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Random backscattering in the parabolic scaling

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Abstract In this paper we revisit the parabolic approximation for wave propagation in random media by taking into account backscattering. We obtain a system of transport equations for the moments of the components of reflection and transmission operators. In the regime in which forward scattering is strong and backward scattering is weak, we obtain closed form expressions for physically relevant quantities related to the reflected wave, such as the beam width, the spectral width and the mean spatial power profile. In particular, we analyze the enhanced backscattering phenomenon, that is, we show that the mean power reflected from an incident quasi-plane wave has a maximum in the backscattered direction. This enhancement can be observed in a small cone around the backscattered direction and we compute the enhancement factor as well as the shape of the enhanced backscattering cone.

Keywords Waves · random media · asymptotic analysis · enhanced backscattering

Mathematics Subject Classification (2000) 76B15 · 35Q99 · 60F05

1 Introduction

Wave propagation in random media has received a lot of attention in recent decades due to a wide range of important applications, for instance, in communication, remote sensing, and imaging. A special regime is encountered in many of these situations, such as for laser propagation in the atmosphere, in which the wave has the form of a beam that propagates along a propagation axis. This situation arises if the propagation distance L and the incoming beam, characterized by a typical wavelength λ and a beam width R, satisfy the parabolic approximation, namely $L \gg R \gg \lambda$ and $R^2 \sim L\lambda$. In this regime and in a homogeneous medium, the paraxial wave equation describes how the wave propagates and spreads out by diffraction.

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In this paper we model the medium as being randomly heterogeneous. The paraxial wave equation in random media has been studied extensively [24,25]. The random fluctuations of the medium can be characterized by two length scales: the longitudinal correlation radius L_z (i.e. the correlation length in the propagation direction) and the transverse correlation radius L_x. Different regimes have been presented and analyzed in the literature, all of them are characterized by the fact that the forward-scattering approximation is used, in the sense that backscattering is neglected and the deformation of the transmitted wave field is then analyzed [6,7,19,20,23]. However, it would be useful to study the backscattered wave, since it may be the only information available in many remote sensing or imaging configurations. In this paper, we address the regime $L_x \sim R$ and $L_z \sim \lambda$ in which conversion from forward-going to backward-going waves is not negligible. In this regime, using diffusion approximation theory, we obtain a system of transport equations for the moments of the components of the reflection and transmission operators. This system captures the full conversion mechanisms between forward- and backward-going waves. The full system is complicated, however, there is a regime in which the analysis can be carried out in detail. It is based on the fact that the conversion rate between two wave components is proportional to the power spectral density of the medium fluctuations evaluated at the difference of the two wavevectors of these wave components. Consequently, if we assume that λ is slightly smaller than L_z, then the conversion rate between backward-going and forward-going waves is smaller than the conversion rate between two forward-going waves with nearby wavevectors. In this regime in which forward scattering can be strong, but backscattering is weak, it is possible to obtain a tractable system of transport equations and to study analytically various physically relevant quantities, such as the beam width, the spectral width, or the spatial power profile of the reflected wave. In addition, it is also possible to identify the enhanced backscattering phenomenon: if a monochromatic quasi-plane wave is incoming with a given incidence angle, then the mean reflected power has a local maximum in the backscattered direction. This enhancement can in fact be observed in a small cone around the backscattered direction. This phenomenon, also called weak localization, is well-known in physics and it has been observed in several experimental contexts, in optics with powder suspensions [30,27], with biological tissues [31], with ultra-cold atoms [16] and in acoustics [26]. It can be explained by diagrammatic expansions [2,28], where the reciprocity principle and interference effects between direct and reverse wave paths play a crucial role. Here we give a mathematical derivation of this phenomenon by an asymptotic analysis in the weak backscattering regime. Namely, we compute the enhancement factor, which is equal to two if forward scattering is strong enough, and we describe the shape of the enhanced backscattering cone.

2 Waves in a random medium

We consider linear acoustic waves propagating in $1+d$ spatial dimensions with heterogeneous and random medium fluctuations. The governing equations are

$$\rho^\varepsilon(z,\mathbf{x})\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + \nabla p^\varepsilon = \mathbf{F}^\varepsilon(t,z,\mathbf{x}), \quad \frac{1}{K^\varepsilon(z,\mathbf{x})}\frac{\partial p^\varepsilon}{\partial t} + \nabla\cdot\mathbf{u}^\varepsilon = 0, \tag{1}$$

where p^ε is the pressure, $\mathbf{u}^\varepsilon$ is the velocity, ρ^ε is the density of the medium, K^ε is the bulk modulus of the medium, and $(z,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$ are the space coordinates. The source is modeled by the forcing term $\mathbf{F}^\varepsilon$. Here we shall focus on propagation through and reflection from a random slab occupying the interval $z \in (0, L)$ with the source $\mathbf{F}^\varepsilon$ located outside of the slab, in the half-space $z > L$. The parameterization is motivated by waves probing for instance the heterogeneous earth and we may think of z as the main probing direction. We shall refer to waves propagating in a direction with a positive z component as right-propagating waves.

2.1 Scaling

We consider a scaling where the random medium fluctuations vary relatively rapidly in space while the "background" medium is constant. We normalize the background bulk modulus and density to one and we consider the following model for the bulk modulus fluctuations

$$\frac{1}{K^\varepsilon(z,\mathbf{x})} = \begin{cases} 1 + \varepsilon^r \nu(z/\varepsilon^2, \mathbf{x}/\varepsilon^{p+\Delta p}) \text{ if } z \in (0, L)\,, \\ 1 \text{ otherwise }\,, \end{cases}$$

where r, p and Δp are nonnegative constants that we discuss below and ε is a small parameter. The dimensionless density is assumed to be constant (and equal to one) for simplicity. The random field $\nu(z, \mathbf{x})$ models the spatial fluctuations of the medium and we assume that it is a zero-mean, stationary, and $1 + d$-dimensional random process and that it satisfies strong mixing conditions in the z-direction. We remark that the medium is specified as being matched at the boundaries of the random slab so that the wave speed in the complement of the slab $z < 0$ and $z > L$ coincides with the background wave speed in the slab $z \in (0, L)$ [9]. We consider a scaling where the central wavelength of the source is of order ε^q and write

$$\mathbf{F}^\varepsilon(t, z, \mathbf{x}) = f\Big(\frac{t}{\varepsilon^q}, \frac{\mathbf{x}}{\varepsilon^p}\Big)\delta(z - z_0)\mathbf{e}_z\,, \tag{2}$$

with q a positive parameter, $z_0 > L$, and $\mathbf{e}_z$ the unit vector pointing in the z-direction. Our objective is to characterize both the transmitted and reflected wave fields. The transmitted wave field is the field observed at the end of the slab (at $z = 0$) while the reflected wave field is the wave field reflected back from the random slab (at $z = L$). Our first task is to identify equations that give a convenient description of coupling between different wave modes. The complex apmlitudes $\check{a}^\varepsilon$ and $\check{b}^\varepsilon$ of the generalized right-propagating and left-propagating modes are defined by

$$\check{a}^\varepsilon(k, z, \mathbf{x}) = \Big(\frac{1}{2\varepsilon^q}\int p^\varepsilon(t, z, \mathbf{x})e^{ikt/\varepsilon^q}dt + \frac{1}{2ik}\int \frac{\partial p^\varepsilon}{\partial z}(t, z, \mathbf{x})e^{ikt/\varepsilon^q}dt\Big)e^{-ikz/\varepsilon^q}\,,$$
$$\check{b}^\varepsilon(k, z, \mathbf{x}) = \Big(\frac{1}{2\varepsilon^q}\int p^\varepsilon(t, z, \mathbf{x})e^{ikt/\varepsilon^q}dt - \frac{1}{2ik}\int \frac{\partial p^\varepsilon}{\partial z}(t, z, \mathbf{x})e^{ikt/\varepsilon^q}dt\Big)e^{ikz/\varepsilon^q}\,.$$

They are such that the pressure field has the form:

$$p^\varepsilon(t, z, \mathbf{x}) = \frac{1}{2\pi}\int \Big(\check{a}^\varepsilon(k, z, \mathbf{x})e^{ikz/\varepsilon^q} + \check{b}^\varepsilon(k, z, \mathbf{x})e^{-ikz/\varepsilon^q}\Big)e^{-ikt/\varepsilon^q}\,dk\,, \tag{3}$$

and they also satisfy the condition that serves to correctly decompose the wave

$$\frac{\partial \check{a}^\varepsilon}{\partial z}e^{ikz/\varepsilon^q} + \frac{\partial \check{b}^\varepsilon}{\partial z}e^{-ikz/\varepsilon^q} = 0\,.$$

In the homogeneous medium with $\nu = 0$ the ansatz (3) gives a decomposition into uncoupled right- and left-propagating modes. In the case in which the medium is layered with $\nu = \nu(z)$ the ansatz gives a decomposition into right- and left-propagating modes that couple via a zero-mean stochastic coupling matrix [1,9]. In the layered case the problem moreover decomposes into mode problems corresponding to a particular lateral slowness, that is, a particular lateral velocity component. We shall see that in the general case with $\nu = \nu(z, \mathbf{x})$ we have a coupling of modes via a zero-mean coupling "matrix", however, in this case this coupling involves in general modes of all lateral directions so that the coupling matrix now becomes a coupling operator.

We next rescale as $\mathbf{x}/\varepsilon^p \to \mathbf{x}$ and obtain the following coupled mode equations:

$$\frac{d\check{a}^\varepsilon}{dz} = (\mathcal{L}_1 + \mathcal{L}_2)\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right)\check{a}^\varepsilon + e^{\frac{-2ikz}{\varepsilon^q}}(\mathcal{L}_1 + \mathcal{L}_2)\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right)\check{b}^\varepsilon\,, \tag{4}$$

$$\frac{d\check{b}^\varepsilon}{dz} = -e^{\frac{2ikz}{\varepsilon^q}}(\mathcal{L}_1 + \mathcal{L}_2)\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right)\check{a}^\varepsilon - (\mathcal{L}_1 + \mathcal{L}_2)\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right)\check{b}^\varepsilon\,, \tag{5}$$

for

$$\mathcal{L}_1\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right) = \frac{i\varepsilon^{r-q}k}{2}\nu\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right)\,, \qquad \mathcal{L}_2\left(\frac{z}{\varepsilon^2}, \frac{\mathbf{x}}{\varepsilon^{\Delta p}}\right) = \frac{i\varepsilon^{q-2p}}{2k}\triangle_\perp\,,$$

with $\triangle_\perp$ the transverse Laplacian. Before we proceed with the analysis of (4-5) we remark that in the white-noise scaling: $r - q = -1$, $q = 2p > 2$ and $\Delta p = 0$, the fast phases $\exp(\pm 2ikz/\varepsilon^q)$ cancel out the coupling terms between $\check{a}^\varepsilon$ and $\check{b}^\varepsilon$ in (4-5) and we obtain the forward or one-way wave approximation corresponding to:

$$\frac{d\check{a}^\varepsilon}{dz} = (\mathcal{L}_1 + \mathcal{L}_2)\left(\frac{z}{\varepsilon^2}, \mathbf{x}\right)\check{a}^\varepsilon\,,$$

which can be written

$$\frac{d\check{a}^\varepsilon}{dz} = \frac{i}{2k}\triangle_\perp\check{a}^\varepsilon + \frac{ik}{2\varepsilon}\nu\Big(\frac{z}{\varepsilon^2}, \mathbf{x}\Big)\check{a}^\varepsilon\,. \tag{6}$$

This is the celebrated Schrödinger or paraxial wave equation. We next make some remarks regarding the scaling that we have set forth.

1) The relative lateral scale of the fluctuations is determined by the parameter Δp. If $\Delta p < 0$ the medium is to leading order layered leading to a situation of the type analyzed in [9]. Here, we extend this analysis by considering the base case situation with $\Delta p = 0$, the situation in which the lateral variation is on the scale of the lateral paraxial spreading scale leading to a delicate interaction between the wave modes. We remark that we will also discuss cases with $\Delta p \neq 0$ below. The parameter Δp characterizes the lateral diversity in the problem and we shall differentiate between the situations: (i) no lateral diversity or layered medium, $\Delta p = -\infty$, (ii) moderate lateral diversity, $-\infty < \Delta p < 0$, a perturbation of the layered situation, (iii) critical lateral diversity which will be our focus, $\Delta p = 0$, (iv) large lateral diversity, $\Delta p > 0$, this regime leads to statistical stability for some important functionals of the wave field, in the context of the Schrödinger equation it was discussed in [19,20].

2) Note that only when we observe the wave field in the parabolic scaling regime corresponding to the lateral scale

$$p = \frac{q}{2}\,, \tag{7}$$

do we observe non-degenerate lateral coupling in the transmitted wave field. This corresponds to the lateral spreading scale $\varepsilon^{q/2}$ of the Schrödinger Green function at a depth of order one and a wavelength of order ε^q. Note that on lateral scales corresponding to $p < q/2$ we observe only the wave front behavior and the problem becomes essentially one-dimensional or layered [9]. Lateral scales corresponding to $p > q/2$ are relatively coarse so that the lateral wave field structure cannot be resolved. We will thus here use the parabolic scaling (7).

3) We shall also use the white-noise scaling $r-q = -1$, corresponding to a potential which in the limit $\varepsilon \to 0$ becomes a Brownian random field in distribution. In this scaling the wave field can be given a weak or distributional characterization. If $r - q < -1$ then the random medium fluctuations become very strong so that the wave field structure cannot be given a generic description. If $r-q > -1$ then the random medium fluctuations are weak and the wave field is not affected by them to leading order.

4) In the parabolic scaling regime we comment on two particular situations.
First, consider the case

$$q = 2p = 2\,. \tag{8}$$

It corresponds to the situation in which the wavelength is on the scale of the random medium fluctuations in the propagation direction. The white-noise scaling then gives $r = 1$, corresponding to a weakly heterogeneous scaling.
Second, if $q = 2p = 1$ then the wavelength is large compared to the scale of the random fluctuations. The white-noise scaling then gives $r = 0$ leading to what we refer to as a strongly heterogeneous scaling.
We remark that if $q = 2p < 1$ then, even with random fluctuations of order one, we are led to the effective medium approximation for the slab. While if $q = 2p > 2$ then the wavelength is small compared to the scale of the random fluctuations leading to a geometrical optics scaling where the wave field interacts strongly with the particular features of medium fluctuations and we shall not pursue this scaling here [22,29]. In this paper we shall focus on the weakly heterogeneous scaling.

5) Finally, we remark that $p + \Delta p = 2$ and $r = 1$ is another particular scaling that can be analyzed, it corresponds to a radiative transfer scaling [21]. In this situation the wave field is not coherent, but the mean incoherent wave intensity can be given a generic description. Here we are interested in scaling regimes that give partially coherent transmission, the situation in which the mean field and wave fluctuations coexist.

In conclusion we shall focus the analysis on the weakly heterogeneous parabolic scaling regime with critical lateral fluctuation length characterized by

$$q = 2p = 2\,, \quad r = 1\,, \quad \Delta p = 0\,. \tag{9}$$

Note that we have made a particular choice of propagation direction z versus lateral directions $\mathbf{x}$. It corresponds in our scaling to a situation with a wave front or beam propagation with the z axis being the propagation direction of the front in the original coordinates in (1) (see Figure 1).

The forward-scattering approximation (6) has played an important role in many applications of wave propagation. However, in a situation with relatively strong medium fluctuations there is a strong longitudinal coupling of modes that is not captured by this approximation. We show next a numerical simulation that illustrates this fact. Consider a random medium half-space with an embedded extended scatterer. The solid line in Figure 2 is the spectrum of the absolute value of the reflected harmonic wave field computed via a finite difference discretization of the Helmholtz equation over a finite domain with non-reflecting boundary conditions approximated by Perfectly Matched

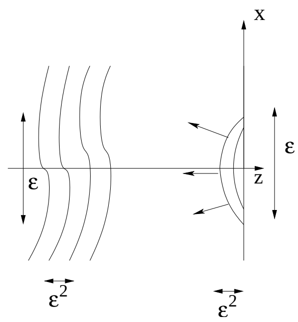

Fig. 1 This figure illustrates the scaling regime discussed in this paper: The typical wavelength is of order ε^2, this is also the case for the longitudinal correlation length. The beam radius is of order ε, this is also the case for the transverse correlation radius. Finally, the propagation distance is of order 1.

Layers. Note that this discretization gives coupling of all unknowns rather than a simple time marching scheme as in the forward-scattering approximation. The dashed line in the left plot is the spectrum that results from using only the uncoupled forward or parabolic approximation. We see that longitudinal scattering is not captured well leading to poor approximation of the wave heterogeneity and an artificially smooth approximation. In the right plot the dashed line corresponds to an approximation of the system (4-5) that captures the important "right" and "left" coupling [13,14]. The latter approximation corresponds to iterative right and left sweeps implementation of (4-5) when the coupling correction associated with $\exp(\pm 2ikz/\varepsilon^q)\mathcal{L}_2$ is neglected, giving convergence after a few iterations. The boundary condition at the depth of the embedded scatterer is implemented via a domain decomposition approach with the inclusion located in a small domain numerically resolved via a discretization of the Helmholtz equation (see [13] for details). Motivated by this computational example we

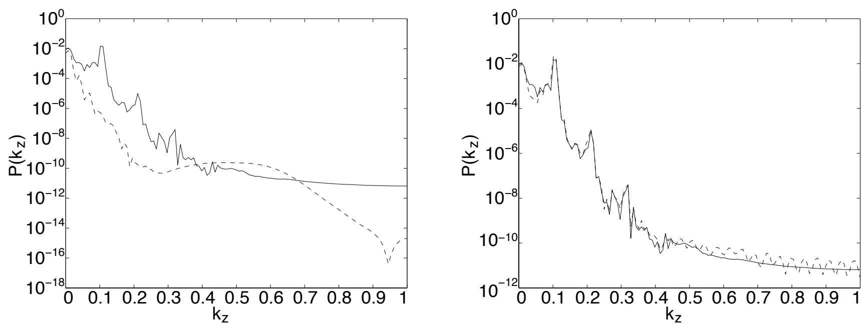

Fig. 2 The figure shows on a log scale: $P(k_z) = |\sum_{j=1}^{N_z} | \int_{-\infty}^{\infty} p(t, \mathbf{0}, z_j) e^{i\omega_0 t}\, dt | e^{2\pi i j k_z}|$, where z_j are the discretization points in the depth direction and $N_z = 1400$, moreover, the wavelength $2\pi c_0/\omega_0$ is $6\pi\Delta z$. The random medium fluctuations are smooth with a Gaussian spectrum, a correlation length of $10\Delta z$ and a contrast in the index of refraction of 5%. The solid line in the two plots corresponds to a discretization of the Helmholtz equation. In the left plot the dashed line derives from a discretization via the uncoupled parabolic approximation and gives an artificially smooth solution, while in the right plot the dashed line corresponds to a discretization via the coupled parabolic approximation system and resolves much better the features of the Helmholtz solution that comes from longitudinal scattering.

derive in the next sections an analytic framework that can be used to understand the longitudinal coupling that is not captured by the forward approximation.

2.2 The boundary conditions

The mode amplitudes $\check{a}^\varepsilon$ and $\check{b}^\varepsilon$ satisfy the system (4-5) in the random slab $z \in (0, L)$. This system can be supplemented by boundary conditions corresponding to the presence of the source term (2) in the plane $z = z_0$, with $z_0 > L$. In the regions $z \in (-\infty, 0)$, $z \in (L, z_0)$ and $z \in (z_0, \infty)$ the medium is homogeneous and the mode amplitudes satisfy the uncoupled paraxial equations

$$\frac{d\check{a}^\varepsilon}{dz} = \frac{i}{2k}\triangle_\perp \check{a}^\varepsilon \,, \qquad \frac{d\check{b}^\varepsilon}{dz} = -\frac{i}{2k}\triangle_\perp \check{b}^\varepsilon \,.$$

Taking into account the fact that there is no source in $(-\infty, 0)$, we find that the right-going mode amplitudes $\check{a}^\varepsilon$ are zero in this half-space. By the continuity of the fields p^ε and $\mathbf{e}_z \cdot \mathbf{u}^\varepsilon$ at $z = 0$, this gives the first boundary condition

$$\check{a}^\varepsilon(k, z = 0, \mathbf{x}) = 0 \,. \tag{10}$$

Taking into account the fact that there is no source in (z_0, ∞), we find that the left-going mode amplitudes $\check{b}^\varepsilon$ are zero in this half-space. The jump conditions accross the source interface $z = z_0$ then give the relations

$$\check{b}^\varepsilon(k, z_0^-, \mathbf{x}) = -\frac{1}{2} e^{ikz_0/\varepsilon^2} \check{f}(k, \mathbf{x}) \,, \quad \check{a}^\varepsilon(k, z_0^+, \mathbf{x}) - \check{a}^\varepsilon(k, z_0^-, \mathbf{x}) = \frac{1}{2} e^{-ikz_0/\varepsilon^2} \check{f}(k, \mathbf{x}) \,.$$

By solving the paraxial wave equation for $\check{b}^\varepsilon$ in the region $z \in (L, z_0)$, we obtain the expression of the complex amplitude of the wave incoming in the random slab at $z = L$:

$$\check{b}^\varepsilon(k, L, \mathbf{x}) = e^{ikz_0/\varepsilon^2} \check{b}(k, \mathbf{x}) \,, \tag{11}$$

$$\check{b}(k, \mathbf{x}) = -\frac{1}{2(2\pi)^d} \int \hat{f}(k, \boldsymbol{\kappa}) e^{\frac{i}{2k}|\boldsymbol{\kappa}|^2 (L - z_0) + i\boldsymbol{\kappa}\cdot\mathbf{x}} d\boldsymbol{\kappa} \,, \tag{12}$$

where the transverse spatial Fourier transform is defined by

$$\hat{f}(k, \boldsymbol{\kappa}) = \int \check{f}(k, \mathbf{x}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}} d\mathbf{x} \,. \tag{13}$$

2.3 Multimode wave equations

We shall make use of an invariant imbedding step and introduce reflection and transmission operators. First, we define the transverse Fourier modes

$$\hat{a}^\varepsilon(k, z, \boldsymbol{\kappa}) = \int \check{a}^\varepsilon(k, z, \mathbf{x}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}} \, d\mathbf{x} \,, \qquad \hat{b}^\varepsilon(k, z, \boldsymbol{\kappa}) = \int \check{b}^\varepsilon(k, z, \mathbf{x}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}} \, d\mathbf{x} \,, \tag{14}$$

and make the ansatz

$$\hat{b}^\varepsilon(k, 0, \boldsymbol{\kappa}) = \int \hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k, z, \boldsymbol{\kappa}, \boldsymbol{\kappa}') \hat{b}^\varepsilon(k, z, \boldsymbol{\kappa}') \, d\boldsymbol{\kappa}' \,, \tag{15}$$

$$\hat{a}^\varepsilon(k, z, \boldsymbol{\kappa}) = \int \hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k, z, \boldsymbol{\kappa}, \boldsymbol{\kappa}') \hat{b}^\varepsilon(k, z, \boldsymbol{\kappa}') d\boldsymbol{\kappa}' \,. \tag{16}$$

For $\hat{b}^\varepsilon(k,L,\boldsymbol{\kappa}')$ giving the incoming wave, $\hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k,L,\boldsymbol{\kappa},\boldsymbol{\kappa}')$ maps this to the wave $\hat{b}^\varepsilon(k,0,\boldsymbol{\kappa})$ transmitted to $z=0$, while $\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,L,\boldsymbol{\kappa},\boldsymbol{\kappa}')$ maps it to the wave $\hat{a}^\varepsilon(k,L,\boldsymbol{\kappa})$ reflected from the random slab at $z=L$.

Using the mode coupling equations (4-5) we find

$$
\begin{aligned}
\frac{d}{dz}\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') &= e^{-\frac{2ikz}{\varepsilon^2}}\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') \\
&\quad + e^{\frac{2ikz}{\varepsilon^2}}\iint \hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}')d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 \\
&\quad + \int \hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}') + \hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}')d\boldsymbol{\kappa}_1\,, \\
\frac{d}{dz}\hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') &= \int \hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}')d\boldsymbol{\kappa}_1 \\
&\quad + e^{\frac{2ikz}{\varepsilon^2}}\iint \hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}')d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\,,
\end{aligned}
\tag{17}
$$

in the parabolic white-noise regime described by (9) and where we have defined

$$
\hat{\mathcal{L}}^\varepsilon\big(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2\big) = -\frac{i}{2k}|\boldsymbol{\kappa}_1|^2\delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2) + \frac{ik}{\varepsilon 2(2\pi)^d}\hat{\nu}\Big(\frac{z}{\varepsilon^2},\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\Big)\,, \tag{18}
$$

with $\hat{\nu}(z,\boldsymbol{\kappa})$ the partial Fourier transform of $\nu(z,\mathbf{x})$ as defined by (13). This system is supplemented by the initial conditions

$$
\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z=0,\boldsymbol{\kappa},\boldsymbol{\kappa}') = 0\,, \qquad \hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k,z=0,\boldsymbol{\kappa},\boldsymbol{\kappa}') = \delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\,, \tag{19}
$$

corresponding to the boundary conditions (10-11). The transmission and reflection operators evaluated at $z=L$ carry all the relevant information about the random medium from the point of view of the transmitted and reflected waves. In [17] an operator Riccati equation for the Dirichlet-to-Neumann map is derived from the exact operator factorization of the Helmholtz equation and used to design a numerical scheme via a finite-dimensional expansion in local eigenfunctions. The situation considered there is a waveguide with general boundary conditions and with macroscale and deterministic medium variations. In [15] the situation with slow transverse modulations of the medium parameters, both for the deterministic and random medium components, is discussed. In terms of ray theory this corresponds to almost straight rays, but with weak random variations. A generalized ray theory is discussed in [22] where the deterministic medium is smooth and varying in a general way on the macroscopic scale, while the microscale variations are one-dimensional and varying relative to a set of level curves, then the rays are general, but deterministic. The latter two formulations are referred to as locally layered media and operator-valued formulations are seth forth in the corresponding formulations. Here, our focus is on a random medium that varies on a microscale in all spatial directions according to the scaling theory outlined above and that is not locally layered. We consider moreover a formulation where the background or deterministic component of the medium is constant so that at the ray theoretical level the picture is simple enabling us to analyze and characterize the interaction of the wave with a $1+d$-dimensional microscale random medium, this is our main objective. In particular we aim at identifying the weak, in a probabilistic sense, asymptotic limit of the distribution-valued stochastic transmission and reflection operators in the scaling limit $\varepsilon\to 0$ with medium fluctuations that are strongly mixing in the propagation direction z and with a finite correlation length in the transverse directions.

The outline of the paper is as follows: In Section 3 we develop our framework for the analysis of the incoherent wave field. In Section 3.1 we derive a framework with generalized transport equations that enables us to analyze and describe the fine scale statistical character of the incoherent fluctuations of the reflected wave field. This leads to a very complicated system and in Subsection 3.2 we identify a simplified system in the regime of weak backscattering. This reduced system enables us, for the first time, to address a number of questions regarding the reflected wave field in the parabolic scaling situation. We consider a number of specific aspects of the incoherent wave field in Sections 4 and 5 corresponding respectively to regimes of small and large Fresnel numbers. In Section 6 we turn our attention to the transmitted wave field. We again derive a system of transport equations that characterizes the transmitted wave field and its statistics. The regime of weak backscattering discussed in Subsection 6.1 again leads to a simplified system of transport equations that can be used for explicit characterization of the transmitted wave field and that goes beyond the regime captured by the classic forward approximation. We use this framework in Section 6.2 to analyze in particular the intensity of the transmitted wave field. In the appendices we present a number of technical proofs.

3 Asymptotic analysis of wave reflections

In this section and in the companion Section 6 we shall identify a set of equations that can be used to determine the spectrum of the reflected and transmitted waves. This provides a novel framework in which a number of applications, in communication and imaging for instance, can be given a complete mathematical analysis. So far this has only been possible in the forward approximation or in the layered case or in the case with waveguides, see [10–12] for a discussion of transport equations in the case of waveguides.

In order to characterize the spectrum of the wave reflection process it is important to identify correlations at nearby frequencies. They will be characterized by a family of transport equations that we give in Subsection 3.1. In Subsection 3.2 we present a dimensionless form of this system in the weak backscattering regime. In Sections 4-5 we shall study the reflected wave in the weak backscattering regime and discuss a few important physical results.

3.1 Generalized transport equations for reflections

The generalized reflection operator $\hat{\boldsymbol{\mathcal{R}}}^\varepsilon$ solves (17) with the initial condition (19). Our objective is now to compute cross moments of the reflection operator using diffusion approximation theory in the limit $\varepsilon \to 0$, in which the phase factors $\exp(\pm 2ikz/\varepsilon^2)$ act as decoupling terms. We have the following result that is proved in Appendix A.

Proposition 1 *Let us introduce some notations. If $\boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}'_p(j) \in \mathbb{R}^d$, $j = 1, \ldots, n_{\mathbf{p}}$, $\boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}'_q(l) \in \mathbb{R}^d$, $l = 1, \ldots, n_{\mathbf{q}}$, then we denote by $\mathbf{p}$ and $\mathbf{q}$ the sets*

$$\mathbf{p} = \big\{ (\boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}'_p(j)) \big\}_{j=1}^{n_{\mathbf{p}}} , \qquad \mathbf{q} = \big\{ (\boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}'_q(l)) \big\}_{l=1}^{n_{\mathbf{q}}} , \tag{20}$$

where $n_{\mathbf{p}}$ stands for the number of pairs of vectors in $\mathbf{p}$ and $n_{\mathbf{q}}$ stands for the number of pairs of vectors in $\mathbf{q}$. We introduce the high-order moments of products of

$\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')$, *the reflection process, at two nearby frequencies:*

$$\mathcal{U}^{\varepsilon}_{\mathbf{p},\mathbf{q}}(k,h,z) = \mathbb{E}\Big[\prod_{j=1}^{n_{\mathbf{p}}} \hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}'_p(j)\Big) \prod_{l=1}^{n_{\mathbf{q}}} \overline{\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}'_q(l)\Big)\Big]. \tag{21}$$

We define the autocorrelation function of the fluctuations of the medium and its Fourier transform by

$$C(z,\mathbf{x}) = \mathbb{E}[\nu(z'+z,\mathbf{x}'+\mathbf{x})\nu(z',\mathbf{x}')]\,, \tag{22}$$

$$\widehat{C}(k,\boldsymbol{\kappa}) = \int_{\mathbb{R}^d}\int_{-\infty}^{\infty} C(z,\mathbf{x}) e^{-i(kz+\boldsymbol{\kappa}\cdot\mathbf{x})} dz d\mathbf{x}\,, \tag{23}$$

$$\widehat{C}^{\pm}(k,\boldsymbol{\kappa}) = 2\int_{\mathbb{R}^d}\int_{0}^{\infty} C(z,\mathbf{x}) e^{\pm ikz - i\boldsymbol{\kappa}\cdot\mathbf{x}} dz d\mathbf{x}\,. \tag{24}$$

The family of Fourier transforms

$$W^{\varepsilon}_{\mathbf{p},\mathbf{q}}(k,\tau,z) = \frac{1}{2\pi}\int e^{-ih[\tau-(n_{\mathbf{p}}+n_{\mathbf{q}})z]} \mathcal{U}^{\varepsilon}_{\mathbf{p},\mathbf{q}}(k,h,z)\, dh\,, \tag{25}$$

converges as $\varepsilon \to 0$ to the solution $W_{\mathbf{p},\mathbf{q}}$ of the system of transport equations

$$\frac{\partial W_{\mathbf{p},\mathbf{q}}}{\partial z} + (n_{\mathbf{p}}+n_{\mathbf{q}})\frac{\partial W_{\mathbf{p},\mathbf{q}}}{\partial \tau} = \frac{i}{2k}\Phi_{\mathbf{p},\mathbf{q}} W_{\mathbf{p},\mathbf{q}} + \frac{k^2}{4(2\pi)^d}\left(\mathcal{L}_W W\right)_{\mathbf{p},\mathbf{q}}\,, \tag{26}$$

with the initial conditions $W_{\mathbf{p},\mathbf{q}}(z=0,k,\tau) = \mathbf{1}_0(n_{\mathbf{p}})\mathbf{1}_0(n_{\mathbf{q}})\delta(\tau)$. Here we have defined

$$\Phi_{\mathbf{p},\mathbf{q}} = -\sum_{j=1}^{n_{\mathbf{p}}}\left(|\boldsymbol{\kappa}_p(j)|^2 + |\boldsymbol{\kappa}'_p(j)|^2\right) + \sum_{l=1}^{n_{\mathbf{q}}}\left(|\boldsymbol{\kappa}_q(l)|^2 + |\boldsymbol{\kappa}'_q(l)|^2\right), \tag{27}$$

$$
\begin{aligned}
&\big(\mathcal{L}_W W\big)_{\mathbf{p},\mathbf{q}} = -\int \Big[n_{\mathbf{p}}\widehat{C}^{+}(2k,\boldsymbol{\kappa}) + n_{\mathbf{q}}\widehat{C}^{-}(2k,\boldsymbol{\kappa}) + (n_{\mathbf{p}}+n_{\mathbf{q}})\widehat{C}(0,\boldsymbol{\kappa})\Big]\, d\boldsymbol{\kappa}\, W_{\mathbf{p},\mathbf{q}} \\
&-\int \widehat{C}(0,\boldsymbol{\kappa})\Big[\sum_{j=1}^{n_{\mathbf{p}}} W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa})\},\mathbf{q}} + \sum_{l=1}^{n_{\mathbf{q}}} W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l)-\boldsymbol{\kappa})}\Big] d\boldsymbol{\kappa} \\
&-\sum_{j_1\neq j_2=1}^{n_{\mathbf{p}}} \int \Big\{ \widehat{C}(2k,\boldsymbol{\kappa}_p(j_1)-\boldsymbol{\kappa}'_p(j_1)) W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_2),\boldsymbol{\kappa}-\boldsymbol{\kappa}_p(j_1)),(\boldsymbol{\kappa}-\boldsymbol{\kappa}'_p(j_1),\boldsymbol{\kappa}'_p(j_2))\},\mathbf{q}} \\
&+\frac{1}{2}\widehat{C}(0,\boldsymbol{\kappa})\Big[W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j_1)),(\boldsymbol{\kappa}_p(j_2)+\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j_2))\},\mathbf{q}} \\
&+2\, W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j_1)),(\boldsymbol{\kappa}_p(j_2),\boldsymbol{\kappa}'_p(j_2)-\boldsymbol{\kappa})\},\mathbf{q}} \\
&+W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_1),\boldsymbol{\kappa}'_p(j_1)-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_p(j_2),\boldsymbol{\kappa}'_p(j_2)+\boldsymbol{\kappa})\},\mathbf{q}}\Big]\Big\} d\boldsymbol{\kappa} \\
&-\sum_{l_1\neq l_2=1}^{n_{\mathbf{q}}} \int \Big\{ \hat{C}(2k,\boldsymbol{\kappa}_q(l_1)-\boldsymbol{\kappa}'_q(l_1)) W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_2),\boldsymbol{\kappa}-\boldsymbol{\kappa}_q(l_1)),(\boldsymbol{\kappa}-\boldsymbol{\kappa}'_q(l_1),\boldsymbol{\kappa}'_q(l_2))\}} \\
&+\frac{1}{2}\widehat{C}(0,\boldsymbol{\kappa})\Big[W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l_1)),(\boldsymbol{\kappa}_q(l_2)+\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l_2))\}} \\
&+2\, W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l_1)),(\boldsymbol{\kappa}_q(l_2),\boldsymbol{\kappa}'_q(l_2)-\boldsymbol{\kappa})\}} \\
&+W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_1),\boldsymbol{\kappa}'_q(l_1)-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_q(l_2),\boldsymbol{\kappa}'_q(l_2)+\boldsymbol{\kappa})\}}\Big]\Big\} d\boldsymbol{\kappa} \\
&+\sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}} \Big\{ \widehat{C}(2k,\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}'_p(j))\delta(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa}_q(l)+\boldsymbol{\kappa}'_q(l)) W_{\mathbf{p}|j,\mathbf{q}|l} \\
&+\int \widehat{C}(0,\boldsymbol{\kappa})\, \Big[W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j))\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l))\}} \\
&+W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa})\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)-\boldsymbol{\kappa})\}} \\
&+W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j))\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)+\boldsymbol{\kappa})\}} \\
&+W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa})\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)+\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l))\}}\Big] d\boldsymbol{\kappa} \\
&+\iiint \widehat{C}(2k,\boldsymbol{\kappa}_1) \\
&\times W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_3),(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_q(l))\}}\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 \Big\}
\end{aligned}
$$

and we have used notations of types:

$$
\begin{aligned}
\mathbf{p}|j' &= \big\{(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j))\big\}_{j=1\neq j'}^{n_{\mathbf{p}}}\,, \qquad \mathbf{q}|l' = \big\{(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l))\big\}_{l=1\neq l'}^{n_{\mathbf{q}}}\,, \\
\mathbf{p}|\{j'|(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\} &= \big\{(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j))\big\}_{j=1\neq j'}^{n_{\mathbf{p}}} \cup (\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\,, \\
\mathbf{q}|\{l'|(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\} &= \big\{(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l))\big\}_{l=1\neq l'}^{n_{\mathbf{q}}} \cup (\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\,.
\end{aligned}
$$

The set of transport equations describes accurately the reflected wave and it is the key tool to analyze various applications with waves in random media. The corresponding transport equations in the layered case with one-dimensional medium fluctuations were first obtained in [1]. They have played a crucial role in the analysis of a wide range of applications and they have been generalized to describe a wide range of propagation scenarios in [9]. The transport equations given in Proposition 1 provide a rigorous tool for studying the multiple scattering effects in a non-layered random medium.

We can now make a few general comments about the system of transport equations and the associated moments $W_{\mathbf{p},\mathbf{q}}$.

1) Consider the set of moments $W_{\mathbf{p},\mathbf{q}}$ such that $n_{\mathbf{p}} - n_{\mathbf{q}} = c$ with c a nonzero integer. These moments form a closed subfamily with each member satisfying a zero initial condition. Therefore, these moments vanish and only moments having the same number of conjugated and unconjugated terms $n_{\mathbf{p}} = n_{\mathbf{q}}$ survive in the limit $\varepsilon \to 0$.

2) Consider the layered case in which $\nu(z,\mathbf{x}) = \nu(z)$ and therefore $\widehat{C}(k,\boldsymbol{\kappa}) = (2\pi)^d \widehat{C}(k)\delta(\boldsymbol{\kappa})$. Under these conditions, the equations (17) for the operator components with different wavevectors are not coupled, $\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')$ is concentrated on $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$. In fact the reflection operator has the form $\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = \hat{\mathcal{R}}^{\varepsilon}(k,z)\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}')$. The analysis of the system for the moments shows that the solution has the form

$$\mathcal{W}_{\mathbf{p},\mathbf{q}}(k,\tau,z) = \begin{cases} w_n(k,\tau,z)\prod_{j=1}^n \delta(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}'_p(j))\prod_{l=1}^n \delta(\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa}'_q(l)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n_{\mathbf{p}} = n_{\mathbf{q}} = n, \\ 0 \text{ otherwise,} \end{cases}$$

where the family $(w_n)_{n\in\mathbb{N}}$ is solution of the closed system of transport equations

$$\frac{\partial w_n}{\partial z} + 2n\frac{\partial w_n}{\partial \tau} = \frac{k^2\widehat{C}(2k)n^2}{4}\left(w_{n+1} + w_{n-1} - 2w_n\right) , \tag{28}$$

with the initial conditions $w_n(k,\tau,z=0) = \mathbf{1}_0(n)\delta(\tau)$. We therefore obtain that the moments of the reflection operator satisfy the system that governs the propagation of one-dimensional waves in random media [9].

3) If the autocorrelation function of the process $\nu(\mathbf{x},z)$ is such that

$$\widehat{C}(2k,\boldsymbol{\kappa}) = 0 \quad \text{for all } \boldsymbol{\kappa} \in \mathbb{R}^d , \tag{29}$$

then there is only coupling in the system of transport equations for indices $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}')$ such that $n_{\mathbf{p}} = n_{\mathbf{p}'}$ and $n_{\mathbf{q}} = n_{\mathbf{q}'}$. Since the initial conditions are zero for all non-empty indices $(\mathbf{p},\mathbf{q})$, the moments $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ are zero as soon as $n_{\mathbf{p}}$ or $n_{\mathbf{q}}$ is positive. In other words $\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = 0$ for all $\boldsymbol{\kappa},\boldsymbol{\kappa}' \in \mathbb{R}^d$ in distribution as $\varepsilon \to 0$. This shows that the forward-scattering approximation is valid as soon as the condition (29) is fulfilled. This approximation is frequently used in the literature. Here we give the necessary and sufficient condition (29) for the validity of this approximation.

4) The reciprocity principle shows that $\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = \hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,-\boldsymbol{\kappa}',-\boldsymbol{\kappa})$ for any $\boldsymbol{\kappa},\boldsymbol{\kappa}' \in \mathbb{R}^d$. This can be seen from (17). Therefore the symmetry relation $W_{\mathbf{p},\mathbf{q}} = W_{\tilde{\mathbf{p}},\tilde{\mathbf{q}}}$ is satisfied for $\tilde{\mathbf{p}}$ (respectively $\tilde{\mathbf{q}}$) obtained from $\mathbf{p}$ (respectively $\mathbf{q}$) by changing some of the pairs $(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j))$ into $(-\boldsymbol{\kappa}'_p(j),-\boldsymbol{\kappa}_p(j))$ (resp. $(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l))$ into $(-\boldsymbol{\kappa}'_q(l),-\boldsymbol{\kappa}_q(l))$).

3.2 The weak backscattering regime

A central quantity that characterizes the backscattered wave field is the cross spectral density

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\overline{\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}}(k,z,\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)\right] = \int W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)}(k,\tau,z)\,d\tau\,. \tag{30}$$

This quantity describes the density of the reflected wave field at the surface $z = L$. We are interested in this quantity in the regime of weak backscattering. This regime derives from the modeling assumption

$$\frac{\widehat{C}(2k,\boldsymbol{\kappa})}{\widehat{C}(0,\mathbf{0})} \le \delta \ll 1\,, \quad \forall\boldsymbol{\kappa} \in \mathbb{R}^d\,. \tag{31}$$

It follows that

$$W_{\mathbf{p},\mathbf{q}} = \begin{cases} 1 & \text{if} \quad n_{\mathbf{p}} = n_{\mathbf{q}} = 0\,, \\ \mathcal{O}(\delta) & \text{if} \quad n_{\mathbf{p}} = n_{\mathbf{q}} = 1\,, \\ \mathcal{O}(\delta^2) & \text{otherwise}\,. \end{cases}$$

Denoting $\mathbf{p}_1 = (\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$ and $\mathbf{q}_1 = (\boldsymbol{\kappa}_3, \boldsymbol{\kappa}_4)$ we have up to terms of order δ^2 the following result.

Proposition 2 *In the weak backscattering regime (31) the limit moments $W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)}$ are given by the system of transport equations*

$$\begin{aligned} &\frac{\partial W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)}}{\partial z} + 2\frac{\partial W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)}}{\partial \tau} \\ &= \frac{i}{2k}\Big[- \big(|\boldsymbol{\kappa}_1|^2 + |\boldsymbol{\kappa}_2|^2\big) + \big(|\boldsymbol{\kappa}_3|^2 + |\boldsymbol{\kappa}_4|^2\big)\Big] W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)} \\ &+ \frac{k^2}{4(2\pi)^d} \int \widehat{C}(0,\boldsymbol{\kappa}) \Big\{ W_{(\boldsymbol{\kappa}_1-\boldsymbol{\kappa},\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3-\boldsymbol{\kappa},\boldsymbol{\kappa}_4)} + W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4-\boldsymbol{\kappa})} \\ &\qquad\qquad + W_{(\boldsymbol{\kappa}_1-\boldsymbol{\kappa},\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4+\boldsymbol{\kappa})} + W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_3+\boldsymbol{\kappa},\boldsymbol{\kappa}_4)} \\ &\qquad\qquad - W_{(\boldsymbol{\kappa}_1-\boldsymbol{\kappa},\boldsymbol{\kappa}_2-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)} - W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3-\boldsymbol{\kappa},\boldsymbol{\kappa}_4-\boldsymbol{\kappa})} \\ &\qquad\qquad - 2W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)} \Big\} d\boldsymbol{\kappa} \\ &+ \frac{k^2}{4(2\pi)^d} \widehat{C}(2k, \boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2)\, \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_4)\, \delta(\tau)\,, \end{aligned} \tag{32}$$

starting from $W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)}(k,\tau,z=0) = 0$.

Therefore $W_{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_4)}$ will be supported on $\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_4 = \mathbf{0}$ so that we can parameterize the solution in terms of three "effective" wavevectors.

From now on we consider a fixed frequency k and omit it in the notation. We introduce the dimensionless autocorrelation function $\mathcal{C}$ of the random medium:

$$\mathbb{E}\left[\nu(z', \mathbf{x}')\nu(z' + z, \mathbf{x}' + \mathbf{x})\right] = \sigma^2 \mathcal{C}\Big(\frac{z}{l_z}, \frac{\mathbf{x}}{l_x}\Big)\,,$$

where l_z (respectively l_x) is the longitudinal (respectively transverse) correlation radius of the random fluctuations, and σ is the standard deviation of the fluctuations. We denote by $\widehat{\mathcal{C}}_K(\boldsymbol{\mu})$ and by $\check{\mathcal{C}}_K(\boldsymbol{\lambda})$ the full and partial Fourier transforms

$$\widehat{\mathcal{C}}_K(\boldsymbol{\mu}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \mathcal{C}(\zeta, \boldsymbol{\lambda}) e^{-iK\zeta - i\boldsymbol{\mu}\cdot\boldsymbol{\lambda}} d\boldsymbol{\lambda} d\zeta\,, \qquad \check{\mathcal{C}}_K(\boldsymbol{\lambda}) = \int_{-\infty}^{\infty} \mathcal{C}(\zeta, \boldsymbol{\lambda}) e^{-iK\zeta} d\zeta\,.$$

By integrating in τ the result of Proposition 2 we obtain the following convergence result.

Proposition 3 *We have as $\varepsilon \to 0$*

$$\mathbb{E}\left[\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(z, \boldsymbol{\kappa}_0' + \boldsymbol{\kappa}, \boldsymbol{\kappa}_0') \overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}(z, \boldsymbol{\kappa}_1' + \boldsymbol{\kappa}', \boldsymbol{\kappa}_1')\right] \stackrel{\varepsilon \to 0}{\longrightarrow} \delta(\boldsymbol{\kappa}' - \boldsymbol{\kappa}) D_{\boldsymbol{\kappa}_0', \boldsymbol{\kappa}_1', \boldsymbol{\kappa}}(z)\,, \tag{33}$$

where the cross spectral density D is of the form

$$\begin{aligned} D_{\boldsymbol{\kappa}_0', \boldsymbol{\kappa}_1', \boldsymbol{\kappa}}(z) = \bar{D} \exp&\left[-i(\boldsymbol{\kappa}_0' - \boldsymbol{\kappa}_1') \cdot (\boldsymbol{\kappa}_0' + \boldsymbol{\kappa}_1' + \boldsymbol{\kappa})\frac{z}{k}\right] \\ &\times \mathcal{D}\left(\frac{z}{L}, (\boldsymbol{\kappa}_0' - \boldsymbol{\kappa}_1') l_x, (\boldsymbol{\kappa}_0' + \boldsymbol{\kappa}_1' + \boldsymbol{\kappa}) l_x, \boldsymbol{\kappa} l_x\right)\,, \end{aligned} \tag{34}$$

with

$$\bar{D} = \frac{k^2\sigma^2 l_z l_x^d L \check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{4(2\pi)^d}\,.$$

The dimensionless cross spectral density $\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})$ solves

$$\begin{aligned}
\frac{d\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})}{d\zeta} &= \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} e^{i\alpha\mathbf{u}\cdot\mathbf{v}\zeta} - \frac{2\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})d\boldsymbol{\mu}\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w}) \\
&+\frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})\Big[e^{i\alpha\boldsymbol{\mu}\cdot\mathbf{v}\zeta}\mathcal{D}(\zeta,\mathbf{u}-\boldsymbol{\mu},\mathbf{v},\mathbf{w}+\boldsymbol{\mu}) + e^{-i\alpha\boldsymbol{\mu}\cdot\mathbf{v}\zeta}\mathcal{D}(\zeta,\mathbf{u}+\boldsymbol{\mu},\mathbf{v},\mathbf{w}+\boldsymbol{\mu})\Big]d\boldsymbol{\mu} \\
&+\frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})\Big[e^{i\alpha\boldsymbol{\mu}\cdot\mathbf{u}\zeta}\mathcal{D}(\zeta,\mathbf{u},\mathbf{v}-\boldsymbol{\mu},\mathbf{w}+\boldsymbol{\mu}) + e^{-i\alpha\boldsymbol{\mu}\cdot\mathbf{u}\zeta}\mathcal{D}(\zeta,\mathbf{u},\mathbf{v}+\boldsymbol{\mu},\mathbf{w}+\boldsymbol{\mu})\Big]d\boldsymbol{\mu} \\
&-\frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})e^{-i\alpha\boldsymbol{\mu}\cdot\mathbf{u}\zeta}\Big[e^{-i\alpha\boldsymbol{\mu}\cdot(\mathbf{v}+\boldsymbol{\mu})\zeta}\mathcal{D}(\zeta,\mathbf{u}+\boldsymbol{\mu},\mathbf{v}+\boldsymbol{\mu},\mathbf{w}) \\
&\qquad +e^{i\alpha\boldsymbol{\mu}\cdot(\mathbf{v}+\boldsymbol{\mu})\zeta}\mathcal{D}(\zeta,\mathbf{u}-\boldsymbol{\mu},\mathbf{v}+\boldsymbol{\mu},\mathbf{w})\Big]d\boldsymbol{\mu}\,, \qquad (35)
\end{aligned}$$

starting from $\mathcal{D}(\zeta=0,\mathbf{u},\mathbf{v},\mathbf{w})=0$. The dimensionless parameters α and β are given by

$$\alpha = \frac{L}{kl_x^2}, \qquad \beta = \frac{k^2 L\sigma^2 l_z}{4}\,. \qquad (36)$$

For a given propagation distance L, the parameter α is the inverse of the Fresnel number at the transverse scale l_x and it characterizes the strength of diffraction at this scale, while the parameter β characterizes the strength of random forward scattering.

Note that the cross spectral density is symmetric in $(\mathbf{u},\mathbf{v})$: $\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w}) = \mathcal{D}(\zeta,\mathbf{v},\mathbf{u},\mathbf{w})$. This can be seen from the structure of the system (35) for $\mathcal{D}$, and this also follows directly from the reciprocity relation $\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(\boldsymbol{\kappa},\boldsymbol{\kappa}') = \hat{\boldsymbol{\mathcal{R}}}^\varepsilon(-\boldsymbol{\kappa}',-\boldsymbol{\kappa})$.

As a first application, we compute the total mean reflected power defined by:

$$P^\varepsilon = \int \mathbb{E}[|\check{a}^\varepsilon(L,\mathbf{x})|^2]\,d\mathbf{x}\,.$$

Corollary 1 *The total mean reflected power P^ε has the limit P as $\varepsilon\to 0$ given by*

$$P = \beta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})\int |\check{b}(\mathbf{x})|^2 d\mathbf{x}\,, \qquad (37)$$

where $\check{b}(\mathbf{x})$ stands for the incoming wave (12).

Proof Using Parseval's formula we obtain

$$\begin{aligned}
P^\varepsilon &= \frac{1}{(2\pi)^d}\int \mathbb{E}[|\hat{a}^\varepsilon(L,\boldsymbol{\kappa})|^2]\,d\boldsymbol{\kappa} \\
&= \frac{1}{(2\pi)^d}\iiint \mathbb{E}\big[\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(L,\boldsymbol{\kappa},\boldsymbol{\kappa}')\overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}'')\big]\hat{b}(\boldsymbol{\kappa}')\bar{\hat{b}}(\boldsymbol{\kappa}'')d\boldsymbol{\kappa}d\boldsymbol{\kappa}'d\boldsymbol{\kappa}''\,,
\end{aligned}$$

where $\hat{b}(\boldsymbol{\kappa})$ stands for the incoming wave. By the convergence (33) the total mean reflected power P^ε has the limit P as $\varepsilon\to 0$ given by

$$P = \frac{1}{(2\pi)^d}\iint D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}-\boldsymbol{\kappa}'}(L)|\hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa}d\boldsymbol{\kappa}' = \frac{1}{(2\pi)^d}\iint D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L)|\hat{b}(\boldsymbol{\kappa}')|^2\,d\boldsymbol{\kappa}d\boldsymbol{\kappa}'\,.$$

Using the identity (34) this can also be written as

$$P = \frac{\bar{D} l_x^{-d}}{(2\pi)^d} \int \mathcal{E}(1, 2\boldsymbol{\kappa}' l_x) |\hat{b}(\boldsymbol{\kappa}')|^2 \, d\boldsymbol{\kappa}' \,,$$

where $\mathcal{E}(\zeta, \mathbf{v}) = \int \mathcal{D}(\zeta, \mathbf{0}, 2\mathbf{v}+\mathbf{w}, \mathbf{w}) d\mathbf{w}$. Then, using the system of coupled differential equations (35), we get that the function $\mathcal{E}(\zeta, \mathbf{v})$ satisfies

$$\frac{d\mathcal{E}(\zeta, \mathbf{v})}{d\zeta} = (2\pi)^d + \frac{\beta}{(2\pi)^d} \int \widehat{\mathcal{C}}_0(\boldsymbol{\mu}) \Big[\mathcal{E}(\zeta, \mathbf{v}+\boldsymbol{\mu}) - \mathcal{E}(\zeta, \boldsymbol{\mu}) \Big] d\boldsymbol{\mu} \,,$$

because all but two of the terms of the right-hand side of (35) cancel each other when taking $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \to (\mathbf{0}, 2\mathbf{v}+\mathbf{w}, \mathbf{w})$ and integrating in $\mathbf{w}$. The initial condition is $\mathcal{E}(\zeta = 0, \mathbf{v}) = 0$ and the solution is the function $\mathcal{E}(\zeta, \mathbf{v}) = (2\pi)^d \zeta$ independent of $\mathbf{v}$. We finally obtain that the total mean reflected power is

$$P = \bar{D} l_x^{-d} \int |\hat{b}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa} = \frac{k^2 \sigma^2 l_z L \check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{4} \int |\check{b}(\mathbf{x})|^2 \, d\mathbf{x} \,.$$

which also reads as (37). □

This corollary shows that the total mean reflected power grows like L by the expression (36) of β. This behavior is expected in the weak backscattering regime, as the proportion of wave energy scattered back increases linearly with the propagation distance. Moreover, the total mean reflected power does not depend on α, which is rather natural since transverse effects do not modify the total longitudinal flux of energy. The reflected power is however proportional to β, which is the strength of random forward scattering.

We remark that in Appendix E we give a simple interpretation of the weak backscattering regime and the associated intensity of the reflected wave in terms of random mirrors and two-phase transmissions.

4 The weak backscattering regime when $\alpha \gg 1$

In Subsection 4.1 we will analyze the system (35) for $\mathcal{D}$ in the limit case $\alpha \to \infty$. This result will allow us to get closed-form expressions for the physically relevant quantities. We will consider the beam width (Subsection 4.2), the spectral width (Subsection 4.3), and the mean power profile (Subsection 4.4). In the last Subsection 4.5 we will present and discuss the enhanced backscattering phenomenon.

4.1 Asymptotic expressions for the cross spectral density

In the next two lemmas we give the asymptotic expressions for the dimensionless cross spectral density in the regime $\alpha \gg 1$.

Lemma 1 *1. There exists* C_β *such that* $\sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^d, \zeta\in[0,1]} |\mathcal{D}(\zeta, \mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_\beta$ *uniformly in* α.
2. If $\mathbf{u}\cdot\mathbf{v} \neq 0$, *then* $\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$.

3. *If* $\mathbf{u} \neq \mathbf{0}$*,* $\mathbf{v} \neq \mathbf{0}$*, and* $\mathbf{u} \cdot \mathbf{v} = 0$*, then*

$$\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} \frac{1 - e^{-2\beta\check{\mathcal{C}}_0(\mathbf{0})\zeta}}{2\beta\check{\mathcal{C}}_0(\mathbf{0})} . \tag{38}$$

4. *If* $\mathbf{u} = \mathbf{0}$ *and* $\mathbf{v} \neq \mathbf{0}$*, then* $\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \mathbf{0}, \mathbf{v}, \mathbf{w}) = \mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})$ *where* $\mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})$ *is the solution of*

$$\frac{d\mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})}{d\zeta} = \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} + \frac{2\beta}{(2\pi)^d} \int \widehat{\mathcal{C}}_0(\boldsymbol{\mu}) \left[\mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w} + \boldsymbol{\mu}) - \mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})\right] d\boldsymbol{\mu} , \tag{39}$$

   starting from $\mathcal{D}_{\mathbf{0}}(\zeta = 0, \mathbf{w}) = 0$*.*
5. *If* $\mathbf{u} \neq \mathbf{0}$ *and* $\mathbf{v} = \mathbf{0}$*, then* $\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \mathbf{u}, \mathbf{0}, \mathbf{w}) = \mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})$*.*
6. *If* $\mathbf{u} = \mathbf{0}$ *and* $\mathbf{v} = \mathbf{0}$*, then*

$$\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \mathbf{0}, \mathbf{0}, \mathbf{w}) = 2\mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w}) - \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} \frac{1 - e^{-2\beta\check{\mathcal{C}}_0(\mathbf{0})\zeta}}{2\beta\check{\mathcal{C}}_0(\mathbf{0})} .$$

This Lemma is proved in Appendix C. By comparing the third and fourth items (or the fifth and sixth items) a sharp transition is noticed from the case $\mathbf{u} = \mathbf{0}$ to $\mathbf{u} \neq \mathbf{0}$. This transition can be studied in detail by looking at small $\mathbf{u}$ of order α^{-1}.

Lemma 2 *1. If* $\mathbf{v} \neq \mathbf{0}$*, then* $\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \alpha^{-1}\boldsymbol{s}, \mathbf{v}, \mathbf{w}) = \mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w})$ *where* $\mathcal{D}_{\boldsymbol{s}}$ *is solution of*

$$\frac{d\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w})}{d\zeta} = \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} e^{i\boldsymbol{s}\cdot\mathbf{v}\zeta} + \frac{\beta}{(2\pi)^d} \int \widehat{\mathcal{C}}_0(\boldsymbol{\mu}) \Big[e^{i\boldsymbol{s}\cdot\boldsymbol{\mu}\zeta} \mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v} - \boldsymbol{\mu}, \mathbf{w} + \boldsymbol{\mu}) + e^{-i\boldsymbol{s}\cdot\boldsymbol{\mu}\zeta} \mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v} + \boldsymbol{\mu}, \mathbf{w} + \boldsymbol{\mu}) - 2\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w}) \Big] d\boldsymbol{\mu} \tag{40}$$

starting from $\mathcal{D}_{\boldsymbol{s}}(\zeta = 0, \mathbf{v}, \mathbf{w}) = 0$*.*

2. *If* $\mathbf{v} = \mathbf{0}$*, then*

$$\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \alpha^{-1}\boldsymbol{s}, \mathbf{0}, \mathbf{w}) = \mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{0}, \mathbf{w}) + \mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w}) - \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} \frac{1 - e^{-2\beta\check{\mathcal{C}}_0(\mathbf{0})\zeta}}{2\beta\check{\mathcal{C}}_0(\mathbf{0})} .$$

Note that, if $\boldsymbol{s} = \mathbf{0}$, then $\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w}) \mid_{\boldsymbol{s}=\mathbf{0}} = \mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})$ as defined by (39), which shows the consistency of the notations. By solving the differential equation (40) we obtain the following integral representation of $\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w})$ valid for all $\boldsymbol{s} \in \mathbb{R}^d$:

$$\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w}) = \int \frac{\check{\mathcal{C}}_{2kl_z}(\boldsymbol{\lambda})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}} \int_0^\zeta e^{i\mathbf{v}\cdot\boldsymbol{s}(\zeta-\zeta')} \times e^{\beta \int_0^{\zeta'} \check{\mathcal{C}}_0(\boldsymbol{\lambda} - \boldsymbol{s}\zeta'') + \check{\mathcal{C}}_0(\boldsymbol{\lambda} + \boldsymbol{s}\zeta'') - 2\check{\mathcal{C}}_0(\mathbf{0}) d\zeta''} d\zeta' d\boldsymbol{\lambda} . \tag{41}$$

In the particular case where $\boldsymbol{s} = \mathbf{0}$ the function $\mathcal{D}_{\boldsymbol{s}}$ is independent of $\mathbf{v}$ and we have

$$\mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w}) = \int \frac{\check{\mathcal{C}}_{2kl_z}(\boldsymbol{\lambda})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}} \int_0^\zeta e^{2\beta[\check{\mathcal{C}}_0(\boldsymbol{\lambda}) - \check{\mathcal{C}}_0(\mathbf{0})]\zeta'} d\zeta' d\boldsymbol{\lambda} .$$

In Appendix D we give a few identities that are useful in the following.

4.2 Beam width

We define the rms (root-mean-squared) width R^ε of the reflected beam by

$$R^{\varepsilon 2} = \frac{\int |\mathbf{x}|^2 \mathbb{E}[|\check{a}^\varepsilon(\mathbf{x},L)|^2] d\mathbf{x}}{\int \mathbb{E}[|\check{a}^\varepsilon(\mathbf{x},L)|^2]\, d\mathbf{x}} . \tag{42}$$

Proposition 4 *The beam width R^ε converges to R as $\varepsilon \to 0$, where R is given by*

$$R^2 = \frac{\iint D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) |\nabla_{\boldsymbol{\kappa}'} \hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa} d\boldsymbol{\kappa}' - \iint \frac{1}{2}(\Delta_{\boldsymbol{\kappa}_0'} + \Delta_{\boldsymbol{\kappa}_1'}) D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) |\hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa} d\boldsymbol{\kappa}'}{\iint D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) |\hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa} d\boldsymbol{\kappa}'}$$
$$+ \frac{\iint i(\nabla_{\boldsymbol{\kappa}_0'} - \nabla_{\boldsymbol{\kappa}_1'}) D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) \mathrm{Im}\big(\hat{b}(\boldsymbol{\kappa}') \overline{\nabla_{\boldsymbol{\kappa}'} \hat{b}(\boldsymbol{\kappa}')}\big) d\boldsymbol{\kappa} d\boldsymbol{\kappa}'}{\iint D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) |\hat{b}(\boldsymbol{\kappa}')|^2\, d\boldsymbol{\kappa} d\boldsymbol{\kappa}'} ., \tag{43}$$

where $D_{\boldsymbol{\kappa}_0',\boldsymbol{\kappa}_1',\boldsymbol{\kappa}}(z)$ is given by (34). If $\check{C}_0$ and $\check{C}_{2kl_z}$ are twice differentiable at $\mathbf{0}$, then we have in the regime $\alpha \gg 1$:

$$R^2 = R_0^2 - \frac{2}{3} \Delta \check{C}_0(\mathbf{0}) l_x^2 \alpha^2 \beta + \frac{4}{3} K_0^2 l_x^4 \alpha^2 + 2 Q_0 l_x^2 \alpha - \frac{\Delta \check{C}_{2kl_z}(\mathbf{0})}{3 \check{C}_{2kl_z}(\mathbf{0})} l_x^2 \alpha^2 , \tag{44}$$

with R_0 (respectively K_0) the rms beam width (respectively spectral width) of the input beam:

$$R_0^2 = \frac{\int |\mathbf{x}|^2 |\check{b}(\mathbf{x})|^2 d\mathbf{x}}{\int |\check{b}(\mathbf{x})|^2 d\mathbf{x}}, \qquad K_0^2 = \frac{\int |\boldsymbol{\kappa}|^2 |\hat{b}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa}}{\int |\hat{b}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa}} , \tag{45}$$

and Q_0 can be refered to as the chirp of the input beam defined by

$$Q_0 = \frac{\int \boldsymbol{\kappa} \cdot \mathrm{Im}\Big(\hat{b}(\boldsymbol{\kappa}) \overline{\nabla_{\boldsymbol{\kappa}} \hat{b}(\boldsymbol{\kappa})}\Big) d\boldsymbol{\kappa}}{\int |\hat{b}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa}} = \frac{-\int \mathbf{x} \cdot \mathrm{Im}\Big(\check{b}(\mathbf{x}) \overline{\nabla_{\mathbf{x}} \check{b}(\mathbf{x})}\Big) d\mathbf{x}}{\int |\check{b}(\mathbf{x})|^2 d\mathbf{x}} . \tag{46}$$

Proof Using Parseval's formula we obtain

$$R^{\varepsilon 2} = \frac{\iiint \mathbb{E}\big[\nabla_{\boldsymbol{\kappa}} \hat{\boldsymbol{\mathcal{R}}}^\varepsilon(L,\boldsymbol{\kappa},\boldsymbol{\kappa}') \overline{\nabla_{\boldsymbol{\kappa}} \hat{\boldsymbol{\mathcal{R}}}^\varepsilon}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}'')\big] \hat{b}(\boldsymbol{\kappa}') \bar{\hat{b}}(\boldsymbol{\kappa}'') d\boldsymbol{\kappa} d\boldsymbol{\kappa}' d\boldsymbol{\kappa}''}{\iiint \mathbb{E}\big[\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(L,\boldsymbol{\kappa},\boldsymbol{\kappa}') \overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}'')\big] \hat{b}(\boldsymbol{\kappa}') \bar{\hat{b}}(\boldsymbol{\kappa}'')\, d\boldsymbol{\kappa} d\boldsymbol{\kappa}' d\boldsymbol{\kappa}''} .$$

By the convergence (33) the beam width R^ε converges to R as $\varepsilon \to 0$, where R is given by (43). By the identity (34) we have

$$\frac{1}{2}(\Delta_{\boldsymbol{\kappa}_0'} + \Delta_{\boldsymbol{\kappa}_1'}) D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(z) = \bar{D} l_x^2 \mathcal{F}_1 \left(\frac{z}{L}, (2\boldsymbol{\kappa}' + \boldsymbol{\kappa}) l_x, \boldsymbol{\kappa} l_x \right) ,$$
$$(\nabla_{\boldsymbol{\kappa}_0'} - \nabla_{\boldsymbol{\kappa}_1'}) D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(z) = \bar{D} l_x \mathcal{F}_2 \left(\frac{z}{L}, (2\boldsymbol{\kappa}' + \boldsymbol{\kappa}) l_x, \boldsymbol{\kappa} l_x \right) ,$$

with

$$\mathcal{F}_1(\zeta,\mathbf{v},\mathbf{w}) = \Big[\Delta_{\mathbf{u}} + \Delta_{\mathbf{v}} - 2\alpha i \zeta \mathbf{v} \cdot \nabla_{\mathbf{u}} - \alpha^2 \zeta^2 |\mathbf{v}|^2 \Big] \mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w}) \mid_{\mathbf{u}=\mathbf{0}} ,$$
$$\mathcal{F}_2(\zeta,\mathbf{v},\mathbf{w}) = [2\nabla_{\mathbf{u}} - 2\alpha i \zeta \mathbf{v}]\, \mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w}) \mid_{\mathbf{u}=\mathbf{0}} .$$

In the limit $\alpha \to \infty$, we obtain by using (84-89):

$$\frac{1}{(2\pi)^d \bar{D} \alpha^2} \int \frac{1}{2} (\Delta_{\boldsymbol{\kappa}_0'} + \Delta_{\boldsymbol{\kappa}_1'}) D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) d\boldsymbol{\kappa} \stackrel{\alpha \to \infty}{\longrightarrow} \frac{1}{3} \frac{\Delta \check{C}_{2kl_z}(\mathbf{0})}{\check{C}_{2kl_z}(\mathbf{0})} - \frac{4}{3} |\boldsymbol{\kappa}'|^2 l_x^2 + \frac{2\beta}{3} \Delta \check{C}_0(\mathbf{0}) ,$$
$$\frac{1}{(2\pi)^d \bar{D} \alpha} \int (\nabla_{\boldsymbol{\kappa}_0'} - \nabla_{\boldsymbol{\kappa}_1'}) D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L) d\boldsymbol{\kappa} \stackrel{\alpha \to \infty}{\longrightarrow} -2i \boldsymbol{\kappa}' l_x ,$$

where we have used the fact that $\int \mathbf{w}\mathcal{D}_{\mathbf{0}}(1,\mathbf{w})d\mathbf{w} = 0$. The first limit holds true (in the sense that the right-hand side is finite) only if $\check{\mathcal{C}}_0$ and $\check{\mathcal{C}}_{2kl_z}$ are twice differentiable at $\mathbf{0}$. The non-differentiable case will be addressed in Section 4.4. Substituting these limits into (43) we obtain that, in the large-α regime, the beam radius is given by (44). □

Using the expressions for α and β, the expression (44) of the beam width also reads:

$$R^2 = R_0^2 - \frac{1}{6}\Delta\check{\mathcal{C}}_0(\mathbf{0})\frac{\sigma^2 l_z L^3}{l_x^2} + \frac{4}{3}\frac{L^2}{k^2}K_0^2 + 2Q_0\frac{L}{k} - \frac{\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{3\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}\frac{L^2}{k^2 l_x^2} .$$

We can interpret all terms in this sum:

1. The first term (with R_0) is the initial beam width.
2. The second term (with $\Delta\check{\mathcal{C}}_0(\mathbf{0})$) is the spreading effect due to random forward scattering; it is the only term (with the initial beam width) that is independent of k (i.e., of the frequency).
3. The third term (with K_0) is due to the natural beam diffraction; this term is independent of the random medium.
4. The fourth term (with Q_0) is a convergence or divergence effect due to the initial beam phase front; this term is independent of the random medium, and it is the only one in the sum that can be negative; the condition $Q_0 < 0$ means that the input beam has an initial phase front that makes it converge, but this convergence is eventually overwhelmed by natural diffraction, and also by spreading induced by random scattering.
5. The last term (with $\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})$) is the spreading induced by random backward scattering.

In the regime $\beta \gg 1$ the main spreading effect is due to random forward scattering and all other effects become negligible. The beam width is of order $\alpha\beta^{1/2}l_x$. It is given explicity by

$$R^2 = -\frac{2}{3}\Delta\check{\mathcal{C}}_0(\mathbf{0})l_x^2\alpha^2\beta = -\frac{1}{6}\Delta\check{\mathcal{C}}_0(\mathbf{0})\frac{\sigma^2 l_z L^3}{l_x^2} , \tag{47}$$

which shows that the beam width increases like $L^{3/2}$.

4.3 Spectral width

We define the rms spectral width K^ε of the reflected beam by

$$K^{\varepsilon 2} = \frac{\int |\boldsymbol{\kappa}|^2 \mathbb{E}[|\hat{a}^\varepsilon(\boldsymbol{\kappa}, L)|^2] d\boldsymbol{\kappa}}{\int \mathbb{E}[|\hat{a}^\varepsilon(\boldsymbol{\kappa}, L)|^2] d\boldsymbol{\kappa}} . \tag{48}$$

Proposition 5 *The spectral width K^ε converges to K as $\varepsilon \to 0$, where K is given by*

$$K^2 = \frac{\iint |\boldsymbol{\kappa}+\boldsymbol{\kappa}'|^2 D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L)|\hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa} d\boldsymbol{\kappa}'}{\iint D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',\boldsymbol{\kappa}}(L)|\hat{b}(\boldsymbol{\kappa}')|^2\, d\boldsymbol{\kappa} d\boldsymbol{\kappa}'} . \tag{49}$$

If $\check{\mathcal{C}}_0$ and $\check{\mathcal{C}}_{2kl_z}$ are twice differentiable at $\mathbf{0}$, then we have in the regime $\alpha \gg 1$:

$$K^2 = K_0^2 - \Delta\check{\mathcal{C}}_0(\mathbf{0})l_x^{-2}\beta - \frac{\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}l_x^{-2} , \tag{50}$$

where K_0 is the spectral width (45) of the incoming beam.

Proof The spectral width is given by

$$K^{\varepsilon 2} = \frac{\iiint |\boldsymbol{\kappa}|^2 \mathbb{E}\big[\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}')\overline{\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}'')\big]\hat{b}(\boldsymbol{\kappa}')\bar{\hat{b}}(\boldsymbol{\kappa}'')d\boldsymbol{\kappa}d\boldsymbol{\kappa}'d\boldsymbol{\kappa}''}{\iiint \mathbb{E}\big[\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}')\overline{\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}}(L,\boldsymbol{\kappa},\boldsymbol{\kappa}'')\big]\hat{b}(\boldsymbol{\kappa}')\bar{\hat{b}}(\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}d\boldsymbol{\kappa}'d\boldsymbol{\kappa}''}\,.$$

By the convergence result (33) the spectral width K^{ε} converges to K given by (49). By (34) and Lemma 1 we get

$$\lim_{\alpha\to\infty} K^2 = l_x^{-2}\frac{\int |\mathbf{w}|^2\mathcal{D}_{\mathbf{0}}(1,\mathbf{w})d\mathbf{w}}{\int \mathcal{D}_{\mathbf{0}}(1,\mathbf{w})d\mathbf{w}} + \frac{\int |\boldsymbol{\kappa}'|^2|\hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa}'}{\int |\hat{b}(\boldsymbol{\kappa}')|^2 d\boldsymbol{\kappa}'}\,,$$

where we have used the fact that $\int \mathbf{w}\mathcal{D}_{\mathbf{0}}(1,\mathbf{w})d\mathbf{w} = \mathbf{0}$. This limit holds true if $\check{\mathcal{C}}_0$ and $\check{\mathcal{C}}_{2kl_z}$ are twice differentiable at $\mathbf{0}$. We can compute the integrals by using (84) and (86) which gives (50). □

Substituting the value of β in the expression (50) of the spectral width, we obtain

$$K^2 = K_0^2 - \frac{1}{4}\Delta\check{\mathcal{C}}_0(\mathbf{0})\frac{k^2L\sigma^2 l_z}{l_x^2} - \frac{\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}\frac{1}{l_x^2}\,.$$

The first term K_0^2 is the initial spectral width (squared). The second term (with $\Delta\check{\mathcal{C}}_0(\mathbf{0})$) is the spectral broadening due to random forward scattering. The third term is the spectral broadening due to random backward scattering. In the regime $\beta \gg 1$ the spectral broadening is dominated by the second term (random forward scattering) and the spectral width grows like $L^{1/2}$.

4.4 Mean reflected power

The mean reflected power is defined by

$$I^{\varepsilon}(\mathbf{x}) = \mathbb{E}[|\check{a}^{\varepsilon}(L,\mathbf{x})|^2].$$

Proposition 6 *The mean reflected power $I^{\varepsilon}(\mathbf{x})$ converges to $I(\mathbf{x})$ as $\varepsilon \to 0$, where*

$$I(\mathbf{x}) = \frac{1}{(2\pi)^{2d}}\iiint \hat{b}(\boldsymbol{\kappa}_0')\bar{\hat{b}}(\boldsymbol{\kappa}_1')e^{i(\boldsymbol{\kappa}_0'-\boldsymbol{\kappa}_1')\cdot\mathbf{x}}D_{\boldsymbol{\kappa}_0',\boldsymbol{\kappa}_1',\boldsymbol{\kappa}}(L)\,d\boldsymbol{\kappa}d\boldsymbol{\kappa}_0'd\boldsymbol{\kappa}_1'\,. \tag{51}$$

This can also be written as

$$I(\mathbf{x}) = \frac{P}{l_x^d}\mathcal{I}\Big(\frac{\mathbf{x}}{l_x}\Big)\,, \tag{52}$$

with P the total mean reflected power given by (37) and

$$\begin{aligned}\lim_{\alpha\to\infty}\alpha^d\mathcal{I}(\alpha\mathbf{y}) = \frac{\bar{D}l_x^{-2d}}{P(2\pi)^d}\iiint_0^1 \big|\hat{b}(\frac{\mathbf{v}}{l_x})\big|^2\frac{\check{\mathcal{C}}_{2kl_z}(\boldsymbol{s}\zeta)}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}e^{i\boldsymbol{s}\cdot(\mathbf{y}-2\mathbf{v}\zeta)}\\ \times e^{\beta\int_0^{2\zeta}\check{\mathcal{C}}_0(\boldsymbol{s}\zeta')-\check{\mathcal{C}}_0(\mathbf{0})d\zeta'}\,d\zeta d\mathbf{v}d\boldsymbol{s}\,.\end{aligned} \tag{53}$$

Proof The mean reflected power is

$$I^\varepsilon(\mathbf{x}) = \frac{1}{(2\pi)^{2d}} \iiiint e^{i(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2)\cdot\mathbf{x}} \mathbb{E}\big[\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(L,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')\overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}(L,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_2')\big] \times \hat{b}(\boldsymbol{\kappa}_1')\bar{\hat{b}}(\boldsymbol{\kappa}_2') d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_1' d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_2' .$$

By Proposition 3 we obtain that the limit of $I^\varepsilon(\mathbf{x})$ as $\varepsilon \to 0$ is (51). Using the dimensionless cross spectral density $\mathcal{D}$ (identity (34)) this can also be written as (52) with

$$\mathcal{I}(\mathbf{y}) = \frac{\bar{D} l_x^{-2d}}{P 2^{3d}\pi^{2d}} \iiint \hat{b}\big(\frac{\mathbf{v}+\mathbf{u}-\mathbf{w}}{2l_x}\big)\bar{\hat{b}}\big(\frac{\mathbf{v}-\mathbf{u}-\mathbf{w}}{2l_x}\big) e^{i\mathbf{u}\cdot\mathbf{y}} \mathcal{D}(1,\mathbf{u},\mathbf{v},\mathbf{w}) e^{-i\alpha\mathbf{u}\cdot\mathbf{v}}\, d\mathbf{u} d\mathbf{v} d\mathbf{w} .$$

In the regime $\alpha \gg 1$, we have seen (in the case in which $\check{\mathcal{C}}$ is smooth) that the beam width is of order $\alpha\beta^{1/2} l_x$, so we look for the beam power profile at this particular scale (in the general case in which $\check{\mathcal{C}}$ is smooth or not). By Lemma 2 we obtain

$$\lim_{\alpha\to\infty} \alpha^d \mathcal{I}(\alpha\mathbf{y}) = \frac{\bar{D} l_x^{-2d}}{P 2^{3d}\pi^{2d}} \iiint \big|\hat{b}\big(\frac{\mathbf{v}-\mathbf{w}}{2l_x}\big)\big|^2 \mathcal{D}_{\boldsymbol{s}}(1,\mathbf{v},\mathbf{w}) e^{i\boldsymbol{s}\cdot(\mathbf{y}-\mathbf{v})} d\boldsymbol{s} d\mathbf{v} d\mathbf{w} .$$

Substituting the expression (41) of $\mathcal{D}_{\boldsymbol{s}}(1,\mathbf{v},\mathbf{w})$ and integrating in $\boldsymbol{\lambda}$ and $\mathbf{w}$ gives (53). □

The regime $\beta \gg 1$ corresponds to strong forward scattering. The asymptotic analysis of this regime shows that we have to distinguish the cases in which $\check{\mathcal{C}}(\boldsymbol{s})$ is smooth at $\mathbf{0}$ or not.

Let us first consider the case in which $\check{\mathcal{C}}(\boldsymbol{s})$ is twice differentiable at $\mathbf{0}$ and can be expanded as $\check{\mathcal{C}}_0(\boldsymbol{s}) \simeq \check{\mathcal{C}}_0(\mathbf{0}) - \frac{1}{2}\check{\mathcal{C}}_0''|\boldsymbol{s}|^2 + o(|\boldsymbol{s}|^2)$, with $\check{\mathcal{C}}_0'' > 0$. This corresponds to a smooth random medium. Since we know that the beam width is of order $\alpha\beta^{1/2}l_x$, we look at the power profile at this particular scale and we obtain

$$\lim_{\beta\to\infty}\lim_{\alpha\to\infty} \alpha^d\beta^{d/2}\mathcal{I}(\alpha\beta^{1/2}\mathbf{y}) = \frac{1}{[\check{\mathcal{C}}_0'']^{d/2}} \mathcal{Q}\Big(\frac{\mathbf{y}}{[\check{\mathcal{C}}_0'']^{1/2}}\Big) ,$$

where

$$\mathcal{Q}(\mathbf{y}) = \frac{1}{(2\pi)^{d/2}} \int_0^1 e^{-\frac{|\mathbf{y}|^2}{2}\frac{3}{8\zeta^3}} \Big(\frac{3}{8\zeta^3}\Big)^{d/2} d\zeta .$$

The dimensionless power profile $\mathcal{Q}(\mathbf{y})$ is normalized so that $\int \mathcal{Q}(\mathbf{y})d\mathbf{y} = 1$. Therefore we can check that the asymptotic expressions satisfy $\int \mathcal{I}(\mathbf{y})d\mathbf{y} = 1$ and $\int I(\mathbf{x})d\mathbf{x} = P$. We can also check that the rms width of the asymptotic profile is (47), using the identity $\Delta\check{\mathcal{C}}_0(\mathbf{0}) = -d\check{\mathcal{C}}_0''$. Moreover, the power profile has a Gaussian tail for large $|\mathbf{y}|$, of the form

$$\mathcal{Q}(\mathbf{y}) \simeq \frac{3^{\frac{d}{2}-2}}{4^{d-2}\pi^{\frac{d}{2}}} \frac{1}{|\mathbf{y}|^2} e^{-\frac{3|\mathbf{y}|^2}{16}} , \qquad |\mathbf{y}| \gg 1 .$$

The local shape of the dimensionless power profile for small $|\mathbf{y}|$ gives a power divergence $\mathcal{Q}(\mathbf{y}) \simeq 2^{-3/4}3^{-2/3}\pi^{-d/2}\Gamma(\frac{d}{2}-\frac{1}{3})|\mathbf{y}|^{2/3-d}$ (at the scale $\alpha\beta^{1/2}\mathbf{y}$). This is given by the contributions of reflections that occur close to the surface $z = L$.

Let us now consider the case in which $\check{\mathcal{C}}_0(\boldsymbol{s})$ is not smooth at $\mathbf{0}$ and has the form $\check{\mathcal{C}}_0(\boldsymbol{s}) = \check{\mathcal{C}}_0(\mathbf{0}) - \check{\mathcal{C}}_0'|\boldsymbol{s}| + o(|\boldsymbol{s}|)$, with $\check{\mathcal{C}}_0' > 0$. This corresponds to a rough random medium, with jumps in the derivative of ν or in ν itself. We find that

$$\lim_{\beta\to\infty}\lim_{\alpha\to\infty} \alpha^d\beta^d\mathcal{I}(\alpha\beta\mathbf{y}) = \frac{1}{[\check{\mathcal{C}}_0']^d} \mathcal{Q}\Big(\frac{\mathbf{y}}{\check{\mathcal{C}}_0'}\Big) ,$$

where

$$\mathcal{Q}(\mathbf{y}) = \frac{1}{2^{2d-1}\pi} \int_0^1 \frac{1}{\left(1+\frac{|\mathbf{y}|^2}{4\zeta^4}\right)^{\frac{d+1}{2}}} \frac{1}{\zeta^{2d}} \, d\zeta$$

is such that $\int \mathcal{Q}(\mathbf{y}) d\mathbf{y} = 1$. We obtain a power law decay at infinity and a power divergence at $\mathbf{0}$:

$$\mathcal{Q}(\mathbf{y}) \overset{|\mathbf{y}|\gg 1}{\simeq} \frac{1}{3\, 2^{d-2}\pi |\mathbf{y}|^{d+1}} , \qquad \mathcal{Q}(\mathbf{y}) \overset{|\mathbf{y}|\ll 1}{\simeq} \frac{1}{2^{\frac{3d}{2}+\frac{5}{4}}\pi} \frac{\Gamma(\frac{5}{8}+\frac{d}{4})\Gamma(\frac{d}{4}-\frac{1}{8})}{\Gamma(\frac{d}{2}+\frac{1}{2})} \frac{1}{|\mathbf{y}|^{d-\frac{1}{2}}} .$$

Note that it is not possible to define a rms beam width in this case, as it is infinite due to the heavy tail $|\mathbf{y}|^{-1-d}$ as $|\mathbf{y}| \to \infty$. However, if we define loosely the beam width as the typical radius of the mean power profile (the full width at half maximum, for instance), then we can claim that the beam width is of order $\alpha\beta l_x$, that is, proportional to L^2 in the physical variables. This can be contrasted with the result obtained in the case of a smooth random medium, where the beam has Gaussian tail and a width of order $L^{3/2}$.

4.5 Enhanced backscattering

Enhanced backscattering (or weak localization) is a well-known phenomenon in physics [2,28] and it has been observed in several experimental contexts [30,27,26,16]. To summarize, when an incoming monochromatic quasi-plane wave is applied with a given incidence angle, the mean reflected power has a local maximum in the backscattered direction, which is usually twice as large as the mean reflected power in the other directions. In this section, we give a mathematical proof of enhanced backscattering and we compute the maximum, the angular width, and the shape of the enhanced backscattering cone.

We will analyze the following experiment: for a given $\boldsymbol{\kappa}_0$, we send a quasi-plane wave of unit power, carrier wavevector $\boldsymbol{\kappa}_0$, and angular aperture much smaller than $\alpha^{-1}(kl_x)^{-1}$, and we record the reflected power in the backscattered direction $-\boldsymbol{\kappa}_0$ or close to it, in a cone of angular aperture of order $\alpha^{-1}(kl_x)^{-1}$. Accordingly we observe

$$|\hat{a}^\varepsilon(L, -\boldsymbol{\kappa}_0 + \alpha^{-1}\boldsymbol{\kappa})|^2 = \Big| \int \hat{\boldsymbol{\mathcal{R}}}^\varepsilon(L, \boldsymbol{\kappa}', -\boldsymbol{\kappa}_0 + \alpha^{-1}\boldsymbol{\kappa}) \hat{b}(\boldsymbol{\kappa}') \, d\boldsymbol{\kappa}' \Big|^2 .$$

If we average with respect to the random medium, and consider the asymptotic regime $\varepsilon \to 0$, we find that we observe in fact

$$P_{\boldsymbol{\kappa},\boldsymbol{\kappa}_0} = \lim_{\varepsilon\to 0} \mathbb{E}\big[|\hat{a}^\varepsilon(L, -\boldsymbol{\kappa}_0 + \alpha^{-1}\boldsymbol{\kappa})|^2\big] = \int D_{\boldsymbol{\kappa}',\boldsymbol{\kappa}',-\boldsymbol{\kappa}_0-\boldsymbol{\kappa}'+\alpha^{-1}\boldsymbol{\kappa}}(L) |\hat{b}(\boldsymbol{\kappa}')|^2 \, d\boldsymbol{\kappa}' .$$

If we take into account the fact that the angular aperture of the input beam is very small, i.e. much smaller than $\alpha^{-1}(kl_x)^{-1}$, then we find that the mean reflected power observed in the relative direction $\boldsymbol{\kappa}$ is

$$P_{\boldsymbol{\kappa},\boldsymbol{\kappa}_0} = D_{\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0,-2\boldsymbol{\kappa}_0+\alpha^{-1}\boldsymbol{\kappa}}(L) . \tag{54}$$

Finally, we average the results over $\boldsymbol{\kappa}_0$ and we obtain the quantity

$$P_{\boldsymbol{\kappa}} = 2^d \int D_{\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0,-2\boldsymbol{\kappa}_0+\alpha^{-1}\boldsymbol{\kappa}}(L) d\boldsymbol{\kappa}_0 . \tag{55}$$

This is the mean power reflected by the random slab $(0, L) \times \mathbb{R}^d$ in the backscattered direction (for $\boldsymbol{\kappa} = \mathbf{0}$) or close to the backscattered direction, in a direction whose angle with respect to the backscattered direction is of order $\alpha^{-1}(kl_x)^{-1}$ (for $\boldsymbol{\kappa} \neq \mathbf{0}$).

Note: the averaging with respect to the random medium is probably not necessary because we expect that the averaging with respect to $\boldsymbol{\kappa}_0$ is sufficient to ensure the averaging with respect to the random fluctuations. A proof of this self-averaging property would require to study higher-order moments, which is beyond the scope of this paper.

Proposition 7 *In the regime $\alpha \gg 1$ the mean power reflected in the direction $\boldsymbol{\kappa}$ relative to the backscattered direction has the form*

$$\lim_{\alpha\to\infty} P_{\boldsymbol{\kappa}} = l_x^{-d} \bar{D} (2\pi)^d \mathcal{P}(\boldsymbol{\kappa} l_x) , \tag{56}$$

with

$$\mathcal{P}(\boldsymbol{s}) = 1 + \int_0^1 e^{2\beta \int_0^\zeta \check{\mathcal{C}}_0(\boldsymbol{s}\zeta') - \check{\mathcal{C}}_0(\mathbf{0}) d\zeta'} d\zeta - \frac{1 - e^{-2\beta \check{\mathcal{C}}_0(\mathbf{0})}}{2\beta \check{\mathcal{C}}_0(\mathbf{0})} . \tag{57}$$

Proof By Lemma 2 we obtain (56) with

$$\mathcal{P}(\boldsymbol{s}) = \frac{1}{(2\pi)^d} \int \mathcal{D}_{\mathbf{0}}(1, \mathbf{w}) + \mathcal{D}_{\boldsymbol{s}}(1, \mathbf{0}, \mathbf{w}) - \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} \frac{1 - e^{-2\beta \check{\mathcal{C}}_0(\mathbf{0})}}{2\beta \check{\mathcal{C}}_0(\mathbf{0})} d\mathbf{w} .$$

The computation of the integral with respect to $\mathbf{w}$ gives (57). □

The mean reflected power in an arbitrary direction out of the small cone around the backscattered direction can be obtained by taking the limit $|\boldsymbol{s}| \to \infty$:

$$\lim_{|\boldsymbol{s}|\to\infty} \mathcal{P}(\boldsymbol{s}) = 1 .$$

The maximum of the enhanced backscattering cone is reached at $\boldsymbol{s} = \mathbf{0}$, that is, for the exact backscattered direction, and this maximum is given by

$$\max_{\boldsymbol{s}\in\mathbb{R}^d} \mathcal{P}(\boldsymbol{s}) = \mathcal{P}(\mathbf{0}) = 2 - \frac{1 - e^{-2\beta \check{\mathcal{C}}_0(\mathbf{0})}}{2\beta \check{\mathcal{C}}_0(\mathbf{0})} , \tag{58}$$

which lies in the interval $(1, 2)$. In the weak forward-scattering regime $\beta \ll 1$, the enhancement factor is equal to 1. In the strong forward-scattering regime $\beta \gg 1$, the enhancement factor is equal to 2. Formula (58) was found in Refs. [4,5] by using diagrammatic expansions. These references are prior to the recent research on enhanced backscattering and weak localization, and formula (58) is in these references called "enhancement of the Born approximation".

The shape $\mathcal{P}(\boldsymbol{s})$ of the cone is given by (57) for any value of β. We can give a more quantitative description in the regime $\beta \gg 1$ but this analysis requires to distinguish the cases in which $\check{\mathcal{C}}_0$ is smooth or not at $\mathbf{0}$.

Let us first consider the case in which $\check{\mathcal{C}}_0(\boldsymbol{s})$ is twice differentiable at $\mathbf{0}$ and can be expanded as $\check{\mathcal{C}}_0(\boldsymbol{s}) \simeq \check{\mathcal{C}}_0(\mathbf{0}) - \frac{1}{2}\check{\mathcal{C}}_0''|\boldsymbol{s}|^2 + o(|\boldsymbol{s}|^2)$, with $\check{\mathcal{C}}_0'' > 0$. We find that

$$\lim_{\beta\to\infty} \mathcal{P}(\beta^{-1/2}\boldsymbol{s}) = \mathcal{Q}\Big([\check{\mathcal{C}}_0'']^{1/2}\boldsymbol{s}\Big), \qquad \mathcal{Q}(\boldsymbol{s}) = 1 + \int_0^1 e^{-\frac{\zeta^3}{3}|\boldsymbol{s}|^2} d\zeta ,$$

For small $|\boldsymbol{s}|$, we have $\mathcal{Q}(\boldsymbol{s}) \simeq 2 - \frac{1}{12}|\boldsymbol{s}|^2$, which shows that the peak is smooth. For large $|\boldsymbol{s}|$, we obtain $\mathcal{Q}(\boldsymbol{s}) \simeq 1 + \frac{2\pi 3^{-7/6}}{\Gamma(2/3)}|\boldsymbol{s}|^{-2/3}$. This shows that the angular aperture of the enhanced backscattering cone is of order

$$A_{\rm EBC} = \frac{1}{[\check{\mathcal{C}}_0'']^{1/2} k l_x \alpha \beta^{1/2}} \sim \frac{l_x}{k L^{3/2} l_z^{1/2} \sigma} .$$

Let us now consider the case in which $\check{\mathcal{C}}_0(\boldsymbol{s})$ is not smooth at $\mathbf{0}$ and has the form $\check{\mathcal{C}}_0(\boldsymbol{s}) = \check{\mathcal{C}}_0(\mathbf{0}) - \check{\mathcal{C}}_0'|\boldsymbol{s}| + o(|\boldsymbol{s}|)$, with $\check{\mathcal{C}}_0' > 0$. We find that

$$\lim_{\beta \to \infty} \mathcal{P}(\beta^{-1}\boldsymbol{s}) = \mathcal{Q}\Big([\check{\mathcal{C}}_0']\boldsymbol{s}\Big) , \qquad \mathcal{Q}(\boldsymbol{s}) = 1 + \int_0^1 e^{-\zeta^2 |\boldsymbol{s}|} \, d\zeta .$$

For small $|\boldsymbol{s}|$ we have $\mathcal{Q}(\boldsymbol{s}) \simeq 2 - \frac{1}{3}|\boldsymbol{s}|$, which shows that the peak is not smooth but has a cusp. For large $|\boldsymbol{s}|$, we obtain $\mathcal{Q}(\boldsymbol{s}) \simeq 1 + \frac{\pi^{1/2}}{2}|\boldsymbol{s}|^{-1/3}$. This shows that the angular aperture of the enhanced backscattering cone is of order

$$A_{\rm EBC} = \frac{1}{[\check{\mathcal{C}}_0'] k l_x \alpha \beta} \sim \frac{l_x}{k^2 L^2 l_z \sigma^2} .$$

Figure 3 plots the enhanced backscattering cone for different values of β and for two different autocorrelation functions.

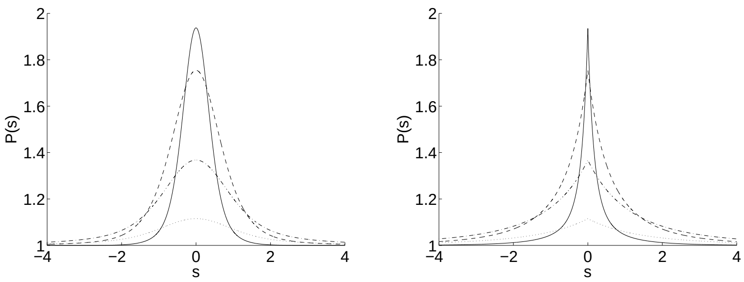

Fig. 3 Enhanced backscattering cone $\mathcal{P}(\boldsymbol{s})$ for a Gaussian autocorrelation function $\check{\mathcal{C}}_0(\boldsymbol{s}) = \check{\mathcal{C}}_0(\mathbf{0})\exp(-|\boldsymbol{s}|^2)$ (left) and an exponential autocorrelation function $\check{\mathcal{C}}_0(\boldsymbol{s}) = \check{\mathcal{C}}_0(\mathbf{0})\exp(-|\boldsymbol{s}|)$ (right). The four peaks represent four different values of the parameter $\bar{\beta} = 2\beta\check{\mathcal{C}}_0(\mathbf{0})$: $\bar{\beta} = 0.25$ (dotted), $\bar{\beta} = 1$ (dot-dashed), $\bar{\beta} = 4$ (dashed), and $\bar{\beta} = 16$ (solid). Note that the enhancement factor (i.e. the maximum of the peak) depends only on $\bar{\beta}$.

Note: we have obtained that the beam width R is of order $\alpha\beta^{1/2} l_x$ (for a smooth medium) or $\alpha\beta l_x$ (for a rough medium), while the angular aperture $A_{\rm EBC}$ of the enhanced backscattering cone is of order $\alpha^{-1}\beta^{-1/2}(kl_x)^{-1}$ (for a smooth medium) or $\alpha^{-1}\beta^{-1}(kl_x)^{-1}$ (for a rough medium). Therefore, the relation $A_{\rm EBC} \sim 1/(kR)$ is always satisfied. This relation is in agreement with the physical interpretation of enhanced backscattering as a constructive interference between pairs of wave "paths" and reversed paths (see Figure 4). The sum of all these constructive interferences should give an enhancement factor of 2 in the backscattered direction. If the reflected wave is observed with an angle A compared to the backscattered direction, then the phase shift between the direct and reversed paths is $ke = kd \sin A$, where d is the typical transverse size of a wave path, which is in our setting of the order of the beam width

R. Therefore, constructive interference is possible if $kRA \leq 1$, which gives the angular aperture of the enhanced backscattering cone. This "path" interpretation is not used in our paper, but we recover the physical result by exploiting our system of transport equations.

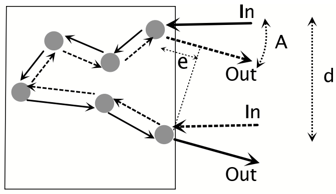

Fig. 4 Physical interpretation of the scattering of a plane wave by a random medium. The output wave in direction A is the superposition of many different scattering paths. One of these paths is plotted as well as the reversed path. The phase difference between the two outgoing waves is $ke = kd \sin A$.

Finally, in this section we have mainly studied the mean reflected power observed in the relative direction $\boldsymbol{\kappa}$ averaged over the incident directions (55). If we are interested in the mean reflected power observed in the relative direction $\boldsymbol{\kappa}$ for a given incident direction $\boldsymbol{\kappa}_0$, given by (54), then we find

$$\lim_{\alpha \to \infty} P_{\boldsymbol{\kappa}, \boldsymbol{\kappa}_0} = \bar{D} \mathcal{P}_{-2\boldsymbol{\kappa}_0 l_x}(\boldsymbol{\kappa} l_x) ,$$

with

$$\mathcal{P}_{\mathbf{w}}(\boldsymbol{s}) = \int \frac{\check{C}_{2kl_z}(\boldsymbol{\lambda})}{\check{C}_{2kl_z}(\mathbf{0})} e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}} \int_0^1 e^{-2\beta\check{C}_0(\mathbf{0})\zeta} \Big[e^{\beta\int_{-\zeta}^{\zeta} \check{C}_0(\boldsymbol{\lambda}+\boldsymbol{s}\zeta')d\zeta'} + e^{2\beta\check{C}_0(\boldsymbol{\lambda})\zeta} - 1 \Big] d\zeta d\boldsymbol{\lambda} .$$

We have in particular

$$\lim_{|\boldsymbol{s}|\to\infty} \mathcal{P}_{\mathbf{w}}(\boldsymbol{s}) = \frac{\widehat{C}_{2kl_z}(\mathbf{w})}{\check{C}_{2kl_z}(\mathbf{0})} \quad \text{and} \quad \mathcal{P}_{\mathbf{w}}(\mathbf{0}) = \int \frac{\check{C}_{2kl_z}(\boldsymbol{\lambda})}{\check{C}_{2kl_z}(\mathbf{0})} e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}} \Big[2 - \frac{1 - e^{-2\beta\check{C}_0(\boldsymbol{\lambda})}}{2\beta\check{C}_0(\boldsymbol{\lambda})} \Big] d\boldsymbol{\lambda} ,$$

which shows that the enhancement factor for large β is 2.

5 The weak backscattering regime when $\alpha \ll 1$

We consider here the regime of large Fresnel number. That is, we analyze the system for $\mathcal{D}$ in the regime $\alpha \ll 1$. This regime can be interpreted as a high frequency situation, respectively, a regime of large transversal correlation radius. We first give the asymptotic behavior of the cross spectral density in the regime $\alpha \ll 1$.

Lemma 3 *We have* $\lim_{\alpha\to 0} \mathcal{D}(\zeta, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathcal{D}^{(0)}(\zeta, \mathbf{w})$ *with*

$$\mathcal{D}^{(0)}(\zeta, \mathbf{w}) = \int \frac{\check{C}_{2kl_z}(\boldsymbol{\lambda})}{\check{C}_{2kl_z}(\mathbf{0})} e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}} \frac{1 - e^{-4\zeta\beta(\check{C}_0(\mathbf{0}) - \check{C}_0(\boldsymbol{\lambda}))}}{4\beta(\check{C}_0(\mathbf{0}) - \check{C}_0(\boldsymbol{\lambda}))} d\boldsymbol{\lambda} . \tag{59}$$

If $\check{\mathcal{C}}_{2kl_z}(\boldsymbol{\lambda})$ and $\check{\mathcal{C}}_0(\boldsymbol{\lambda})$ are twice differentiable in $\boldsymbol{\lambda}$, then we have $\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w}) = \mathcal{D}^{(0)}(\zeta,\mathbf{w}) + i\alpha\mathbf{u}\cdot\mathbf{v}\mathcal{D}^{(1)}(\zeta,\mathbf{w}) + O(\alpha^2)$ with

$$\mathcal{D}^{(1)}(\zeta,\mathbf{w}) = \int \frac{\check{\mathcal{C}}_{2kl_z}(\boldsymbol{\lambda})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}} \frac{e^{-4\zeta\beta(\check{\mathcal{C}}_0(\mathbf{0})-\check{\mathcal{C}}_0(\boldsymbol{\lambda}))} - 1 + 4\zeta\beta(\check{\mathcal{C}}_0(\mathbf{0}) - \check{\mathcal{C}}_0(\boldsymbol{\lambda}))}{\big[4\beta(\check{\mathcal{C}}_0(\mathbf{0}) - \check{\mathcal{C}}_0(\boldsymbol{\lambda}))\big]^2}\, d\boldsymbol{\lambda}\,. \tag{60}$$

In the following we give the expansions of the beam width, spectral width, and mean reflected power profile for small α. We will assume that $\check{\mathcal{C}}_0$ and $\check{\mathcal{C}}_{2kl_z}$ are twice differentiable at $\mathbf{0}$.

Beam width

We consider the rms width of the reflected beam R^ε defined by (42). From the expression (43) of the limit $R = \lim_{\varepsilon\to 0} R^\varepsilon$ it follows using the expressions for $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ that

$$R^2 \overset{\alpha \ll 1}{\simeq} R_0^2 + 2Q_0 l_x^2 \alpha + O(\alpha^2)\,.$$

The only noticeable effect is the convergence or divergence due to the initial phase front of the beam characterized by Q_0 given by (46). Random scattering plays no role here, and diffraction is not yet noticeable either (they both arise at order α^2).

Spectral width

The spectral width is defined by (48) and it converges to K given by (49) as $\varepsilon \to 0$. In the regime $\alpha \ll 1$ we obtain

$$K^2 \overset{\alpha \ll 1}{\simeq} K_0^2 - 2\Delta\check{\mathcal{C}}_0(\mathbf{0}) l_x^{-2} \frac{\beta}{(2\pi)^d} - \frac{\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} l_x^{-2} + O(\alpha^2)\,.$$

By comparing with (50) we can see that the small and large α limits of the spectral spreading almost coincide, up to a factor 2 in the term due to random forward scattering (the one with $\Delta\check{\mathcal{C}}_0(\mathbf{0})$). These results show that random scattering has a strong effect of order one on the spectral width in the regime $\alpha \ll 1$, but this has little influence (or order α^2) on the beam width. This remark will be confirmed by the study of the mean reflected power.

Mean reflected power

Recall that in the limit $\varepsilon \to 0$, the mean reflected power is given by (52). In the regime $\alpha \ll 1$ we have then

$$\begin{aligned}\mathcal{I}(\mathbf{y}) \overset{\alpha \ll 1}{\simeq} & \frac{\bar{D} l_x^{-3d}}{2^{3d}\pi^{2d}} \iiint \hat{b}\Big(\frac{\mathbf{v}+\mathbf{u}-\mathbf{w}}{2l_x}\Big)\bar{\hat{b}}\Big(\frac{\mathbf{v}-\mathbf{u}-\mathbf{w}}{2l_x}\Big) e^{i\mathbf{u}\cdot\mathbf{y}} \\ & \times \Big\{ \mathcal{D}^{(0)}(1,\mathbf{w}) + i\alpha\mathbf{u}\cdot\mathbf{v}[\mathcal{D}^{(1)}(1,\mathbf{w}) - \mathcal{D}^{(0)}(1,\mathbf{w})] + O(\alpha^2)\Big\}\, d\mathbf{u}d\mathbf{v}d\mathbf{w}\,,\end{aligned}$$

and find

$$\mathcal{I}(\mathbf{y}) \overset{\alpha \ll 1}{\simeq} \frac{l_x^d}{P}\Big[\big|\check{b}(l_x\mathbf{y})\big|^2 - \alpha l_x^2 \mathrm{Im}\Big(\bar{\check{b}}\Delta\check{b}(l_x\mathbf{y})\Big) + O(\alpha^2)\Big]\,.$$

Once again, the only noticeable effect is the convergence or divergence due to the initial phase front of the beam.

6 Generalized transport equations in transmission

In this section we characterize the spectrum of the transmitted wave. This characterization is a generalization of the results in the reflection case presented in Subsection 3.1 and leads to modified transport equations for the cross moments of reflection and transmission coefficients. In Section 6.1 we present a dimensionless form of this system in the regime of weak backscattering. In Section 6.2 we give an application by computing the spatial power profile of the transmitted wave. First we state the main theoretical result that is proved in Appendix B:

Proposition 8 *Using the same notations as in Proposition 1 we introduce the moments of products of* $\hat{\boldsymbol{\mathcal{R}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')$*, the reflection operator, and* $\hat{\boldsymbol{\mathcal{T}}}^\varepsilon(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}')$*, the transmission operator, at two nearby frequencies:*

$$\mathcal{U}_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'),\varepsilon}(k,h,z;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b) = \mathbb{E}\Big[\prod_{j=1}^{n_{\mathbf{p}}} \hat{\boldsymbol{\mathcal{R}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}_p'(j)\Big) \tag{61}$$

$$\times \prod_{l=1}^{n_{\mathbf{q}}} \overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}_q'(l)\Big) \hat{\boldsymbol{\mathcal{T}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_a'\Big) \overline{\hat{\boldsymbol{\mathcal{T}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_b, \boldsymbol{\kappa}_b'\Big)\Big].$$

The family of Fourier transforms

$$W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'),\varepsilon}(k,\tau,z) = \frac{1}{2\pi}\int e^{-ih[\tau-(n_{\mathbf{p}}+n_{\mathbf{q}}+1)z]} \mathcal{U}_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'),\varepsilon}(k,h,z)\, dh\,, \tag{62}$$

converges as $\varepsilon \to 0$ *to the solution* $W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}$ *of the system of transport equations*

$$\frac{\partial W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}}{\partial z} + (n_{\mathbf{p}}+n_{\mathbf{q}}+1)\frac{\partial W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}}{\partial \tau} = \frac{i}{2k}\Phi_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} + \frac{k^2}{4(2\pi)^d}\big(\mathcal{L}_W^T W\big)_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} \tag{63}$$

with the initial conditions:

$$W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}(k,\tau,z=0;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b) = \mathbf{1}_0(n_{\mathbf{p}})\mathbf{1}_0(n_{\mathbf{q}})\delta(\boldsymbol{\kappa}_a'-\boldsymbol{\kappa}_a)\delta(\boldsymbol{\kappa}_b'-\boldsymbol{\kappa}_b)\delta(\tau)\,.$$

Here we have defined

$$\Phi_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} = \Phi_{\mathbf{p},\mathbf{q}} - |\boldsymbol{\kappa}_a'|^2 + |\boldsymbol{\kappa}_b'|^2\,, \tag{64}$$

$$
\begin{aligned}
\big(\mathcal{L}_W^T W\big)_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)} &= \big(\mathcal{L}_W W^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}\big)_{\mathbf{p},\mathbf{q}} + \int \widehat{C}(0,\boldsymbol{\kappa})\Big(W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}'_a+\boldsymbol{\kappa},\boldsymbol{\kappa}'_b+\boldsymbol{\kappa})} - W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}\Big)\,d\boldsymbol{\kappa}\\
&+\sum_{j=1}^{n_{\mathbf{p}}}\int \widehat{C}(0,\boldsymbol{\kappa})\Big(-W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)+\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j))\},\mathbf{q}}^{(\boldsymbol{\kappa}'_a+\boldsymbol{\kappa},\boldsymbol{\kappa}'_b)} - W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa})\},\mathbf{q}}^{(\boldsymbol{\kappa}'_a+\boldsymbol{\kappa},\boldsymbol{\kappa}'_b)}\\
&+W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j))\},\mathbf{q}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b+\boldsymbol{\kappa})} + W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)+\boldsymbol{\kappa})\},\mathbf{q}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b+\boldsymbol{\kappa})}\Big)d\boldsymbol{\kappa}\\
&+\sum_{l=1}^{n_{\mathbf{q}}}\int \widehat{C}(0,\boldsymbol{\kappa})\Big(W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l))\}}^{(\boldsymbol{\kappa}'_a+\boldsymbol{\kappa},\boldsymbol{\kappa}'_b)} + W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)+\boldsymbol{\kappa})\}}^{(\boldsymbol{\kappa}'_a+\boldsymbol{\kappa},\boldsymbol{\kappa}'_b)}\\
&-W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)+\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l))\}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b+\boldsymbol{\kappa})} - W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)-\boldsymbol{\kappa})\}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b+\boldsymbol{\kappa})}\Big)d\boldsymbol{\kappa}\\
&-\sum_{j=1}^{n_{\mathbf{p}}}\widehat{C}(2k,\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}'_p(j))\int W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}+\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_a)\},\mathbf{q}}^{(\boldsymbol{\kappa}+\boldsymbol{\kappa}'_p(j),\boldsymbol{\kappa}'_b)}\,d\boldsymbol{\kappa}\\
&-\sum_{l=1}^{n_{\mathbf{q}}}\widehat{C}(2k,\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa}'_q(l))\int W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}+\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_b)\}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}+\boldsymbol{\kappa}'_q(l))}\,d\boldsymbol{\kappa}\\
&-\int \widehat{C}(2k,\boldsymbol{\kappa})d\boldsymbol{\kappa}\,W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)} + \iiint \widehat{C}(2k,\boldsymbol{\kappa}_1)\,W_{\mathbf{p}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_a),\mathbf{q}\cup(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}'_b)}^{(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3)}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3\\
&+\sum_{j=1}^{n_{\mathbf{p}}}\iiint \widehat{C}(2k,\boldsymbol{\kappa}_1)\,W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_b)}^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3\\
&+\sum_{l=1}^{n_{\mathbf{q}}}\iiint \widehat{C}(2k,\boldsymbol{\kappa}_1)\,W_{\mathbf{p}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_a),\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}'_q(l))\}}^{(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_b)}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3\,.
\end{aligned}
$$

This set of transport equations describes accurately the transmitted wave field and it is the key tool to analyze various applications for waves in random media. The corresponding transport equations in the layered case are presented in [9].

6.1 The transmission system in the weak backscattering regime

We are again interested in the weak backscattering regime introduced in Section 3.2. When we take into account terms of order zero and one (in δ) in the weak backscattering regime, we obtain:

$$
\begin{aligned}
\frac{\partial W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)}}{\partial z} + \frac{\partial W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)}}{\partial \tau} &= \frac{i}{2k}\big(|\boldsymbol{\kappa}'_1|^2 - |\boldsymbol{\kappa}'_0|^2\big)W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)} - \frac{k^2\check{C}(2k,\mathbf{0})}{4}W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)}\\
&+\frac{k^2}{4(2\pi)^d}\int \widehat{C}(0,\boldsymbol{\kappa})\left(W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0+\boldsymbol{\kappa},\boldsymbol{\kappa}'_1+\boldsymbol{\kappa})} - W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)}\right)d\boldsymbol{\kappa}\,, \quad (65)
\end{aligned}
$$

with $W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)}(k,\tau,z=0;\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_1) = \delta(\tau)\delta(\boldsymbol{\kappa}_0-\boldsymbol{\kappa}'_0)\delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}'_1)$. Note that $W_{\emptyset,\emptyset}^{(\boldsymbol{\kappa}'_0,\boldsymbol{\kappa}'_1)}$ will be supported on $\boldsymbol{\kappa}_0-\boldsymbol{\kappa}'_0-\boldsymbol{\kappa}_1+\boldsymbol{\kappa}'_1=\mathbf{0}$ so that we again can parameterize the solution in terms of three "effective" wavevectors as in the analysis of the cross spectral density of the reflected wave. By integrating the system (65) in τ we obtain the following convergence result.

Proposition 9 *We have as $\varepsilon \to 0$*

$$\mathbb{E}\Big[\hat{\boldsymbol{\mathcal{T}}}^{\varepsilon}(k,z,\boldsymbol{\kappa}_0'+\boldsymbol{\kappa},\boldsymbol{\kappa}_0')\overline{\hat{\boldsymbol{\mathcal{T}}}^{\varepsilon}}(k,z,\boldsymbol{\kappa}_1'+\boldsymbol{\kappa}',\boldsymbol{\kappa}_1')\Big] \stackrel{\varepsilon\to 0}{\longrightarrow} \delta(\boldsymbol{\kappa}'-\boldsymbol{\kappa})D^T_{\boldsymbol{\kappa}_0',\boldsymbol{\kappa}_1',\boldsymbol{\kappa}}(z)\,, \tag{66}$$

where the cross spectral density D^T is of the form

$$D^T_{\boldsymbol{\kappa}_0',\boldsymbol{\kappa}_1',\boldsymbol{\kappa}}(z) = l_x^d \mathcal{D}^T\Big(\frac{z}{L},(\boldsymbol{\kappa}_0'-\boldsymbol{\kappa}_1')l_x,(\boldsymbol{\kappa}_0'+\boldsymbol{\kappa}_1'+\boldsymbol{\kappa})l_x,\boldsymbol{\kappa}l_x\Big) \\ \times \exp\Big[i(|\boldsymbol{\kappa}_1'|^2-|\boldsymbol{\kappa}_0'|^2)\frac{z}{2k}\Big]\exp\Big(-\frac{k^2\sigma^2 l_z \check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{4}z\Big)\,, \tag{67}$$

and the dimensionless cross spectral density $\mathcal{D}^T(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})$ is solution of

$$\frac{d\mathcal{D}^T(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})}{d\zeta} = \frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})\Big[e^{i\alpha\boldsymbol{\mu}\cdot\mathbf{u}\zeta}\mathcal{D}^T(\zeta,\mathbf{u},\mathbf{v}-\boldsymbol{\mu},\mathbf{w}+\boldsymbol{\mu})-\mathcal{D}^T(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})\Big]d\boldsymbol{\mu}\,,$$

starting from $\mathcal{D}^T(\zeta=0,\mathbf{u},\mathbf{v},\mathbf{w})=\delta(\mathbf{w})$.

It is remarkable that the first-order correction due to weak backscattering is a simple frequency-dependent attenuation in (67). The density $\mathcal{D}^T$ takes a simple form in the regime $\alpha \gg 1$.

Lemma 4 *1. If $\mathbf{u}\neq\mathbf{0}$, then $\lim_{\alpha\to\infty}\mathcal{D}^T(\zeta,\mathbf{u},\mathbf{v},\mathbf{w}) = e^{-\beta\check{\mathcal{C}}_0(\mathbf{0})\zeta}\delta(\mathbf{w})$.*
2. $\lim_{\alpha\to\infty}\mathcal{D}^T(\zeta,\alpha^{-1}\boldsymbol{s},\mathbf{v},\mathbf{w}) = \mathcal{D}^T_{\boldsymbol{s}}(\zeta,\mathbf{w})$ where $\mathcal{D}^T_{\boldsymbol{s}}(\zeta,\mathbf{w})$ is solution of

$$\frac{d\mathcal{D}^T_{\boldsymbol{s}}(\zeta,\mathbf{w})}{d\zeta} = \frac{\beta}{(2\pi)^d}\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})\Big[e^{i\boldsymbol{\mu}\cdot\boldsymbol{s}\zeta}\mathcal{D}^T_{\boldsymbol{s}}(\zeta,\mathbf{w}+\boldsymbol{\mu})-\mathcal{D}^T_{\boldsymbol{s}}(\zeta,\mathbf{w})\Big]\,d\boldsymbol{\mu}\,,$$

starting from $\mathcal{D}^T_{\boldsymbol{s}}(\zeta=0,\mathbf{w})=\delta(\mathbf{w})$.

It is possible to obtain an integral representation for $\mathcal{D}^T_{\boldsymbol{s}}$:

$$\mathcal{D}^T_{\boldsymbol{s}}(\zeta,\mathbf{w}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{w}\cdot\boldsymbol{\lambda}}e^{\beta\int_0^\zeta \check{\mathcal{C}}_0(\boldsymbol{\lambda}-\zeta'\boldsymbol{s})-\check{\mathcal{C}}_0(\mathbf{0})d\zeta'}\,d\boldsymbol{\lambda}\,. \tag{68}$$

6.2 The transmitted power profile in the weak backscattering regime

The results of the previous subsection allow us to address various physically relevant problems. For instance, we can compute the mean power profile of the transmitted wave. In the limit $\varepsilon\to 0$, the mean transmitted power $I^T(\mathbf{x}) = \lim_{\varepsilon\to 0}\mathbb{E}[|\check{b}^\varepsilon(0,\mathbf{x})|^2]$ is given by

$$I^T(\mathbf{x}) = \frac{1}{(2\pi)^{2d}}\iiint \hat{b}(\boldsymbol{\kappa}_0')\overline{\hat{b}(\boldsymbol{\kappa}_1')}e^{i(\boldsymbol{\kappa}_0'-\boldsymbol{\kappa}_1')\cdot\mathbf{x}}D^T_{\boldsymbol{\kappa}_0',\boldsymbol{\kappa}_1',\boldsymbol{\kappa}}(L)d\boldsymbol{\kappa}d\boldsymbol{\kappa}_0'd\boldsymbol{\kappa}_1'\,.$$

Using the dimensionless cross spectral density $\mathcal{D}^T$ this can also be written as

$$I^T(\mathbf{x}) = \frac{P^T}{l_x^d}\mathcal{I}^T\Big(\frac{\mathbf{x}}{l_x}\Big)\,,$$

where P^T (respectively P_0) is the total mean transmitted power (respectively total incoming power) given by

$$P^T = P_0\exp\Big(-\frac{k^2\sigma^2 l_z\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{4}L\Big),\qquad P_0 = \int|\check{b}(\mathbf{x})|^2\,d\mathbf{x}\,,$$

and the dimensionless transmitted power profile is

$$\mathcal{I}^T(\mathbf{y}) = \frac{l_x^{-d}}{2^{3d}\pi^{2d}P_0} \iiint \hat{b}\Big(\frac{\mathbf{v}+\mathbf{u}-\mathbf{w}}{2l_x}\Big)\overline{\hat{b}}\Big(\frac{\mathbf{v}-\mathbf{u}-\mathbf{w}}{2l_x}\Big)e^{i\mathbf{u}\cdot\mathbf{y}} \times \mathcal{D}^T(1,\mathbf{u},\mathbf{v},\mathbf{w})e^{-\frac{i}{2}\alpha\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})}\, d\mathbf{u}d\mathbf{v}d\mathbf{w}\,.$$

In the regime $\alpha \gg 1$ we find by using Lemma 4 that

$$\lim_{\alpha\to\infty} \alpha^d \mathcal{I}^T(\alpha\mathbf{y}) = \frac{l_x^{-d}}{(2\pi)^{2d}P_0} \iiint \mathcal{D}^T_{\boldsymbol{s}}(\mathbf{w})e^{i\boldsymbol{s}\cdot(\mathbf{y}-\mathbf{v})}\big|\hat{b}\big(\frac{\mathbf{v}}{l_x}\big)\big|^2\, d\boldsymbol{s}d\mathbf{v}d\mathbf{w}\,.$$

Substituting the expression (68) of $\mathcal{D}^T_{\boldsymbol{s}}(\mathbf{w})$, we obtain

$$\lim_{\alpha\to\infty} \alpha^d \mathcal{I}^T(\alpha\mathbf{y}) = \frac{l_x^{-d}}{(2\pi)^{2d}P_0} \iint e^{i\boldsymbol{s}\cdot(\mathbf{y}-\mathbf{v})}\big|\hat{b}\big(\frac{\mathbf{v}}{l_x}\big)\big|^2 e^{\beta\int_0^1 \check{\mathcal{C}}_0(\zeta\boldsymbol{s})-\check{\mathcal{C}}_0(\mathbf{0})\,d\zeta} d\boldsymbol{s}d\mathbf{v}\,.$$

If $\beta \ll 1$, meaning that random forward scattering is weak, then

$$\lim_{\beta\to0}\lim_{\alpha\to\infty} \alpha^d \mathcal{I}^T(\alpha\mathbf{y}) = \frac{l_x^{-d}}{(2\pi)^{d}P_0}\big|\hat{b}\big(\frac{\mathbf{y}}{l_x}\big)\big|^2\,,$$

which is the standard formula that can be obtained by a stationary phase argument applied on the expression of the transmitted power in a homogeneous medium:

$$\mathcal{I}^T(\alpha\mathbf{y})\,|_{\rm homo} = \frac{l_x^{-d}}{(2\pi)^{2d}P_0}\Big|\int \hat{b}\big(\frac{\mathbf{v}}{l_x}\big)e^{i\alpha(\mathbf{v}\cdot\mathbf{y}-\frac{|\mathbf{v}|^2}{2})}d\mathbf{v}\Big|^2\,.$$

The asymptotic regime $\beta \gg 1$ corresponds to strong forward scattering. The analysis of this regime requires to distinguish the cases in which $\check{\mathcal{C}}_0(\boldsymbol{s})$ is smooth at $\mathbf{0}$ or not.

Let us first consider the case in which $\check{\mathcal{C}}_0(\boldsymbol{s})$ is twice differentiable at $\mathbf{0}$ and can be expanded as $\check{\mathcal{C}}_0(\boldsymbol{s}) \simeq \check{\mathcal{C}}_0(\mathbf{0}) - \frac{1}{2}\check{\mathcal{C}}_0''|\boldsymbol{s}|^2 + o(|\boldsymbol{s}|^2)$, with $\check{\mathcal{C}}_0'' > 0$. We obtain that the transmitted wave has a Gaussian profile

$$\lim_{\beta\to\infty}\lim_{\alpha\to\infty} \alpha^d\beta^{d/2}\mathcal{I}(\alpha\beta^{1/2}\mathbf{y}) = \frac{1}{[\check{\mathcal{C}}_0'']^{d/2}}\mathcal{Q}\Big(\frac{\mathbf{y}}{[\check{\mathcal{C}}_0'']^{1/2}}\Big)\,, \qquad \mathcal{Q}(\mathbf{y}) = \frac{3^{d/2}}{(2\pi)^{d/2}}e^{-\frac{3|\mathbf{y}|^2}{2}}\,.$$

The profile $\mathcal{Q}$ is such that $\int \mathcal{Q}(\mathbf{y})d\mathbf{y} = 1$. The beam width is therefore of the order of $\alpha\beta^{1/2}l_x$, which shows that it is proportional to $L^{3/2}$ in the physical variables. This $L^{3/2}$-scaling was first obtained in the physical literature in Ref. [8] and confirmed mathematically in Ref. [7].

Let us now consider the case in which $\check{\mathcal{C}}_0(\boldsymbol{s})$ is not smooth at $\mathbf{0}$ and has the form $\check{\mathcal{C}}_0(\boldsymbol{s}) = \check{\mathcal{C}}_0(\mathbf{0}) - \check{\mathcal{C}}_0'|\boldsymbol{s}| + o(|\boldsymbol{s}|)$, with $\check{\mathcal{C}}_0' > 0$. We find that

$$\lim_{\beta\to\infty}\lim_{\alpha\to\infty} \alpha^d\beta^{d}\mathcal{I}(\alpha\beta\mathbf{y}) = \frac{1}{[\check{\mathcal{C}}_0']^{d}}\mathcal{Q}\Big(\frac{\mathbf{y}}{\check{\mathcal{C}}_0'}\Big)\,, \qquad \mathcal{Q}(\mathbf{y}) = \frac{2^d}{d\pi}\frac{1}{\big(1+4|\mathbf{y}|^2\big)^{\frac{d+1}{2}}}\,.$$

The profile $\mathcal{Q}$ is such that $\int \mathcal{Q}(\mathbf{y})d\mathbf{y} = 1$. Therefore, if the random medium is rough, then the transmitted beam has a large width, of the order of $\alpha\beta l_x$ (proportional to L^2), and it has a heavy tail decaying as $|\mathbf{y}|^{-1-d}$. These results are in agreement with those reported in [7], and they can be contrasted with the ones obtained in the case of a smooth random medium.

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A Derivation of generalized transport equations

We introduce the family of products of $\hat{\boldsymbol{\mathcal{R}}}^\varepsilon$:

$$U^\varepsilon_{\mathbf{p},\mathbf{q}}(k,h,z) = \prod_{j=1}^{n_\mathbf{p}} \hat{\boldsymbol{\mathcal{R}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}'_p(j)\Big) \prod_{l=1}^{n_\mathbf{q}} \overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}'_q(l)\Big)\,.$$

It now follows from (17) that $U^\varepsilon_{\mathbf{p},\mathbf{q}}$ satisfies an evolution equation of the form

$$\frac{\partial U^\varepsilon_{\mathbf{p},\mathbf{q}}}{\partial z} = \mathcal{H}^\varepsilon(U^\varepsilon)_{\mathbf{p},\mathbf{q}}\,, \tag{69}$$

with the initial conditions $U^\varepsilon_{\mathbf{p},\mathbf{q}}(k,\tau,z=0) = \mathbf{1}_0(n_\mathbf{p})\mathbf{1}_0(n_\mathbf{q})$. Here $\mathcal{H}^\varepsilon\left(U^\varepsilon\right)_{\mathbf{p},\mathbf{q}}$ is a finite sum of integral operators acting on $U^\varepsilon_{\mathbf{p}^{(1)},\mathbf{q}^{(1)}},\ldots,U^\varepsilon_{\mathbf{p}^{(m)},\mathbf{q}^{(m)}}$ where the index sets $\mathbf{p}^{(1)},\mathbf{q}^{(1)},\ldots,\mathbf{p}^{(m)},\mathbf{q}^{(m)}$ are obtained from $\mathbf{p}$ and $\mathbf{q}$ by one or two replacements. We have explicitly

$$\begin{aligned}
\mathcal{H}^\varepsilon(U^\varepsilon)_{\mathbf{p},\mathbf{q}}(k,h,z) &= e^{-i(2kz/\varepsilon^2+hz)} \sum_{j=1}^{n_\mathbf{p}} U^\varepsilon_{\mathbf{p}|j,\mathbf{q}} \hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}'_p(j)\Big) \\
&+ e^{i(2kz/\varepsilon^2+hz)} \sum_{j=1}^{n_\mathbf{p}} \iint U^\varepsilon_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1),(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}} \hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big)\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 \\
&+ \sum_{j=1}^{n_\mathbf{p}} \int \Big\{ U^\varepsilon_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}} \hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_p(j), \boldsymbol{\kappa}_1\Big) \\
&\qquad + U^\varepsilon_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1)\},\mathbf{q}} \hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}'_p(j)\Big)\Big\}\, d\boldsymbol{\kappa}_1 \\
&+ e^{i(2kz/\varepsilon^2-hz)} \sum_{l=1}^{n_\mathbf{q}} U^\varepsilon_{\mathbf{p},\mathbf{q}|l} \overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}'_q(l)\Big) \\
&+ e^{-i(2kz/\varepsilon^2-hz)} \sum_{l=1}^{n_\mathbf{q}} \iint U^\varepsilon_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1),(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_q(l))\}} \overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big)\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 \\
&+ \sum_{l=1}^{n_\mathbf{q}} \int \Big\{ U^\varepsilon_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_q(l))\}} \overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_q(l), \boldsymbol{\kappa}_1\Big) \\
&\qquad + U^\varepsilon_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1)\}} \overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}'_q(l)\Big)\Big\}\, d\boldsymbol{\kappa}_1\,.
\end{aligned} \tag{70}$$

A.1 The homogeneous propagator equations

In order to eliminate the h-dependence in the coefficients of (70) we introduce the Fourier transform $V^\varepsilon_{\mathbf{p},\mathbf{q}}(k,\tau,z)$ of $U^\varepsilon_{\mathbf{p},\mathbf{q}}(k,h,z)$ defined by

$$V^\varepsilon_{\mathbf{p},\mathbf{q}}(k,\tau,z) = \frac{1}{2\pi}\int e^{-ih(\tau-(n_\mathbf{p}+n_\mathbf{q})z)} U^\varepsilon_{\mathbf{p},\mathbf{q}}(k,\tau,z)\, dh\,, \tag{71}$$

and observe that

$$\hat{\mathcal{L}}^\varepsilon\Big(k \pm \frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big) \simeq \hat{\mathcal{L}}^\varepsilon(k, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)\,,$$

as $\varepsilon \to 0$. We then find to leading order

$$\frac{\partial V^\varepsilon_{\mathbf{p},\mathbf{q}}}{\partial z} + (n_\mathbf{p} + n_\mathbf{q})\frac{\partial V^\varepsilon_{\mathbf{p},\mathbf{q}}}{\partial \tau} = \mathcal{H}^\varepsilon_V(V^\varepsilon)_{\mathbf{p},\mathbf{q}}\,, \tag{72}$$

with the initial conditions $V^\varepsilon_{\mathbf{p},\mathbf{q}}(k,\tau,z=0) = \mathbf{1}_0(n_\mathbf{p})\mathbf{1}_0(n_\mathbf{q})\delta(\tau)$. Here $\mathcal{H}^\varepsilon_V\left(V^\varepsilon\right)_{\mathbf{p},\mathbf{q}}$ is a finite sum of integral operators acting on $V^\varepsilon_{\mathbf{p}^{(1)},\mathbf{q}^{(1)}}, \ldots, V^\varepsilon_{\mathbf{p}^{(m)},\mathbf{q}^{(m)}}$ where the index sets $\mathbf{p}^{(1)}, \mathbf{q}^{(1)}, \ldots, \mathbf{p}^{(m)}, \mathbf{q}^{(m)}$ are obtained from $\mathbf{p}$ and $\mathbf{q}$ by one or two replacements. We have explicitly

$$\begin{aligned}
\mathcal{H}^\varepsilon_V(V^\varepsilon)_{\mathbf{p},\mathbf{q}}(z,k,\tau) &= e^{-i2kz/\varepsilon^2}\sum_{j=1}^{n_\mathbf{p}} V^\varepsilon_{\mathbf{p}|j,\mathbf{q}}\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)) \\
&+ e^{i2kz/\varepsilon^2}\sum_{j=1}^{n_\mathbf{p}}\iint V^\varepsilon_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1),(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}}\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 \\
&+ \sum_{j=1}^{n_\mathbf{p}}\int\Big\{V^\varepsilon_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}}\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1) \\
&\qquad + V^\varepsilon_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1)\},\mathbf{q}}\hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_p(j))\Big\}\,d\boldsymbol{\kappa}_1 \\
&+ e^{i2kz/\varepsilon^2}\sum_{l=1}^{n_\mathbf{q}} V^\varepsilon_{\mathbf{p},\mathbf{q}|l}\overline{\hat{\mathcal{L}}^\varepsilon}(k,z,\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)) \\
&+ e^{-i2kz/\varepsilon^2}\sum_{l=1}^{n_\mathbf{q}}\iint V^\varepsilon_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1),(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_q(l))\}}\overline{\hat{\mathcal{L}}^\varepsilon}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 \\
&+ \sum_{l=1}^{n_\mathbf{q}}\int\Big\{V^\varepsilon_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_q(l))\}}\overline{\hat{\mathcal{L}}^\varepsilon}(k,z,\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1) \\
&\qquad + V^\varepsilon_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1)\}}\overline{\hat{\mathcal{L}}^\varepsilon}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_q(l))\Big\}\,d\boldsymbol{\kappa}_1\,.
\end{aligned} \tag{73}$$

A.2 Transport Equations

We next apply the diffusion approximation to get transport equations for the moments (see [9] for background material on and related applications of the diffusion approximation). Observe that the function $\mathcal{H}^\varepsilon_V$ is linear and the random coefficients are rapidly fluctuating in view of (18) and (17). The coefficients of order ε^{-1} are centered and fluctuate on the scale ε^2, moreover they are assumed to be rapidly mixing, giving a white-noise scaling situation. We can thus apply diffusion approximation results to obtain transport equations for the moments $\mathbb{E}[V^\varepsilon_{\mathbf{p},\mathbf{q}}]$ in the limit $\varepsilon \to 0$:

$$W_{\mathbf{p},\mathbf{q}}(k,\tau,z) = \lim_{\varepsilon\to 0}\mathbb{E}\big[V^\varepsilon_{\mathbf{p},\mathbf{q}}(k,\tau,z)\big]\,.$$

We then obtain from (72) that $W_{\mathbf{p},\mathbf{q}}$ solves the infinite-dimensional system of partial differential equations

$$\frac{\partial W_{\mathbf{p},\mathbf{q}}}{\partial z} + (n_\mathbf{p} + n_\mathbf{q})\frac{\partial W_{\mathbf{p},\mathbf{q}}}{\partial \tau} = \frac{i}{2k}\Phi_{\mathbf{p},\mathbf{q}}W_{\mathbf{p},\mathbf{q}} + \mathcal{H}_W\left(W\right)_{\mathbf{p},\mathbf{q}}\,, \tag{74}$$

with the initial conditions $W_{\mathbf{p},\mathbf{q}}(k,\tau,z=0) = \mathbf{1}_0(n_\mathbf{p})\mathbf{1}_0(n_\mathbf{q})\delta(\tau)$ and $\Phi_{\mathbf{p},\mathbf{q}}$ defined by (27). The first term to the right in (74) is the contributions of the scattering terms in (18). The source term has the form

$$\mathcal{H}_W\,(W)_{\mathbf{p},\mathbf{q}} = \sum_{k=1}^{6} \mathbf{I}_k\,, \tag{75}$$

and we next identify the coupling terms $\mathbf{I}_k$. We remark that in applying the diffusion approximation there is coupling only between terms whose rapid phase modulations $\exp[\pm 2ikz/\varepsilon^2]$ compensate each other.

There are 8 terms in the expression for $\mathcal{H}_V^\varepsilon$ in (73), we label the first four terms associated with the index set $\mathbf{p}$ by $1_p,\ldots,4_p$. The last four terms associated with the index set $\mathbf{q}$, are labeled by $1_q,\ldots,4_q$. First, we consider the cross interaction of the terms 1_p and 2_p and also the corresponding combination 1_q and 2_q that is associated with complex conjugate coefficients. We label their contribution by the term $\mathbf{I}_1$ which is given by

$$\begin{aligned}\mathbf{I}_1 = &-\frac{k^2}{4(2\pi)^d}\Big\{ n_{\mathbf{p}}\int \widehat{C}^{+}(2k,\boldsymbol{\kappa})\,d\boldsymbol{\kappa}\,W_{\mathbf{p},\mathbf{q}} + \sum_{j_1\neq j_2=1}^{n_{\mathbf{p}}} \widehat{C}(2k,\boldsymbol{\kappa}_p(j_1)-\boldsymbol{\kappa}_p'(j_1))\\ &\times\int W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_2),\boldsymbol{\kappa}-\boldsymbol{\kappa}_p(j_1)),(\boldsymbol{\kappa}-\boldsymbol{\kappa}_p'(j_1),\boldsymbol{\kappa}_p'(j_2))\},\mathbf{q}}\,d\boldsymbol{\kappa}\\ &+n_{\mathbf{q}}\int \widehat{C}^{-}(2k,\boldsymbol{\kappa})\,d\boldsymbol{\kappa}\,W_{\mathbf{p},\mathbf{q}} + \sum_{l_1\neq l_2=1}^{n_{\mathbf{q}}} \widehat{C}(2k,\boldsymbol{\kappa}_q(l_1)-\boldsymbol{\kappa}_q'(l_1))\\ &\times\int W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_2),\boldsymbol{\kappa}-\boldsymbol{\kappa}_q(l_1)),(\boldsymbol{\kappa}-\boldsymbol{\kappa}_q'(l_1),\boldsymbol{\kappa}_q'(l_2))\}}\,d\boldsymbol{\kappa}\Big\}\,,\end{aligned}$$

where the autocorrelation function of the fluctuations and its Fourier transform are defined by (22-24). Next, we consider the cross interaction of the terms 1_p and 2_p with the terms 1_q and 2_q. We label their contribution by the term $\mathbf{I}_2$ which is given by

$$\begin{aligned}\mathbf{I}_2 = &\frac{k^2}{4(2\pi)^d}\Big\{\sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}} \widehat{C}(2k,\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}_p'(j))\delta(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}_p'(j)-\boldsymbol{\kappa}_q(l)+\boldsymbol{\kappa}_q'(l))W_{\mathbf{p}|j,\mathbf{q}|l}\\ &+\sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}}\iiint \hat{C}(2k,\boldsymbol{\kappa}_1)\\ &\times W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_2),(\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_p'(j))\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_3),(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_q'(l))\}}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3\Big\}\,.\end{aligned}$$

We have completed the analysis of the terms associated with phase modulation of the form $\exp[\pm 2ikz/\varepsilon^2]$ and consider now the terms without a fast phase modulation. Consider first the interaction of the terms 3_p, 4_p 3_q and 4_q with themselves. We label this contribution by $\mathbf{I}_3$, it is given by

$$\begin{aligned}\mathbf{I}_3 = &-\frac{k^2}{4(2\pi)^d}\Big\{ n_{\mathbf{p}}\int \widehat{C}(0,\boldsymbol{\kappa})\,d\boldsymbol{\kappa}\,W_{\mathbf{p},\mathbf{q}}\\ &+\frac{1}{2}\sum_{j_1\neq j_2=1}^{n_{\mathbf{p}}}\int \widehat{C}(0,\boldsymbol{\kappa})W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}_p'(j_1)),(\boldsymbol{\kappa}_p(j_2)+\boldsymbol{\kappa},\boldsymbol{\kappa}_p'(j_2))\},\mathbf{q}}\,d\boldsymbol{\kappa}\\ &+\frac{1}{2}\sum_{j_1\neq j_2=1}^{n_{\mathbf{p}}}\int \widehat{C}(0,\boldsymbol{\kappa})W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_1),\boldsymbol{\kappa}_p'(j_1)-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_p(j_2),\boldsymbol{\kappa}_p'(j_2)+\boldsymbol{\kappa})\},\mathbf{q}}\,d\boldsymbol{\kappa}\\ &+n_{\mathbf{q}}\int \widehat{C}(0,\boldsymbol{\kappa})\,d\boldsymbol{\kappa}\,W_{\mathbf{p},\mathbf{q}}\\ &+\frac{1}{2}\sum_{l_1\neq l_2=1}^{n_{\mathbf{q}}}\int \widehat{C}(0,\boldsymbol{\kappa})W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}_q'(l_1)),(\boldsymbol{\kappa}_q(l_2)+\boldsymbol{\kappa},\boldsymbol{\kappa}_q'(l_2))\}}\,d\boldsymbol{\kappa}\\ &+\frac{1}{2}\sum_{l_1\neq l_2=1}^{n_{\mathbf{q}}}\int \widehat{C}(0,\boldsymbol{\kappa})W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_1),\boldsymbol{\kappa}_q'(l_1)-\boldsymbol{\kappa}),(\boldsymbol{\kappa}_q(l_2),\boldsymbol{\kappa}_q'(l_2)+\boldsymbol{\kappa})\}}\,d\boldsymbol{\kappa}\Big\}\,.\end{aligned}$$

Next, we deal with the cross interaction between the terms 3_p, 4_p and correspondingly between 3_q and 4_q. We label this contribution by $\mathbf{I}_4$ and obtain

$$\begin{aligned}\mathbf{I}_4 = -\frac{k^2}{4(2\pi)^d}\Big\{ &\sum_{j=1}^{n_{\mathbf{p}}} \int \widehat{C}(0,\boldsymbol{\kappa})\, W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa})\},\mathbf{q}}\, d\boldsymbol{\kappa} \\ &+ \sum_{j_1\neq j_2=1}^{n_{\mathbf{p}}} \int \widehat{C}(0,\boldsymbol{\kappa}) W_{\mathbf{p}|\{j_1,j_2|(\boldsymbol{\kappa}_p(j_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j_1)),(\boldsymbol{\kappa}_p(j_2),\boldsymbol{\kappa}'_p(j_2)-\boldsymbol{\kappa})\},\mathbf{q}}\, d\boldsymbol{\kappa} \\ &+ \sum_{l=1}^{n_{\mathbf{q}}} \int \widehat{C}(0,\boldsymbol{\kappa})\, W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l)-\boldsymbol{\kappa})\}}\, d\boldsymbol{\kappa} \\ &+ \sum_{l_1\neq l_2=1}^{n_{\mathbf{q}}} \int \widehat{C}(0,\boldsymbol{\kappa}) W_{\mathbf{p},\mathbf{q}|\{l_1,l_2|(\boldsymbol{\kappa}_q(l_1)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l_1)),(\boldsymbol{\kappa}_q(l_2),\boldsymbol{\kappa}'_q(l_2)-\boldsymbol{\kappa})\}}\, d\boldsymbol{\kappa}\Big\}.\end{aligned}$$

Now we consider the cross interaction between the terms 3_p, 3_q and correspondingly between 4_p and 4_q. We label this contribution by $\mathbf{I}_5$ and obtain

$$\begin{aligned}\mathbf{I}_5 = \frac{k^2}{4(2\pi)^d}\Big\{ &\sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}} \int \widehat{C}(0,\boldsymbol{\kappa})\, W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j))\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_{ql}(l)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l))\}}\, d\boldsymbol{\kappa} \\ &+ \sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}} \int \widehat{C}(0,\boldsymbol{\kappa})\, W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa})\},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)-\boldsymbol{\kappa})\}}\, d\boldsymbol{\kappa}\Big\}.\end{aligned}$$

Finally, we analyze the cross interaction between the terms 3_p, 4_q and correspondingly between 4_p and 3_q. We label this contribution by $\mathbf{I}_6$ and obtain

$$\begin{aligned}\mathbf{I}_6 = \frac{k^2}{4(2\pi)^d}\Big\{ &\sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}} \int \widehat{C}(0,\boldsymbol{\kappa})\, W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}'_p(j)),\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_q(l)+\boldsymbol{\kappa})}\, d\boldsymbol{\kappa} \\ &+ \sum_{j=1}^{n_{\mathbf{p}}}\sum_{l=1}^{n_{\mathbf{q}}} \int \widehat{C}(0,\boldsymbol{\kappa})\, W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_p(j)-\boldsymbol{\kappa}),\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)+\boldsymbol{\kappa},\boldsymbol{\kappa}'_q(l))}\, d\boldsymbol{\kappa}\Big\}.\end{aligned}$$

Finally, we substitute the expressions for $I_1,\ldots,I_6$ in (75) to obtain the transport equations (26).

B Derivation of transmission transport equations

We consider next the wave field that is transmitted through the random medium and develop a family of transport equations that generalizes the one we derived above for the characterization of the reflected field. The transmitted field can be characterized by the transmission operator as shown in (15) and the transmission and reflection operators solve (17). In order to obtain a closed system of transport equations we introduce the quantities

$$U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}(k,h,z;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b) = \hat{\boldsymbol{\mathcal{T}}}^{\varepsilon}\Big(k+\frac{\varepsilon^2 h}{2},z,\boldsymbol{\kappa}_a,\boldsymbol{\kappa}'_a\Big)\overline{\hat{\boldsymbol{\mathcal{T}}}^{\varepsilon}}\Big(k-\frac{\varepsilon^2 h}{2},z,\boldsymbol{\kappa}_b,\boldsymbol{\kappa}'_b\Big)U^{\varepsilon}_{\mathbf{p},\mathbf{q}}(k,h,z)\,.$$

Then we find, using (69),

$$
\begin{aligned}
&\frac{\partial U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z} = \mathcal{H}^\varepsilon(U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon})_{\mathbf{p},\mathbf{q}} + U^\varepsilon_{\mathbf{p},\mathbf{q}}\overline{\hat{\boldsymbol{\mathcal{T}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_b, \boldsymbol{\kappa}'_b\Big)\\
&\Big\{\int \hat{\boldsymbol{\mathcal{T}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_1\Big)\hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}'_a\Big)\, d\boldsymbol{\kappa}_1 + e^{i(2kz/\varepsilon^2+hz)}\\
&\times \iint \hat{\boldsymbol{\mathcal{T}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_1\Big)\hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big)\hat{\boldsymbol{\mathcal{R}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}'_a\Big)\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\Big\}\\
&+U^\varepsilon_{\mathbf{p},\mathbf{q}}\hat{\boldsymbol{\mathcal{T}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_a, \boldsymbol{\kappa}'_a\Big)\Big\{\int \overline{\hat{\boldsymbol{\mathcal{T}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_b, \boldsymbol{\kappa}_1\Big)\overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}'_b\Big)\, d\boldsymbol{\kappa}_1\\
&+e^{-i(2kz/\varepsilon^2-hz)}\iint \overline{\hat{\boldsymbol{\mathcal{T}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_b, \boldsymbol{\kappa}_1\Big)\overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big)\\
&\times \overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}'_b\Big)\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\Big\},
\end{aligned}
$$

with $\mathcal{H}^\varepsilon$ defined in (70). We remark that the family of coefficients $U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}(k,h,z;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b)$ for fixed $\boldsymbol{\kappa}_a$ and $\boldsymbol{\kappa}_b$ form a closed sub-family, which allows us to rewrite the previous system as

$$
\frac{\partial U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z} = \mathcal{H}^\varepsilon(U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon})_{\mathbf{p},\mathbf{q}} + \Delta_1\mathcal{H}^\varepsilon(U^\varepsilon)^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}_{\mathbf{p},\mathbf{q}} + \Delta_2\mathcal{H}^\varepsilon(U^\varepsilon)^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}_{\mathbf{p},\mathbf{q}}, \tag{76}
$$

for

$$
\begin{aligned}
&\Delta_1\mathcal{H}^\varepsilon(U^\varepsilon)^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}_{\mathbf{p},\mathbf{q}} = \int \hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}'_a\Big)U^{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}\, d\boldsymbol{\kappa}_1\\
&\quad+\int \overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}'_b\Big)U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}_1),\varepsilon}_{\mathbf{p},\mathbf{q}}\, d\boldsymbol{\kappa}_1\\
&\Delta_2\mathcal{H}^\varepsilon(U^\varepsilon)^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}_{\mathbf{p},\mathbf{q}}\\
&\quad= e^{i(2kz/\varepsilon^2+hz)}\iint \hat{\boldsymbol{\mathcal{R}}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}'_a\Big)\hat{\mathcal{L}}^\varepsilon\Big(k+\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big)U^{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\\
&\quad+e^{-i(2kz/\varepsilon^2-hz)}\iint \overline{\hat{\boldsymbol{\mathcal{R}}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}'_b\Big)\overline{\hat{\mathcal{L}}^\varepsilon}\Big(k-\frac{\varepsilon^2 h}{2}, z, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\Big)U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}_1),\varepsilon}_{\mathbf{p},\mathbf{q}}\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\,.
\end{aligned}
$$

B.1 Homogeneous propagator equations in the transmission case

In order to eliminate the h-dependence in the coefficients of (76) we introduce the transformation

$$
V^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}(k,\tau,z) = \frac{1}{2\pi}\int e^{-ih(\tau-(n_{\mathbf{p}}+n_{\mathbf{q}}+1)z)}U^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}(k,h,z)\, dh\,. \tag{77}
$$

We then obtain from (76) that $V^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}$ solves the infinite-dimensional system of partial differential equations

$$
\frac{\partial V^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z} + (n_{\mathbf{p}}+n_{\mathbf{q}}+1)\frac{\partial V^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial \tau} = \widetilde{\mathcal{H}}^\varepsilon_V(V^\varepsilon)^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}_{\mathbf{p},\mathbf{q}}, \tag{78}
$$

with the initial conditions $V^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b),\varepsilon}_{\mathbf{p},\mathbf{q}}(k,\tau,z=0;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b) = \mathbf{1}_0(n_{\mathbf{p}})\mathbf{1}_0(n_{\mathbf{q}})\delta(\boldsymbol{\kappa}_a-\boldsymbol{\kappa}'_a)\delta(\boldsymbol{\kappa}_b-\boldsymbol{\kappa}'_b)\delta(\tau)$. We decompose the source term as

$$
\widetilde{\mathcal{H}}^\varepsilon_V = \mathcal{H}^\varepsilon_V + \Delta_1\mathcal{H}^\varepsilon_V + \Delta_2\mathcal{H}^\varepsilon_V\,, \tag{79}
$$

with $\mathcal{H}_V^\varepsilon$ defined in (73) and where the transmission specific source terms are

$$\begin{aligned}\Delta_1\mathcal{H}_V^\varepsilon(V^\varepsilon)_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} &= \int \hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_a')V_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_b'),\varepsilon}\,d\boldsymbol{\kappa}_1\\ &+\int \overline{\hat{\mathcal{L}}^\varepsilon}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_b')V_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_1),\varepsilon}\,d\boldsymbol{\kappa}_1\,, \qquad (80)\\ \Delta_2\mathcal{H}_V^\varepsilon(V^\varepsilon)_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} &= e^{i2kz/\varepsilon^2}\iint \hat{\mathcal{L}}^\varepsilon(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)V_{\mathbf{p}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_a'),\mathbf{q}}^{(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_b'),\varepsilon}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\\ &+e^{-i2kz/\varepsilon^2}\iint \overline{\hat{\mathcal{L}}^\varepsilon}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2)V_{\mathbf{p},\mathbf{q}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_b')}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_1),\varepsilon}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2\,. \qquad (81)\end{aligned}$$

B.2 Transport equations

We now apply the diffusion approximation to get transport equations for the moments that are relevant in the transmission case. That is, we deduce transport equations for the moments $\mathbb{E}[V_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'),\varepsilon}]$ in the limit $\varepsilon \to 0$:

$$W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}(k,\tau,z;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b) = \lim_{\varepsilon\to 0}\mathbb{E}[V_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'),\varepsilon}(k,\tau,z;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b)]\,.$$

We obtain from (78) that $W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}$ solves the infinite-dimensional system of partial differential equations

$$\begin{aligned}\frac{\partial W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}}{\partial z} + (n_{\mathbf{p}}+n_{\mathbf{q}}+1)\frac{\partial W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}}{\partial \tau} &= \frac{i}{2k}\Phi_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} + \mathcal{H}_W(W^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')})_{\mathbf{p},\mathbf{q}}\\ &+\Delta\mathcal{H}_W(W)_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}\,, \qquad (82)\end{aligned}$$

with the initial conditions $W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}(k,\tau,z=0;\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b) = \mathbf{1}_0(n_{\mathbf{p}})\mathbf{1}_0(n_{\mathbf{q}})\delta(\boldsymbol{\kappa}_a-\boldsymbol{\kappa}_a')\delta(\boldsymbol{\kappa}_b-\boldsymbol{\kappa}_b')\delta(\tau)$. Here $\Phi_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')}$ is defined by (64), the source term $\mathcal{H}_W$ is defined in (75) and the specific transmission source term has the form

$$\Delta\mathcal{H}_W\,(W)_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} = \sum_{k=1}^{4}\tilde{\mathbf{I}}_k\,, \qquad (83)$$

and we next identify the coupling terms $\tilde{\mathbf{I}}_k$.

First, we consider the terms that correspond to the interaction of the terms $\Delta_1\mathcal{H}_V^\varepsilon$ in (80) with themselves. This contribution is

$$\tilde{\mathbf{I}}_1 = \frac{k^2}{4(2\pi)^d}\left\{-\int \widehat{C}(0,\boldsymbol{\kappa})\,d\boldsymbol{\kappa}\,W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b')} + \int \widehat{C}(0,\boldsymbol{\kappa})\,W_{\mathbf{p},\mathbf{q}}^{(\boldsymbol{\kappa}_a'+\boldsymbol{\kappa},\boldsymbol{\kappa}_b'+\boldsymbol{\kappa})}\,d\boldsymbol{\kappa}\right\}.$$

Then, we consider the cross interaction of the terms in $\Delta_2\mathcal{H}_V^\varepsilon$ in (81). This gives the contribution

$$\tilde{\mathbf{I}}_2 = \frac{k^2}{4(2\pi)^d}\iiint \widehat{C}(2k,\boldsymbol{\kappa}_1)\,W_{\mathbf{p}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_a'),\mathbf{q}\cup(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}_b')}^{(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3)}\,d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3\,.$$

The terms in $\Delta_1\mathcal{H}_V^\varepsilon$ interact with those in $\mathcal{H}_V^\varepsilon$ having no phase modulation and give the following contribution to the diffusion approximation

$$\begin{aligned}\tilde{\mathbf{I}}_3 = \frac{k^2}{4(2\pi)^d}\Big\{\int \widehat{C}(0,\boldsymbol{\kappa})\Big[\sum_{j=1}^{n_{\mathbf{p}}}\Big(&-W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)+\boldsymbol{\kappa},\boldsymbol{\kappa}_p'(j))\},\mathbf{q}}^{(\boldsymbol{\kappa}_a'+\boldsymbol{\kappa},\boldsymbol{\kappa}_b')} - W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_p'(j)-\boldsymbol{\kappa})\},\mathbf{q}}^{(\boldsymbol{\kappa}_a'+\boldsymbol{\kappa},\boldsymbol{\kappa}_b')}\\ &+W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa},\boldsymbol{\kappa}_p'(j))\},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'+\boldsymbol{\kappa})} + W_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_p'(j)+\boldsymbol{\kappa})\},\mathbf{q}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'+\boldsymbol{\kappa})}\Big)\\ &+\sum_{l=1}^{n_{\mathbf{q}}}\Big(W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa},\boldsymbol{\kappa}_q'(l))\}}^{(\boldsymbol{\kappa}_a'+\boldsymbol{\kappa},\boldsymbol{\kappa}_b')} + W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_q'(l)+\boldsymbol{\kappa})\}}^{(\boldsymbol{\kappa}_a'+\boldsymbol{\kappa},\boldsymbol{\kappa}_b')}\\ &-W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l)+\boldsymbol{\kappa},\boldsymbol{\kappa}_q'(l))\}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'+\boldsymbol{\kappa})} - W_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_q'(l)-\boldsymbol{\kappa})\}}^{(\boldsymbol{\kappa}_a',\boldsymbol{\kappa}_b'+\boldsymbol{\kappa})}\Big)\Big]d\boldsymbol{\kappa}\Big\}\,.\end{aligned}$$

Finally, we consider the cross interaction of the terms in $\Delta_2\mathcal{H}_V^\varepsilon$ with those in $\mathcal{H}_V^\varepsilon$. This gives the contribution

$$\begin{aligned}
\tilde{\mathbf{I}}_4 = \frac{k^2}{4(2\pi)^d}\Big\{ & -\sum_{j=1}^{n_{\mathbf{p}}} \widehat{C}(2k,\boldsymbol{\kappa}_p(j)-\boldsymbol{\kappa}'_p(j)) \int W^{(\boldsymbol{\kappa}+\boldsymbol{\kappa}'_p(j),\boldsymbol{\kappa}'_b)}_{\mathbf{p}|\{j|(\boldsymbol{\kappa}+\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}'_a)\},\mathbf{q}}\, d\boldsymbol{\kappa} \\
& -\sum_{l=1}^{n_{\mathbf{q}}} \widehat{C}(2k,\boldsymbol{\kappa}_q(l)-\boldsymbol{\kappa}'_q(l)) \int W^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}+\boldsymbol{\kappa}'_q(l))}_{\mathbf{p},\mathbf{q}|\{l|(\boldsymbol{\kappa}+\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}'_b)\}}\, d\boldsymbol{\kappa} - \int \widehat{C}(2k,\boldsymbol{\kappa})\, d\boldsymbol{\kappa}\, W^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}'_b)}_{\mathbf{p},\mathbf{q}} \\
& +\sum_{j=1}^{n_{\mathbf{p}}} \iiint \widehat{C}(2k,\boldsymbol{\kappa}_1)\, W^{(\boldsymbol{\kappa}'_a,\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)}_{\mathbf{p}|\{j|(\boldsymbol{\kappa}_p(j),\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}'_p(j))\},\mathbf{q}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_b)}\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 \\
& +\sum_{l=1}^{n_{\mathbf{q}}} \iiint \widehat{C}(2k,\boldsymbol{\kappa}_1)\, W^{(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_b)}_{\mathbf{p}\cup(\boldsymbol{\kappa}_2,\boldsymbol{\kappa}'_a),\mathbf{q}|\{l|(\boldsymbol{\kappa}_q(l),\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_3),(\boldsymbol{\kappa}_3,\boldsymbol{\kappa}'_q(l))\}}\, d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 \Big\}.
\end{aligned}$$

We can now assemble the terms in the source term $\Delta\mathcal{H}_W$ for the transport equation, and this completes the proof of Proposition 8.

C Proofs of technical lemmas

We first give the proof of Lemma 1. The first item follows from Gronwall's lemma. For the proof of the second item, let us consider the set

$$A_\alpha = \Big\{(\mathbf{u},\mathbf{v}) \in \mathbb{R}^{2d} \text{ s.t. } |\mathbf{u}\cdot\mathbf{v}| \geq \alpha^{-1/2},\ |\mathbf{u}| \geq \alpha^{-1/2},\ |\mathbf{v}| \geq \alpha^{-1/2}\Big\}\,.$$

By considering the integral form of the system (35) and the inequality

$$\sup_{(\mathbf{u},\mathbf{v})\in A_\alpha} \Big|\int_0^\zeta e^{i\alpha\mathbf{u}\cdot\mathbf{v}\zeta'}d\zeta'\Big| = \sup_{(\mathbf{u},\mathbf{v})\in A_\alpha}\Big|\frac{e^{i\alpha\mathbf{u}\cdot\mathbf{v}\zeta}-1}{\alpha\mathbf{u}\cdot\mathbf{v}}\Big| \leq \frac{2}{\sqrt{\alpha}}\,,$$

we obtain the estimate

$$\begin{aligned}
\sup_{(\mathbf{u},\mathbf{v})\in A_\alpha,\mathbf{w}\in\mathbb{R}^d} |\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})| \leq \frac{2}{\sqrt{\alpha}} &+ C\int_0^\zeta \sup_{(\mathbf{u},\mathbf{v})\in A_\alpha,\mathbf{w}\in\mathbb{R}^d}|\mathcal{D}(\zeta',\mathbf{u},\mathbf{v},\mathbf{w})|d\zeta' \\
&+C\int |\check{\mathcal{C}}_0(\boldsymbol{\mu})|\Big[\mathbf{1}_{A^c_\alpha}(\mathbf{u}+\boldsymbol{\mu},\mathbf{v}) + \mathbf{1}_{A^c_\alpha}(\mathbf{u},\mathbf{v}+\boldsymbol{\mu}) \\
&\quad +\mathbf{1}_{A^c_\alpha}(\mathbf{u}+\boldsymbol{\mu},\mathbf{v}+\boldsymbol{\mu}) + \mathbf{1}_{A^c_\alpha}(\mathbf{u}-\boldsymbol{\mu},\mathbf{v}+\boldsymbol{\mu})\Big]\, d\boldsymbol{\mu}\,.
\end{aligned}$$

Using the fact $\check{\mathcal{C}}_0(\boldsymbol{\mu})$ is integrable and applying the dominated convergence theorem, the last term of the right-hand side converges to 0 as $\alpha\to\infty$ since the indicator functions converge to zero almost surely with respect to the Lebesgue measure over $\mathbb{R}^d$. Therefore, applying Gronwall's lemma, we get

$$\lim_{\alpha\to\infty} \sup_{(\mathbf{u},\mathbf{v})\in A_\alpha,\mathbf{w}\in\mathbb{R}^d,\zeta\in[0,1]} |\mathcal{D}(\zeta,\mathbf{u},\mathbf{v},\mathbf{w})| = 0\,.$$

If we consider a fixed pair $(\mathbf{u},\mathbf{v})\in\mathbb{R}^{2d}$ such that $\mathbf{u}\cdot\mathbf{v}\neq 0$, then $(\mathbf{u},\mathbf{v})\in A_\alpha$ for α large enough, which proves the second item of the lemma.

We now consider the third point of the lemma. Let us consider a pair $(\mathbf{u},\mathbf{v})\in\mathbb{R}^{2d}$ such that $\mathbf{u}\cdot\mathbf{v}=0$, $\mathbf{u}\neq\mathbf{0}$, and $\mathbf{v}\neq\mathbf{0}$. In this case, $(\mathbf{u}-\boldsymbol{\mu})\cdot\mathbf{v} = -\boldsymbol{\mu}\cdot\mathbf{v}\neq 0$ for almost every $\boldsymbol{\mu}$ (with respect to the Lebesgue measure over $\mathbb{R}^d$). Therefore, by using the dominated convergence theorem and the second item of the lemma, we obtain

$$\Big|\int \widehat{\mathcal{C}}_0(\boldsymbol{\mu})e^{i\alpha\boldsymbol{\mu}\cdot\mathbf{v}\zeta}\mathcal{D}(\zeta,\mathbf{u}-\boldsymbol{\mu},\mathbf{v},\mathbf{w}+\boldsymbol{\mu})d\boldsymbol{\mu}\Big| \leq \int |\widehat{\mathcal{C}}_0(\boldsymbol{\mu})||\mathcal{D}(\zeta,\mathbf{u}-\boldsymbol{\mu},\mathbf{v},\mathbf{w}+\boldsymbol{\mu})|d\boldsymbol{\mu} \stackrel{\alpha\to\infty}{\longrightarrow} 0\,.$$

Using this estimate, we obtain by taking the limit $\alpha \to \infty$ in (35) that $\lim_{\alpha\to\infty} \mathcal{D}(\zeta, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is the solution of

$$\frac{d\mathcal{D}}{d\zeta} = \frac{\widehat{\mathcal{C}}_{2kl_z}(\mathbf{w})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} - 2\beta\check{\mathcal{C}}_0(\mathbf{0})\mathcal{D}\,,$$

starting from $\mathcal{D}(\zeta = 0, \mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$. Solving this differential equation then gives (38). The forth point is obtained by the same strategy. The fifth point of the lemma follows from the reciprocity identity $\mathcal{D}(\zeta, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathcal{D}(\zeta, \mathbf{v}, \mathbf{u}, \mathbf{w})$. The proofs of Lemmas 2 and 4 follow the same lines.

D Some useful identities in the regime $\alpha \gg 1$

By integrating in $\mathbf{w}$ the expression (41) of $\mathcal{D}_{\boldsymbol{s}}$ we find identities that are used in Section 4:

$$\frac{1}{(2\pi)^d}\int \mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})d\mathbf{w} = \zeta\,, \tag{84}$$

$$\frac{1}{(2\pi)^d}\int \mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w})d\mathbf{w} = \int_0^\zeta e^{i\mathbf{v}\cdot\boldsymbol{s}(\zeta-\zeta')}e^{2\beta\int_0^{\zeta'}\check{\mathcal{C}}_0(\boldsymbol{s}\zeta'')-\check{\mathcal{C}}_0(\mathbf{0})d\zeta''}d\zeta'\,. \tag{85}$$

Moreover, if $\check{\mathcal{C}}_0$ and $\check{\mathcal{C}}_{2kl_z}$ are twice differentiable at $\mathbf{0}$, then we get

$$\frac{1}{(2\pi)^d}\int |\mathbf{w}|^2\mathcal{D}_{\mathbf{0}}(\zeta, \mathbf{w})d\mathbf{w} = -\frac{\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}\zeta - \beta\Delta\check{\mathcal{C}}_0(\mathbf{0})\zeta^2\,, \tag{86}$$

$$\frac{1}{(2\pi)^d}\int \nabla_{\boldsymbol{s}}\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w})\mid_{\boldsymbol{s}=\mathbf{0}} d\mathbf{w} = \frac{i}{2}\mathbf{v}\zeta^2\,, \tag{87}$$

$$\frac{1}{(2\pi)^d}\int \Delta_{\boldsymbol{s}}\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}, \mathbf{w})\mid_{\boldsymbol{s}=\mathbf{0}} d\mathbf{w} = -\frac{1}{3}|\mathbf{v}|^2\zeta^3 + \frac{\beta}{6}\Delta\check{\mathcal{C}}_0(\mathbf{0})\zeta^4\,, \tag{88}$$

$$\frac{1}{(2\pi)^d}\int \Delta_{\boldsymbol{s}}\mathcal{D}_{\boldsymbol{s}}(\zeta, \mathbf{v}+\mathbf{w}, \mathbf{w})\mid_{\boldsymbol{s}=\mathbf{0}} d\mathbf{w} = \frac{1}{3}\Big(\frac{\Delta\check{\mathcal{C}}_{2kl_z}(\mathbf{0})}{\check{\mathcal{C}}_{2kl_z}(\mathbf{0})} - |\mathbf{v}|^2\Big)\zeta^3 + \frac{\beta}{3}\Delta\check{\mathcal{C}}_0(\mathbf{0})\zeta^4\,. \tag{89}$$

If $\check{\mathcal{C}}$ is not twice differentiable, then the four integrals (86-89) diverge.

E Interpretation of the weak backscattering regime in terms of a random mirror

The purpose of this short section is to give an elementary picture of the weak backscattering regime. We first consider the situation in which a wave is incoming from the right half-space (L, ∞) and impinges on a slab of random medium $[L_M, L]$, with $L_M \in [0, L]$. At the plane $z = L_M$ an inhomogeneous mirror is inserted, with the impedance $Z(\mathbf{x})$, so that the boundary condition at $z = L_M$ reads

$$\check{a}^\varepsilon(k, L_M, \mathbf{x}) = R(\mathbf{x})\check{b}^\varepsilon(k, L_M, \mathbf{x})e^{-2ik\frac{L_M}{\varepsilon^2}}\,,$$

where $R(\mathbf{x}) = (Z(\mathbf{x}) - 1)/(Z(\mathbf{x}) + 1)$ is the reflection coefficient of the mirror. If $R(\mathbf{x}) = -1$, then we deal with the standard reflecting boundary condition $\hat{p}^\varepsilon\mid_{z=L_M} = 0$. If $R(\mathbf{x}) \neq -1$, then we deal with the generalized reflection condition $\frac{\partial\hat{p}^\varepsilon}{\partial z} + ik\frac{1-R}{1+R}\hat{p}^\varepsilon\mid_{z=L_M} = 0$. In the following, we consider the case of a random mirror in which $R(\mathbf{x})$ is a zero-mean random stationary process with the autocorrelation function $\psi(\mathbf{x}) = \mathbb{E}[R(\mathbf{x}' + \mathbf{x})R(\mathbf{x}')]$.

In the forward-scattering approximation, the Fourier transform $\hat{a}^\varepsilon(k, L, \boldsymbol{\kappa})$ of the reflected wave $\check{a}^\varepsilon(k, L, \mathbf{x})$ is given by

$$\hat{a}^\varepsilon(k, L, \boldsymbol{\kappa}) = \int \hat{\boldsymbol{\mathcal{R}}}^\varepsilon_{L_M}(k, L, \boldsymbol{\kappa}, \boldsymbol{\kappa}')\hat{b}(k, \boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'\,,$$

where $\hat{b}$ is the Fourier transform of the input beam and $\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}$ is the solution of

$$\frac{d}{dz}\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}') = \int \hat{\mathcal{L}}^{\varepsilon}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}') + \hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}(k,z,\boldsymbol{\kappa},\boldsymbol{\kappa}_1)\hat{\mathcal{L}}^{\varepsilon}(k,z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}')d\boldsymbol{\kappa}_1, \qquad z\in[L_M,L],$$

starting from $\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}(k,z=L_M,\boldsymbol{\kappa},\boldsymbol{\kappa}') = \hat{R}(\boldsymbol{\kappa}-\boldsymbol{\kappa}')e^{-2ikL_M/\varepsilon^2}$. Using a diffusion approximation theorem, we obtain that

$$\frac{1}{2\pi}\int \mathbb{E}\Big[\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}\big(k+\frac{\varepsilon^2 h}{2},z,\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'\big)\overline{\hat{\boldsymbol{\mathcal{R}}}^{\varepsilon}_{L_M}}\big(k-\frac{\varepsilon^2 h}{2},z,\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1'\big)\Big]e^{-ih(\tau-2z)}dh \xrightarrow{\varepsilon\to 0} W^{L_M}_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}(k,\tau,z),$$

where $W^{L_M}_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}$ solves the system (32) in $z\in[L_M,L]$ without the last source term (the one with $\widehat{C}(2k,\cdot)$), but with the non-zero initial condition

$$W^{L_M}_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}(k,\tau,z=L_M) = (2\pi)^d\hat{\psi}(\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_0')\delta(\tau)\delta(\boldsymbol{\kappa}_0-\boldsymbol{\kappa}_0'-\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_1').$$

Here, we have used the fact that $\mathbb{E}[\hat{R}(\boldsymbol{\kappa})\overline{\hat{R}(\boldsymbol{\kappa}')}] = (2\pi)^d\hat{\psi}(\boldsymbol{\kappa})\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}')$. By Duhamel's principle, it is possible to express the solution $W_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}$ of the system (32) in the weak backscattering regime as the superposition of solutions $W^{L_M}_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}$ of the systems in the presence of random mirrors at L_M, if we choose the impedance of the random mirror such that

$$\hat{\psi}(\boldsymbol{\kappa}) = \frac{k^2}{4(2\pi)^{2d}}\widehat{C}(2k,\boldsymbol{\kappa}),$$

and if we average out over the mirror position L_M between 0 and L:

$$W_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}(k,\tau,L) = \frac{1}{L}\int_0^L W^{L_M}_{(\boldsymbol{\kappa}_0,\boldsymbol{\kappa}_0'),(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_1')}(k,\tau,L)\,dL_M.$$

This establishes a correspondence between the two problems. This statement is only valid for a fixed frequency k and for the second-order moments of the wave field, and it would require further work to establish it in the time-domain (note that the impedance of the equivalent random mirror is found to depend on the frequency k). It confirms the naive interpretation of the weak backscattering regime: the wave propagates first in the forward-scattering approximation, it is reflected at some random position, and it propagates back in the forward-scattering approximation. However, it should be stressed that correlations between the forward and backward propagations have to be taken into account. Indeed the wave propagates in the same medium in both ways, and an approach based on two independent propagation steps leads to wrong predictions (in particular, reciprocity is violated and the enhanced backscattering phenomenon cannot be captured).

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