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Light propagation in square law media with random imperfections

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Abstract

This paper investigates the deformation of the field transmitted through a square law medium waveguide. We consider the situation where the center of the waveguide randomly oscillates around the optical axis or the radius of the waveguide randomly pulsates. The random perturbations are small, but the waveguide is long, which gives rise to a macroscopic effect of the inhomogeneities. This effect is characterized by coupling mechanisms between optical modes, which tend to strengthen high order modes. Precise expressions for the transmitted wave are derived which exhibit remarkable regimes, where behaviors such as shift, spreading or even focusing of the field can be observed. Numerical simulations are in good agreement with the theoretical results. ©2000 Elsevier Science B.V. All rights reserved.

1. Introduction

The design of devices capable of guiding light over large distances is of current interest in optics, especially for telecommunications purposes [1]. Indeed, because of diffraction, any light beam spreads out while it propagates in homogeneous medium. It is therefore necessary to compensate for this natural mechanism. The main idea of a waveguide is to guide a beam of light by employing a variation of the index of refraction in the transverse direction so as to cause the light to travel along a well-defined channel. The dependence of the index of refraction on the transverse direction may be continuous or discontinuous, but the essential element is that the index of refraction is maximal in the channel along which one wishes to guide the light. As the name indicates, a square law medium waveguide achieves the guidance effect by a continuous variation of the index of refraction, which is maximal at the center of the guide and decreases quadratically with increasing vbvb distance from the center. These waveguides describe with very good accuracy the modern optical glass fibers with graded index of refraction [2]. The perfect waveguide has been extensively studied [2,3]. The study of random waveguide has become an essential need because such problems naturally arise in many configurations [3]. The perturbations may originate from different physical reasons: misalignments, geometrical imperfections, impurities and so on. In most cases they are small but their cumulative effects may become important after a long propagation length. Furthermore, they are unpredictable so that the statistical approach is convenient to study their effects. Literature contains a lot of papers and reviews

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which deal with the transmittivity of random waveguides. They are mainly concerned with the mean power content in the propagating modes. The most common method vbvb consists in deriving and solving effective coupled modes equations [3–6]. McLaughlin considers in [7] the propagation of a Gaussian beam in a strongly focusing medium, which is subject to random deformations of the beam axis. He derives the average intensity and the intensity fluctuations on the beam axis. Other methods such as ray-mode analysis [8], path-integration analysis [9] or random matrix theory [10] can also be applied to this problem. Besides, different questions have been addressed: statistical properties of the cumulative phase [11], influence of absorption [12], or else the degree of polarization [13]. Influence of backscattering is also examined vbvb in [14] in a particular case (rectangular waveguide with two propagating modes). In our paper we shall give a mathematical approach of the propagation of light through a random square law medium. We aim at deriving precise expressions for both the modal representation and the wave itself at the output of the random waveguide for any input wave. Furthermore, the complete distribution is sought, and we shall see that the mean behavior may be very different from the typical behavior of the wave, since the mean is imposed by exceptional but remarkable realizations. The analysis puts into evidence the usual scales [15]: wavelength, amplitude and correlation radius of the random perturbations, length and radius of the waveguide. We study the asymptotic behavior of the transmitted wave in the framework introduced by Papanicolaou and its co-authors based on the separation of these scales. Our main aim is to exhibit vbvb the asymptotic regime which corresponds to the case where the amplitudes of the random fluctuations go to zero and the size of the system goes to infinity. We then describe explicitly the effective random evolution of the field. The paper is organized as follows. Section 2 with the wave description of light inside the perfect waveguide, while we state our main convergence result about the transmitted field through a random waveguide in Sections 3 and 4, devoted respectively to the study of random perturbations of the axis and of the core radius of the waveguide. We finally compare the theoretical asymptotic results with numerical simulations of the perturbed propagation equation in Section 5.

2. Light propagation in a perfect waveguide

2.1. Propagation equation

Physically speaking we are dealing in this paper with the square law medium in paraxial approximation. The square law medium is a medium with an index of refraction of the form:

$$n^2(x, y) = n_0^2 - n_0 n_1 (x^2 + y^2), \tag{1}$$

where $n_0 > 0$ is the maximal index at the center of the waveguide and $n_1 > 0$ is the quadratic decrease rate of the index per unit length. For sufficiently large $x^2 + y^2$, the squared index n^2 becomes negative, which can occur for instance in ionized gases. Although this is not the case for waveguides, Eq. (1) is considered as a good first approximation for graded index waveguides, as long as the field distribution is close to the optical axis [2]. Furthermore, square law media have the physically relevant property that the center of intensity of paraxial light moves according to the law of geometrical optics (see Section 3.6 of [3]). We shall also recall that the normal modes of such media are the well-known Gauss–Hermite functions (see Section 7.2 of [3]), which will appear very convenient for our purpose. Within the parabolic approximation, the propagation of the field is governed by the following Schrödinger equation, sometimes called Fock equation [3]:

$$-2\mathrm{i}kn_0 \frac{\partial E}{\partial z} - k^2 n_0 n_1 (x^2 + y^2) E + \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0, \tag{2}$$

where k is the free space wavenumber. In order to transform this equation into a standard and adimensional form, we multiply the spatial coordinates x and y by $k^{1/2}(n_0 n_1)^{1/4}$ and z by $n_0^{1/2} n_1^{-1/2}$, so that (2) now reads:

$$-2\mathrm{i}\frac{\partial E}{\partial z} - (x^2+y^2)E + \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0. \tag{3}$$

2.2. Normal modes of square law media

Eq. (3) admits a family of normal modes indexed by two integers p and q:

$$f_{p,q}(x, y) = f_p(x) f_q(y), \tag{4}$$

where the real-valued functions f_p are the so-called Hermite–Gaussian functions:

$$f_p(x) = \frac{1}{\sqrt{2^p \sqrt{\pi}\, p!}} H_p\,(x)\, \mathrm{e}^{-x^2/2}, \qquad H_p(x) = (-1)^p \mathrm{e}^{x^2} \frac{\mathrm{d}^p}{\mathrm{d}x^p} \mathrm{e}^{-x^2}. \tag{5}$$

The family $(f_{p,q})_{p,q\in\mathbb{N}}$ is complete in the following sense ([16], Prop. 1.5.7).

Proposition 2.1.

1. *The normalized Hermite–Gaussian functions* $(f_p)_{p\in\mathbb{N}}$ (*resp.* $(f_{p,q})_{p,q\in\mathbb{N}}$) *are a complete orthonormal set in the physical space* $L^2(\mathbb{R},\mathbb{C})$ (resp. $L^2(\mathbb{R}^2,\mathbb{C})$).

$$\int_{\mathbb{R}} f_p f_{p'}(x)\,\mathrm{d}x = \delta_{pp'}, \qquad \int_{\mathbb{R}^2} f_{p,q} f_{p',q'}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \delta_{pp'}\delta_{qq'}.$$

2. $(x, y, z) \mapsto \mathrm{e}^{\mathrm{i}(p+q+1)z} f_{p,q}(x, y)$ *is a solution of* (3).

Furthermore, by applying the recursion relations amongst the Hermite polynomials ([17], Eq. (8.952)), we get relations between the Hermite–Gaussian functions that will be useful in the following:

$$\frac{\partial f_p}{\partial x} = -\frac{\sqrt{p+1}}{\sqrt{2}} f_{p+1} + \frac{\sqrt{p}}{\sqrt{2}} f_{p-1}, \qquad x f_p = \frac{\sqrt{p+1}}{\sqrt{2}} f_{p+1} + \frac{\sqrt{p}}{\sqrt{2}} f_{p-1}. \tag{6}$$

We define the eigenstate decomposition as the map $\Theta : E \in L^2(\mathbb{R},\mathbb{C}) \mapsto (c_p)_{p\in N}$, where c_p is defined by

$$\Theta(E)_p := c_p = \int_{\mathbb{R}} f_p(x) E(x)\,\mathrm{d}x. \tag{7}$$

By Proposition 2.1, Θ is an isometry from $L^2(\mathbb{R},\mathbb{C})$ onto l^2, the space of all the sequences $(c_p)_{p\in\mathbb{N}}$ from $\mathbb{N}$ into $\mathbb{C}$ which are square integrable.

3. Square law media with randomly perturbed axis

3.1. Formulation of the problem

In this section we look for the evolution of the field E in a waveguide whose quadratic varying index of refraction is affected by small perturbations. More exactly we assume that the center of the waveguide oscillates around the optical axis according to $\varepsilon\mathbf{m}(z)$, where ε is a small parameter which characterizes the amplitudes of the fluctuations and $\mathbf{m} = (m_x, m_y)$ is a $\mathbb{R}^2$-valued random process (see Fig. 1). The perturbed equation which governs the evolution of the field is

$$-2\mathrm{i}\frac{\partial E}{\partial z} - (x - \varepsilon m_x(z))^2 E - (y - \varepsilon m_y(z))^2 E + \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0. \tag{8}$$

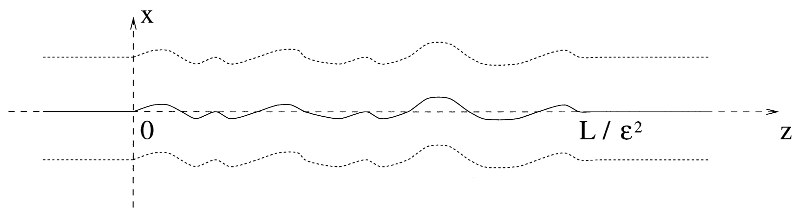

Fig. 1. Perturbation of the center of the waveguide.

3.2. Scales and hypotheses

We assume that the amplitudes of the fluctuations are of order $\varepsilon \ll 1$, and that the perturbed region, i.e. the support of the function $\mathbf{m}$, is of order ε^{-2}. The function $\mathbf{m} = (m_x, m_y)$ is assumed to be almost surely bounded, zero-mean, stationary and ergodic process. In order to simplify the presentation of the forthcoming results, we shall assume that the real processes m_x and m_y are independent and identically distributed. However, this hypothesis does not alter qualitatively the results nor the corresponding arguments. The rigorous proof actually requires that the random process $\mathbf{m}$ has "enough decorrelation", more exactly that it fulfills the technical mixing condition "$\mathbf{m}$ is ϕ-mixing, with $\phi \in L^{1/2}(\mathbb{R}^+)$" (see [18], Section 4.6.2). The function $\phi(z)$ is roughly the normalized correlation function $\mathbb{E}[m(0)m(z)]/\mathbb{E}[m(0)^2]$.

In order to simplify the presentation of the main results, we shall first assume that only one transversal coordinate is present, say *x*. We shall then present and discuss the results of the physically relevant problem with the two transversal coordinates *x* and *y* in Section 3.6. Accordingly we first consider the equation:

$$-2\mathrm{i}\frac{\partial E}{\partial z} - (x - \varepsilon m_x(z))^2 E + \frac{\partial^2 E}{\partial x^2} = 0. \tag{9}$$

3.3. Evolution of the modal decomposition

We aim at studying the evolution of the normalized field $E^\varepsilon(x,z) := E(x, z/\varepsilon^2)$. Since Θ is an isometry from L^2 onto l^2, it is equivalent to study the evolution of its modal decomposition, i.e. the corresponding normalized coefficients c^ε:

$$c_p^\varepsilon(z) := \Theta(E^\varepsilon(.,z))_p \mathrm{e}^{-\mathrm{i}(p+1/2)z/\varepsilon^2}, \tag{10}$$

$$E^\varepsilon(x,z) = \sum_{p=0}^{\infty} c_p^\varepsilon(z) \mathrm{e}^{\mathrm{i}(p+1/2)z/\varepsilon^2} f_p(x). \tag{11}$$

Substituting the expression (11) into Eq. (9), and using the relations (6), we get that the equation which governs the evolution of c^ε is

$$\frac{\mathrm{d}c_p^\varepsilon}{\mathrm{d}z} = \frac{1}{\varepsilon} m_x\left(\frac{z}{\varepsilon^2}\right)\frac{\mathrm{i}}{\sqrt{2}}\left(\sqrt{p}\,c_{p-1}^\varepsilon \mathrm{e}^{-\mathrm{i}z/\varepsilon^2} + \sqrt{p+1}\,c_{p+1}^\varepsilon \mathrm{e}^{\mathrm{i}z/\varepsilon^2}\right) + m_x^2\left(\frac{z}{\varepsilon^2}\right)\frac{\mathrm{i}}{2} c_p^\varepsilon. \tag{12}$$

We consider also the infinite-dimensional system of linear differential equations starting from $c(0) = c_0$:

$$\begin{aligned} \mathrm{d}c_p = {} & \sqrt{\gamma_{1,x}}\left(\sqrt{p}\,c_{p-1} - \sqrt{p+1}\,c_{p+1}\right)\mathrm{d}W_z^1 + \mathrm{i}\sqrt{\gamma_{1,x}}\left(\sqrt{p}\,c_{p-1} + \sqrt{p+1}\,c_{p+1}\right)\mathrm{d}W_z^2 \\ & + \left(\frac{\mathrm{i}}{2}\sigma_x^2 - \gamma_{1,x}(2p+1)\right) c_p \mathrm{d}z, \end{aligned} \tag{13}$$

where W^1 and W^2 are independent standard Brownian motions, σ_x^2 is the variance of the random variable m_x and $\gamma_{j,x}$ is the integrated covariance:

$$\sigma_x^2 = \mathbb{E}[m_x(0)^2], \quad \gamma_{j,x} = \frac{1}{2}\int_0^\infty \mathbb{E}[m_x(0)m_x(z)]\cos(jz)\,\mathrm{d}z. \tag{14}$$

$\gamma_{j,x}$ is nonnegative because it is proportional to the j-frequency evaluation of the spectral density function by the Wiener–Khintchine theorem [19]. We can now state our main convergence result.

Proposition 3.1.
1. *There exists a unique solution* c *of* Eq. (13).
2. *The processes* c^ε *converge in distribution as continuous functions from* $[0,\infty)$ *into* l^2 *to the diffusion Markov process* c *solution of* (13) *as* $\varepsilon \to 0$.

This proposition was established in the framework of the quantum harmonic oscillator in [20]. The interested reader can find in this paper the complete proof. It consists essentially in justifying the substitutions of independent Brownian motions for the processes $\varepsilon^{-1}m_x(z/\varepsilon^2)\cos(z/\varepsilon^2)$ and $\varepsilon^{-1}m_x(z/\varepsilon^2)\sin(z/\varepsilon^2)$ when taking the limit $\varepsilon \to 0$ in Eq. (12). Taking into account Stratonovitch corrections we then get Eq. (13). These techniques based on a martingale approach to some limit theorems in the diffusion-approximation regime are now well known and extensively reviewed in literature [15,18]. Eq. (13) consists of a closed-form but complicated description of the asymptotic evolution of the modal decomposition of the field. In the framework of randomly perturbed square law media, we are concerned with the description of the wave at the output of the waveguide. From this viewpoint Proposition 3.1 is the starting point of our study.

3.4. Asymptotic evolution of the field

We now give remarkable properties of the asymptotic system (13). First we want to emphasize that the assertion "*E^ε converge as a continuous function from* $[0,\infty)$ *into* L^2" does not hold true, because of the fast varying phases of the modes $\exp(\mathrm{i}(p+\frac{1}{2})\frac{z}{\varepsilon^2})$. However, the field presents the remarkable property that it reproduces itself in the perfect waveguide in regularly spaced planes. The interval between the planes is of length 2π. As a consequence the corresponding discontinuous field in the planes $\mathcal{P}_k = \{(2\pi k, x), x \in \mathbb{R}\}$, $k \in \mathbb{N}$, defined by [1]

$$\tilde{E}^\varepsilon(x,z) = E\left(x, 2\pi\left[\frac{z}{2\pi\varepsilon^2}\right]\right) \tag{15}$$

possesses nice convergence properties in the space of the càd-làg functions **D** equipped with the Skorohod topology (**D** is the space of the so-called right-continuous functions with left limits. See [21] (Ch. 3) for an introduction to this space and its properties).

Proposition 3.2. *$\tilde{E}^\varepsilon$ converges in distribution in* $\mathbf{D}([0,\infty), L^2)$ *to* $\tilde{E}(x,z) = \sum_{p=0}^\infty c_p(z) f_p(x)$ *which is the unique solution of*

$$\mathrm{d}\tilde{E} = -\sqrt{2\gamma_{1,x}}\frac{\partial \tilde{E}}{\partial x}\mathrm{d}W_z^1 + \mathrm{i}\sqrt{2\gamma_{1,x}}x\tilde{E}\,\mathrm{d}W_z^2 + \left(\mathrm{i}\frac{\sigma_x^2}{2}\tilde{E} - \gamma_{1,x}x^2\tilde{E} + \gamma_{1,x}\frac{\partial^2 \tilde{E}}{\partial x^2}\right)\mathrm{d}z \tag{16}$$

starting from $\tilde{E}(x,0) = E_0(x)$.

If we get rid off the linear phase and set: $\hat{E}(x,z) = \tilde{E}(x,z)\exp(-\mathrm{i}\sigma_x^2 z/2)$, then we come to the surprising conclusion that the mean field $E_{\mathrm{mean}} := \mathbb{E}[\hat{E}]$ satisfies a diffusion equation (of the type heat flow with cooling), while we have started with a Schrödinger equation:

$$\frac{\partial E_{\mathrm{mean}}}{\partial z} = -\gamma_{1,x}x^2 E_{\mathrm{mean}} + \gamma_{1,x}\frac{\partial^2}{\partial x^2}E_{\mathrm{mean}}, \quad E_{\mathrm{mean}}(x,0) = E_0(x).$$

[1] $[\tau]$ stands for the integral part of a real number τ.

But the more striking point is that we can give a closed-form expression for the solution of Eq. (16). Indeed $\tilde{E}$ is found to be

$$\tilde{E}(x,z) = E_0\left(x - \sqrt{2\gamma_{1,x}}W_z^1\right)\text{expi}\left(x\sqrt{2\gamma_{1,x}}W_z^2 - 2\gamma_{1,x}\int_0^z W_{z'}^2 \mathrm{d}W_{z'}^1 + \frac{\sigma_x^2 z}{2}\right). \tag{17}$$

In particular, the intensity profile in the plane at normalized distance z of the input plane is simply the initial profile shifted by $\sqrt{2\gamma_{1,x}}W_z^1$: $\left|\tilde{E}(x,z)\right|^2 = \left|E_0\left(x - \sqrt{2\gamma_{1,x}}W_z^1\right)\right|^2$.

More generally, let $\delta \in [0, 2\pi]$. If we denote by $\mathcal{P}_{\delta,k}$, $k \in \mathbb{N}$, the sequence of planes $\mathcal{P}_{\delta,k} = \{(x, \delta + 2\pi k), x \in \mathbb{R}\}$ and by $\tilde{E}_\delta^\varepsilon$ the corresponding fields which lie in these planes:

$$\tilde{E}_\delta^\varepsilon(x,z) = E\left(x, \delta + 2\pi\left[\frac{z}{2\pi\varepsilon^2}\right]\right), \tag{18}$$

then $\tilde{E}_\delta^\varepsilon$ converges in distribution $\mathbf{D}([0,\infty), L^2)$ to $\tilde{E}_\delta$:

$$\tilde{E}_\delta(x,z) = \sum_{p=0}^{\infty} c_p(z)\mathrm{e}^{\mathrm{i}(p+12)\delta} f_p(x).$$

The asymptotic field $\tilde{E}_\delta$ can be derived from $\tilde{E}$ through the following equation, in which z is frozen:

$$-2\mathrm{i}\frac{\partial \tilde{E}_\delta}{\partial \delta} - x^2\tilde{E}_\delta + \frac{\partial^2 \tilde{E}_\delta}{\partial x^2} = 0, \qquad \tilde{E}_\delta(x,z)\Big|_{\delta=0} = \tilde{E}(x,z). \tag{19}$$

The general solution $\tilde{E}_\delta(x,z)$ is found to be

$$\tilde{E}_\delta(x,z) = E_{0,\delta}\left(x - x_\delta(z)\right)\exp\left(\mathrm{i}\phi_\delta(x,z)\right), \tag{20}$$

where

$$x_\delta(z) = c_\delta\sqrt{2\gamma_{1,x}}W_z^1 - s_\delta\sqrt{2\gamma_{1,x}}W_z^2, \qquad \phi_\delta(x,z) = x(s_\delta\sqrt{2\gamma_{1,x}}W_z^1 + c_\delta\sqrt{2\gamma_{1,x}}W_z^2)$$
$$-c_\delta s_\delta\gamma_{1,x}(W_{z^2}^1 - W_{z^2}^2) + 2c_\delta^2\gamma_{1,x}\int_0^z W_{z'}^2\mathrm{d}W_{z'}^1 - 2s_\delta^2\gamma_{1,x}\int_0^z W_{z'}^1\mathrm{d}W_{z'}^2 + \frac{\sigma_x^2 z}{2},$$

and $c_\delta = \cos(\delta)$ and $s_\delta = \sin(\delta)$. Besides $E_{0,\delta}$ stands for the field in the plane $\mathcal{P}_{\delta,0}$ which results from the propagation of the initial field E_0 in the perfect waveguide from 0 to δ:

$$E_{0,\delta}(x) = \sum_{p=0}^{\infty} c_{0_p}\mathrm{e}^{\mathrm{i}(p+1/2)\delta} f_p(x).$$

Eq. (20) gives the complete description in the full space of the field in the asymptotic framework $\varepsilon \to 0$. The longitudinal coordinate divides into two parts: the macroscopic scale z and the microscopic scale δ. Note also that for any δ, the random process $x_\delta(z)$ which represents the position of the center of the field obeys the distribution of a Brownian motion with variance $2\gamma_{1,x}$.

3.5. The Hamiltonian of the field

We have already seen that the field E^ε does not converge as a process with respect to the longitudinal coordinate, but only a stroboscopic version, $\tilde{E}^\varepsilon$ or $\tilde{E}_\delta^\varepsilon$, does. However we can explicitly compute the variations of an infinity of quantities which are constants of motion for the unperturbed waveguide. We deal in the following proposition with the most important of them.

Proposition 3.3. *Let us denote the Hamiltonian by*

$$H_x = \int \left|\frac{\partial E}{\partial x}\right|^2 + x^2|E|^2\,\mathrm{d}x, \tag{21}$$

where E is the solution of (8) *starting from* $E|_{z=0} = E_0$. *The quantity* $H_x(./\varepsilon^2)$ *converges as a continuous function from* $[0,\infty)$ *into* $\mathbb{R}^+$ *to the process* $\bar{H}_x$ *given by*

$$\bar{H}_x(z) = H_x^0 + 2\gamma_{1,x}N_0\left(W_z^{1^2} + W_z^{2^2}\right) + \sqrt{8\gamma_{1,x}}\left(N_{1,x}W_z^1 + N_{2,x}W_z^2\right), \tag{22}$$

whose mean value is

$$\mathbb{E}[\bar{H}_x(z)] = H_x^0 + 4\gamma_{1,x}zN_0, \tag{23}$$

where H_x^0 *corresponds to* E_0, N_0 *is the squared* L^2*-norm* $\|E_0\|^2$ *which is conserved,* $N_{1,x} = \int x|E_0|^2\mathrm{d}x$ *and* $N_{2,x} = \mathrm{Im}\left(\int \frac{\partial E_0}{\partial x}E_0^*\,\mathrm{d}x\right)$.

Proof. The process $\bar{H}_x$ can be expressed in terms of the modal coefficients c_p as

$$\bar{H}_x(z) = \sum_p (2p+1)|c_p|^2(z). \tag{24}$$

After some algebra using Eq. (13) we get the expression of the Hamiltonian in terms of the Brownian motions W^j. □

3.6. Interpretation of the results with two transversal coordinates

Let us consider the physically relevant problem with the two transversal coordinates *x* and *y*. The corresponding solution is a direct product of two solutions derived here above. Let $\delta \in [0, 2\pi]$. If we denote by $\mathcal{P}_{\delta,k}$, $k \in \mathbb{N}$, the sequence of planes $\mathcal{P}_{\delta,k} = \left\{(x, y, \delta + 2\pi k), (x, y) \in \mathbb{R}^2\right\}$ and by $\tilde{E}_\delta^\varepsilon$ the corresponding fields which lie in these planes:

$$\tilde{E}_\delta^\varepsilon(x, y, z) = E\left(x, y, \delta + 2\pi\left[\frac{z}{2\pi\varepsilon^2}\right]\right), \tag{25}$$

then $\tilde{E}_\delta^\varepsilon$ converges in distribution to $\tilde{E}_\delta$:

$$\tilde{E}_\delta(x, y, z) = E_{0,\delta}\left(x - x_\delta(z), y - y_\delta(z)\right)\exp\left(\mathrm{i}\phi_\delta(x, y, z)\right), \tag{26}$$

where

$$\begin{aligned} x_\delta(z) &= c_\delta\sqrt{2\gamma_{1,x}}W_z^1 - s_\delta\sqrt{2\gamma_{1,x}}W_z^2, \qquad y_\delta(z) = c_\delta\sqrt{2\gamma_{1,y}}W_z^3 - s_\delta\sqrt{2\gamma_{1,y}}W_z^4,\\ \phi_\delta(x, y, z) &= x(s_\delta\sqrt{2\gamma_{1,x}}W_z^1 + c_\delta\sqrt{2\gamma_{1,x}}W_z^2) + y(s_\delta\sqrt{2\gamma_{1,y}}W_z^3 + c_\delta\sqrt{2\gamma_{1,y}}W_z^4)\\ &\quad -c_\delta s_\delta\left(\gamma_{1,x}(W_z^{1^2} - W_z^{2^2}) + \gamma_{1,y}(W_z^{3^2} - W_z^{4^2})\right) + 2c_\delta^2\left(\gamma_{1,x}\int_0^z W_{z'}^2\,\mathrm{d}W_{z'}^1 - \gamma_{1,y}\int_0^z W_{z'}^4\,\mathrm{d}W_{z'}^3\right)\\ &\quad -2s_\delta^2\left(\gamma_{1,x}\int_0^z W_{z'}^1\,\mathrm{d}W_{z'}^2 - \gamma_{1,y}\int_0^z W_{z'}^3\,\mathrm{d}W_{z'}^4\right) + \frac{(\sigma_x^2 + \sigma_y^2)z}{2}, \end{aligned}$$

W^j, $j = 1, \ldots, 4$ are four independent standard Brownian motions, and $c_\delta = \cos(\delta)$ and $s_\delta = \sin(\delta)$. Besides $E_{0,\delta}$ stands for the field in the plane $\mathcal{P}_{\delta,0}$ which results from the propagation of the initial field E_0 in the perfect waveguide from 0 to δ:

$$E_{0,\delta}(x, y) = \sum_{p,q=0}^{\infty} c_{0_{p,q}} \mathrm{e}^{\mathrm{i}(p+q+1)\delta} f_{p,q}(x, y).$$

Eq. (26) gives the complete description of the field in the asymptotic framework $\varepsilon \to 0$. The longitudinal coordinate is divided into two parts: the macroscopic scale z and the microscopic scale δ. Note also that for any δ, the random process $(x_\delta(z), y_\delta(z))$ which represents the position of the center of the field obeys the distribution of a two-dimensional Brownian motion with variance $(2\gamma_{1,x}, 2\gamma_{1,y})$.

In [7] the very same problem we consider in this section is addressed. By choosing as an initial wave the fundamental mode $\exp(-(x^2 + y^2)/2)$ and substituting the ansatz:

$$\exp(\mathrm{i}z)\exp\left(-\frac{x^2+y^2}{2} + \mathrm{i}\mathrm{e}^{\mathrm{i}z}(\Delta_x(z)x + \Delta_y(z)y) + \mathrm{i}\Delta(z)\right),$$

into Eq. (8) McLaughin obtains a closed form equation for the set of parameters $(\Delta_x, \Delta_y, \Delta)$. He then computes the mean intensity on the beam axis $x = y = 0$ and the corresponding fluctuations. Here we have proven that the same kind of results can be obtained for any initial wave: the output wave can be described by a finite number of random parameters. We are able to describe explicitly the statistical distribution of the output wave. We exhibit that the primary effect of the random fluctuations of the axis is to involve a random shift of the initial intensity profile, and this shift behaves like a Brownian motion. This result is useful for telecommunication applications, since square law media are a good first approximation for graded index fibers. The explicit formulae that we have obtained allow to predict for instance the random propagation distance at which the incoming localized profile will collide with the cladding of the fiber. Over an elementary interval $[2\pi[z/(2\pi\varepsilon^2)], 2\pi[z/(2\pi\varepsilon^2)] + 2\pi)$ the farthest point that the maximum of the intensity profile visits is at distance $r(z)$ from the fiber axis:

$$r(z)^2 = \sup_{\delta\in[0,2\pi)} \left(x_\delta(z)^2 + y_\delta(z)^2\right)$$

and a straightforward calculation shows that it is equal to

$$r(z)^2 = r_0(z)^2 + \left[r_0(z)^4 - 4\gamma_{1,x}\gamma_{1,y}\left(W_z^2 W_z^3 - W_z^1 W_z^4\right)^2\right]^{1/2},$$

$$r_0(z)^2 = \gamma_{1,x}(W_z^{1^2} + W_z^{2^2}) + \gamma_{1,y}(W_z^{3^2} + W_z^{4^2}).$$

The radius $r(z)$ is bounded above by $r_0(z)$ and below by $r_0(z)/2$, so we can claim that it behaves like the modulus of a four-dimensional Brownian motion.

4. Square law media with randomly perturbed radius

4.1. Formulation of the problem

We assume in this section that the center of the waveguide is perfect and coincides with the optical axis z, but the radius of the waveguide oscillates around its mean value according to $\varepsilon\mathbf{m}(z)$, where ε is a small parameter which characterizes the amplitudes of the fluctuations and $\mathbf{m} = (m_x, m_y)$ is a $\mathbb{R}^2$-valued random process (see Fig. 2). The random process $\mathbf{m}$ satisfies the same hypothesis as in Section 3.2. The perturbed equation which governs the evolution of the field is

$$-2\mathrm{i}\frac{\partial E}{\partial z} - x^2(1 + \varepsilon m_x(z))E - y^2(1 + \varepsilon m_y(z))E + \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = 0. \quad (27)$$

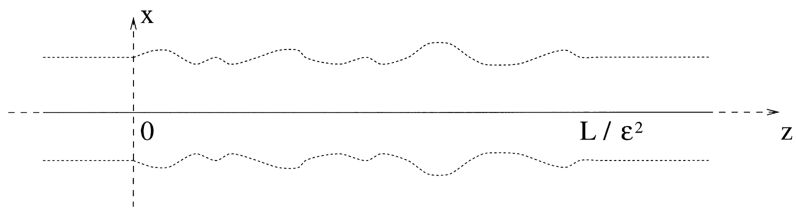

Fig. 2. Perturbation of the radius of the waveguide.

The strategy is the same as in Section 3, and the proofs are very similar. The derivations of the results is carried out with one transversal dimension. We shall then make a direct product to compute the solution with two transversal dimensions in Section 4.5. Accordingly we consider the problem:

$$-2\mathrm{i}\frac{\partial E}{\partial z} - x^2(1+\varepsilon m_x(z))E + \frac{\partial^2 E}{\partial x^2} = 0. \tag{28}$$

Denoting by c^ε the normalized modal decomposition (10) of the field, the equation which governs their coupled evolutions is obtained by substituting the expression (11) into Eq. (28):

$$\frac{\mathrm{d}c_p^\varepsilon}{\mathrm{d}z} = \frac{1}{\varepsilon} m_x\left(\frac{z}{\varepsilon^2}\right)\frac{\mathrm{i}}{4}\left(\sqrt{p(p-1)}c_{p-2}^\varepsilon \mathrm{e}^{-2\mathrm{i}z/\varepsilon^2} + (2p+1)c_p^\varepsilon + \sqrt{(p+1)(p+2)}c_{p+2}^\varepsilon \mathrm{e}^{2\mathrm{i}z/\varepsilon^2}\right). \tag{29}$$

4.2. Statement of the convergence result

We consider the infinite-dimensional system of linear differential equations starting from $c(0) = c_0$:

$$\begin{aligned}\mathrm{d}c_p = {} & \frac{\sqrt{\gamma_{2,x}}}{2\sqrt{2}}\left(\sqrt{(p+1)(p+2)}c_{p+2} - \sqrt{p(p-1)}c_{p-2}\right)\mathrm{d}W_z^1 + \frac{\mathrm{i}\sqrt{\gamma_{2,x}}}{2\sqrt{2}}\left(\sqrt{(p+1)(p+2)}c_{p+2}\right.\\ & \left.+\sqrt{p(p-1)}c_{p-2}\right)\mathrm{d}W_z^2 + \frac{\mathrm{i}\sqrt{\gamma_{0,x}}}{2}(2p+1)c_p\,\mathrm{d}W_z^3 - \frac{\gamma_{0,x}}{8}(4p^2+4p+1)c_p\,\mathrm{d}z\\ & -\frac{\gamma_{2,x}}{4}(p^2+p+1)c_p\,\mathrm{d}z,\end{aligned} \tag{30}$$

where W^j, $j=1,\dots,3$ are independent standard Brownian motions. $\gamma_{j,x}$ is the integrated covariance of the process m_x defined by (14). We denote by $\tilde{E}^\varepsilon(x,z)$ the field in the plane $2\pi[z/(2\pi\varepsilon^2)]$ defined by (15).

Proposition 4.1.
1. *There exists a unique solution c of* Eq. (30).
2. *The processes c^ε converge in distribution as continuous functions from* $[0,\infty)$ *into l^2 to the diffusion Markov process c solution of* (30) *as* $\varepsilon \to 0$.
3. *$\tilde{E}^\varepsilon$ converges in distribution in* $\mathbf{D}([0,\infty), L^2)$ *to $\tilde{E}$ which is the unique solution of*

$$\begin{aligned}\mathrm{d}\tilde{E} = {} & \frac{\sqrt{\gamma_{2,x}}}{2\sqrt{2}}\left(1+2x\frac{\partial}{\partial x}\right)\tilde{E}\,\mathrm{d}W_z^1 + \frac{\mathrm{i}\sqrt{\gamma_{2,x}}}{2\sqrt{2}}\left(x^2+\frac{\partial^2}{\partial x^2}\right)\tilde{E}\,\mathrm{d}W_z^2\\ & +\frac{\gamma_{2,x}}{16}\left(-1+4x\frac{\partial}{\partial x}+2x^2\frac{\partial^2}{\partial x^2}-\frac{\partial^4}{\partial x^4}-x^4\right)\tilde{E}\,\mathrm{d}z\\ & +\frac{\mathrm{i}\sqrt{\gamma_{0,x}}}{2}\left(-x^2+\frac{\partial^2}{\partial x^2}\right)\tilde{E}\,\mathrm{d}W_z^3 + \frac{\gamma_{0,x}}{8}\left(2+4x\frac{\partial}{\partial x}+2x^2\frac{\partial^2}{\partial x^2}-\frac{\partial^4}{\partial x^4}-x^4\right)\tilde{E}\,\mathrm{d}z,\end{aligned} \tag{31}$$

starting from $\tilde{E}(x,0) = E_0(x)$.

4.3. Asymptotic evolution of the field

Very fortunately Eq. (31) can be solved for any initial condition $E_0 \in L^2$. Indeed, still denoting by c_0 the modal decomposition of E_0, we have

$$\tilde{E}(z,x) = \sqrt{a_x(z)}\mathrm{e}^{-\mathrm{i}b_x(z)x^2}\sum_{p=0}^{\infty} c_{0_p} f_p(a_x(z)x)\mathrm{e}^{-\mathrm{i}(2p+1)\psi_x(z)}, \tag{32}$$

where the coefficients a_x, b_x, and ψ_x, obey the following stochastic differential equations:

$$\mathrm{d}a_x = \sqrt{\frac{\gamma_{2,x}}{2}} a_x \mathrm{d}W_z^1 + \sqrt{2\gamma_{2,x}} a_x b_x \mathrm{d}W_z^2 + 2\sqrt{\gamma_{0,x}} a_x b_x \mathrm{d}W_z^3 + \gamma_{2,x}\left(2a_x b_x^2 - \frac{1}{4}a_x^5\right)\mathrm{d}z$$
$$+\gamma_{0,x}\left(\frac{1}{2}a_x + 4a_x b_x^2 - \frac{1}{2}a_x^5\right)\mathrm{d}z, \tag{33}$$

$$\mathrm{d}b_x = \sqrt{2\gamma_{2,x}} b_x \mathrm{d}W_z^1 - \sqrt{\frac{\gamma_{2,x}}{8}}(1 + a_x^4 - 4b_x^2)\mathrm{d}W_z^2 + \frac{1}{2}\sqrt{\gamma_{0,x}}(1 - a_x^4 + 4b_x^2)\mathrm{d}W_z^3$$
$$+\gamma_{2,x}\left(\frac{1}{2}b_x + 2b_x^3 - \frac{3}{2}a_x^4 b_x\right)\mathrm{d}z + \gamma_{0,x}\left(b_x + 4b_x^3 - 3a_x^4 b_x\right)\mathrm{d}z, \tag{34}$$

$$\mathrm{d}\psi_x = \sqrt{\frac{\gamma_{2,x}}{8}} a_x^2\,\mathrm{d}W_z^2 + \frac{1}{2}\sqrt{\gamma_{0,x}} a_x^2\,\mathrm{d}W_z^3 + \frac{\gamma_{2,x}}{4} a_x^2 b_x\,\mathrm{d}z + \frac{\gamma_{0,x}}{2} a_x^2 b_x\,\mathrm{d}z, \tag{35}$$

starting from $a_x(0) = 1$, $b_x(0) = 0$, $\psi_x(0) = 0$. In particular, if E_0 is a single-mode beam f_p, then the intensity profile at the normalized distance z is simply: $|\tilde{E}(z,x)|^2 = a_x(z)|E_0(a_x(z)x)|^2$, which is the initial profile rescaled by the factor $a_x(z)$.

Furthermore, the fields $\tilde{E}_\delta^\varepsilon$ in the intermediate planes $\delta + 2\pi[z/(2\pi\varepsilon^2)]$ defined by (18) converge in distribution in $\mathbf{D}([0,\infty), L^2)$ to $\tilde{E}_\delta$, whose expression can be derived from $\tilde{E}$ through Eq. (19). The general solution $\tilde{E}_\delta(x,z)$ is found to be

$$\tilde{E}_\delta(z,x) = \sqrt{a_x(\delta,z)}\mathrm{e}^{-\mathrm{i}b_x(\delta,z)x^2}\sum_{p=0}^{\infty} c_{0_p} f_p(a_x(\delta,z)x)\mathrm{e}^{-\mathrm{i}(2p+1)\psi_x(\delta,z)}, \tag{36}$$

where

$$a_x(\delta,z) = \frac{a_x(z)}{\sqrt{(\cos(\delta) + 2b_x(z)\sin(\delta))^2 + a_x(z)^4\sin(\delta)^2}}, \tag{37}$$

$$b_x(\delta,z) = \frac{b_x(z)\cos(2\delta) + (4b_x(z)^2 + a_x(z)^4 - 1)(\sin(2\delta)/4)}{(\cos(\delta) + 2b_x(z)\sin(\delta))^2 + a_x(z)^4\sin(\delta)^2}, \tag{38}$$

$$\psi_x(\delta,z) = \psi_x(z) - \frac{\delta}{2}. \tag{39}$$

Giving an explicit representation in terms of the Brownian motions W^j of the joint distributions of all the coefficients a_x, b_x and ψ_x is a difficult problem. Nevertheless we can give simple representations of the distributions of the variables $a_x(\delta,z)$ and $b_x(\delta,z)$ in terms of two auxiliary Brownian motions.

Proposition 4.2.

1. *The processes* $(a_x(\delta,.), b_x(\delta,.))$ *obey the same distribution as* $(a_x(.), b_x(.))$.

2. *For any δ and z, the variables $(a_x^2(\delta,z), b_x(\delta,z))$ obey the distributions of*

$$a_x^2(z) = \exp\left(\sqrt{2\gamma_{2,x}}W_z^{1,x} - \gamma_{2,x}z\right), \tag{40}$$

$$b_x(z) = \sqrt{\frac{\gamma_{2,x}}{2}}\exp\left(\sqrt{2\gamma_{2,x}}W_z^{1,x} - \gamma_{2,x}z\right) \times \int_0^z \exp\left(-\sqrt{2\gamma_{2,x}}W_{z'}^{1,x} + \gamma_{2,x}z'\right) \mathrm{d}W_{z'}^{2,x}, \tag{41}$$

where $W^{1,x}$ and $W^{1,y}$ are independent standard Brownian motions.

3. *The probability that at a given point z the value of $a_x^2(z)$ excesses the value $M \geq 0$ is*

$$\mathbb{P}\left(a_x^2(z) \geq M\right) = \mathrm{erfc}\left(\frac{\ln(M) + \gamma_{2,x}z}{\sqrt{2\gamma_{2,x}z}}\right),$$

where $\mathrm{erfc}(t) = \frac{1}{\sqrt{2\pi}}\int_t^\infty \mathrm{e}^{-s^2/2}\mathrm{d}s$.

Proof. The first and second points are technical and referred to the Appendix A. The third point is then a straightforward corollary. □

An exponential martingale such as (40) presents very special properties. Let us discuss one of them, which concerns the fundamental difference between its long-range mean behavior and long-range typical behavior. For large *z* it is well known that the typical value of $W_z^{1,x}$ is of the order of $\sqrt{z}$, so that the typical value of the random variable $a_x^2(z)$ is of the order of $\exp\left(-\gamma_{2,x}z \pm \sqrt{2\gamma_{2,x}z}\right)$. In this exponential the first term prevails for large *z*, so that $a_x^2(z)$ is exponentially small. Nevertheless, the mean value $\mathbb{E}[a_x^2(z)] = 1$, whatever *z*. This means that the *mean* value does not correspond at all to a *typical* value that can be reached by the random variable $a_x^2(z)$. The mean value is actually imposed by very few events which represent large deviations of the Brownian motion. These exceptional events are enhanced by the exponential, which gives $\mathbb{E}[\exp\sqrt{2\gamma_{2,x}}W_z^{1,x}] = \exp(\gamma_{2,x}z)$.

4.4. The Hamiltonian of the field

Proposition 4.3. *Let us consider the Hamiltonian H_x defined by* (21). *The quantity $H_x(./\varepsilon^2)$ converges as a continuous function from $[0,\infty)$ into $\mathbb{R}^+$ to the process $\bar{H}_x$ given by*

$$\bar{H}_x(z) = H_x^0\left(\frac{a_x^2(z)}{2} + \frac{1+4b_x^2(z)}{2a_x^2(z)}\right), \tag{42}$$

where H_x^0 corresponds to E_0. Its mean value and second moment are

$$\mathbb{E}[\bar{H}_x(z)] = H_x^0\mathrm{e}^{2\gamma_{2,x}z}, \qquad \mathbb{E}[\bar{H}_x(z)^2] = H_x^{0^2}\left(\frac{2}{3}\mathrm{e}^{6\gamma_{2,x}z} + \frac{1}{3}\right). \tag{43}$$

Proof. The mean of $\bar{H}_x$ can be computed directly since we get from the expression (24) that it satisfies a closed-form differential equation $\frac{\mathrm{d}\mathbb{E}[\bar{H}_x]}{\mathrm{d}z} = 2\gamma_{2,x}\mathbb{E}[\bar{H}_x]$. The expression of $\bar{H}_x$ in terms of a_x, b_x can be obtained from the expression (32). We then are able to compute $\mathbb{E}[H_x(z)^p]$ for any *p* by Itô's formula (see also Eqs. ((A.5))-(A.7) in Appendix A). □

The Hamiltonian has a very large variance, much larger than the square of its mean, which shows that it has very large fluctuations with respect to its mean value. This can be also put into evidence by comparing the typical value of the Hamiltonian with its mean value. $a_x^2(z)$ is an exponential martingale which decays as $\exp(-\gamma_{2,x}z)$ for $z \gg 1$, since $W_z^{1,x} \sim \sqrt{z} \ll z$ with very high probability. Furthermore $b_x(z)$ is roughly equal to $\sqrt{\gamma_{2,x}/2}\exp(-\gamma_{2,x}z)\int_0^z \exp(\gamma_{2,x}z')\mathrm{d}W_{z'}^{2,x}$ which obeys a normal distribution with mean 0 and variance 1/4 for

$z \gg 1$. By substituting these approximations into Eq. (42), we get that the typical behavior of the Hamiltonian $\bar{H}_x$ is for large z:

$$\frac{1}{z}\ln \bar{H}_x(z) \simeq \gamma_{2,x} \quad \text{for } z \gg 1. \tag{44}$$

The typical exponential behavior (44) is different from the exponential behavior of the mean Hamiltonian (43), and the departure originates from large deviations of the Brownian motion $W^{1,x}$.

4.5. Interpretation of the results with two transversal coordinates

Let us consider the physically relevant problem with two transversal coordinates x and y. If we denote by $\tilde{E}_\delta^\varepsilon$ the fields which lie in the planes $\mathcal{P}_{\delta,k}$ and which are defined by (25), then $\tilde{E}_\delta^\varepsilon$ converges in distribution to $\tilde{E}_\delta$:

$$\begin{aligned}\tilde{E}_\delta(z,x,y) &= \sqrt{a_x(\delta,z)a_y(\delta,z)}\mathrm{e}^{-\mathrm{i}b_x(\delta,z)x^2-\mathrm{i}b_y(\delta,z)y^2}\\ &\quad\times\sum_{p,q=0}^{\infty} c_{0_{p,q}} f_{p,q}(a_x(\delta,z)x, a_y(\delta,z)y)\mathrm{e}^{-\mathrm{i}(2p+1)\psi_x(\delta,z)-\mathrm{i}(2q+1)\psi_y(\delta,z)},\end{aligned} \tag{45}$$

where (a_x, b_x, ψ_x) and (a_y, b_y, ψ_y) are independent and identically distributed. They obey the same distributions as the ones described by Eqs. (37)–(39). In particular, if E_0 is a single-mode beam $f_{p,q}$, then the intensity profile at the normalized distance z is simply

$$|\tilde{E}_\delta(z,x,y)|^2 = a_x(\delta,z)a_y(\delta,z)|E_0(a_x(\delta,z)x, a_y(\delta,z)y)|^2,$$

which is the initial profile rescaled by the two-dimensional factor $(a_x(\delta,z), a_y(\delta,z))$. From Proposition 4.2 we can deduce the statistical distribution of the (local) maximal intensity of the field:

$$I_{\max}(\delta,z) := \sup_{x,y}\left|\tilde{E}_\delta(x,y,z)\right|^2$$

which simply reads as $I_{\max}(\delta,z) = a_x(\delta,z)a_y(\delta,z)I_{0,\max}$, where $I_{0,\max} := \sup_{x,y}|E_0(x,y)|^2$.

This also shows that the distribution of $I_{\max}(\delta,z)$ depends only on z, and we shall denote $I_{\max}(z) := I_{\max}(0,z)$. Since the sum of two independent standard Brownian motions is a Brownian motion with variance 2, we get immediately the following corollary.

Corollary 4.4. *Let us assume that the incident wave is a single-mode beam.*

1. *The normalized maximal intensity $\bar{I}_{\max}(z) := I_{\max}(z)/I_{0,\max}$ of the field in the plane z obeys the distribution of*

$$\bar{I}_{\max}(z) = \exp\left(\sqrt{\gamma_2}W_z - \gamma_2 z\right), \tag{46}$$

   where W is a standard one-dimensional Brownian motion, and $\gamma_2 = \left(\gamma_{2,x}+\gamma_{2,y}\right)/2$.
2. *The probability that at a given point z the value of $\bar{I}_{\max}$ excesses the value $M \geq 0$ is*

$$\mathbb{P}\left(\bar{I}_{\max}(z) \geq M\right) = \mathrm{erfc}\left(\frac{\ln(M)+\gamma_2 z}{\sqrt{\gamma_2 z}}\right),$$

   where $\mathrm{erfc}(t) = \frac{1}{\sqrt{2\pi}}\int_t^\infty \mathrm{e}^{-s^2/2}\mathrm{d}s$.

Note that $\bar{I}_{\max}$ is not an exponential martingale, and in particular that $\mathbb{E}[\bar{I}_{\max}] = \exp(-\gamma_2 z/2)$. Note also that the typical value of $\bar{I}_{\max}$, obtained by neglecting the contributions of the Brownian motions in the exponential term in Eq. (46), is of order $\exp(-\gamma_2 z)$. However, it is possible that the intensity profile becomes at some distance very narrow and very high. This anomalous focusing is quite surprising. In our framework the propagation is linear and no pathology occurs, since the variations of the radius of the core of the waveguide are very smooth and weak. This is a purely random phenomenon due to large deviations of the Brownian motions, which is canceled if the output field is averaged over many realizations. The averaged value of $\bar{I}_{\max}(z)$ shows indeed that the mean maximal intensity decays exponentially at rate $\exp(-\gamma_2 z/2)$.

In Ref. [22] a situation mathematically similar to Eq. (27) is considered. The initial state is assumed to be Gaussian and adopting a Gaussian-like ansatz Bulatov et al. exhibit a closed-form set of differential equations for the parameters of the ansatz. Here we have proven that the output wave can be described by a finite number of random parameters whatever the initial wave. We exhibit that the primary effect of random fluctuations of the radius of the waveguide is to involve a random focusing or defocusing of the initial intensity profile, and the inverse square radius behaves like an exponential martingale which goes to zero with high probability, but whose mean value is constant. The quantitative results we have derived here above are of interest for telecommunication purposes, since they allow to predict the spreading of localized pulses in graded index fibers.

5. Numerical simulations

The results in the previous sections are theoretically valid in the limit case $\varepsilon \to 0$, where the amplitudes of the perturbations go to zero and the length of the random waveguide goes to infinity. In this section we aim at showing that the asymptotic behaviors of the field can be easily observed in numerical simulations in the case where ε is small, more precisely smaller than any other characteristic scale of the problem. In order to simplify the problem we shall consider in this section that the transverse space has dimension 1, which corresponds to neglecting the y-direction. It involves no loss of generality, since the phenomena we have put into evidence in the above sections can be divided into independent phenomena in both orthogonal directions. We use a fourth-order split-step method to simulate the one-dimensional perturbed linear Schrödinger equations:

$$-2\mathrm{i}\frac{\partial E}{\partial z} - x^2 E - V^{\varepsilon}(x,z)E + \frac{\partial^2 E}{\partial x^2} = 0. \tag{47}$$

We choose the following models for the perturbation $V^{\varepsilon}(x,z)$ which correspond respectively to Eqs. (9) and (28):

$$\text{Random axis case}: \quad V^{\varepsilon}(x,z) = -2\varepsilon m_x(z)x + \varepsilon^2 m_x(z)^2,$$

$$\text{Random radius case}: \quad V^{\varepsilon}(x,z) = \varepsilon m_x(z)x^2.$$

This numerical algorithm provides accurate and stable solutions to a large class of linear and nonlinear partial differential equations [23].

Let Δz be the elementary length step and h be the elementary transverse spatial step. We denote by $(E_0(jh))_{j=-K,\dots,K-1}$ the initial wave solution. By induction we compute $E^{n+1} := (E(jh,(n+1)\Delta z))_{j=-K,\dots,K-1}$ from $E^n := (E(jh,n\Delta z))_{j=-K,\dots,K-1}$:

$$\text{first step}: \quad E^{n+1/3} = A(\Delta z/2)E^n,$$

$$\text{second step}: \quad E^{n+2/3} = B(z,\Delta z)E^{n+1/3},$$

$$\text{third step}: \quad E^{n+1} = A(\Delta z/2)E^{n+2/3},$$

where $A(\Delta z/2)$ is the linear operator generated by $-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial x^2}$ and B is simply a scalar multiplication in the physical space by $\exp\frac{\mathrm{i}}{2}\left(x^2 + V^{\varepsilon}(x,z)\right)\Delta z$. The first and third steps can be solved using a Fourier transform. Indeed, in Fourier space the effect of the exponential operator $A(\Delta z/2)$ is a scalar multiplication by $\exp \mathrm{i}(k^2\Delta z/4)$. To sum up, the split step algorithm involves a sequence of steps that include free-space propagation over a half-step, then the ‘perturbed’ guiding correction, and free space propagation over the final half-step. Practically, it is easy to see that the two back-to-back free-space half-steps can be combined into a single free-space step over Δz. The error of the split-step algorithm comes from the splitting process, because the operators A and B do not commute. It is well known that the method vbvb described here above is of second-order [24]. However, we can use a standard method which transforms a second-order method into a fourth-order method [25]. Indeed, if $C(\Delta z)E(x,z)$ is a second-order approximation to $E(x,z+\Delta z)$, then $C(\alpha\Delta z)C(-\beta\Delta z)C(\alpha\Delta z)E(x,z)$ is a fourth-order approximation to $E(x,z+\Delta z)$, if we take care to choose $\alpha = 1/(2-2^{1/3})$ and $\beta = 2^{1/3}/(2-2^{1/3})$. The drawback of this method is that we implicitly impose periodicity on the solutions because of the Fourier transform that is used on a finite interval. We can control it by imposing boundaries of the computational domain which absorb outgoing waves. This can be readily achieved by adding a complex potential which is smooth so as to reduce reflections. We choose to substitute the complex potential vbvb $\tilde{V}(x,z) = V^\varepsilon(x,z) + \mathrm{i}V_{\mathrm{abs}}(x)$ for the random potential $V^\varepsilon(x,z)$, where

$$V_{\mathrm{abs}}(x) = \begin{cases} V_{\mathrm{abs\,max}} \sin^2\left(\dfrac{\pi}{2}\dfrac{x_\ell - x}{x_\ell - X_\ell}\right) & \text{if } X_\ell \le x < x_\ell, \\ 0 & \text{if } x_\ell \le x < x_{\mathrm{r}}, \\ V_{\mathrm{abs\,max}} \sin^2\left(\dfrac{\pi}{2}\dfrac{x - x_{\mathrm{r}}}{X_{\mathrm{r}} - x_{\mathrm{r}}}\right) & \text{if } x_{\mathrm{r}} \le x < X_{\mathrm{r}}, \end{cases}$$

where X_ℓ (resp. X_{r}) is the left (resp. right) end of the computational domain, and $[X_\ell, x_\ell]$ (resp. $[x_{\mathrm{r}}, X_{\mathrm{r}}]$) is the left (resp. right) absorbing slab. The L^2-norm of the field is theoretically preserved by the split-step method up to the machine accuracy in case of a truly periodic situation without absorption. The domain that we consider is not periodic, and the outgoing wave is absorbed by an imaginary potential at the boundaries of the domain. However the size of the shifting domain is taken so that the distortion imposed by the non-periodicity has a negligible effect. We adopt in this section the following model for the perturbation:

$$m_x(z) = u_l \quad \text{if } l + z_0 \le \frac{z}{r_{\mathrm{c}}} < l + z_0 + 1,$$

where $(u_l)_{l=0,\dots,Z-1}$ is a sequence of independent and identically distributed variables, which obey uniform distributions over the interval $[-1/2, 1/2]$, and z_0 is a random variable independent of $u_.$ which also obeys an uniform distribution over $[-1/2, 1/2]$. r_{c} is the so-called correlation radius of the random process m. The autocorrelation function of the ergodic process m is equal to $\mathbb{E}[m_x(0)m_x(z)] = \frac{1}{12}(1 - z/r_{\mathrm{c}})\,1_{z \le r_{\mathrm{c}}}$, so that the power spectral density defined by (14) is

$$\gamma_{j,x} = \frac{1}{24}\frac{1 - \cos(jr_{\mathrm{c}})}{j^2 r_{\mathrm{c}}},$$

and $\gamma_{0,x} = r_{\mathrm{c}}/48$. The quantity Z/r_{c} which is equal to the length of the random waveguide will be chosen so large that we can observe the effect of the small perturbation εm_x. We measure the L^2-norm, the Hamiltonian and the intensity profile of the transmitted solution, that we can compare with the corresponding data of the incident field. We present results corresponding to simulations where the initial wave at $z = 0$ is the fundamental Gaussian mode $E_0(x) = \pi^{-1/4}\mathrm{e}^{-x^2/2}$, whose L^2-norm $N_0^{1/2}$ and Hamiltonian H_x^0 are equal to 1. We have first simulated the homogeneous Schrödinger equation (Eq. (47) with $V^\varepsilon \equiv 0$) which admits as an exact solution $E_0(x)\mathrm{e}^{\mathrm{i}z/2}$. We can therefore check the accuracy of the numerical method, since we can see that the computed solution maintains a very close resemblance to the initial field (data not shown), while the L^2-norm and the Hamiltonian are almost constant.

The other simulations are carried out with different realizations of the random perturbation with $\varepsilon = 0.1$. Fig. 3 plots the evolutions of the Hamiltonians for different values of the correlation radius r_{c} in the cases of random axis or random radius and for one particular realization of the random perturbation. As predicted by the asymptotic theory, the effect of the perturbation for $r_{\mathrm{c}} = 2\pi$ in the random axis case and for $r_{\mathrm{c}} = \pi$ in the random radius

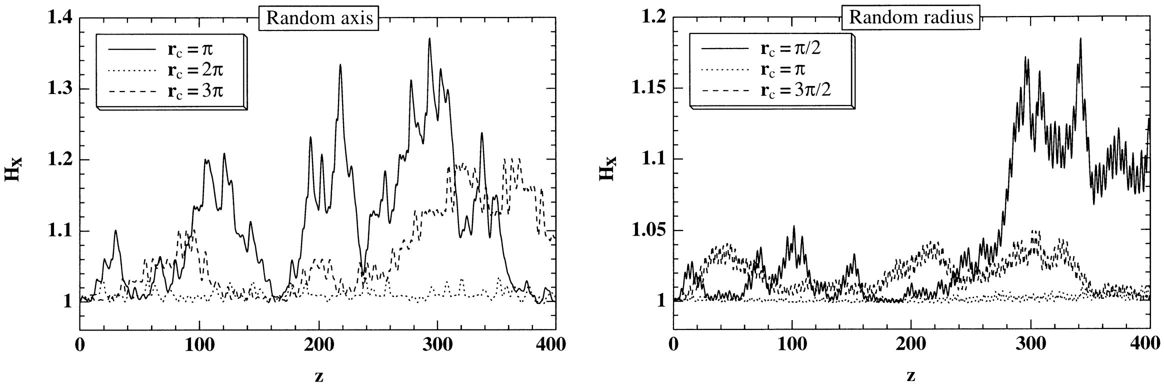

Fig. 3. Evolutions of the Hamiltonian of the field for different correlation radii. The left figure corresponds to the random axis case: $V^\varepsilon(x,z) = -2\varepsilon m(z)x + \varepsilon^2 m(z)^2$ with $\varepsilon = 0.1$. The right figure corresponds to the random radius case: $V^\varepsilon(x,z) = \varepsilon m(z)x^2$ with $\varepsilon = 0.1$.

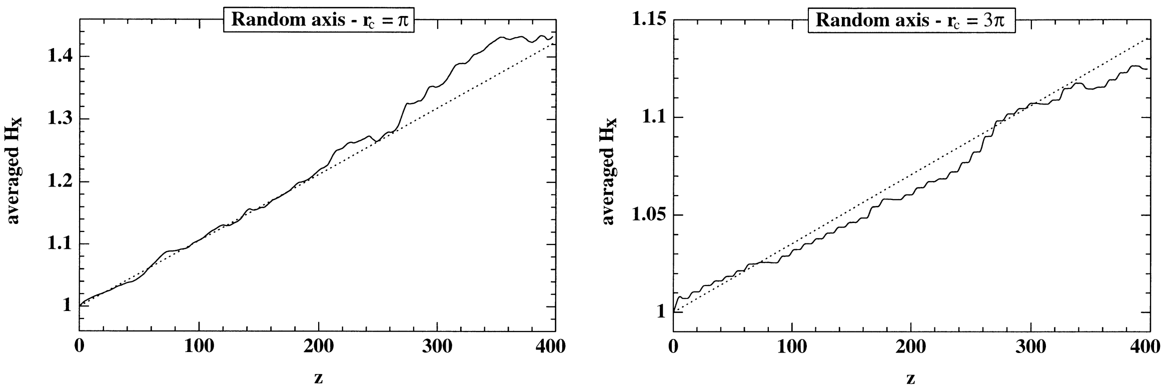

Fig. 4. Evolutions of the Hamiltonian of the field averaged over 100 realizations of the random perturbation with $\varepsilon = 0.1$ (solid line) and theoretical behaviors for the corresponding asymptotic situations. Random axis case: $V^\varepsilon(x,z) = -2\varepsilon m(z)x + \varepsilon^2 m(z)^2$. For $r_c = \pi$ we have $\gamma_{1,x} = 1/(12\pi)$ and the theoretical mean Hamiltonian is $\mathbb{E}[\bar{H}_x(z)] = H_x^0 + 4\gamma_{1,x} N_0 \varepsilon^2 z \simeq 1 + 1.06 \times 10^{-3}\, z$. For $r_c = 3\pi$ we have $\gamma_{1,x} = 1/(36\pi)$ and the theoretical mean Hamiltonian is $\mathbb{E}[\bar{H}_x(z)] \simeq 1 + 3.54 \times 10^{-4}\, z$.

case is almost zero. Indeed these particular values for the correlation radius correspond to $\gamma_{1,x} = 0$ and $\gamma_{2,x} = 0$, respectively.

In Figs. 4 and 5 we present the simulated evolutions of the Hamiltonian of the field averaged over 100 realizations and compare them with the mean theoretical evolutions given by (23) (random axis case) and (43) (random radius case) in the scale z/ε^2. It thus appears that the numerical simulations are in very good agreement with the theoretical results.

Fig. 6 plots the intensity profiles of the solutions at different depths corresponding to one of the simulations, which shows that the wave keeps the basic form of the fundamental mode although it is shifted in some random direction in the random axis case and it is re-scaled by some random factor in the random radius case. As predicted by the asymptotic results, we sometimes observe that the intensity profile in the random radius case is narrower and higher than the fundamental mode. All these observations confirm that the asymptotic theory describes with accuracy the propagation of the field through a random waveguide for small perturbations and large length.

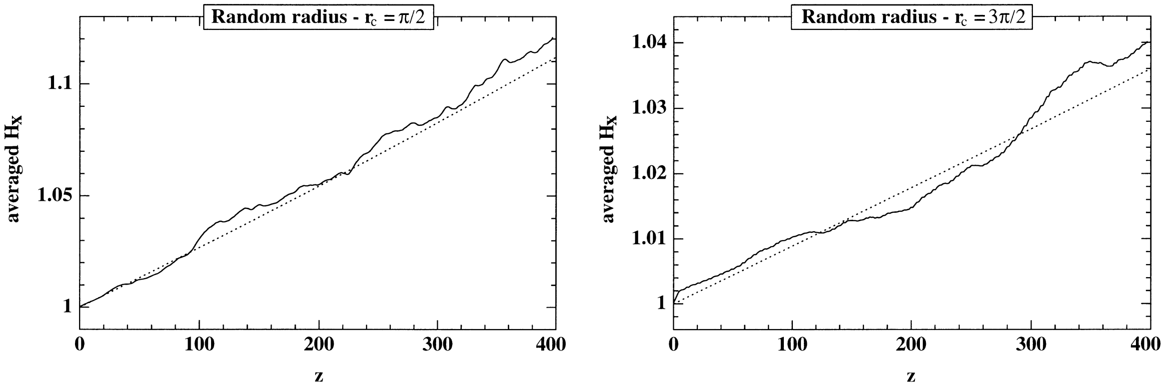

Fig. 5. Evolutions of the Hamiltonian of the field averaged over 100 realizations of the random perturbation with $\varepsilon = 0.1$ (solid line) and theoretical behaviors for the corresponding asymptotic situations. Random radius case: $V^\varepsilon(x,z) = \varepsilon m(z)x^2$. For $r_c = \pi/2$ we have $\gamma_{2,x} = 1/(24\pi)$ and the theoretical mean Hamiltonian is $\mathbb{E}[\bar{H}_x(z)] = H_x^0 \exp(2\gamma_{2,x}\varepsilon^2 z) \simeq \exp(2.66 \times 10^{-4} z)$. For $r_c = 3\pi/2$ we have $\gamma_{1,x} = 1/(72\pi)$ and the theoretical mean Hamiltonian is $\mathbb{E}[\bar{H}_x(z)] \simeq \exp(8.84 \times 10^{-5} z)$.

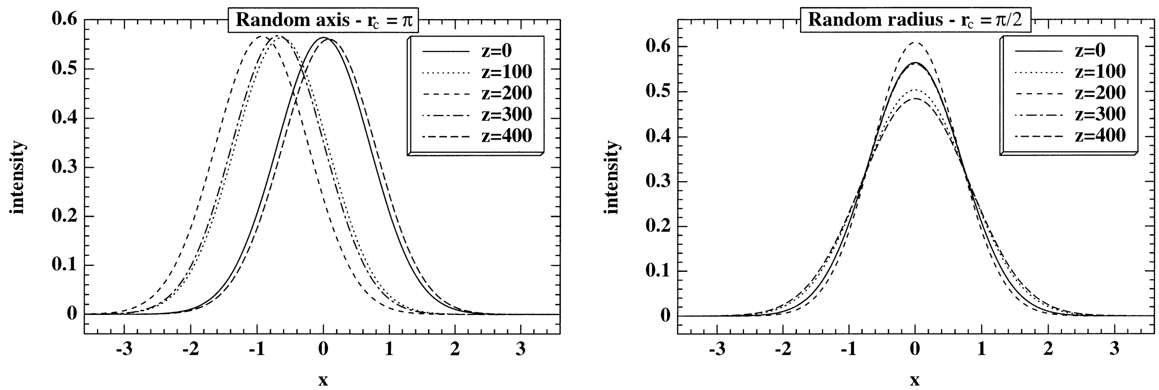

Fig. 6. Intensity profile of the incident field (solid line) and at different depths (dotted and dashed lines). The left figure corresponds to the random axis case: $V^\varepsilon(x,z) = -2\varepsilon m(z)x + \varepsilon^2 m(z)^2$ with $\varepsilon = 0.1$ and $r_c = \pi$. The profiles have the same Gaussian shape with the same radius, but the centers randomly oscillate around $x = 0$. The right figure corresponds to the random radius case: $V^\varepsilon(x,z) = \varepsilon m(z)x^2$ with $\varepsilon = 0.1$ and $r_c = \pi/2$. The profiles have the same Gaussian shape with center at $x = 0$, but the radius randomly oscillates. At lengths $z = 100$ and $z = 400$, the profiles have spread out, at length $z = 300$ the profile is almost exactly the initial one, and at length $z = 200$ the profile is narrower and higher.

6. Conclusion

We have analyzed in this paper the effects of different kinds of random perturbations on the propagation of waves in square law media which involve specific alterations of the wave. Random misalignments of the waveguide give rise to shifts of the wave with respect to the optical axis, and the shifts obey the statistical distributions of Brownian motions. Random perturbations of the radius of the core make the field pulsate. The intensity profile tends to spread out, but large deviations of the governing spreading process may involve a local focusing of the profile and an enhancement of the maximal intensity. The precise results we have derived are a priori limited to square law media. For such media the normal modes are explicit and tabulated formulae are available, so that we are able to perform exact calculations and compute closed-form expressions for the outgoing waves. Nevertheless we feel that the results demonstrated in this paper could be generalized to most of the graded index fibers. Indeed the basic assumption which requires the existence of a complete set of normal modes for the unperturbed waveguide holds true for most of them. The remainder of the study then consists in technical developments to exhibit and analyze the coupling mechanisms between the modes during the propagation of the wave in the perturbed waveguide.

Appendix A

We aim at identifying the distributions of the variables $(a_x(z), b_x(z))$ which satisfy the system of coupled stochastic differential equations (33),(34). We first apply Itô's formula ([21], Theorem V-2-9) to $a_x(z)^n b_x(z)^p$ and take the expectation:

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}[a_x^n b_x^p]}{\mathrm{d}z} &= \gamma_{2,x}\left\{(n+p)(n+p+1)\mathbb{E}[a_x^n b_x^{p+2}] - \left(\frac{n}{4}+\frac{np}{2}+\frac{p(p+2)}{2}\right)\mathbb{E}[a_x^{n+4}b_x^p]\right\}\\ &+\gamma_{2,x}\left\{\left(\frac{n(n-1)}{4}+\frac{p^2}{2}+\frac{np}{2}\right)\mathbb{E}[a_x^n b_x^p] + \frac{p(p-1)}{16}\mathbb{E}[a_x^n b_x^{p-2}]\right\}\\ &+\gamma_{2,x}\left\{\frac{p(p-1)}{8}\mathbb{E}[a_x^{n+4}b_x^{p-2}] + \frac{p(p-1)}{16}\mathbb{E}[a_x^{n+8}b_x^{p-2}]\right\}\\ &+\gamma_{0,x}\left\{2(n+p)(n+p+1)\mathbb{E}[a_x^n b_x^{p+2}] - \left(\frac{n}{2}+np+p^2+2p\right)\mathbb{E}[a_x^{n+4}b_x^p]\right\}\\ &+\gamma_{0,x}\left\{\left(\frac{n}{4}+np+p^2\right)\mathbb{E}[a_x^n b_x^p] + \frac{p(p-1)}{8}\mathbb{E}[a_x^n b_x^{p-2}]\right\}\\ &+\gamma_{0,x}\left\{-\frac{p(p-1)}{4}\mathbb{E}[a_x^{n+4}b_x^{p-2}] + \frac{p(p-1)}{8}\mathbb{E}[a_x^{n+8}b_x^{p-2}]\right\}. \end{aligned} \tag{A.1}$$

A first method to compute $\mathbb{E}[a_x(z)^n b_x(z)^p]$ consists in studying carefully the system (A.1), but we can avoid these technical developments and find a way to simplify (A.1) by using the following trick. We compute in the asymptotic framework $\varepsilon \to 0$ the evolution of the fundamental Gaussian mode $\mathrm{e}^{-x^2/2}$ from the initial plane $z = 0$ to the plane $2\pi[z/(2\pi\varepsilon^2)] + \delta$ by two ways.

1. The first way consists in going first from 0 to $2\pi[z/(2\pi\varepsilon^2)]$. We then find the field:

$$\tilde{E}^{\mathrm{gauss}}(x, y, z) = \sqrt{a_x(z)}\exp\left(-\frac{a_x(z)^2x^2}{2} - \mathrm{i}b_x(z)x^2 - \mathrm{i}\psi_x(z)\right).$$

   The processes (a_x, b_x) are solutions of (33) and (34), with Brownian motions W^j which correspond to the limit configuration associated with the random process **m** over the interval $(0, 2\pi[z/(2\pi\varepsilon^2)])$. We then get the expression of the field in the plane labeled (z, δ), whose macroscopic longitudinal coordinate is z and microscopic one is δ, by solving

$$-2\mathrm{i}\frac{\partial \tilde{E}_\delta^{\mathrm{gauss}}}{\partial\delta} - x^2\tilde{E}_\delta^{\mathrm{gauss}} + \frac{\partial^2\tilde{E}_\delta^{\mathrm{gauss}}}{\partial x^2} = 0, \qquad \tilde{E}_\delta^{\mathrm{gauss}}(x, z)\Big|_{\delta=0} = \tilde{E}^{\mathrm{gauss}}(x, z). \tag{A.2}$$

2. The second way consists in going first from 0 to δ. We then find the field $\mathrm{e}^{-x^2/2+\mathrm{i}\delta/2}$. Then we go from δ to the plane (z, δ) and find that the field is

$$\tilde{E}_\delta^{\mathrm{gauss}}(x, z) = \sqrt{a_x(\delta, z)}\exp\left(-\frac{a_x(\delta, z)^2x^2}{2} - \mathrm{i}b_x(\delta, z)x^2 - \mathrm{i}\psi_x(z) + \mathrm{i}\frac{\delta}{2}\right). \tag{A.3}$$

The processes $(a_x(\delta, .), b_x(\delta, .))$ are solutions of (33) and (34), with Brownian motions $W^{\delta,j}$ which correspond for each δ to the limit configuration associated with the random process $\mathbf{m}$ over the interval $(\delta, \delta + 2\pi[z/(2\pi\varepsilon^2)])$. Substituting (A.3) into (A.2), we get that (a_x, b_x) satisfy the equations:

$$\frac{\partial a_x(\delta, z)}{\partial \delta} = -2a_x(\delta, z)b_x(\delta, z), \ \frac{\partial b_x(\delta, z)}{\partial \delta} = \frac{1}{2}\left(a_x(\delta, z)^4 - 4b_x(\delta, z)^2 - 1\right),$$

which implies in particular that, for any integer n' and p':

$$a_x^{n'} b_x^{p'}(\delta, z) - a_x^{n'} b_x^{p'}(0, z) = \int_0^\delta \left(\frac{p'}{2}(a_x^{n'+4} b_x^{p'-1} - a_x^{n'} b_x^{p'-1}) - 2(n' + p')a_x^{n'} b_x^{p'+1}\right)(s, z)\,\mathrm{d}s. \tag{A.4}$$

The Brownian motions $W^{\delta,j}$ depend on δ, since they correspond to the limit configuration associated with the random process $\mathbf{m}$ over the interval $(\delta, \delta + 2\pi[z/(2\pi\varepsilon^2)])$. Nevertheless, the distribution of $\mathbf{m}$ is translationally invariant, so that the $W^{\delta,j}$ and consequently $(a_x(\delta, .), b_x(\delta, .))$ obey the same distributions as W^j and $(a_x(.), b_x(.))$, respectively, for every δ. Taking the expectation in (A.4) with $(n' = n, \ p' = p - 1)$ and $(n' = n + 4, \ p' = p - 1)$ yields that the integrands have mean zero and consequently:

$$\mathbb{E}\left[\frac{p-1}{2} a_x^{n+4} b_x^{p-2}\right] = \mathbb{E}\left[2(n + p - 1)a_x^n b_x^p + \frac{p-1}{2} a_x^n b_x^{p-2}\right],$$
$$\mathbb{E}\left[\frac{p-1}{2} a_x^{n+8} b_x^{p-2}\right] = \mathbb{E}\left[2(n + p + 3)a_x^{n+4} b_x^p + \frac{p-1}{2} a_x^{n+4} b_x^{p-2}\right].$$

By inserting into (A.1) we get a simplified version of the system satisfied by the moments of $(a_x(z), b_x(z))$:

$$\frac{\mathrm{d}\mathbb{E}[a_x(z)^n b_x(z)^p]}{\mathrm{d}z} = \gamma_{2,x}\left(\frac{n(n-2)}{4} + np + p(p-1)\right)\mathbb{E}[a_x(z)^n b_x(z)^p] + \gamma_{2,x}\frac{p(p-1)}{4}\mathbb{E}[a_x(z)^n b_x(z)^{p-2}]. \tag{A.5}$$

On the other hand, one can check by direct computation that the processes defined by (40) and (41) satisfy Eq. (A.5) for every n and p. Consequently they obey the same distributions as $(a_x(z), b_x(z))$, which proves the result. In practice $\mathbb{E}[a_x(z)^n b_x(z)^p]$ is computed recursively with respect to p. One finds in particular that $\mathbb{E}[a_x(z)^n b_x(z)^p] = 0$ for any odd integer p and

$$\mathbb{E}[a_x(z)^n] = \mathrm{e}^{\gamma_{2,x}[n(n-2)/4]z}, \tag{A.6}$$

$$\mathbb{E}[a_x(z)^n b_x(z)^2] = \frac{\mathrm{e}^{\gamma_{2,x}2(n+1)z} - 1}{4(n+1)}\mathrm{e}^{\gamma_{2,x}[n(n-2)/4]z}. \tag{A.7}$$

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