBED NONLINEAR SCHRÖDINGER EQUATION

For more detail about the following statements and their proofs we refer to [Ablowitz and Segur [1981], Manakov, Novikov, Pitaevskii, and Zakharov [1984]]. Let us consider the unperturbed NLSE:

$$iu_z + \frac{1}{2}u_{tt} + |u|^2 u = 0. \tag{10.1}$$

The inverse scattering transform consists in a linearization of this nonlinear equation. It is based on the fact that $u(z,.)$ can be characterized by a set of spectral data of the operator $L(u(z,.))$ in which u plays the role of a potential:

$$L(u) = iP\frac{\partial}{\partial t} + Q(u), \text{ with } P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Q(u) = \begin{pmatrix} 0 & u^* \\ -u & 0 \end{pmatrix}.$$

Essential spectrum of the operator $L(u)$. Let us consider the spectral problem associated with the operator $L(u)$:

$$L(u)\psi = \lambda\psi, \qquad \psi = \psi_1\mathbf{e_1} + \psi_2\mathbf{e_2}, \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.2}$$

We introduce the so-called Jost functions f and g, defined as the eigenfunctions of $L(u)$ associated with the real eigenvalue λ which satisfy the following boundary conditions:

$$f(t,\lambda) \overset{t\to+\infty}{\longrightarrow} \mathbf{e}_2 e^{i\lambda t}, \quad g(t,\lambda) \overset{t\to-\infty}{\longrightarrow} \mathbf{e}_1 e^{-i\lambda t}.$$

If we denote by $\bar{\psi}$ the vector $(\psi_2^*, -\psi_1^*)$ associated with a vector ψ solution of (10.2), then $\bar{\psi}$ is a solution of $L\bar{\psi} = \lambda^*\bar{\psi}$. In the case of a real eigenvalue, ψ and $\bar{\psi}$ are linearly independent and form a base of the space of the solutions of (10.2). It can then be proved that the Jost functions are related by:

$$g(t,\lambda) = a(\lambda)\bar{f}(t,\lambda) + b(\lambda)f(t,\lambda), \quad f(t,\lambda) = -a(\lambda)\bar{g}(t,\lambda) + b^*(\lambda)g(t,\lambda).$$

Multiplying the first identity by the vector $\bar{f}^*$, we get an explicit representation of the coefficient a as the Wronskian of f and g:

$$a(\lambda) = g_1(t,\lambda)f_2(t,\lambda) - g_2(t,\lambda)f_1(t,\lambda).$$

Substituting the second identity into the first one we also get the conservation relation $|a|^2 + |b|^2 = 1$.

Point spectrum of the operator $L(u)$. It is possible to define an analytic continuation of $a(\lambda)$ over the upper complex half-plane. A noticeable feature then appears. If λ_r is a zero of $a(\lambda)$, then f and g are linearly dependent, so there exists a coefficient ρ_r such that $g(t, \lambda_r) = \rho_r f(t, \lambda_r)$. The corresponding eigenfunction is bounded and decays exponentially as $t \to +\infty$ (because $|f| \sim e^{-\mathrm{Im}\lambda_r t}$) and as $t \to -\infty$ (because $|g| \sim e^{+\mathrm{Im}\lambda_r t}$). Thus λ_r is an element of the point spectrum of $L(u)$. It can then be proved that the set $(a(\lambda), b(\lambda), \lambda_r, \rho_r, a'(\lambda_r))$ characterizes the Jost functions f and g as well as the solution u. The inverse transform is essentially based on the resolution of the linear integro-differential Gelfand-Levitan-Marchenko (GLM) equation, whose entries are constituted by the set $(a, b, \lambda_r, \rho_r, a'(\lambda_r))$:

$$K_1(s,t) = \Phi^*(s+t) - \int_s^\infty K_1(s,t'') \int_s^\infty \Phi^*(t+t')\Phi(t'+t'')dt'dt'' \quad (10.3)$$

$$\text{where } \Phi(t) = -\sum_r \frac{i\rho_r}{a'(\lambda_r)} e^{i\lambda_r t} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{b(\lambda)}{a(\lambda)} e^{i\lambda t} d\lambda. \quad (10.4)$$

We then obtain u by the formula $u(t) = -2iK_1^*(t,t)$. The study of the inverse problem associated with the operator $L(u)$ has not yet been completely achieved. In particular the precise characterization of the spectral data which lead to well-defined potentials u has not yet been completed. However, in the case where the initial condition u_0 is rapidly decaying so that it satisfies $t \mapsto |t|^n |u_0|(t) \in L^1$ for any n, the inverse scattering can be rigorously achieved [Ablowitz and Segur [1981]].

Evolution of the scattering data. The evolution equations of the scattering data are simple and uncoupled for the unperturbed NLSE:

$$\partial_z a(z,\lambda) = 0, \quad \partial_z b(z,\lambda) = -2i\lambda^2 b(z,\lambda), \quad \partial_z \rho_r(z) = -2i\lambda_r^2 \rho_r.$$

To sum up, the scattering transform involves the following operations:

$$\begin{array}{ccc} u(z_0,t) & \overset{\text{direct scatt.}}{\longrightarrow} & (a,b,\lambda_r,\rho_r,a'(\lambda_r))\,(z_0) \\ \text{NLSE} \downarrow & & \downarrow \text{uncoupled evolution equations} \\ u(z,t) & \overset{\text{inverse scatt.}}{\longleftarrow} & (a,b,\lambda_r,\rho_r,a'(\lambda_r))\,(z) \end{array}$$

Soliton. The soliton solution of the NLSE is:

$$u_S(z,t) = 2\nu \frac{\exp i\left(2\mu(t-2\mu z) + 2(\nu^2+\mu^2)z\right)}{\cosh\left(2\nu(t-2\mu z)\right)}. \quad (10.5)$$

The scattering coefficients are: $a(\lambda) = (\lambda - \lambda_S)/(\lambda - \lambda_S^*)$, where $\lambda_S = \mu + i\nu$ is the unique zero of a in the upper complex half-plane, while $b = 0$.

Conserved quantities. There exists an infinite number of quantities which are preserved by the unperturbed NLSE [Manakov, Novikov, Pitaevskii, and Zakharov [1984]]. They can be represented as functionals of the solution u or in terms of the scattering data. We present only two of them. The energy of the wave $\mathcal{E} = \int |u|^2 dt$ can be written as

$$\mathcal{E} = \sum_r 2i(\lambda_r^* - \lambda_r) + \int n(\lambda) d\lambda, \tag{10.6}$$

where $n(\lambda) = -\pi^{-1} \ln |a(\lambda)|^2$. The Hamiltonian $\mathcal{H} = (1/2) \int |u_t|^2 - |u|^4 dt$ can also be expressed as

$$\mathcal{H} = \sum_r \frac{4i}{3} ({\lambda_r^*}^3 - {\lambda_r}^3) + 2 \int \lambda^2 n(\lambda) d\lambda. \tag{10.7}$$

10.2. PERTURBATION THEORY FOR THE NLSE

We consider a perturbed NLSE that reads in dimensionless units as:

$$i u_z + \frac{1}{2} u_{tt} + |u|^2 u = \varepsilon R(u). \tag{10.8}$$

In the framework described in Section 3 u represents the complex envelope of the electromagnetic field in a cubic nonlinear medium. The initial condition corresponds to a single-soliton state. If ε is small we can assume that the propagating wave consists of one soliton plus radiation. The random perturbation $R(u)$ induces variations of the spectral data. The evolution of the continuous component corresponding to radiation can be found from the evolution equations of the Jost coefficients [Karpman [1979]]:

$$\frac{\partial a(\lambda, z)}{\partial z} = -\varepsilon \left(a(\lambda, t) \bar{\gamma}(\lambda, z) + b(\lambda, z) \gamma(\lambda, z) \right), \tag{10.9}$$

$$\frac{\partial b(\lambda, z)}{\partial z} = -2i\lambda^2 b(\lambda, z) + \varepsilon \left(a(\lambda, z) \gamma^*(\lambda, z) + b(\lambda, z) \bar{\gamma}(\lambda, z) \right), \tag{10.10}$$

where $\gamma(\lambda, z) = \int dt R(u)^* f_2^2 + R(u) f_1^2$ and $\bar{\gamma}(\lambda, z) = \int dt R(u) f_1 {f_2}^* - R(u)^* {f_1}^* f_2$. The evolutions of the soliton parameters can then be derived from the conservation of some integrals of motion, such as the energy.

We can obtain the expression of the total wave by the following way. The function Φ which appears in (10.4) is equal to the sum $\Phi_S + \Phi_L$ with:

$$\Phi_S(z,t) = \frac{-i\rho_S}{a'(\lambda_S)} e^{i\lambda_S t}, \quad \Phi_L(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{b(z,\lambda)}{a(z,\lambda)} e^{i\lambda t} d\lambda.$$

Φ_L plays the role of a perturbation of Φ. The kernel K associated with the total wave $u(z,\cdot)$ can be represented as the sum $K_S + K_L$, where:

$$K_S(s,t) = \frac{\nu \exp\left((\nu + i\mu)(s-t)\right)}{\cosh\left(2\nu(s - t_S)\right)} \begin{pmatrix} -i \exp i\left(2\mu(t_S - s) - \phi_S\right) \\ -\exp\left(2\nu(t_S - s)\right) \end{pmatrix},$$

where $\lambda_S = \mu + i\nu$,

$$t_S = \frac{1}{2\nu} \ln\left(\frac{1}{2\nu}\left|\frac{\rho_S}{a'(\lambda_S)}\right|\right), \quad \phi_S = -\arg\left(\frac{\rho_S}{a'(\lambda_S)}\right) + 2\mu t_S. \qquad (10.11)$$

Neglecting the terms of higher order, K_{L1} is solution of:

$$K_{L1}(s,t) + \int_s^\infty K_{L1}(s,t'') \int_s^\infty \Phi_S^*(t+t')\Phi_S(t'+t'')dt'dt'' = \Psi(s,t),$$

$$\Psi(s,t) = \Phi_L^*(s+t) - \int_s^\infty K_{S1}(s,t'') \int_s^\infty \left[\Phi_S^*(t+t')\Phi_L(t'+t'') + \Phi_L^*(t+t')\Phi_S(t'+t'')\right] dt'dt''.$$

Following the IST method, we obtain the transmitted wave by the formula $u(z,t) = -2iK_1^*(t,t)$, which establishes that the total wave is given by the sum $u(z,t) = u_S(z,t) + u_L(z,t)$, where u_S is a soliton of energy 4ν and velocity 4μ:

$$u_S(z,t) = 2\nu \frac{\exp i\left(2\mu(t - t_S) + \phi_S\right)}{\cosh\left(2\nu(t - t_S)\right)}, \qquad (10.12)$$

which shows that t_S and ϕ_S are respectively the position and the phase of the soliton. u_L admits the following expression:

$$\begin{aligned} u_L(z,t) = \; & \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{b}{a}(\lambda) \frac{\left(\lambda - \mu + i\nu \tanh\left(2\nu(t - t_S)\right)\right)^2}{(\lambda - \mu + i\nu)^2} e^{2i\lambda t} d\lambda, \\ & - \frac{\nu^2}{i\pi} \frac{\exp 2i\left(2\mu(t - t_S) + \phi_S\right)}{\cosh^2\left(2\nu(t - t_S)\right)} \int_{-\infty}^{\infty} \frac{b^*}{a^*}(\lambda) \frac{1}{(\lambda - \mu - i\nu)^2} e^{-2i\lambda t} d\lambda. \end{aligned} \qquad (10.13)$$

§ 11. Appendix 2. The Inverse Scattering Transform for the Manakov system

11.1. THE UNPERTURBED MANAKOV SYSTEM

We consider the Manakov system that reads in dimensionless units as:

$$iu_z + \frac{1}{2}u_{tt} + \left(|u|^2 + |v|^2\right) u = 0, \tag{11.1}$$

$$iv_z + \frac{1}{2}v_{tt} + \left(|u|^2 + |v|^2\right) v = 0. \tag{11.2}$$

This model can be derived from the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\int \left(|u_t|^2 + |v_t|^2\right) - \left(|u|^2 + |v|^2\right)^2 dt. \tag{11.3}$$

Direct transform: the scattering problem. The scattering problem associated with the Manakov system is the following eigenvalue problem [Manakov [1974]]:

$$\partial_t |f\rangle = \mathcal{L}|f\rangle, \qquad \mathcal{L} := \begin{pmatrix} -i\lambda & iu^* & iv^* \\ iu & i\lambda & 0 \\ iv & 0 & i\lambda \end{pmatrix}. \tag{11.4}$$

If $\lambda \in \mathbb{R}$ then we can construct two sets of special solutions of (11.4), denoted by $|\phi_i(t,\lambda)\rangle$ and $|\psi_i(t,\lambda)\rangle$ $(i = 1, 2, 3)$ with the asymptotic behavior:

$$|\phi_i\rangle_j = \delta_{ij}\exp(-iI_j\lambda t), \quad t \to -\infty, \qquad\qquad |\psi_i\rangle_j = \delta_{ij}\exp(-iI_j\lambda t), \quad t \to \infty,$$

with $I_1 = 1$ and $I_2 = I_3 = -1$. The kets $|\phi_i\rangle$ and $|\psi_i\rangle$ are known as the Jost functions. They both represent a complete set of solutions to the problem (11.4) for the eigenvalue $\lambda \in \mathbb{R}$. Hence we may write the elements of one basis in terms of the other basis:

$$|\phi_i(t,\lambda)\rangle = \sum_{j=1}^{3} \alpha_{ij}(\lambda)|\psi_j(t,\lambda)\rangle.$$

From the equivalence $\alpha_{ij}(\lambda) = \langle\psi_j|\phi_i\rangle$ we derive the conservation relations for $i = 1, 2, 3$: $\sum_{l=1}^{3}|\alpha_{il}|^2 = 1$, while for $i \neq j$ $\sum_{l=1}^{3}\alpha_{il}^*\alpha_{jl} = 0$.

The functions $|\phi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ can be analytically continued in the upper complex half-plane $\mathrm{Im}(\lambda) > 0$, whereas the same holds true in the lower half-plane for the functions $|\phi_2\rangle$, $|\psi_3\rangle$, and $|\psi_1\rangle$. The analyticity of the Jost functions implies the analyticity of $\alpha_{11}(\lambda) = \langle\psi_1|\phi_1\rangle$ in $\mathrm{Im}(\lambda) > 0$. Let us denote by λ_k, $k = 1, ..., N$ the zeros of the function α_{11}. For such an eigenvalue λ_k we have:

$$|\phi_1(t, \lambda_k)\rangle = \rho_{k2}|\psi_2(t, \lambda_k)\rangle + \rho_{k3}|\psi_3(t, \lambda_k)\rangle.$$

Since $|\phi_1(t, \lambda_k)\rangle$ decays exponentially to 0 as $t \to -\infty$ and $|\psi_l(t, \lambda_k)\rangle$, $l = 2, 3$, both decay exponentially to 0 as $t \to \infty$, we get that the zeros of the function α_{11} correspond to the discrete eigenvalues of the problem (11.4).

The main point of the IST method is that the solutions u and v that play the roles of potentials in the eigenvalue problem are completely characterized by the set of scattering data:

$$\mathcal{S} = \left\{ (\lambda_k, \gamma_{2k}, \gamma_{3k}), k = 1, ..., N, \frac{\alpha_{12}}{\alpha_{11}}(\lambda), \frac{\alpha_{13}}{\alpha_{11}}(\lambda), \lambda \in \mathbb{R} \right\},$$

where $\gamma_{lk} = \rho_{lk}/\alpha'_{11}(\lambda_k)$, $l = 2, 3$. As in the case of the scalar NLSE, the evolution equations of the scattering data are uncoupled:

$$\partial_z \alpha_{11}(\lambda) = 0, \quad \lambda \in \mathbb{R}, \tag{11.5}$$

$$\partial_z \alpha_{1l}(\lambda) = -2i\lambda^2 \alpha_{1l}(\lambda), \quad \lambda \in \mathbb{R}, \quad l = 2, 3, \tag{11.6}$$

$$\partial_z \gamma_{lk} = -2i\lambda_k^2 \gamma_{lk} \tag{11.7}$$

The inverse transform: the GLM equation. The inverse transform proposed by Manakov [1974] is based on concepts used to solve standard Riemann-Hilbert problems. Since global results are often difficult to obtain from such equations, we prefer to go to a Gel'fand-Levitan-Marchenko representation. The inverse problem then reads as the resolution of a system of linear integro-differential equations, whose entries are constituted by the set $\mathcal{S}$ of scattering data:

$$\mathbf{K}^{(1)}(s, t) = \sum_{j=2}^{3} F_j(s + t)\mathbf{e}_j + \int_s^\infty \mathbf{K}^{(j)}(s, t')F_j(t' + t)dt', \tag{11.8}$$

$$\mathbf{K}^{(j)}(s, t) = F_j^*(s + t)\mathbf{e}_1 - \int_s^\infty \mathbf{K}^{(1)}(s, t')F_j^*(t' + t)dt', \quad j = 2, 3 \tag{11.9}$$

$$F_l(t) = -\sum_r i\gamma_r e^{i\lambda_r t} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\alpha_{1l}(\lambda)}{\alpha_{11}(\lambda)} e^{i\lambda t} d\lambda, \tag{11.10}$$

where $\mathbf{e}_j = (\delta_{ij})_{i=1,2,3}$. We can get the Jost functions $|\psi_j\rangle$ from the kernels $\mathbf{K}^{(j)}$ solutions of the GLM equations:

$$|\psi_j\rangle(t,\lambda) = \mathbf{e_j} e^{-I_j i\lambda t} + \int_t^\infty \mathbf{K}^{(j)}(t,s) e^{-iI_j\lambda s} ds, \quad j = 1,2,3.$$

with $I_1 = 1$ and $I_2 = I_3 = -1$. We then obtain (u,v) by the formula

$$u(t) = -2iK_1^{(2)}(t,t)^*, \qquad v(t) = -2iK_1^{(3)}(t,t)^*. \tag{11.11}$$

Accordingly it is sufficient to solve the two following equations:

$$\begin{aligned} K_1^{(j)}(s,t) &= F_j^*(s+t) - \sum_{l=2}^{3} \int_s^\infty K_1^{(l)}(s,t') G_{lj}(s,t,t') dt', \quad j = 2,3, \\ G_{jl}(s,t,t') &= \int_s^\infty F_j(t'+t'') F_l^*(t''+t) dt''. \end{aligned} \tag{11.12}$$

Soliton. The soliton solution of the Manakov system is:

$$\begin{pmatrix} u_S \\ v_S \end{pmatrix} = 2\nu \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \frac{\exp i\left(2\mu(t-2\mu z) + 2(\nu^2+\mu^2)z\right)}{\cosh\left(2\nu(t-2\mu z)\right)}. \tag{11.13}$$

The scattering coefficients are: $\alpha_{11}(\lambda) = (\lambda - \lambda_S)/(\lambda - \lambda_S^*)$, where $\lambda_S = \mu + i\nu$ is the unique zero of α_{11} in the upper complex half-plane, while $\alpha_{12} = \alpha_{13} = 0$.

Conserved quantities. Conserved quantities can be worked out as in any integrable system. The total energy and Hamiltonian [defined by (11.3)], and more generally all conserved quantities, can be expressed in terms of scattering data. Let us define

$$n(\lambda) := \frac{1}{\pi} \log\left(1 - \left|\frac{\alpha_{12}}{\alpha_{11}}\right|^2(\lambda) - \left|\frac{\alpha_{13}}{\alpha_{11}}\right|^2(\lambda)\right), \tag{11.14}$$

for $\lambda \in \mathbb{R}$. The energy and Hamiltonian can be decomposed into the sums of continuous parts and discrete parts:

$$\mathcal{E} = \int |u|^2 + |v|^2 dt = \int_{\mathbb{R}} n(\lambda) d\lambda + 4\sum_{j=1}^{J} \nu_j, \tag{11.15}$$

$$\mathcal{H} = 2\int_{\mathbb{R}} \lambda^2 n(\lambda) d\lambda + 8\sum_{j=1}^{J} \left(\nu_j \mu_j^2 - \frac{\nu_j^3}{3}\right). \tag{11.16}$$

11.2. PERTURBATION THEORY FOR THE MANAKOV SYSTEM

We consider a perturbed Manakov system that reads in dimensionless units as:

$$iu_z + \frac{1}{2}u_{tt} + \left(|u|^2 + |v|^2\right) u = \varepsilon R_u(u, v), \qquad (11.17)$$

$$iv_z + \frac{1}{2}v_{tt} + \left(|u|^2 + |v|^2\right) v = \varepsilon R_v(u, v). \qquad (11.18)$$

In the framework described in Section 5.1 u and v represent the complex envelopes of the two orthogonal polarizations of a transverse electromagnetic field in a cubic nonlinear medium.

The evolution equations of the Jost coefficients are ($n = 1, 2, 3$):

$$\frac{d\alpha_{1n}}{dz} = i \int u_z \phi_{11} \psi_{n2}^* + u_z^* \phi_{12} \psi_{n1}^* + v_z \phi_{11} \psi_{n3}^* + v_z^* \phi_{13} \psi_{n1}^* dt. \qquad (11.19)$$

Assume that the total wave consists of one soliton $\lambda_S = \mu + i\nu$ plus radiation. The functions F_2 and F_3 which appear in (11.12) are equal to the sum $F_{Sj} + F_{Lj}$ with:

$$F_{Sj}(t) = -i\gamma_{Sj} e^{i\lambda_S t}, \quad F_{Lj}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\alpha_{1j}(\lambda)}{\alpha_{11}(\lambda)} e^{i\lambda t} d\lambda.$$

Assume also that the radiation is small enough so that $F_{Lj}(t)$ is small. As a consequence F_{Lj} plays the role of a perturbation of F_{Sj}. The kernels $K_1^{(2)}$ and $K_1^{(3)}$ associated with the total wave u, v can be represented as the sum $K_1^{(2)} = K_S^{(2)} + K_L^{(2)}$ and $K_1^{(3)} = K_S^{(3)} + K_L^{(3)}$, where:

$$K_S^{(2)}(s,t) = -2i\nu \cos(\theta) e^{i(-\mu(s+t-2t_S)-\phi_{S2})} \frac{\exp\left(-\nu(s+t-2t_S)\right)}{1+\exp\left(-4\nu(s-t_S)\right)},$$

$$K_S^{(3)}(s,t) = -2i\nu \sin(\theta) e^{i(-\mu(s+t-2t_S)-\phi_{S3})} \frac{\exp\left(-\nu(s+t-2t_S)\right)}{1+\exp\left(-4\nu(s-t_S)\right)},$$

where $\theta = \arctan\left(|\gamma_{S3}/\gamma_{S2}|\right)$ is the soliton angle,

$$t_S = \frac{1}{4\nu} \ln\left(\frac{|\gamma_{S2}|^2 + |\gamma_{S3}|^2}{4\nu^2}\right), \qquad \phi_{Sj} = -\arg\left(\gamma_{Sj}\right) + 2\mu T_s$$

are the soliton center and phases, respectively. Neglecting the terms of higher order, $K_L^{(2)}$ and $K_L^{(3)}$ are solutions of:

$$K_L^{(j)}(s,t) \;=\; \psi_L^{(j)}(s,t) - \sum_{l=2}^{3} \int_s^\infty K_L^{(l)}(s,t') G_{SS}^{(lj)}(s,t,t') dt' \quad j = 2, 3,$$

where

$$\psi_L^{(j)}(s,t) = F_{Lj}^*(s+t) - \sum_{l=2}^{3} \int_s^\infty K_S^{(l)}(s,t')(G_{LS}^{(lj)} + G_{SL}^{(lj)})(s,t,t')dt', \qquad j = 2,3,$$

and

$$G_{XY}^{(jl)}(s,t,t') = \int_s^\infty F_{Xj}(t'+t'')F_{Yl}^*(t''+t)dt''.$$

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