Collective oscillations of one-dimensional Bose-Einstein gas in a time-varying trap potential and atomic scattering length

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The collective oscillations of one-dimensional (1D) repulsive Bose gas with external harmonic confinement in two different regimes are studied. The first regime is the mean-field regime when the density is high. The second regime is the Tonks-Girardeau regime when the density is low. We investigate the resonances under periodic modulations of the trap potential and the effective nonlinearity. Modulations of the effective nonlinear coefficient result from modulations of the atomic scattering length by the Feshbach resonance method or variations of the transverse trap frequency. In the mean-field regime we predict bistability in the *nonlinear oscillations* of the condensate. In the Tonks-Girardeau regime the resonance has the character of a *linear parametric* resonance. In the case of rapid strong modulations of the nonlinear coefficient we find analytical expressions for the nonlinearity managed soliton width and the frequency of the slow secondary oscillations near the fixed point. We confirm the analytical predictions by direct numerical simulations of the 1D Gross-Pitaevskii equation and the effective nonlinear Schrödinger equation with quintic nonlinearity and trap potential.

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I. INTRODUCTION

Low-dimensional Bose-Einstein condensates (BEC's) in highly asymmetric traps have recently been achieved, which opens new possibilities in the investigation of clouds of bosonic and fermionic gases [1–3]. Low-dimensional bosonic systems have many remarkable properties which distinguish them from three-dimensional (3D) systems. One of them is the growth of the interaction when the density is decreased. As a result the system enters into the Tonks-Girardeau (TG) regime. The properties of the Bose gas then coincide with the gas of free fermions. This follows from the exact solution to the problem of hard-core bosons with repulsive interaction obtained by Lieb and Liniger [4,5]. The opposite regime with weak interaction (high-density case) is the mean-field (MF) regime [6].

The TG and MF regimes can be characterized by the parameter γ which is equal to the ratio of the interaction energy and the kinetic energy of the ground state of gas—i.e., $\gamma = mg_{1D}/(\hbar^2 n_{1D})$. Here m is the atomic mass, g_{1D} is the one-dimensional coupling constant, and n_{1D} is the 1D density. The case $\gamma \ll 1$ corresponds to high densities, when the description by the mean-field theory is valid. The case $\gamma \gg 1$ corresponds to the strong repulsive interaction—the Tonks-Girardeau regime. Modern experiments with Bose gas in highly elongated traps now are in the region $\gamma \sim 1$ [7].

One of the important phenomena for experiments is collective oscillations of the Bose gas in different regimes. It is particularly interesting to investigate the dynamics of breathing and dipole modes for MF and TG regimes, as well as the crossover between them. The theoretical predictions for the frequencies in harmonic longitudinal traps $V(z)=m\omega_z^2 z^2/2$ are obtained in Refs. [8–12]. It is shown that the frequency of oscillations in the mean-field regime is $\sqrt{3}\omega_z$, while it is $2\omega_z$ in the TG regime. The last value coincides with the one observed for the thermal gas. For the 3D cigar the frequency is $\sqrt{5/2}\omega_z$. These investigations involve the combination of the exact solution by Lieb and Liniger, the local density approximation, the hydrodynamic equations, and the extended nonlinear Schrödinger equation. Note that information for oscillations in the high-dimensional regimes can be found in the review in [13]. In this work we shall study resonances in the oscillations of 1D Bose gas in both regimes. The periodic and random modulations of the trap potential and the atomic scattering length are subject to our analysis.

The MF regime is described by the 1D Gross-Pitaevskii (GP) equation with two-body interaction

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{zz} + V(z,t)\psi + g_{1D}\Gamma(t)|\psi|^2\psi. \qquad (1)$$

This equation is derived from the 3D GP equation in a strongly anisotropic external potential. The dynamics in the radial direction is then averaged out [6] and the longitudinal profile of the wave function satisfies Eq. (1). The wave function is normalized so that the number of atoms in the BEC is $N=\int|\psi(t,x)|^2dx$. Here $V(z,t)$ is the longitudinal trapping potential, which is assumed in this work to be harmonic, $V(z,t)=m\omega_z^2 z^2 F(t)/2$. The function $F(t)$ describes the variation in time of the trap. The effective nonlinear coupling constant is g_{1D}. In the case of a harmonic transverse trap potential $m\omega_\perp^2(x^2+y^2)/2$, Eq. (1) is valid under the assumption $\omega_\perp \gg \omega_z$ and we have $g_{1D}=2\hbar a_s\omega_\perp$. The function $\Gamma(t)$

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describes the variation in time of the effective nonlinearity. A first method to vary the effective nonlinearity is to modulate in time the transverse trap width or equivalently the transverse frequency $\omega_\perp$. The modulations of the transverse frequency impose variations of the BEC density in the (x,y) plane, which in turn involve variations of the nonlinear interaction [14]. Nonlinear resonances in 2D BEC's for such modulations have been studied in [15,16]. A second method to vary the effective nonlinearity is to modulate the atomic scattering length by the so-called Feshbach resonance technique [17].

Theoretical and experimental studies have demonstrated that variation of the *s*-wave scattering length, including a possibility to change its sign, can be achieved by using the Feshbach resonance

$$a(t) = a_s\left[1 + \frac{\Delta}{B_0 - B(t)}\right],$$

where a_s is the value of the scattering length far from resonance, $B(t)$ is the time-dependent external magnetic field, Δ is the width of the resonance, and B_0 is the resonant value of the magnetic field. Feshbach resonances have been observed in ^{23}Na at 853 and 907 G [17], in ^{7}Li at 725 G [18], and in ^{85}Rb at 164 G with $\Delta = 11$ G [19]. In the case of resonance dynamics where a_s is slowly varying and keeps a constant sign, atom losses are negligible. However, atom losses may be important when crossing the resonance [17,20]. This is the case of the ^{23}Na condensate where it is necessary to cross the Feshbach resonance to change the sign of the atomic scattering length. The approach developed in this paper should then be modified to take into account this phenomenon. If we are not close to the resonance, losses are small and in a first approximation they can be taken into account by a time-varying number of atoms in the effective variational equation for the width. This should lead to damped secondary oscillations with a time scale larger than the oscillation time. However, atoms losses can be minimized down to a negligible level by certain experimental controls that have been implemented in particular in the ^{85}Rb case [21–24]. Furthermore, in Ref. [18] it is demonstrated in the case of ^{7}Li that a change of sign of the scattering length can be obtained without crossing the resonance by the so-called coupled-channel method. Our study of the nonlinear management is triggered by these experimental achievements.

According to Kolomeisky *et al.* [25] the TG regime is described by the nonlinear Schrödinger equation with quintic nonlinearity and trap potential:

$$i\hbar\phi_t = -\frac{\hbar^2}{2m}\phi_{zz} + V(z,t)\phi + \frac{\pi^2\hbar^2}{2m}|\phi|^4\phi. \tag{2}$$

It is known that this model does not capture every aspect of the dynamics of an atomic gas in the TG regime; in particular, it overestimates the coherence in interference patterns at a small number of particles [26]. However, Eq. (2) has been shown to reproduce the collective spectrum of a gas in the TG regime within a local density approximation [11,27]. Furthermore, we shall show that it also gives the correct frequency of oscillations as predicted theoretically and observed experimentally [7]. As our attention is focused on resonance phenomena, Eq. (2) seems to be a good model, both physically relevant and mathematically tractable. Using the time-dependent variational method or the hydrodynamic approach we shall derive the equation for the TG gas width and study resonances in oscillations under periodic and random modulations of the trap potential.

The paper is organized as follows. In Secs. II–IV we analyze the nonlinear resonances in gas oscillations in the mean-field regime using the 1D mean-field GP equation. We apply a time-dependent variational approach and introduce action-angle variables in Sec. II. Sections III and IV are devoted to the resonances driven by periodic and random modulations of the trap frequency and the nonlinearity, respectively. In Sec. V we address the same problems for the 1D TG regime.

II. MEAN-FIELD CASE

A. Variational approach

We first put Eq. (1) into dimensionless form by setting $t' = \omega_z t$, $x = z/l_z$, $l_z = \sqrt{\hbar/(m\omega_z)}$, and $u = \sqrt{2|a_s|\omega_\perp/\omega_z}\psi$. In the following we omit primes, so the mean-field GP equation reads

$$iu_t + \frac{1}{2}u_{xx} - \frac{1}{2}\tilde{F}(t)x^2u - \tilde{\Gamma}(t)|u|^2u = 0, \tag{3}$$

where $\tilde{F}(t) = F(t/\omega_z)$ and $\tilde{\Gamma}(t) = \Gamma(t/\omega_z)\mathrm{sgn}(a_s)$. We apply a variational approach using the Gaussian ansatz

$$u(t,x) = A\exp\left(-\frac{x^2}{2a^2(t)} - i\frac{b(t)x^2}{2} - i\phi(t)\right). \tag{4}$$

Note that the equation for the gas center of mass is decoupled from the equations for oscillations, so we did not take it into consideration. The ansatz yields a closed-form evolution equation for the width

$$a_{tt} = \frac{1}{a^3} - f(t)a + \frac{\gamma(t)}{a^2}, \tag{5}$$

where $f(t) = \tilde{F}(t)$, $\gamma(t) = P\tilde{\Gamma}(t)$, and $P = \int |u|^2 dx/\sqrt{2\pi} = (2|a_s|\omega_\perp N)/(\sqrt{2\pi}\omega_z l_z)$. In the repulsive case $a_s > 0$ and in the absence of modulations $F \equiv 1$, $\Gamma \equiv 1$, we have $f = 1$ and $\gamma = P = N/N^*$ with $N^* = (\sqrt{2\pi}\omega_z l_z)/(2a_s\omega_\perp)$. If the number of atoms is large enough $N \gg N^*$, then $P \gg 1$ so that we can neglect $1/a^3$ in Eq. (5) and the fixed point is given by

$$a_g = P^{1/3}. \tag{6}$$

In the dimensional variables we get

$$L_c = [(\sqrt{2}a_s l_z^2\omega_\perp N)/(\sqrt{\pi}\omega_z)]^{1/3},$$

which agrees (up to a numerical multiplicative constant) with the Thomas-Fermi value for the 1D BEC width.

B. Action-angle variables

We assume in this section that $\gamma(t) = \gamma_0 + \tilde{\gamma}(t)$ where the average nonlinear coefficient $\gamma_0 > 0$ which corresponds to the repulsive case, while $\tilde{\gamma}$ represents a zero-mean periodic or random component. Similarly, we take $f(t)=1+\tilde{f}(t)$. The unperturbed problem consists in taking $\tilde{\gamma}(t)=\tilde{f}(t)=0$. Assume that γ_0 is large so that the kinetic term $1/a^3$ can be neglected in Eq. (5). The energy E of the unperturbed BEC is given by

$$E(t) = \frac{1}{2}a_t^2(t) + U(a(t)), \quad U(a) = \frac{1}{2}a^2 + \frac{\gamma_0}{a}. \qquad (7)$$

In the absence of fluctuations the energy E is an integral of motion. The BEC width obeys a simple dynamics with Hamiltonian structure

$$H(p,q) = \frac{1}{2}p^2 + U(q), \qquad (8)$$

with $q=a$ and $p=a_t$. The potential U possesses a unique minimum $a_g=\gamma_0^{1/3}$ which is a stable fixed point with oscillation frequency $\omega=\sqrt{3}$. The corresponding ground state has energy $E_g=U(a_g)=(3/2)\gamma_0^{2/3}$.

If the initial conditions $(a(0),a_t(0))$ correspond to an energy above E_g, then the orbit of the motion is closed, corresponding to periodic oscillations. In order to explicate the periodic structure of the variables a and a_t, we introduce the action-angle variables. The orbits are determined by the energy imposed by the initial conditions

$$E = \frac{1}{2}a_t^2(0) + U(a(0)).$$

For $E>E_g$, we introduce $a_1(E)<a_2(E)$, the extremities of the orbit of a for the energy E. They are the positive solutions of the cubic equation $U(a)=E$ and they are given by $(j=1,2)$

$$a_j(E) = -2\left(\frac{2E}{3}\right)^{1/2}\cos\left(\frac{\xi+(-1)^j 2\pi}{3}\right),$$

$$\xi = \arccos\left[\left(\frac{E_g}{E}\right)^{3/2}\right].$$

The action I is defined as a function of the energy E by

$$\mathcal{I}(E) = \frac{1}{2\pi}\oint p\,dq = \frac{1}{\pi}\int_{a_1(E)}^{a_2(E)}\sqrt{2E-2U(b)}\,db. \qquad (9)$$

The motion described by Eq. (8) is periodic, with period

$$\mathcal{T}(E) = \oint\frac{dq}{p} = 2\int_{a_1(E)}^{a_2(E)}\frac{db}{\sqrt{2E-2U(b)}}, \qquad (10)$$

or else $\mathcal{T}(E)=2\pi(d\mathcal{I}/dE)(E)$. The angle ϕ is defined as a function of E and a by

$$\phi(E,a) = -\int^a\frac{\partial p}{\partial I}dq = -\frac{2\pi}{\mathcal{T}(E)}\int^a\frac{db}{\sqrt{2E-2U(b)}}.$$

The transformation $(E,a)\to(I,\phi)$ can be inverted to give the functions $\mathcal{E}(I)$ and $\mathcal{A}(I,\phi)$. The BEC width oscillates between the minimum value $a_1(E)$ and the maximum value $a_2(E)$. The energy E as well as the action I are constant and determined by the initial conditions, so the evolution of the BEC width is governed by

$$a(t) = \mathcal{A}(\mathcal{I}(E),\phi(t)),$$

$$\phi(t) = \phi(0) - \frac{2\pi}{\mathcal{T}(E)}t.$$

For E close to the ground-state energy E_g, we have

$$a_j(E) = a_g + (-1)^j\sqrt{\frac{2}{3}(E-E_g)}, \quad j=1,2, \qquad (11)$$

$$\mathcal{T}(E) = \frac{2\pi}{\sqrt{3}}, \quad \mathcal{I}(E) = \frac{E-E_g}{\sqrt{3}}, \quad \mathcal{A}(I,\phi) = a_g + \sqrt{2I}\cos(\phi). \qquad (12)$$

For large energies $E\gg E_g$, we have

$$a_1(E) = a_g\frac{2E_g}{3E}, \quad a_2(E) = a_g\sqrt{\frac{3E}{E_g}}, \qquad (13)$$

$$\mathcal{T}(E) = \pi, \quad \mathcal{I}(E) = \frac{E}{2}, \quad \mathcal{A}(I,\phi) = \sqrt{2I}\sqrt{1+\cos(\phi)}. \qquad (14)$$

III. RESONANCES IN THE MEAN-FIELD CASE DRIVEN BY A TIME-VARYING NONLINEARITY

In this section we address the role of a time-varying nonlinearity and assume that the trap is stationary $\tilde{f}(t)=0$. We shall mainly focus our attention on the periodic management $\Gamma(t)=\Gamma_0+\Gamma_1\sin(\Omega_g t)$, but we shall also consider random fluctuations of the effective nonlinearity in the 1D GP equation.

A. High-frequency nonlinear management

We shall first address the case where the oscillation frequency of the nonlinear management is much higher than the trapping frequency—i.e., $\Omega_g\gg\omega_z$. We must also assume $\Omega_g\ll\omega_\perp$ to prevent from exciting the transverse modes. In such a case the influence of the nonlinear management is negligible unless the nonlinear management amplitude is large. The problem of nonlinearity management for BEC's, the so-called Feshbach resonance (FR) management, has already been considered for 1D BEC's in Refs. [24,28] and for 2D BEC's in Refs. [29,30]. In Ref. [24] the authors were the first ones to propose a technique of FR management, based on a time-periodic change of the magnitude and sign of the scattering length by a resonantly tuned ac magnetic field. The FR management resembles the so-called dispersion-management (DM) technique in fiber optics, which is based on a periodic concatenation of fibers with opposite signs of the group-velocity dispersion. The DM technique has been shown to support robust breathing pulses, the so-called DM solitons celebrated in optics [31]. The FR technique is shown in Refs.

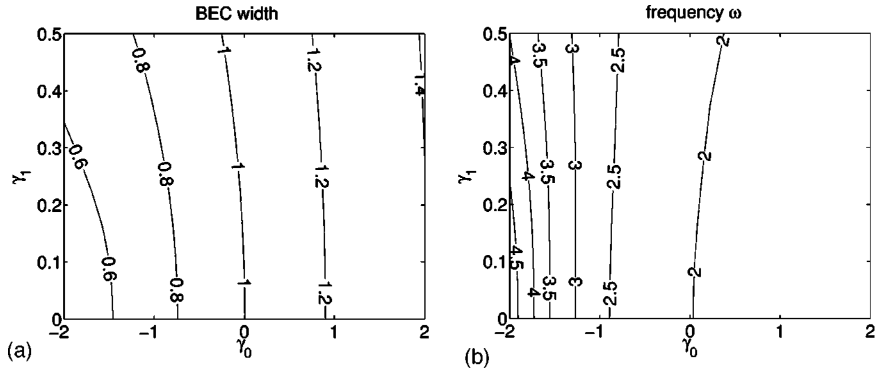

FIG. 1. BEC width (a) and oscillation frequency (b) as predicted by the theoretical model (in dimensionless units). For $\gamma_0 \gg 1$ the BEC width is $\gamma_0^{1/3}$ and the oscillation frequency is $\sqrt{3}$.

[24,28] to drive stable localized structures, named FR-managed matter-wave solitons. Here, in distinction from [24,28], we give analytical expressions for the fixed point (corresponding to the FR managed soliton) and the frequency of slow secondary oscillations. More precisely, we introduce the small parameter $\delta = \omega_z/\Omega_g$ and we assume that the nonlinear management amplitude is large compared to the average value. We write accordingly

$$\gamma(t) = \gamma_0 + \frac{\gamma_1}{\delta}\sin\left(\frac{t}{\delta}\right)$$

and perform an asymptotic analysis $\delta \to 0$ following the Kapitsa averaging theorem. We expand $a(t) = a_0(t) + \delta a_1(t, t/\delta) + \cdots$. We substitute this form into Eq. (5) and get a compatibility condition which reads

$$a_{0_{tt}} = \frac{1}{a_0^3} - a_0 + \frac{\gamma_0}{a_0^2} + \frac{\gamma_1^2}{a_0^5}, \tag{15}$$

while the first-order corrective term can be expressed as

$$a_1(t,\tau) = \frac{\gamma_1}{a_0^2(t)}\sin(\tau).$$

Note that the initial conditions for the slowly varying envelope are

$$a_0(0) = a(0), \quad a_0'(0) = a'(0) - \frac{\gamma_1}{a(0)^2},$$

where $a(0)$ and $a'(0)$ are the initial values of the width and its time derivative. The ground state can then be analytically studied. Let us first deal with the case $\gamma_0 = 0$. We introduce the critical value $\gamma_c = \sqrt[4]{4/27} \simeq 0.62$. It is found out that, if $|\gamma_1| \leqslant \gamma_c$, then Eq. (15) admits a unique fixed point describing the width of the ground state:

$$a_g^2 = \frac{2}{\sqrt{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{\gamma_1^2}{\gamma_c^2}\right)\right].$$

If $|\gamma_1| > \gamma_c$, then

$$a_g^2 = \left(\frac{\gamma_1^2}{2}\right)^{1/3}\left[\left(1 + \sqrt{1 - \frac{\gamma_c^4}{\gamma_1^4}}\right)^{1/3} + \left(1 - \sqrt{1 - \frac{\gamma_c^4}{\gamma_1^4}}\right)^{1/3}\right],$$

and a_g increases with γ_1 and goes from the value 1 for $\gamma_1 = 0$ to the asymptotic behavior $a_g \sim \gamma_1^{2/3}$ for large γ_1. The linear stability analysis of the effective equation (15) shows that the fixed point is stable. If the initial condition is close to this point, then the width a oscillates around the value a_g with the oscillation frequency $\omega = \sqrt{6 - 2/a_g^4}$ which increases from 2 for $\gamma_1 = 0$ to its limit value $\sqrt{6}$. In the general case $\gamma_0 \neq 0$, $\gamma_1 \neq 0$, there exists a unique fixed point which is the unique positive zero of the equation $a^2 - a^6 + \gamma_0 a^3 + \gamma_1^2 = 0$. It is plotted in Fig. 1(a), and the corresponding oscillation frequency is plotted in Fig. 1(b).

We have carried out numerical simulations of the ordinary differential equation (ODE) model (5) and the partial differential equation (PDE) model (3) to check the theoretical predictions of the asymptotic analysis $\delta \to 0$ (see Fig. 2). Note that stable BEC's can be achieved with a negative (i.e., attractive) or positive (i.e., repulsive) average nonlinear coefficient γ_0.

Let us estimate the parameters for a realistic experiment. For the ^{85}Rb condensate with $a_s = -0.5$ nm, $\omega_\perp = 2\pi \times 360$ Hz, $\omega_z = 2\pi \times 14.4$ Hz, $N = 10^4$, $l_z = 100\ \mu$m, $\Omega_g = 10\omega_z$, $\Gamma_0 = 1$, and $\Gamma_1 = 2$, we find that the condensate width is $a_g \approx 0.7 l_z$ and the frequency of the secondary oscillations is $\omega \approx 2.6\omega_z$.

B. Resonant nonlinear management

In this section we address the case where the periodic nonlinear management is resonant or close to resonant. We shall focus our attention to the particularly interesting case where the number of atoms N is large, which in turn implies that the dimensionless parameter γ_0 is large (say, at least 5). We shall see that a periodic modulation of the nonlinear coefficient

$$\gamma(t) = \gamma_0 + \gamma_1 \sin(\Omega t)$$

may dramatically modify the dynamics, and this phenomenon will be noticeable when the dimensionless parameter γ_1 is of order 1. If $\gamma_0 \geqslant 5$, then Eq. (5) can be simplified into

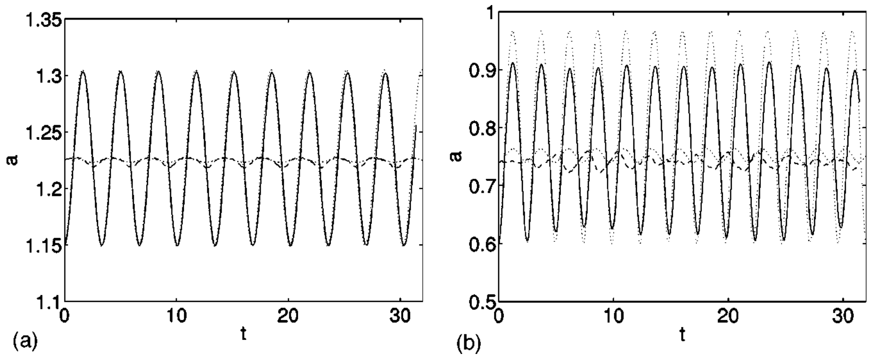

FIG. 2. BEC width in the presence of high-frequency periodic modulation of the nonlinear coefficient. Here $\gamma(t)=\gamma_0+2\sin(10t)$ with $\gamma_0=1$ (a) and $\gamma_0=-1$ (b). The solid and dashed lines stand for full numerical simulations of the PDE system with two different initial conditions (dashed lines: theoretical ground states), while the dotted lines represent the theoretical predictions.

$$a_{tt}=-a+\frac{\gamma(t)}{a^2}. \tag{16}$$

In the absence of periodic modulation $\gamma_1=0$, the ground state is $a_g=\sqrt[3]{\gamma_0}$ and the oscillation frequency of the BEC is $\omega=\sqrt{3}$ in dimensionless units. We now consider a periodic modulation of the nonlinear coefficient with an amplitude γ_1 smaller than γ_0 and with frequency Ω of the same order as ω. We expand $a=a_g+\tilde{a}$ where $\tilde{a}$ satisfies

$$\tilde{a}_{tt}+\omega^2\tilde{a}+\delta\tilde{a}^2+\lambda\tilde{a}^3=\epsilon\sin(\Omega t)+\epsilon_2\sin(\Omega t)\tilde{a},$$

with $\delta=-3\gamma_0^{-1/3}$, $\lambda=4\gamma_0^{-2/3}$, $\epsilon=-\gamma_1\gamma_0^{-2/3}$, and $\epsilon_2=-2\gamma_1\gamma_0^{-1}$. This is the equation for a nonlinear oscillator with external and parametric drives. The analysis of this problem can be carried out by applying the standard method described in Ref. [32]. We consider the harmonic expansion of $\tilde{a}$:

$$\tilde{a}=\tilde{a}_1\cos(\nu)+\tilde{a}_0+\tilde{a}_2\cos(2\nu),$$

where $\nu(t)=\Omega t+\theta(t)$. We substitute this ansatz into Eq. (16) and we collect the terms with the same harmonic. This yields a system of differential equations that give the zeroth and second harmonics in terms of the first one and a compatibility condition which reads as a system of two differential equations for the first harmonic $\tilde{a}_1$ and the slow phase θ:

$$\partial_t\tilde{a}_1=\left[-\frac{\epsilon}{2\Omega}\left(1+\frac{\epsilon_2^2\cos^2(\theta)}{9\omega^4}\right)+\frac{\epsilon_2\delta}{\omega^2\Omega}\left(\frac{7\tilde{a}_1^2}{24}+\frac{\epsilon\cos(\theta)\tilde{a}_1}{18\omega^2}\right)-\frac{7\epsilon_2^2\tilde{a}_1}{24\omega^2\Omega}\cos(\theta)\right]\sin(\theta), \tag{17}$$

$$\begin{aligned}\partial_t\theta=&\frac{\omega^2-\Omega^2}{2\Omega}-\frac{\delta\epsilon_2\epsilon}{6\omega^4\Omega}\cos^2(\theta)-\frac{5\epsilon_2^2}{24\omega^2\Omega}\cos^2(\theta)\\&-\frac{\epsilon}{2\Omega}\left(1-\frac{\epsilon_2^2\cos^2(\theta)}{9\omega^4}\right)\frac{\cos(\theta)}{\tilde{a}_1}\\&+\left(-\frac{\delta^2\epsilon}{9\omega^4\Omega}+\frac{5\delta\epsilon_2}{8\omega^2\Omega}\right)\cos(\theta)\tilde{a}_1+\left(\frac{3\lambda}{8\Omega}-\frac{5\delta^2}{12\omega^2\Omega}\right)\tilde{a}_1^2.\end{aligned} \tag{18}$$

The first harmonic is dominant so that the BEC width evolution can be described at first order as $a(t)=a_g+\tilde{a}_1(t)\cos[\Omega t+\theta(t)]$. The stability analysis shows very interesting features. It appears that the strongest resonance occurs when the nonlinear management has frequency

$$\Omega_c=\omega\sqrt{1+\frac{1}{2}\left(\frac{\gamma_1}{\gamma_0}\right)^{2/3}}, \tag{19}$$

which is above the oscillation frequency of the BEC. Resonance is still noticeable for a modulation frequency Ω in a vicinity of Ω_c with bandwidth of order $\omega(\gamma_1/\gamma_0)^{2/3}$. More precisely, if $\Omega<\Omega_c$, then there exists a unique fixed point to the system (17) and (18). The value of $\tilde{a}_1$ corresponding to this fixed point is

$$a_{osc}=\frac{1}{3}\left[27\gamma_1+3\sqrt{24\gamma_0^2(\omega^2-\Omega^2)^3+81\gamma_1^2}\right]^{1/3}-\frac{2\gamma_0^{2/3}(\omega^2-\Omega^2)}{\left[27\gamma_1+3\sqrt{24\gamma_0^2(\omega^2-\Omega^2)^3+81\gamma_1^2}\right]^{1/3}}.$$

A linear stability analysis shows that this fixed point is stable. If $\Omega>\Omega_c$, then there are three fixed points. A bistability in the condensate oscillations occurs. The intermediate fixed point is always unstable. The upper fixed point is the continuation of the fixed point exhibited in the case $\Omega<\Omega_c$. It can be observed by increasing slowly and carefully the modulation frequency from a frequency below Ω_c to a frequency above Ω_c. The lower fixed point is the one that is observed when imposing without any particular precaution a modulation with frequency $\Omega>\Omega_c$ or when perturbing the metastable upper fixed point. The value of the stable fixed point is

$$a_{osc}=2\sqrt{\frac{2}{3}\gamma_0^{2/3}(\Omega^2-\omega^2)}\cos\left(\frac{\xi_{osc}}{3}\right),$$

$$\xi_{osc}=\arccos\left(\frac{|\gamma_1|}{\left[\frac{2}{3}\gamma_0^{2/3}(\Omega^2-\omega^2)\right]^{3/2}}\right).$$

Thus the system encounters a jump in the oscillation amplitude when the modulation frequency crosses the value Ω_c.

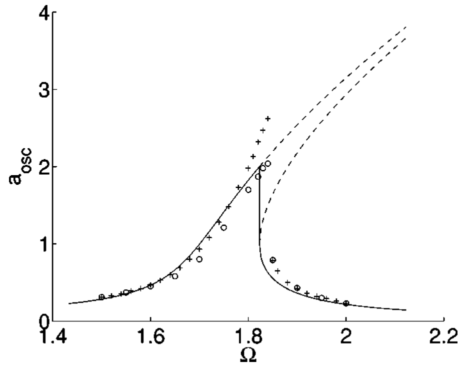

FIG. 3. Oscillation amplitude as predicted by the theoretical model (solid lines). Here $\gamma_0=10$ and $\gamma_1=1$. Comparisons with simulations with the ODE model (crosses) and with the PDE model (circles).

Evaluating the two expressions of a_{osc} just above and just below the critical frequency Ω_c, we get that the amplitude of the jump is $a_{osc}(\Omega_c^-)-a_{osc}(\Omega_c^+)=|\gamma_1|^{1/3}$.

In Fig. 3 we plot the values of the oscillation amplitude as a function of the frequency of the nonlinear management in the case $\gamma_0=10$ and $\gamma_1=1$. The theoretical prediction is that the modulation frequency driving the most resonant response is $\Omega_c \simeq 1.823$ which is above the eigenfrequency $\omega=\sqrt{3} \simeq 1.732$. We have carried out numerical experiments with the ODE model (5) and the full PDE model (3) to check this prediction. The results for the frequency $\Omega=1.5$ are plotted in Fig. 4. For each frequency we can detect the amplitude of the oscillations and report on Fig. 3 to compare with the theoretical predictions. We have numerically determined that the frequency driving the most important resonance is $\simeq 1.84$ which is indeed above the eigenfrequency $\sqrt{3}$ and very close to the theoretical prediction Ω_c. The numerical simulations confirm the jump in the oscillation amplitude when crossing the critical frequency.

Let us estimate parameters for a realistic experiment. The magnetic trap can be taken with parameters $\omega_\perp=2\pi \times 10^3$ Hz, $\omega_z=2\pi\times 10$ Hz, and the number of atoms of ^{85}Rb $N=5\times 10^4$. For the external field $B=159$ G, $a_s=0.8$ nm (repulsive gas). Then, for $\Omega=18.25$ Hz, we should observe large oscillations with $a_{osc}\sim 2.1 l_z$, while at $\Omega=15$ Hz we should observe much smaller oscillations with $a_{osc}\sim 0.3 l_z$. Practically, if we start from the fixed point a_g given by Eq. (6), we should observe oscillations of the BEC width with the maximum a_g+2a_{osc}.

C. Random fluctuations of the nonlinear coefficient

We assume here that the scattering length or the transverse frequency $\omega_\perp$ is randomly varying, inducing random fluctuations of the nonlinear coefficient of the GP equation

$$\gamma(t) = \gamma_0 + \gamma_1 \eta(t),$$

where $\eta(t)$ is a normalized random noise with standard deviation of order 1 and $\gamma_1 > 0$ represents the amplitude of the random fluctuations. This model also presents an interest for the study of nonlinear management schemes in spatial optical solitons if the widths of the nonlinear layers in arrays of waveguides are randomly distributed [34]. We use the angle-action formalism introduced in Sec. II B. In the presence of perturbations, the motion of a is not purely oscillatory, because the energy and action are slowly varying in time. We adopt the action-angle formalism, because it allows us to separate the fast scale of the locally periodic motion and the slow scale of the evolution of the action. We assume that γ_1 is smaller than γ_0 and introduce the dimensionless parameter $\delta=\gamma_1/\gamma_0$. Thus, after rescaling $t=\delta^2\tau$, the action-angle variables satisfy the differential equations

$$\frac{dI}{d\tau} = \frac{1}{\delta}\eta\left(\frac{\tau}{\delta^2}\right)h_\phi(I,\phi),$$

$$\frac{d\phi}{d\tau} = -\frac{1}{\delta^2}\omega(I) - \frac{1}{\delta}\eta\left(\frac{\tau}{\delta^2}\right)h_I(I,\phi),$$

where $h(I,\phi)=-\gamma_0/\mathcal{A}(I,\phi)$ and $\omega(I)=2\pi/\mathcal{T}(\mathcal{E}(I))$ are smooth functions and h is periodic with respect to ϕ with period 2π. Applying a standard diffusion-approximation theorem [35] establishes that, for small δ, $(I(\tau))_{\tau\geq 0}$ has the statistical distribution of a diffusion Markov process characterized by the self-adjoint infinitesimal generator

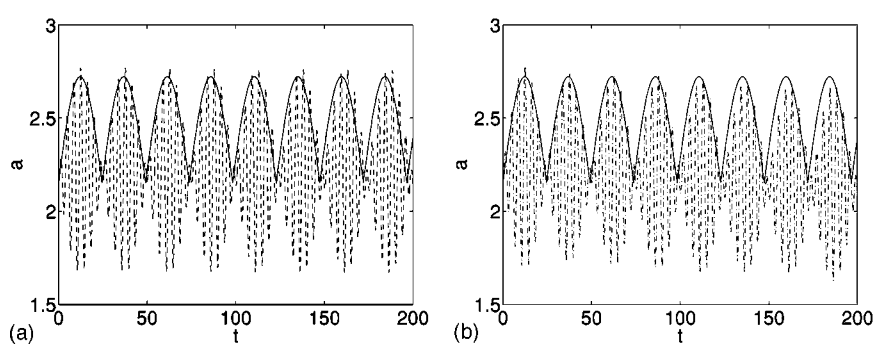

FIG. 4. BEC width versus time for a nonlinear management with frequency $\Omega=1.5$. We compare the theoretical envelope of the oscillation with numerical simulations of the ODE model (a) and the PDE model (b). Here $\gamma_0=10$ and $\gamma_1=1$. For the simulation of the PDE model the initial state is the Gaussian ansatz with $a_g=2.15$, which corresponds to the theoretical fixed point in the absence of nonlinear management. The solid lines represent the theoretical slowly varying envelope $\overline{a}(t)=a_g+2a_{osc}|\sin(\omega_{osc}t/2)|$.

$$\mathcal{L} = \frac{1}{2}\frac{\partial}{\partial I}A(I)\frac{\partial}{\partial I},$$

where

$$A(I) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} h_\phi(I,\phi)h_\phi(I,\phi - \omega(I)t)\langle \eta(0)\eta(t)\rangle dt d\phi$$

and the brackets stand for a statistical averaging with respect to the distribution of the noise η. This means in particular that the probability density function of I satisfies the Fokker-Planck equation $\partial_\tau p = \mathcal{L}p$, $p(\tau=0,I)=\delta(I-I_0)$, where I_0 is the initial value of the action at time 0. As long as the energy remains close to the the one of the ground state, we can use the asymptotic expansions (12) to expand the effective diffusion coefficient $A(I)$. In the original time scale we then establish that the growth of the action is

$$I(t) = \left(\sqrt{I_0} + \frac{\sqrt{\alpha_c}\gamma_1}{\omega\gamma_0^{2/3}}W_t\right)^2,$$

where $\omega=\sqrt{3}$, W_t is a standard Brownian motion (i.e., a Gaussian process with zero-mean and standard deviation $\sqrt{t}$), and

$$\alpha_c = \int_0^{\infty}\langle \eta(0)\eta(t)\rangle \cos(\omega t)dt. \tag{20}$$

If the BEC is in the ground state at time 0, then the BEC width a oscillates with eigenfrequency $\omega=\sqrt{3}$ between the values a_- and a_+ which evolve slowly as

$$a_\pm(t) = \gamma_0^{1/3} \pm \frac{\sqrt{2\alpha_c}\gamma_1}{\omega\gamma_0^{2/3}}|W_t|.$$

As a result, taking into account the periodic modulations, the rms BEC width is $[\mathrm{rms}^2(a)(t)] = \alpha_c\gamma_1^2 t/(\omega^2\gamma_0^{4/3})$ where the root mean square (rms) is defined by $[\mathrm{rms}(a)] = \langle (a-\langle a\rangle)^2\rangle^{1/2}$.

In the white noise case $\langle \eta(0)\eta(t)\rangle = 2\sigma^2\delta(t)$, there exists a direct way to compute the growth of the rms amplitude. It consists in writing a closed-form system for the second-order moments of the BEC width and its derivative. We expand $a=a_g+\tilde{a}$ and denote $\tilde{b}=a_t$. By considering the column vector $X(t) = (\langle \tilde{a}^2\rangle, \langle \tilde{a}\tilde{b}\rangle, \langle \tilde{b}^2\rangle)$, we get by applying Itô's calculus that X satisfies the closed-form system

$$\frac{dX}{dt} = M_0 X + M_1 X + V_0, \tag{21}$$

where

$$M_0 = \begin{pmatrix} 0 & 2 & 0 \\ -\omega^2 & 0 & 1 \\ 0 & -2\omega^2 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8\alpha_c\gamma_1^2\gamma_0^{-2} & 0 & 0 \end{pmatrix},$$

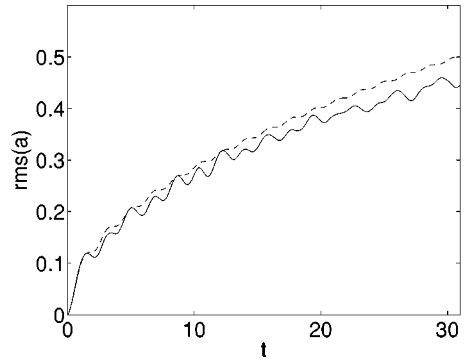

FIG. 5. Root mean square of the BEC width in the presence of random fluctuations of the nonlinear coefficient. The theoretical result [Eq. (22)] is plotted as the dashed line. The results of numerical simulations of the PDE model are plotted as the solid line and correspond to the averaging of 1000 different realizations of the random noise η.

$$V_0 = \begin{pmatrix} 0 \\ 0 \\ 2\alpha_c\gamma_1^2\gamma_0^{-4/3} \end{pmatrix},$$

$\omega=\sqrt{3}$, $\alpha_c=\sigma^2$, V_0 is the source term, and M_1 is a stochastic resonance term. It is negligible during the first steps of the dynamics and becomes important only when the rms amplitude becomes of the order of $a_g=\gamma_0^{1/3}$. However, in that case, the linearization procedure is not valid anymore. Integrating Eq. (21) by neglecting M_1 we get

$$[\mathrm{rms}^2(a)(t)] = \frac{\alpha_c\gamma_1^2}{\gamma_0^{4/3}\omega^2}\left(t - \frac{\sin(2\omega t)}{2\omega}\right). \tag{22}$$

In Fig. 5 we compare the statistical predictions with a set of numerical simulations of the PDE model. We have taken $\gamma_0=\gamma_1=10$. We have adopted a stepwise constant model for $\eta(t)$. Here η is constant over elementary intervals with duration t_c and takes random values uniformly distributed between -1 and 1. With $t_c=0.03$ we have $\alpha_c=0.005$. The initial BEC width is $a_g=\gamma_0^{1/3}\simeq 2.15$. The diffusive growth in agreement with Eq. (22) is noticeable.

IV. RESONANCES IN THE MEAN-FIELD CASE DRIVEN BY A TIME-VARYING POTENTIAL TRAP

We first focus our attention on the periodic management $f(t)=1+f_1\sin(\Omega t)$, and second we address random fluctuations of the trap.

A. High-frequency periodic modulation of the trap

We shall first address the case where the oscillation frequency of the trap modulation is higher than the trapping frequency—i.e., $\Omega \gg \omega=\sqrt{3}$. In such a case the influence of the modulation is negligible unless its amplitude is large. We introduce the small parameter $\delta=1/\Omega$ and assume that the trap modulation amplitude is large, of order δ^{-1}. We write accordingly

$$f(t) = 1 + \frac{f_1}{\delta}\sin\left(\frac{t}{\delta}\right)$$

and perform an asymptotic analysis $\delta \to 0$ following the same line as in Sec. III A. We get that the slowly varying envelope of the BEC width obeys the effective equation

$$a_{0tt} = \frac{\gamma_0}{a_0^2} - a_0 - \frac{f_1^2}{2\Omega^2}a_0,$$

with the initial conditions

$$a_0(0) = a(0), \quad a_0'(0) = a'(0) - \frac{f_1 a(0)}{\Omega}.$$

The high-frequency modulation thus involves a shift of the effective potential which in turn implies a shift of the ground state (see also [36]).

B. Resonances due slow periodic variations of the trap potential

The study is similar to the nonlinear management, and qualitatively the same conclusion holds true, especially concerning the bistable diagram. We shall only point out the main differences. First, the strongest resonance occurs when the periodic modulation of the trap has the frequency

$$\Omega_c = \omega\sqrt{1 + \frac{1}{2}|f_1|^{2/3}}, \tag{23}$$

which is above the eigenfrequency of the BEC. Note that Ω_c does not depend on the nonlinear coefficient γ_0 and is proportional to the eigenfrequency of the BEC. The oscillation amplitude of the BEC turns out to be also proportional to the BEC width. The resonant bandwidth is of order $|f_1|^{2/3}\omega$. More precisely, if $\Omega < \Omega_c$, then the oscillation amplitude of the BEC is

$$\frac{a_{osc}}{a_g} = \frac{1}{3}[27|f_1| + 3\sqrt{24(\omega^2-\Omega^2)^3 + 81 f_1^2}]^{1/3} - \frac{2(\omega^2-\Omega^2)}{[27|f_1| + 3\sqrt{24(\omega^2-\Omega^2)^3 + 81 f_1^2}]^{1/3}},$$

while for $\Omega > \Omega_c$,

$$\frac{a_{osc}}{a_g} = 2\sqrt{\frac{2}{3}(\Omega^2-\omega^2)}\cos\left(\frac{\xi_{osc}}{3}\right),$$

$$\xi_{osc} = \arccos\left(\frac{|f_1|}{\left[\frac{2}{3}(\Omega^2-\omega^2)\right]^{3/2}}\right).$$

Evaluating the two expressions of a_{osc} around the critical frequency Ω_c, we get that the amplitude of the jump is $a_{osc}(\Omega_c^-) - a_{osc}(\Omega_c^+) = |f_1|^{1/3}\gamma_0^{1/3}$.

C. Random fluctuations of the trap

We consider in this section a random modulation of the trap

$$f(t) = 1 + f_1\eta(t),$$

with $0 < f_1 \ll 1$ and η is a normalized random noise. We once again use the action-angle formalism. We carry out an asymptotic analysis similar to the one presented in Sec. III C, with the small parameter f_1 and the function h given by

$$h(I,\phi) = \frac{1}{2}\mathcal{A}^2(I,\phi). \tag{24}$$

We then get the statistical distribution of the slow evolution of the action in terms of a Brownian motion,

$$I(t) = \left(\sqrt{I_0} + \frac{\sqrt{\alpha_c}}{\sqrt{2\omega}} f_1\gamma_0^{1/3}W_t\right)^2,$$

where $\omega = \sqrt{3}$ and α_c is given by Eq. (20). As a consequence the BEC width a oscillates with frequency ω between the values a_- and a_+ which evolve slowly as

$$a_\pm(t) = a_g \pm \frac{\sqrt{\alpha_c}}{\omega} f_1\gamma_0^{1/3}|W_t|.$$

This means that the BEC spreads out at the diffusive rate $[\mathrm{rms}^2(a)(t)] = \alpha_c f_1^2\gamma_0^{2/3}t/(2\omega^2)$. The doubling of the width is observed after a time of order $\omega^2/(\alpha_c f_1^2)$. In the case of an optical trap imposed by a laser field whose intensity is fluctuating the typical fluctuation level is of the order of $\alpha_c \sim 0.01$. If the trap frequency $\omega_z = 2\pi \times 300$ Hz, then we predict that the doubling of the width should be observed after a time of the order of a few seconds.

V. RESONANCES IN THE TONKS-GIRARDEAU REGIME

As is shown in Ref. [25] the variety of properties of the hard-core Bose gas with repulsive interaction in the dilute regime can be described by the nonlinear Schrödinger (NLS) equation with quintic nonlinearity:

$$i\hbar\phi_t = -\frac{\hbar^2}{2m}\phi_{zz} + V(z,t)\phi + \frac{\pi^2\hbar^2}{2m}|\phi|^4\phi, \tag{25}$$

where $V(z,t)$ is the time-dependent trap potential. In principle it can include an anharmonic part together with the harmonic component. Below we will restrict ourselves to the harmonic case $V(z,t) = m\omega_z^2 z^2 F(t)/2$. The wave function is normalized to the number of atoms, $\int|\phi|^2 dz = N$. This equation takes correctly into account the dependence on the density of the energy of the ground state of 1D Bose gas and reproduces correctly the collective modes [11]. It was shown in [26] by means of numerical simulations for a small number of atoms that the interference effects are overestimated by this equation. It should be noted that the nonlinear coefficient does not depend on the scattering length—i.e., the details of the interaction. Accordingly this parameter cannot be managed. Introducing $x = z/l_z$, $t' = t\omega_z$, $u = \sqrt{\pi l_z}\phi/2^{1/4}$, and $l_z = \sqrt{\hbar/(m\omega)}$ we can write the equation in the dimensionless form

$$iu_t = -\frac{1}{2}u_{xx} + \frac{1}{2}f(t)x^2 u + |u|^4 u, \tag{26}$$

where we have dropped the primes and $f(t)=F(t/\omega_z)$. As can be expected the change in the exponent of the nonlinear term induces strong differences between the MF regime and the TG regime. The quintic nonlinearity is especially interesting in the 1D case as it represents the critical nonlinearity for the NLS equation. Indeed, for a given dimension d, there exists a critical exponent $\sigma=2/d$ for the nonlinear term $|u|^{2\sigma}u$ which separates two different regimes [37]. These regimes are different in terms of global existence, blowup, instability growth, etc. In the 1D case, this exponent is $\sigma=2$ (quintic NLS equation); in the 2D case, it is $\sigma=1$ (cubic NLS equation). In the BEC framework, this leads for the repulsive gas to the parametric instability in a trapped 1D quintic system similarly to the parametric instability observed in a trapped 2D cubic system. For the attractive gas this leads to the collapse in a 1D quintic system with a number of atoms $N>N_c$ as well as the collapse in a 2D cubic system for $N>N_c$.

A. Variational approach

We apply the variational approach with the Gaussian ansatz (4). We find that the equation for the phase is decoupled from the equation for the atomic cloud width that reads

$$a_{tt} + f(t)a = \frac{\tilde{C}}{a^3}, \tag{27}$$

where $C=\int|u|^2dx=(\pi/\sqrt{2})N$ and $\tilde{C}=1+(4C^2)/(3^{3/2}\pi)$. The stationary value of the BEC width is given by the fixed point of Eq. (27):

$$a_g = \tilde{C}^{1/4}.$$

In the dimensional variables the BEC width is $L_c=[(2\pi)^{1/4}/3^{3/8}]\sqrt{N}l_z$. Linearizing near this solution the variational equation (27), we obtain the frequency of oscillations, $2\omega_z$, which coincides with the hydrodynamic calculations based on the local field approximation (see the next section). Equation (27) belongs to the so-called Ermakov-Penney equations [12,38,39] and the solution is

$$a(t) = \sqrt{\beta^2 + \frac{\tilde{C}}{W^2}c^2}, \tag{28}$$

where the functions β and c are linearly independent solutions to the equation

$$y_{tt} + f(t)y = 0, \tag{29}$$

$W=\beta c_t - c\beta_t$ is the constant Wronskian, and $\beta(0)=a(0)$, $\beta'(0)=a'(0)$, $c(0)=0$, and $c'(0)=1$. For a periodic function f, Eq. (29) is the so-called Hill equation which has been extensively studied [40]. So, in spite of the nonlinear character of Eq. (27), the resonant response of gas has a linear character and, in particular, the frequency of oscillations does not depend on the amplitude.

B. Hydrodynamic approach

Equation (26) can be cast in the form of Landau hydrodynamic equations by setting $u=\sqrt{\rho}\exp(i\theta)$ [11,41]. Furthermore, we get the Thomas-Fermi solution by neglecting the kinetic energy term with respect to the interaction term, so that the equations read

$$\rho_t = -(\rho v)_x, \tag{30}$$

$$v_t = -vv_x - f(t)x + 2\rho\rho_x, \tag{31}$$

where the velocity field v is defined by $\theta(t,x)=\int^x v(x',t)dx'$. The equilibrium profile for ρ corresponds to a stationary solution of the form $u(t,x)=\exp(-i\mu t)\tilde{u}(x)$ where the chemical potential μ is related to the normalized number of atoms, $C=\int|u|^2dx$, through the identity $\mu=\sqrt{2}C/\pi=N$. The equilibrium profile has a finite extension

$$\rho(t,x) = \frac{\mu}{a(t)}\sqrt{1-\frac{x^2}{2a^2(t)}} \tag{32}$$

for $x\in(-\sqrt{2}a,\sqrt{2}a)$, the velocity field is $xb(t)$, and a and b satisfy the coupled equations

$$a_t = ab, \tag{33}$$

$$b_t = -b^2 - f(t) + \frac{\mu^2}{a^4}. \tag{34}$$

Accordingly a satisfies the closed-form equation

$$a_{tt} + f(t)a = \frac{\tilde{C}}{a^3}, \tag{35}$$

where $\tilde{C}=\mu^2=2C^2/\pi^2$. Note that we have normalized the density profile so that $a/\sqrt{2}$ is the rms width, which is the same as for the Gaussian ansatz (4). We can thus compare the result (27) obtained with the variational approach using the Gaussian ansatz with the result (35) obtained with the hydrodynamic approach. Taking into account that the hydrodynamic approach is derived in the framework of a large number of atoms to neglect the kinetic term, we get that both approaches give the same effective equation, up to a small mismatch in the numerical value of $\tilde{C}\simeq 0.245C^2$ (variational approach) and $\tilde{C}\simeq 0.203C^2$ (hydrodynamic approach). This departure originates from the fact that the two stationary profiles do not coincide. Note that the stationary point is $\tilde{C}^{1/4}$, so that the difference is around 5% which is negligible in practical situations. Eventually, the BEC dynamics is found to be governed by the same effective equation according to both approaches.

C. Periodic modulations of the trap potential

The dynamics of the Hill equation driven by a periodic modulation $f(t)=1+f_1\sin(\Omega t)$ is characterized by a parametric resonance phenomenon studied in [32]. In particular the stability of the solutions to the Hill equation depends on the parameters f_1 and Ω. The theoretical prediction is that the

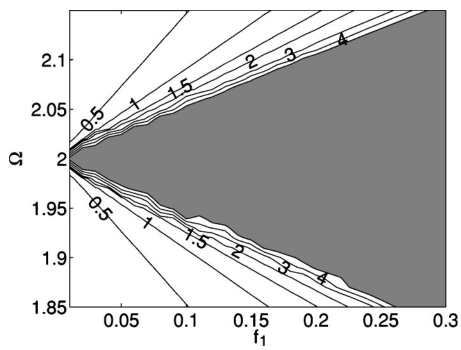

FIG. 6. Contour levels of the oscillation amplitude of the BEC width in the TG regime in the presence of periodic modulation of the trap potential $f(t)=1+f_1\sin(\Omega t)$. The gray area corresponds to the configurations where a blowup of the solution has been numerically observed.

stationary solution a_g is unstable when the modulation frequency is close enough to the eigenfrequency of the BEC or, more precisely,

$$|\Omega-2| \leqslant \frac{|f_1|}{2}. \quad (36)$$

We have performed numerical simulations of the PDE (26) to check this theoretical prediction. We have initiated the PDE with the initial condition given by the theoretically stationary Gaussian profile with $C=5\pi/\sqrt{2}$. The initial width is then $a_g \simeq 2.36$. Note in Fig. 6 the presence of a cone of instability in the (f_1,Ω) landscape, in full agreement with Eq. (36). We have found numerically that the dynamics is unstable if $|\Omega-2|\leqslant 0.55|f_1|$.

Note that the equation for the oscillations of a cloud of nonineracting fermions in a time-dependent elongated trap has the same form as Eq. (27) (see, for example, [33]). As a consequence the linear parametric resonance for the width of a fermionic cloud at the same frequency exists. In that sense we can say that the Fermi-Bose mapping still exists for the Tonks gas in an oscillating trap potential.

D. Random modulations of the trap potential

We examine in this section the effects of random modulations of the trap potential of the form $f(t)=1+\eta(t)$ where η is a zero-mean stochastic process. The dynamics then exhibits stochastic resonance as shown, for instance, in [42,43]. Contrarily to the periodic case, we always observe an exponential growth of the oscillations of the BEC width, unless the random modulation has a vanishing power spectral density in the vicinity of the resonant eigenfrequency 2. More precisely, we get that a periodically oscillates between the values $a_g \pm a_{osc}(t)$ with a_{osc}, which grows exponentially as

$$a_{osc}(t) \sim \exp\left(\frac{\alpha_c t}{4} + \frac{\sqrt{\alpha_c}}{2}W_t\right), \quad (37)$$

where $\alpha_c = \int_0^\infty \langle \eta(0)\eta(s)\rangle \cos(2s)ds$ and W_t is a Brownian motion. If η is a white noise $\langle \eta(0)\eta(t)\rangle = 2\sigma^2\delta(t)$, then we simply have $\alpha_c=\sigma^2$. The long-time behavior of a is dominated by the deterministic exponential growth term $\exp(\alpha_c t/4)$ with very high probability because $W_t \sim \sqrt{t}$. Note, however, that taking the expectation of Eq. (37) yields a different exponential growth rate

$$\langle a_{osc}^2\rangle^{1/2}(t) \sim \exp\left(\frac{\alpha_c}{2}t\right). \quad (38)$$

This is due to the fact that some exceptional realizations of the random fluctuations may induce very strong oscillations, and these exceptional realizations actually impose the value of the expected value.

In the white noise case $\langle \eta(0)\eta(t)\rangle = 2\sigma^2\delta(t)$, using the same linearization procedure as in Sec. III C we can get precise expressions for the rms amplitude as long as stochastic resonance can be neglected:

$$[\mathrm{rms}(a)(t)] = \frac{\sigma a_g}{\omega}\sqrt{t - \frac{\sin(2\omega t)}{2\omega}}, \quad (39)$$

where $\omega=2$ is the eigenfrequency. The simplest way to take into account stochastic resonance is to multiply the previous expression by the exponential damping term $\exp(\sigma^2 t/2)$. There exists a more accurate way based on Itô's calculus. Assume that the initial state is $a(0)=a_0$, $a'(0)=0$. The column vector $X=(\langle a^2\rangle,\langle ab\rangle,\langle b^2\rangle)^T$ satisfies the closed system

$$\frac{dX}{dt} = MX, \quad M = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 2\sigma^2 & -2 & 0 \end{pmatrix}, \quad (40)$$

starting from $X(0)=V_0=(a_0^2,0,\widetilde{C}/a_0^2)^T$. As a first application we can compute the exact expression of the largest eigenvalue of M which governs the exponential growth of the modulation $\langle a^2\rangle$:

$$\lambda_{\max} = \frac{(54\sigma^2 + 6\sqrt{48+81\sigma^4})^{2/3} - 12}{(54\sigma^2 + 6\sqrt{48+81\sigma^4})^{1/3}}.$$

Note that we recover formula (38) by expanding this expression for $\sigma \ll 1$: $\lambda_{\max} \simeq \sigma^2 + O(\sigma^6)$. A straightforward numerical integration of Eq. (38) gives the exact evolution of $\langle a^2\rangle$. We have performed numerical simulations of the PDE system (26) with a random modulation of the trap potential to check the predictions obtained with the variational approach. We have taken the model where $\eta(t)$ is stepwise constant over elementary intervals with duration t_c and takes random values uniformly distributed between −1 and 1. With $t_c = 0.06$ we then have $\alpha_c = 0.01$. We compare the results of numerical simulations with the theoretical predictions in Fig. 7. We can see that Eq. (39) efficiently predicts the initial growth of the oscillation amplitude, but it is necessary to take into account stochastic resonance when the amplitude becomes larger.

VI. CONCLUSION

In this work we have considered the resonances in collective oscillations of 1D Bose gas under time-dependent variations of the trap potential and the effective nonlinearity. Two

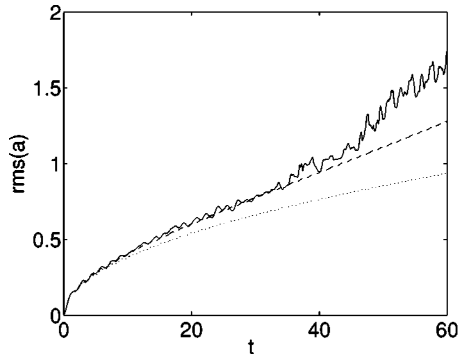

FIG. 7. Root mean square of the BEC width obtained from the averaging over a set of 1000 numerical simulations (solid line) and compared with the theoretical growth rate without stochastic resonance (dotted line) and with stochastic resonance (dashed line). The initial state is the Gaussian ansatz with $C=5\pi/\sqrt{2}$ and $a(0)=a_g=2.36$.

regimes have been studied—the mean-field regime and the Tonks-Girardeau regime. The analysis shows that in the mean-field regime the resonances are *nonlinear* and bistability exists in the vicinity of a critical frequency which is significantly above the eigenfrequency of the BEC. The dynamics is then characterized by stable oscillations with large amplitudes which depend on the frequency detuning between the frequency of the breathing mode and the modulation frequency. This type of dynamics is also predicted for the modulations of the trap as well as for the modulations of the atomic scattering length. In the Tonks-Girardeau regime the theory based on the nonlinear Schrödinger equation with quintic nonlinearity predicts *linear parametric* resonance in the gas oscillations. The effect reflects the Bose-Fermi map existing for the Bose gas in this regime [11]. We also study the oscillations under random variations of the trap potential and effective nonlinearity. The analysis shows that the dynamics is nonlinear in the mean-field regime. In the Tonks-Girardeau regime the study predicts stochastic parametric resonance. We also investigate the dynamics of 1D Bose gas in the mean-field regime under rapid and strong modulations of the atomic scattering length. This problem has recently attracted great attention since the dynamically stable nonlinearity managed atomic matter solitons can be generated. We find the analytical expressions for the stationary value of the width and frequency of the slow secondary oscillations of the width. All theoretical predictions turn out to be well supported by direct numerical simulations of the 1D GP equation and the quintic nonlinear Schrödinger equation.

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