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OPTIMAL TRANSMISSION THROUGH A RANDOMLY PERTURBED WAVEGUIDE IN THE LOCALIZATION REGIME

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Abstract. We demonstrate that increased power transmission through a random single-mode or multi-mode channel can be obtained in the localization regime by optimizing the spatial wave front or the time pulse profile of the source. The idea is to select and excite the few modes or the few frequencies whose transmission coefficients are anomalously large compared to the typical exponentially small value. We prove that time reversal is optimal for maximizing the transmitted intensity at a given time or space, while iterated time reversal is optimal for maximizing the total transmitted energy. The statistical stability of the optimal transmitted intensity and energy is also obtained.

1. **Introduction.** It is known that the total power transmission coefficient of a multi-mode random system decays to zero when the length of the system becomes large. The decay rate is algebraic in the diffusive regime and exponential in the localization regime [1]. It is also known that the transmission through a multi-mode random system in the diffusive regime is the result of a small number of open channels with a transmission coefficient close to one [3]. It is therefore possible to increase the transmission by shaping the wave front so as to excite the open channels. Following this idea an optimization algorithm has been proposed and implemented in [18] that maximizes the intensity transmitted in a diffraction limited spot. The experiments carried out in [18] were in excellent quantitative agreement with random matrix theory that predicts that, with perfectly shaped wave fronts, the total power transmission coefficient of a multi-mode random system tends to a universal value of 2/3, regardless of the length of the system [10].

The results reported so far were obtained theoretically and experimentally in the diffusive regime, far away from the localization regime. In our paper we address the localization regime and show that a qualitatively similar picture holds. We exhibit that the mode power transmission coefficients have a bimodal distribution, so that most of them are exponentially small, but a small (but not negligible) number of them are of order one. In particular the total power transmission coefficient depends only on these exceptional modes.

We first address the time-harmonic problem for a random multi-mode channel and study the optimization of the spatial wave front. We show that the maximal intensity delivered at a target point of the output of the channel is obtained by using a time-reversal scheme and that it is a self-averaging quantity in the limit of a

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large number of modes. The total power transmission coefficient obtained with this optimal illumination is also self-averaging, it depends on the ratio of the length of the system over the localization length, and it becomes equal to $1/4$ when the length of the system is larger than the localization length. This is in dramatic contrast with the case of a uniform illumination in which the total power transmission coefficient decays exponentially with the length of the system. We also show that the maximal total power delivered at the output of the channel is obtained by using an iterated time-reversal scheme, which gives after $2q+1$ iterations a total power transmission coefficient of $(1-1/(4q+2))^2$ in the strong localization regime.

Second we address the time-dependent problem for a random single-mode channel and study the optimization of the source profile. We show that the maximal intensity delivered at a prescribed time is obtained by using a time-reversal scheme, and that the total transmitted energy can be optimized by using an iterated time-reversal scheme. The n-th order iterative time-reversal scheme gives a total transmitted energy equal to $(1-1/(2n))^2$ in the strong localization regime. Therefore the analysis of the multi-mode time-harmonic problem on the one hand and of the single-mode time-dependent problem on the other hand shows that the exponential decay due to the wave localization can be overcome by optimizing the illumination.

The paper is organized as follows. In Section 2 we list the hypotheses that the random multi-mode system should satisfy so that our theory can be applied. In Section 3 we present two models that satisfy the set of hypotheses. In Section 4 we study in detail the statistics of the mean mode power transmission coefficient in the localization regime and exhibit its bimodal structure. In Section 5 we describe the statistics (mean and fluctuations) of the power transmission using optimal illumination and make the connection with time-reversal. This optimal illumination is designed to maximize the intensity delivered at a target point. In Section 6 we show that the total power transmission can be optimized by using iterated time reversal. Finally, in Section 7 we introduce and study a time-dependent single-mode problem in which an optimized pulse profile is used to maximize the intensity transmitted through the random system at a given time. We show that time-reversal is optimal for the maximization of the transmitted intensity at the prescribed time while iterated time-reversal is optimal for the maximization of the total transmitted energy.

2. **A multi-mode waveguide model.**

We consider a randomly perturbed waveguide or system with N channels or modes. The transverse section of the system is $\Omega \subset \mathbb{R}^d$. Each mode is characterized by its unperturbed transverse spatial profile $(\phi_j(\boldsymbol{x}))_{\boldsymbol{x}\in\Omega}$ and a random transmission coefficient T_j. If the input profile is $\psi_{\rm in}(\boldsymbol{x})$, then the output profile is

$$\psi_{\rm out}(\boldsymbol{y}) = \sum_{j=1}^N \alpha_j T_j \phi_j(\boldsymbol{y}), \qquad \alpha_j = \int_\Omega \overline{\phi_j(\boldsymbol{x})} \psi_{\rm in}(\boldsymbol{x}) d\boldsymbol{x}.$$

The orthonormality property of the (unperturbed) modes means that, for all $j, j' = 1, \ldots, N$,

$$\int_\Omega \phi_j(\boldsymbol{x}) \overline{\phi_{j'}(\boldsymbol{x})} d\boldsymbol{x} = \mathbf{1}_{j=j'}.$$

Let us assume that we have an array of n regularly spaced emission points $(\boldsymbol{x}_k)_{k=1,\ldots,n}$ in the input plane of the waveguide and an array of p regularly spaced control points $(\boldsymbol{y}_l)_{l=1,\ldots,p}$ in the output plane of the waveguide.

Hypothesis 1: continuum approximation and discrete orthonormality property. The input and output arrays are regular and dense enough so that the discrete orthonormality property holds. If δx, resp. δy, is the spacing between the emission, resp. control, points, then $n = |\Omega|/\delta x^d$, $p = |\Omega|/\delta y^d$, and for all $j, j' = 1, \dots, N$,

$$\frac{|\Omega|}{n} \sum_{k=1}^{n} \phi_j(\boldsymbol{x}_k) \overline{\phi_{j'}(\boldsymbol{x}_k)} = \mathbf{1}_{j=j'}, \qquad \frac{|\Omega|}{p} \sum_{l=1}^{p} \phi_j(\boldsymbol{y}_l) \overline{\phi_{j'}(\boldsymbol{y}_l)} = \mathbf{1}_{j=j'}. \tag{1}$$

Hypothesis 2: uniform distribution of eigenfunctions. The number N of modes is large enough so that, for almost every $\boldsymbol{y} \in \Omega$,

$$\frac{|\Omega|}{N} \sum_{j=1}^{N} |\phi_j(\boldsymbol{y})|^2 = 1 + o_{N \to \infty}(1), \tag{2}$$

and

$$\frac{|\Omega|^2}{N^2} \sum_{j=1}^{N} |\phi_j(\boldsymbol{y})|^4 = o_{N \to \infty}(1). \tag{3}$$

The hypothesis (2) is fulfilled for very general waveguides in a weak sense [7, 19]. The condition (2) indicates that the intensity of an equipartitioned field (whose energy is distributed over all modes) is well distributed spatially in the Cesaro mean, and the condition (3) prevents one mode from carrying all the energy (the situation in which $|\phi_{j_0}(\boldsymbol{y})|^2 \simeq N$ and $|\phi_j(\boldsymbol{y})|^2 \simeq 0$ for $j \neq j_0$ is rejected).

Hypothesis 3: localization regime. The mode power transmission coefficients $|T_j|^2$ are independent and identically distributed. The distribution exhibits an exponential decay as a function of the length of the system and it is described in Section 4.

The hypotheses 1, 2, and 3 can be regarded as natural hypotheses for a random multi-mode waveguide in the localization regime. The only point that can be considered as special is the independence of the mode power transmission coefficients. This is not what could be expected from a random matrix theory approach in which level repulsion involves correlation [11, 17, 1]. However this hypothesis is fulfilled by realistic models that we describe in the next section and for which a complete analysis can be carried out, so it deserves our attention.

3. **Random waveguide models.**

In this section we present two models of random multi-mode waveguides that satisfy the three hypotheses listed above.

3.1. **Metallic planar waveguides.**

In this section we study propagation of optical modes in dielectric films. We consider the basic problem of TE (transversal electric) mode propagation in slab dielectric waveguides [2]. We assume that the slab waveguide has thickness D and is located in the region $\boldsymbol{x} = (x, y, z) \in [-D/2, D/2] \times \mathbb{R}^2$. Its axis is z and the slab is infinite in the y-direction. The slab is switched between two metallic slabs for $x > D/2$ and for $x < -D/2$. We restrict ourselves to the y-independent case and consider time-harmonic waves $\boldsymbol{E}(\boldsymbol{x}) e^{-i\omega t}$ which depend only on x and z and which solve

$$\Delta \boldsymbol{E}(\boldsymbol{x}) + k_0^2(\omega) n^2(\boldsymbol{x}) \boldsymbol{E}(\boldsymbol{x}) = 0, \tag{4}$$

for $x \in (-D/2, D/2)$, where $k_0(\omega) = \omega/c$ is the vacuum wavenumber. The wave also satisfies continuity conditions of the tangential components of the field at the dielectric interfaces.

We begin by studying the properties of guided modes in a perfect waveguide, whose core has a homogeneous index of refraction equal to n_0. A mode is a monochromatic wave whose complex amplitude $\boldsymbol{E}(\boldsymbol{x})$ is solution of the time-harmonic wave equation (4) with $n(\boldsymbol{x}) = n_0$. Limiting ourselves to waves with phase front normal to the waveguide axis z, we have $\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}(x)e^{i\beta z}$. We are looking for TE modes $\boldsymbol{E} = (0, E_y, 0)$ with field component $E_y = \phi(x)e^{i\beta z}$. The scalar field ϕ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + (k^2(\omega) - \beta^2)\phi = 0, \tag{5}$$

where $k(\omega)$ is the homogeneous wavenumber $k(\omega) = n_0 k_0(\omega)$. In the two metallic slabs $x > D/2$ and $x < -D/2$ the electric field is zero. Because of the need to match E_y at $x = -D/2$ and $x = D/2$ the field ϕ solution of (5) satisfies the Dirichlet boundary conditions $\phi(-D/2) = \phi(D/2) = 0$. There exists solutions only for some values of β. Thus the metallic planar waveguide can only support a finite number of confined TE modes, $E_j(x,z) = \phi_j(x)e^{i\beta_j(\omega)z}$, where

$$\phi_j(x) = \begin{cases} \frac{\sqrt{2}}{\sqrt{D}} \cos\Big(\frac{j\pi x}{D}\Big), & \text{if } j \text{ is odd}, \\ \frac{\sqrt{2}}{\sqrt{D}} \sin\Big(\frac{j\pi x}{D}\Big), & \text{if } j \text{ is even}, \end{cases} \tag{6}$$

and $\beta_j(\omega) > 0$ satisfies the dispersion relation

$$\beta_j^2(\omega) + \frac{\pi^2 j^2}{D^2} = k^2(\omega). \tag{7}$$

There exists N guided modes, where N is

$$N = \Big[\frac{k(\omega)D}{\pi}\Big], \tag{8}$$

and $[x]$ is the integer part of a real number x. Note that the mode profiles satisfy the orthonormality property

$$\int_{-D/2}^{D/2} \phi_j(x)\overline{\phi_{j'}(x)}dx = \mathbf{1}_{j=j'}.$$

Moreover, we have

$$\begin{aligned} \frac{D}{N}\sum_{j=1}^N |\phi_j(x))|^2 &= 1 - \frac{1}{N}\sum_{j=1}^N \cos\Big(\frac{2j\pi x}{D}\Big)(-1)^j \\ &= 1 + \frac{1}{N}\frac{\cos\big(\frac{\pi N x}{D} + \frac{\pi N}{2}\big)\sin\big(\frac{\pi(N-1)x}{D} + \frac{\pi(N-1)}{2}\big)}{\cos\big(\frac{\pi x}{D}\big)}, \end{aligned}$$

which shows that the mode profiles satisfy the uniformity property (2):

$$\frac{D}{N}\sum_{j=1}^N |\phi_j(x)|^2 = 1 + O\big(\frac{1}{N}\big) \qquad \forall x \in (-D/2, D/2).$$

The condition (3) is readily fulfilled since ϕ_j is bounded by $\sqrt{2/D}$. Therefore the modes satisfy the hypotheses 1 and 2 listed in Section 2.

From now on we consider the case in which a section $z \in [0, L]$ of randomly perturbed waveguide is sandwiched in between two homogeneous waveguides. We assume that a monochromatic wave is incoming from the left and has the following form at the entrance of the random waveguide:

$$\psi_{\rm in}(x) = \sum_{j=1}^{N} \phi_j(x)\alpha_j, \tag{9}$$

where $(\alpha_j)_{j=1,\dots,N}$ is the decomposition of the incident wave on the propagating modes. We assume that the medium inside the section $z \in [0, L]$ of the waveguide is affected by small random inhomogeneities, so that its index of refraction has the representation:

$$n^2(\boldsymbol{x}) = n_0^2\big(1 + m(z)\big). \tag{10}$$

The random coefficient $m(z)$ which describes the inhomogeneities is assumed to be a zero-mean stationary random process with strong mixing properties so that we can use averaging techniques for stochastic differential equations as presented in [4, Chapter 6]. We denote by γ the power spectral density of the fluctuations, which is the Fourier transform of the autocorrelation function:

$$\gamma(k) = \int_{-\infty}^{\infty} \cos(2kz)\mathbb{E}[m(0)m(z)]dz. \tag{11}$$

We consider here that the typical amplitude of the random fluctuations of the medium is small and that the length of the perturbed waveguide is much longer than the wavelength. In this weakly scattering regime the fluctuations of the index of refraction induce a random coupling between forward and backward modes, which results in a decay of the mode power transmission coefficients. More precisely we have the following proposition [5].

Proposition 1. *The transmitted wave has the following form:*

$$\psi_{\rm out}(x) = \sum_{j=1}^{N} \alpha_j T_j(L)\phi_j(x). \tag{12}$$

In the homogeneous case the transmission coefficients are

$$T_j(L) = e^{i\beta_j(\omega)L}.$$

In the weakly scattering regime the mode power transmission coefficients $\big(\mathcal{T}_j(L)\big)_{L\geq 0}$ *defined by*

$$\mathcal{T}_j(L) = |T_j(L)|^2, \qquad j = 1, \dots, N,$$

are independent diffusion Markov processes with generators

$$\mathcal{L}_j = \frac{1}{L_{\rm loc}^{(j)}(\omega)}\Big(\tau^2(1-\tau)\frac{\partial^2}{\partial\tau^2} - \tau^2\frac{\partial}{\partial\tau}\Big), \qquad \frac{1}{L_{\rm loc}^{(j)}(\omega)} = \frac{k^4(\omega)\gamma(\beta_j(\omega))}{4\beta_j^2(\omega)}. \tag{13}$$

As we will see in Section 4 we can observe an exponential localization of the modes. Therefore the modes satisfy the hypothesis 3 in Section 2.

3.2. **An array of single-mode waveguides.** Let us consider an array of N single-mode randomly perturbed optical fibers. The injection of the light into this array is done by sending a wave field through a converging lens and light is collected in the far field through a similar system (see Figure 1). In order to simplify the presentation we consider again a two-dimensional system (one dimension for the propagation axis and one transverse dimension). We denote by D the diameter of the emission array in the near field of the input lens, by λ the wavelength, by F the focal length of the lens, and by a the distance between the fibers. The transverse

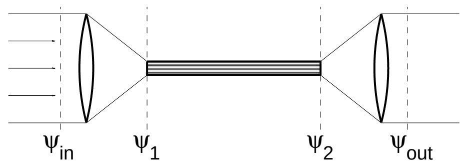


FIGURE 1. An array of single-mode waveguides. The input field is injected in the array by a lens and the field emerging from the array is collected by a similar lens.

position of the j-th fiber is $x_j = (j - N/2)a$, $j = 1, \dots, N$. If we denote by $\psi_{\rm in}(x)$ the wavefield at the entrance plane of the input lens, then the field at the entrance of the jth fiber is (up to a multiplicative constant):

$$\psi_1(x_j) = \int_{-D/2}^{D/2} \psi_{\rm in}(x) \exp\Big(i2\pi \frac{x_j x}{\lambda F}\Big) dx,$$

the field at the output of the jth fiber is

$$\psi_2(x_j) = T_j \psi_1(x_j),$$

where T_j is the transmission coefficient of the jth optical fiber. The output field collected by the output lens is (up to a multiplicative constant):

$$\psi_{\rm out}(y) = \sum_{j=1}^N \psi_2(x_j) \exp\Big(- i2\pi \frac{x_j y}{\lambda F}\Big).$$

Therefore we have

$$\psi_{\rm out}(y) = \sum_{j=1}^N \alpha_j T_j \phi_j(y), \qquad \alpha_j = \int_{-D/2}^{D/2} \overline{\phi_j(x)} \psi_{\rm in}(x) dx,$$

with

$$\phi_j(x) = \frac{1}{\sqrt{D}} \exp\Big(- i2\pi \frac{x_j x}{\lambda F}\Big) = \frac{e^{i\pi QNx/D}}{\sqrt{D}} \exp\Big(- i2\pi Q j \frac{x}{D}\Big),$$

where $Q = Da/(F\lambda)$. If we assume that Q is an integer, then the modes $\phi_j(x)$ satisfy the orthonormality property

$$\int_{-D/2}^{D/2} \phi_j(x) \overline{\phi_{j'}(x)} dx = \mathbf{1}_{j=j'},$$

and the uniformity property (2): for all $y \in [-D/2, D/2]$,

$$\frac{D}{N} \sum_{j=1}^N |\phi_j(y)|^2 = 1.$$

Moreover the condition (3) is fulfilled since $|\phi_j(x)| = 1/\sqrt{D}$. The mode power transmission coefficients $\mathcal{T}_j(L) = |T_j(L)|^2$ are independent since they are associated to different fibers and they are identically distributed according to the distribution described in Proposition 1 in the case $N = 1$ (single-mode waveguide).

Proposition 2. *The transmitted wave has the form*

$$\psi_{\rm out}(x) = \sum_{j=1}^N \alpha_j T_j(L) \phi_j(x). \tag{14}$$

In the homogeneous case the transmission coefficients are equal to

$$T_j(L) = e^{i\beta_1(\omega)L},$$

where L is the length of the fiber and $\beta_1(\omega)$ is the reduced wavenumber for the fundamental (and unique) mode.

In the weakly scattering regime the mode power transmission coefficients $\big(\mathcal{T}_j(L)\big)_{L\geq 0}$, $j=1,\ldots,N$, are independent and identically distributed diffusion Markov processes with generator

$$\mathcal{L} = \frac{1}{L_{\rm loc}(\omega)} \Big(\tau^2(1-\tau) \frac{\partial^2}{\partial \tau^2} - \tau^2 \frac{\partial}{\partial\tau}\Big), \qquad \frac{1}{L_{\rm loc}(\omega)} = \frac{k^4(\omega)\gamma(\beta_1(\omega))}{4\beta_1^2(\omega)}. \tag{15}$$

Therefore the modes satisfy the three hypotheses listed in Section 2.

4. **The distribution of the mode power transmission coefficient.**

As a function of L, the mode power transmission coefficient $\mathcal{T}(L)$ is a Markov process with the infinitesimal generator given by

$$\mathcal{L} = \frac{1}{L_{\rm loc}} \Big(\tau^2(1-\tau) \frac{\partial^2}{\partial \tau^2} - \tau^2 \frac{\partial}{\partial\tau}\Big). \tag{16}$$

The initial condition for the power transmission coefficient is $\mathcal{T}(L=0)=1$. This result has been known for a long time for one-dimensional random systems [8, 1, 13].

4.1. **The moments of the mode power transmission coefficient.**

The mode power transmission coefficient $\mathcal{T}(L)$ is a random variable whose moments are given in [4, Section 7.1.5]:

$$\mathbb{E}\big[\mathcal{T}^n(L)\big] = \xi_n\Big(\frac{L}{L_{\rm loc}}\Big), \tag{17}$$

where

$$\xi_n(l) = \exp\Big(-\frac{l}{4}\Big) \int_0^\infty e^{-\mu^2 l} \frac{2\pi\mu \sinh(\pi\mu)}{\cosh^2(\pi\mu)} \zeta_n(\mu) d\mu, \tag{18}$$

with $\zeta_1(\mu)=1$ and, for $n\geq 2$,

$$\zeta_n(\mu) = \prod_{j=1}^{n-1} \frac{1}{j^2}\left[\mu^2 + (j-\frac{1}{2})^2\right].$$

The large-l behavior of the functions $\xi_n(l)$ is

$$\xi_n(l) \overset{l\gg 1}{\simeq} \frac{\pi^{5/2}\zeta_n(0)}{2l^{3/2}} \exp\Big(-\frac{l}{4}\Big). \tag{19}$$

We can remark that

$$\frac{\xi_n(l)}{\xi_1(l)} \overset{l\gg 1}{\simeq} \zeta_n(0), \qquad \zeta_n(0) = \prod_{j=1}^{n-1}\Big(1-\frac{1}{2j}\Big)^2, \tag{20}$$

which shows that the decay is the same for all moments of the mode power transmission coefficient. We have in particular

$$\frac{\mathbb{E}\big[\mathcal{T}^2(L)\big]}{\mathbb{E}\big[\mathcal{T}(L)\big]} = \frac{\xi_2(L/L_{\rm loc})}{\xi_1(L/L_{\rm loc})} \overset{L\gg L_{\rm loc}}{\simeq} \frac{1}{4}, \tag{21}$$

which is a key theoretical result that will be used in the analysis of transmission enhancement by optimal illumination in Section 5.

4.2. **The probability density function of the mode power transmission coefficient.** The computation of the probability density function of $\mathcal{T}(L)$ can be carried out using the Mehler-Fock transform. Here we denote by $P_{-1/2+i\mu}(\eta)$, $\eta \geq 1$, $\mu \geq 0$, the Legendre function of the first kind, which is the solution of

$$\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta}P_{-1/2+i\mu}(\eta) = -\Big(\mu^2+\frac{1}{4}\Big)P_{-1/2+i\mu}(\eta), \tag{22}$$

starting from $P_{-1/2+i\mu}(1)=1$. It has the integral representation

$$P_{-1/2+i\mu}(\eta) = \frac{\sqrt{2}}{\pi}\cosh(\pi\mu)\int_0^\infty \frac{\cos(\mu s)}{\sqrt{\cosh(s)+\eta}}ds. \tag{23}$$

The probability density of the mode power transmission coefficient $\mathcal{T}(L)$ is [4, Section 7.6.1]

$$p\Big(\tau,\frac{L}{L_{\rm loc}}\Big) = \frac{2}{\tau^2}\int_0^\infty \mu\tanh(\mu\pi)P_{-1/2+i\mu}\Big(\frac{2}{\tau}-1\Big)\exp\Big[-\big(\mu^2+\frac{1}{4}\big)\frac{L}{L_{\rm loc}}\Big]d\mu.$$

This expression can be approximated when $L \gg L_{\rm loc}$, but we must pay attention to the range of values of τ for the approximation as we now discuss.

The distribution of $\mathcal{T}(L)$ is concentrated in a logarithmic sense around the value [4, Section 7.1.6]

$$\frac{1}{L}\ln\mathcal{T}(L) \stackrel{L\to\infty}{\longrightarrow} -\frac{1}{L_{\rm loc}} \quad \text{almost surely},$$

and more precisely

$$\sqrt{L}\Big(\frac{1}{L}\ln\mathcal{T}(L)+\frac{1}{L_{\rm loc}}\Big) \stackrel{L\to\infty}{\longrightarrow} \mathcal{N}\Big(0,\frac{2}{L_{\rm loc}}\Big) \quad \text{in distribution}.$$

A log-normal approximation for the pdf can be derived from this result:

$$p(\tau,l) \stackrel{l\gg 1}{\simeq} \frac{1}{2\sqrt{\pi}l^{1/2}\tau}\exp\Big(-\frac{(\ln(\tau)+l)^2}{4l}\Big). \tag{24}$$

This expression is valid when $\tau \ll 1$.

The probability that $\mathcal{T}(L)$ is of order one is small but not completely negligible. In fact these rare events are important and they determine the values of the moments of the mode power transmission coefficient. For any $\tau_0>0$, we have

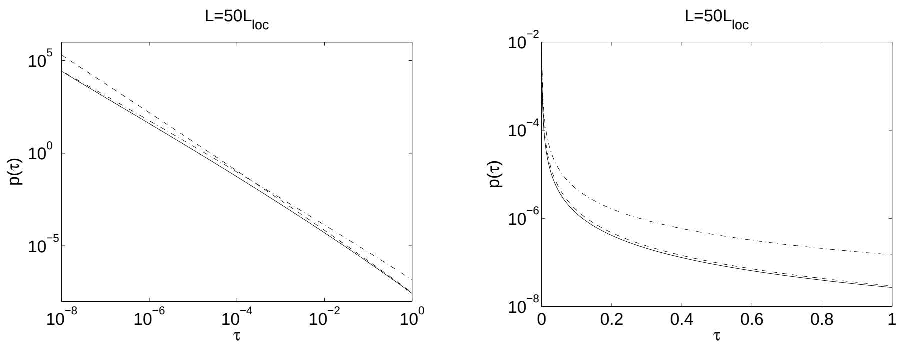
$$p(\tau,l) \stackrel{l\gg 1}{\simeq} p_\infty(\tau) := \frac{\sqrt{\pi}}{l^{3/2}}\exp\Big(-\frac{l}{4}\Big)\frac{1}{\tau^{3/2}}K(1-\tau), \tag{25}$$

uniformly for $\tau \in [\tau_0,1]$, where K is the complete elliptic integral of the first kind defined by

$$K(m) = \int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2\theta}}d\theta.$$

On Figure 2a where the pdf is plotted with a logarithmic scale in τ, we can check that the log-normal approximation gives the right value for the pdf of $\mathcal{T}(L)$ for the small values of τ.

On Figure 2b where the pdf is plotted with a linear scale in τ, we can check that the log-normal approximation over-estimates the probability that $\mathcal{T}(L)$ is of order one, while the approximation $p_\infty(\tau)$ is indeed correct for τ of order one.



(a) **(b)**

FIGURE 2. Pdf of the mode power transmission coefficient. The solid line is the exact expression. The dashed line is the approximation $p_\infty(\tau)$ given by (25) which is valid for τ of order one. The dot-dashed line is the log-normal approximation (24) which is valid for small τ. The horizontal axis is logarithmic in picture a (so that the behavior of the pdf for small values of τ can be observed) and linear in picture b (so that the behavior of the pdf for values of τ of order one can be observed).

Note also that $p_\infty(\tau)$ is not integrable (there is a divergence at 0 which is not contradictory since the approximation $p(\tau, l) \simeq p_\infty(\tau)$ as $l \gg 1$ is valid only for $\tau > 0$). However, for any $n \geq 1$,

$$\int_0^1 \tau^n p_\infty(\tau) d\tau = \frac{\pi^{5/2}\zeta_n(0)}{2l^{3/2}} \exp\Big(-\frac{l}{4}\Big).$$

By comparing with (19) this result confirms the assertion that the rare events for which $\mathcal{T}(L)$ is of order one determine the values of the moments of the mode power transmission coefficient.

These results are qualitatively similar to, but quantitatively different from, the ones found by Dorokhov [3] in the diffusive regime. In the diffusive regime most of the channels have exponentially small transmission, while there are a few open channels with transmission of order one whose distribution is proportional to

$$p_0(\tau) = \frac{1}{\tau\sqrt{1-\tau}}. \tag{26}$$

This result can be proved by using random matrix theory [10]. In this case, the ratio of the first two moments of the transmission coefficient is given by

$$\frac{\mathbb{E}\big[\mathcal{T}^2(L)\big]}{\mathbb{E}\big[\mathcal{T}(L)\big]} \overset{\text{diff.}}{\simeq} \frac{\int_0^1 \tau^2 p_0(\tau) d\tau}{\int_0^1 \tau p_0(\tau) d\tau} = \frac{2}{3}. \tag{27}$$

To summarize, in the localization regime, the distribution of $\mathcal{T}(L)$ is bimodal: with a probability close to one (of the order of $1 - \exp(-l/4)$, where $l = L/L_{\rm loc}$), it is of the order of $\exp(-l)$; with a probability of the order of $\exp(-l/4)$ it is of order one. This result can be interpreted as a manifestation in the localization regime of the maximal fluctuations theory known in the diffusive regime [9, 16].

5. **Optimization of the transmitted intensity.** In this section we want to optimize the illumination in the input plane of the waveguide so as to maximize the transmitted intensity at a target point in the output plane. The situation is the one described in Section 2. If we use the illumination $(E_k)_{k=1,\dots,n}$, then the input field is

$$\psi_{\rm in}(\boldsymbol{x}) = \sum_{k=1}^n E_k \psi_\delta(\boldsymbol{x}-\boldsymbol{x}_k), \qquad \psi_\delta(\boldsymbol{x}) = \frac{1}{(\delta x)^d} \mathbf{1}_{[-\delta x/2,\delta x/2]^d}(\boldsymbol{x}),$$

and the wave field at the control point $\boldsymbol{y}_l$ $(l=1,\dots,p)$ is:

$$\psi_{\rm out}(\boldsymbol{y}_l) = \sum_{k=1}^n M_{lk} E_k,$$

where $\mathbf{M}$ is the $p\times n$ transmission matrix between the input and output arrays

$$M_{lk} = \sum_{j=1}^N \phi_j(\boldsymbol{y}_l) T_j \overline{\phi_j(\boldsymbol{x}_k)}.$$

The total incident power can be expressed in terms of the illumination vector $(E_k)_{k=1,\dots,n}$ as:

$$\mathcal{P}_{\rm in} := \int_\Omega |\psi_{\rm in}(\boldsymbol{x})|^2 d\boldsymbol{x} = \frac{n}{|\Omega|} \sum_{k=1}^n |E_k|^2.$$

The field received in the output plane using the illumination $(E_k)_{k=1,\dots,n}$ is:

$$\psi_{\rm out}(\boldsymbol{y}) = \sum_{j=1}^N \phi_j(\boldsymbol{y}) T_j \Big(\sum_{k=1}^n \overline{\phi_j(\boldsymbol{x}_k)} E_k \Big),$$

and the total transmitted power is defined by

$$\mathcal{P}_{\rm out} := \int_\Omega \big|\psi_{\rm out}(\boldsymbol{y})\big|^2 d\boldsymbol{y}.$$

Let us fix some $l_0 \in \{1,\dots,p\}$. If we want to maximize the intensity transmitted at $\boldsymbol{y}_{l_0}$ amongst all possible illuminations with unit incident power, then we need to maximize the functional

$$(E_k)_{k=1,\dots,n} \mapsto \Big| \sum_{k=1}^n M_{l_0 k} E_k \Big|^2 \text{ with the constraint } \sum_{k=1}^n |E_k|^2 = \frac{|\Omega|}{n}.$$

Using the Cauchy-Schwarz inequality, we have

$$\Big| \sum_{k=1}^n M_{l_0 k} E_k \Big|^2 \leq \Big[\sum_{k=1}^n |M_{l_0 k}|^2 \Big] \Big[\sum_{k=1}^n |E_k|^2 \Big],$$

with equality if and only if $E_k = c\overline{M_{l_0 k}}$ for all $k=1,\dots,n$ and for some constant c. Therefore the optimal illumination vector with unit incident power is

$$E_k^{\rm opt} = \frac{\sqrt{|\Omega|}}{\sqrt{n}} \frac{\overline{M_{l_0 k}}}{\big[\sum_{m=1}^n |M_{l_0 m}|^2\big]^{1/2}}, \quad k=1,\dots,n, \tag{28}$$

and the maximal intensity obtained at the control point $\boldsymbol{y}_{l_0}$ with this optimal illumination vector is

$$I_{l_0}^{\rm opt} := |\psi_{\rm out}^{\rm opt}(\boldsymbol{y}_{l_0})|^2 = \frac{|\Omega|}{n} \sum_{k=1}^n |M_{l_0 k}|^2.$$

The optimal illumination vector $(E_k^{\rm opt})_{k=1,\ldots,n}$ is (proportional to) the complex conjugate of the transfer function from $\boldsymbol{x}_k$ to $\boldsymbol{y}_{l_0}$, which means that a time-reversal operation or a phase-conjugation mirror achieves the optimal illumination: If the point $\boldsymbol{y}_{l_0}$ at the output plane emits a wave that is recorded by the array $(\boldsymbol{x}_k)_{k=1,\ldots,n}$ at the input plane, then the time-harmonic transfer vector is $(M_{kl_0})_{k=1,\ldots,n}$ and the optimal illumination vector (28) is proportional to the complex conjugate of the transfer vector. As we will see in Section 7 in which we address a time-dependent problem, the correct interpretation is in terms of time-reversal rather than phase conjugation.

Of course the optimal illumination vector depends on the target point $\boldsymbol{y}_{l_0}$ on which we want to transmit power. However, the total transmitted power is also increased by using this illumination, as we will see in Proposition 3. The output field received in the output plane with the illumination $(E_k^{\rm opt})_{k=1,\ldots,n}$ is denoted by $\psi_{\rm out}^{\rm opt}$ and the total transmitted power by $\mathcal{P}_{\rm out}^{\rm opt}$. Since the total incident power has been normalized to one, the quantity $\mathcal{P}_{\rm out}^{\rm opt}$ is the total power transmission coefficient.

In the following proposition we compare the transmitted intensity at $\boldsymbol{y}_{l_0}$ and the total transmitted power using the optimal illumination with the ones obtained with uniform illumination. We define a uniform illumination with unit incident power as an input field $\psi_{\rm in}(\boldsymbol{x})$ of the form

$$\psi_{\rm in}(\boldsymbol{x}) = \sum_{j=1}^N \alpha_j \phi_j(\boldsymbol{x}),$$

where the coefficients $(\alpha_j)_{j=1,\ldots,N}$ are uniformly distributed over the unit sphere in $\mathbb{C}^N$.

Proposition 3. *With the hypotheses 1, 2, and 3 and with $N \gg 1$.*

1. Using a uniform illumination, the intensity transmitted to $\boldsymbol{y}_{l_0}$ is a random quantity with expectation and variance given by

$$\mathbb{E}\big[I_{l_0}^{\rm unif}\big] = \frac{\xi_1(L/L_{\rm loc})}{|\Omega|}, \qquad {\rm Var}\big(I_{l_0}^{\rm unif}\big) = \frac{\xi_1^2(L/L_{\rm loc})}{|\Omega|^2}, \tag{29}$$

where the function ξ_1 is defined by (18). The total power transmission coefficient is a statistically stable quantity given by

$$\mathcal{P}_{\rm out}^{\rm unif} = \xi_1(L/L_{\rm loc}). \tag{30}$$

2. Using the optimal illumination calibrated with the target point $\boldsymbol{y}_{l_0}$, the intensity transmitted to $\boldsymbol{y}_{l_0}$ and the total power transmission coefficient are self-averaging quantities given by

$$I_{l_0}^{\rm opt} = \frac{N\xi_1(L/L_{\rm loc})}{|\Omega|}, \qquad \mathcal{P}_{\rm out}^{\rm opt} = \frac{\xi_2(L/L_{\rm loc})}{\xi_1(L/L_{\rm loc})}, \tag{31}$$

where the functions ξ_1 and ξ_2 are defined by (18).

This proposition is proved in the Appendix. In fact it can be proved that $I_{l_0}^{\rm unif}$ has an exponential distribution as $N \to \infty$, which corresponds to speckle pattern statistics. Using the optimal illumination the intensity observed at $\boldsymbol{y}_{l_0}$ is (in the limit $N \to \infty$) deterministic. Heuristically, this can be explained as follows. The transmitted field at $\boldsymbol{y}_{l_0}$ is the incoherent superposition of N complex-valued modes with zero mean and averaged intensity $\xi_1/(N|\Omega|)$. The central limit theorem then predicts that the transmitted field at $\boldsymbol{y}_{l_0}$ is a complex Gaussian field with mean zero and variance $\xi_1/|\Omega|$, which gives an exponential distribution with mean $\xi_1/|\Omega|$ for the transmitted intensity $I_{l_0}^{\rm unif}$. Using the optimal illumination, the random phases between the modes are canceled at $\boldsymbol{y}_{l_0}$. Therefore the transmitted field at $\boldsymbol{y}_{l_0}$ is the coherent superposition of N modes with averaged amplitude $\sqrt{\xi_1/(N|\Omega|)}$. The law of large numbers then predicts that the transmitted field at $\boldsymbol{y}_{l_0}$ is the deterministic value $\sqrt{N\xi_1/|\Omega|}$, which gives (31).

By optimizing the transmission to one control point, the intensity transmitted to that point is multiplied by N while the total power transmission coefficient is enhanced by the factor $\xi_2/\xi_1^2(L/L_{\rm loc})$ (which is always larger than 1 and all the larger as $L/L_{\rm loc}$ is larger, as we will see below). We can be more precise in the distribution of the total power transmission. Let us introduce the correlation function

$$C(\boldsymbol{y},\boldsymbol{y}_{l_0}) = \left| \frac{\sum_{j=1}^N \phi_j(\boldsymbol{y})\overline{\phi_j(\boldsymbol{y}_{l_0})}}{\sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^2} \right|^2 .$$

The correlation function is maximal and equal to one for $\boldsymbol{y}=\boldsymbol{y}_{l_0}$ and decays to 0 for $\boldsymbol{y}$ far from $\boldsymbol{y}_{l_0}$. Note that

$$\frac{1}{|\Omega|}\int_\Omega C(\boldsymbol{y},\boldsymbol{y}_{l_0}) d\boldsymbol{y} = \frac{1}{N}, \qquad \frac{1}{p}\sum_{l=1}^p C(\boldsymbol{y}_l,\boldsymbol{y}_{l_0}) = \frac{1}{N},$$

which indicates that the width of the peak around $\boldsymbol{y}_{l_0}$ is of the order of $|\Omega|^{1/d}/N^{1/d}$ where $|\Omega|^{1/d}$ is the diameter of the domain Ω. For instance, in the two examples introduced in Section 3 we have

$$C(y,y_{l_0}) = {\rm sinc}^2\Big(\frac{\pi N(y-y_{l_0})}{D}\Big),$$

where D is the diameter of the domain $\Omega=[-D/2,D/2]$.

Proposition 4. *Using the optimal illumination calibrated with the target point* $\boldsymbol{y}_{l_0}$*, the mean intensity transmitted to* $\boldsymbol{y}_l$ *is given by*

$$\mathbb{E}\big[I_l^{\rm opt}\big] = \frac{N\xi_1(L/L_{\rm loc})}{|\Omega|} C(\boldsymbol{y}_l,\boldsymbol{y}_{l_0}) + \frac{\xi_2(L/L_{\rm loc})-\xi_1^2(L/L_{\rm loc})}{\xi_1(L/L_{\rm loc})|\Omega|}. \tag{32}$$

In Eq. (32) the first term of the right-hand side dominates when $\boldsymbol{y}_l$ is close to $\boldsymbol{y}_{l_0}$.

This proposition is proved in the Appendix. It shows that the intensity enhancement is maximal at $\boldsymbol{y}_{l_0}$ and affects also neighboring points. The width of the enhancement peak is small, of the order of $|\Omega|^{1/d}/N^{1/d}$. We can also observe a modification of the intensity transmitted to the points far from $\boldsymbol{y}_{l_0}$ (i.e. the background intensity). There are two regimes (see Figure 4a):

- If $L<2.06L_{\rm loc}$ (which corresponds to the case in which $(\xi_2-\xi_1^2)/\xi_1<\xi_1$), then the intensities transmitted to the points far from $\boldsymbol{y}_{l_0}$ using the optimal illumination is reduced compared to the uniform illumination. In the picture $\boldsymbol{y}_l \mapsto \mathbb{E}[I_l^{\rm opt}]$ (see Figure 3a) we can therefore observe an enhanced peak around $\boldsymbol{y}_{l_0}$ and a reduced background (compared to the uniform illumination). In this case the optimal illumination has modified the phases of the modes to favor the transmitted intensity at $\boldsymbol{y}_{l_0}$ and this reduces the intensity transmitted to other points.
- If $L>2.06L_{\rm loc}$ (which corresponds to the case in which $(\xi_2-\xi_1^2)/\xi_1>\xi_1$), then the intensities transmitted to the points far from $\boldsymbol{y}_{l_0}$ using the optimal illumination are increased compared to the ones obtained with uniform illumination. In the picture $\boldsymbol{y}_l \mapsto \mathbb{E}[I_l^{\rm opt}]$ (see Figure 3b) we can therefore observe an enhanced peak around

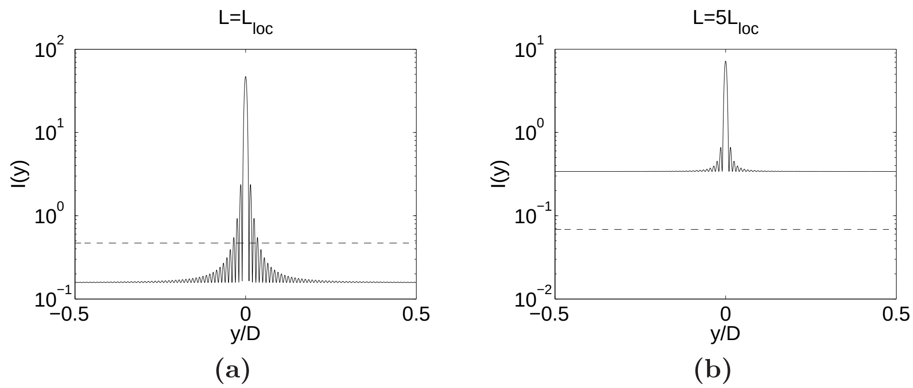


FIGURE 3. Mean transmitted intensity profiles using the optimal illumination calibrated with $y_{l_0} = 0$ (solid lines) and using the uniform illumination (dashed lines) in the case of a multimode waveguide with diameter $D = 1$ and $N = 100$. The peak intensity at y_{l_0} is always larger with the optimal illumination than with the uniform illumination. The background intensity with the optimal illumination is smaller (resp. larger) than the background intensity with the uniform illumination when $L < 2.06L_{\rm loc}$ (resp. $L > 2.06L_{\rm loc}$).

$\boldsymbol{y}_{l_0}$ and an enhanced background, although the optimal illumination algorithm does not look after such an enhancement. It is the result of the analysis that the optimal illumination process calibrated for a target point also benefits to the other points if scattering is strong enough, while it removes intensity from the other points if scattering is weak.

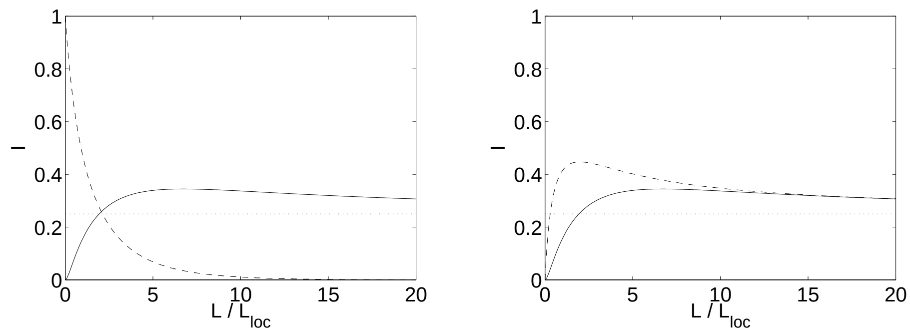
The total transmitted power using the optimal illumination is ξ_2/ξ_1 which is always larger than the total transmitted power ξ_1 obtained with the uniform illumination. The contribution of the enhanced peak around $\boldsymbol{y}_{l_0}$ to the total transmitted power is equal to ξ_1, while the contribution of the background to the total transmitted power is $(\xi_2 - \xi_1^2)/\xi_1$. In the strong localization regime $L \gg L_{\rm loc}$, since $\xi_2 \gg \xi_1^2$, the enhancement of the background gives the main contribution to the total transmission power enhancement.

Remark. Proposition 4 is also valid in the homogeneous case. In this case we have $|T_j|^2 = 1$ and we find that the transmitted intensity profile is:

$$I_l^{\rm opt} = \frac{N}{|\Omega|} C(\boldsymbol{y}_l, \boldsymbol{y}_{l_0}). \tag{33}$$

This result is not surprising. Indeed we have remarked that the optimal illumination process is equivalent to a time-reversal operation, and (33) is the expression of transverse spatial profile of the time-reversed refocused wave field in a homogeneous waveguide [6, 4].

Remark. We have seen that the intensity transmitted at the target point $\boldsymbol{y}_{l_0}$ is self-averaging, as well as the total transmitted power. The background intensity, that is, the intensity at a point $\boldsymbol{y}_l$ far from the enhanced peak around $\boldsymbol{y}_{l_0}$ is, however, a random function of the point $\boldsymbol{y}_l$, with a correlation radius of the order of



(a) **(b)**

FIGURE 4. Picture a: Mean background transmitted intensity for $|\Omega| = 1$. The dashed line is the mean background transmitted intensity ξ_1 with uniform illumination and the solid line is the mean background transmitted intensity $(\xi_2 - \xi_1^2)/\xi_1$ with optimal illumination, which converges to $1/4$ as $L/L_{\rm loc} \to \infty$. Picture b: Mean and standard deviation of the background transmitted intensity for $|\Omega| = 1$ and for optimal illumination. The solid line is the mean $(\xi_2 - \xi_1^2)/\xi_1$ and the dashed line is the standard deviation $\sqrt{\xi_2^2/\xi_1^2 - \xi_1^2}$.

$|\Omega|^{1/d}/N^{1/d}$. The expectation and variance of the background intensity are:

$$\mathbb{E}\big[I_l^{\rm opt}\big] = \frac{\xi_2(L/L_{\rm loc}) - \xi_1^2(L/L_{\rm loc})}{\xi_1(L/L_{\rm loc})|\Omega|}, \qquad {\rm Var}\big(I_l^{\rm opt}\big) = \frac{\xi_2^2(L/L_{\rm loc}) - \xi_1^4(L/L_{\rm loc})}{\xi_1^2(L/L_{\rm loc})|\Omega|^2}.$$

The standard deviation of the fluctuations of $I_l^{\rm opt}$ is always larger than its expectation, although both converge to the value $1/(4|\Omega|)$ as $L/L_{\rm loc} \to \infty$ (see Figure 4b). The fact that the relative standard deviation (ratio of the standard deviation over the expectation) is larger than one shows that the background intensity is not the classical speckle pattern but has stronger spatial fluctuations.

Using (21) and Proposition 3 we obtain the following corollary.

Corollary 1. *In the same conditions as in Proposition 3, if, additionally,* $L \gg L_{\rm loc}$*, then*

$$\mathcal{P}_{\rm out}^{\rm opt} \simeq \frac{1}{4}.$$

Note that, in the diffusive regime, we would use (27) and obtain the value $2/3$. What is really surprising is that, in the localization regime, the exponential decay of the transmission can be canceled by the optimal illumination process. We have just proved that we can obtain a total power transmission coefficient of $1/4$ in this regime by a time-reversal operation.

6. **Optimization of the total transmitted power.** In this section we want to optimize the illumination in the input plane of the waveguide so as to maximize the total transmitted power in the output plane. The situation is the one described in Section 2. We have seen in the previous section that the illumination that maximizes the intensity delivered at one target point in the output plane allows

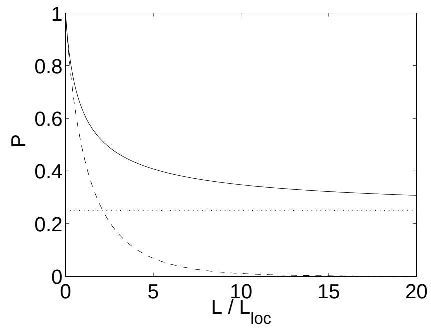


FIGURE 5. Total transmitted power. The dashed line is the total transmitted power ξ_1 with uniform illumination and the solid line is the total transmitted power ξ_2/ξ_1 with optimal illumination, which converges to $1/4$ as $L/L_{\rm loc} \to \infty$.

also a significant enhancement of the total transmitted power. We will see in this section that it is possible to do even better.

If we want to maximize the total transmitted power amongst all possible illuminations with unit incident power, then we need to maximize the functional

$$(E_k)_{k=1,\ldots,n} \mapsto \mathcal{P}_{\rm out} = \frac{|\Omega|}{p} \sum_{l=1}^{p} \Big| \sum_{k=1}^{n} M_{lk} E_k \Big|^2 \text{ with the constraint } \sum_{k=1}^{n} |E_k|^2 = \frac{|\Omega|}{n}.$$

We first note that the total transmitted power has the expression

$$\mathcal{P}_{\rm out} = \frac{|\Omega|}{p} \overline{\boldsymbol{E}}^t \overline{\mathbf{M}}^t \mathbf{M} \boldsymbol{E}, \tag{34}$$

in which the time-reversal operator

$$\mathbf{K}_{\rm TR} = \overline{\mathbf{M}}^t \mathbf{M}$$

appears. The time-reversal operator is the transfer operator from the input array to itself corresponding to the following time-reversal experiment:
i) emit from the input array an illumination $\boldsymbol{E}$ and record at the output array the field $\boldsymbol{F} = (F_l)_{l=1,\ldots,p}$. The recorded field is $\boldsymbol{F} = \mathbf{M}\boldsymbol{E}$.
i)) time-reverse the field $\boldsymbol{F}$, reemit it from the output array towards the input array, record the field transmitted at the input array, and time-reverse it. The resulting field is $\boldsymbol{E}_{\rm TR} = \overline{\mathbf{M}}^t \mathbf{M} \boldsymbol{E} = \mathbf{K}_{\rm TR} \boldsymbol{E}$.
Note that, if the input array and output array are sampled in the same way, then $\mathbf{M}^t = \mathbf{M}$ by reciprocity. In such a case, in order to measure the time-reversal operator $\mathbf{K}_{\rm TR}$, it is sufficient to carry out the first step of the time-reversal experiment which gives the matrix $\mathbf{M}$.

Using the discrete orthonormality property (1) we have

$$(\mathbf{K}_{\rm TR})_{k'k} = \frac{p}{|\Omega|} \sum_{j=1}^{N} \phi_j(\boldsymbol{x}_{k'}) |T_j|^2 \overline{\phi_j(\boldsymbol{x}_k)}, \qquad k, k' = 1, \ldots, n. \tag{35}$$

From (34) and (35) we obtain the result that the optimal illumination is

$$E_k^{\rm opt} = \frac{|\Omega|}{n}\phi_{j_0}(\boldsymbol{x}_k) \text{ where } j_0 \text{ is such that } |T_{j_0}|^2 = \max_{j=1,\dots,N} |T_j|^2. \tag{36}$$

In other words we need to choose an illumination that corresponds to the mode (or one of the modes) with the maximal mode power transmission coefficient.

We next show that an iterated time-reversal procedure can (almost) achieve this optimal illumination. Let us consider the following illumination:

$$E_k^{(q)} = \sqrt{\frac{|\Omega|}{n}} \frac{\big(\mathbf{K}_{\rm TR}^q \overline{\mathbf{M}}^t\big)_{kl_0}}{\big[\sum_{k'=1}^n |\big(\mathbf{K}_{\rm TR}^q \overline{\mathbf{M}}^t\big)_{k'l_0}|^2\big]^{1/2}}, \qquad k=1,\dots,n, \tag{37}$$

where $q \in \mathbb{N}$. This is the result of a series of q TR operations from the input array to itself using the illumination $(E_k)_{k=1,\dots,n} = (\overline{M}_{kl_0})_{k=1,\dots,n}$ used in the previous section to maximize the intensity transmitted at the target point $\boldsymbol{y}_{l_0}$. The denominator is simply a normalizing factor to get an input field with unit power. Using the illumination $(E_k^{(q)})_{k=1,\dots,n}$ we obtain the total transmitted power

$$\mathcal{P}_{\rm out}^{(q)} = \frac{\sum_{j=1}^N |T_j|^{4q+4} |\phi_j(\boldsymbol{y}_{l_0})|^2}{\sum_{j=1}^N |T_j|^{4q+2} |\phi_j(\boldsymbol{y}_{l_0})|^2}.$$

From the independence of the mode power transmission coefficients and the hypothesis (2) we obtain the following result.

Proposition 5. *Using the illumination (37) the total transmitted power is, in the regime $N \gg 1$, self-averaging and given by*

$$\mathcal{P}_{\rm out}^{(q)} = \frac{\xi_{2q+2}(L/L_{\rm loc})}{\xi_{2q+1}(L/L_{\rm loc})},$$

where the functions ξ_q are defined by (18).

If, additionally, $L \gg L_{\rm loc}$, then

$$\mathcal{P}_{\rm out}^{(q)} \simeq \Big(1 - \frac{1}{4q+2}\Big)^2.$$

This shows that a single time-reversal step increases the total transmitted power from 1/4 to 25/36, and that an iteration of the time-reversal procedure tends to give a total power transmission close to one.

For completeness, we can add that the intensity delivered at the point $\boldsymbol{y}_{l_0}$ using the illumination (37) is self-averaging and given by:

$$I_{l_0}^{(q)} = I_{l_0}^{(0)} \frac{\xi_{q+1}^2(L/L_{\rm loc})}{\xi_1(L/L_{\rm loc})\xi_{2q+1}(L/L_{\rm loc})}, \qquad I_{l_0}^{(0)} = \frac{N\xi_1(L/L_{\rm loc})}{|\Omega|}.$$

The intensity $I_{l_0}^{(q)}$ is smaller than the maximal intensity $I_{l_0}^{(0)}$ obtained with the illumination $(E_k^{(0)})_{k=1,\dots,n}$ that has been shown in the previous section to maximize the intensity delivered at $\boldsymbol{y}_{l_0}$. In particular, if $L \gg L_{\rm loc}$, then

$$\frac{I_{l_0}^{(q)}}{I_{l_0}^{(0)}} \simeq \prod_{j=1}^q \Big(\frac{1-\frac{1}{2j}}{1-\frac{1}{2q+2j}}\Big)^2.$$

This shows that a single time-reversal step increases the total transmitted power from 1/4 to 25/36, and it also reduces the intensity delivered at $\boldsymbol{y}_{l_0}$ by a factor 1/9.

7. **The one-dimensional time-dependent problem.** The multi-mode time-harmonic problem studied in the previous sections is almost equivalent to a single-mode time-dependent problem, as we now explain. In this section we consider a random single-mode channel, such as a randomly perturbed single-mode optical fiber or acoustic waveguide. The problem is to maximize the transmitted intensity at a given time. There is no transverse spatial aspect involved here. We assume that we can use a source whose bandwidth $[\omega_0 - B/2, \omega_0 + B/2]$ is fixed and that it can deliver a fixed total energy, say 1. The input time-dependent field has the form

$$\psi_{\rm in}(t) = \frac{1}{2\pi} \int_{\omega_0 - B/2}^{\omega_0 + B/2} e^{-i\omega t} \hat{\psi}_{\rm in}(\omega) d\omega,$$

where $\hat{\psi}_{\rm in}$ has support in $[\omega_0 - B/2, \omega_0 + B/2]$ and can be chosen by the user with the constraint:

$$\int |\psi_{\rm in}(t)|^2 dt = 1 \Longleftrightarrow \int_{\omega_0 - B/2}^{\omega_0 + B/2} |\hat{\psi}_{\rm in}(\omega)|^2 d\omega = 2\pi.$$

The output time-dependent field is

$$\psi_{\rm out}(t) = \frac{1}{2\pi} \int_{\omega_0 - B/2}^{\omega_0 + B/2} e^{-i\omega t} \hat{\psi}_{\rm in}(\omega) T(\omega, L) d\omega,$$

where $T(\omega, L)$ is the frequency-dependent transmission coefficient of the random channel and L is the length of the channel.

In the case of a homogeneous channel with length L we have

$$T(\omega, L) = \exp\big(i\beta_1(\omega) L\big), \tag{38}$$

where $\beta_1(\omega)$ is the wavenumber of the fundamental (and unique) mode of the channel.

In the case of a random channel with random fluctuations of the index of refraction with standard deviation σ and correlation length l_c, the joint statistics of $T(\omega, L)$ at different frequencies ω is known [4, Chapter 7]. In particular, for a fixed frequency, the moments of the mode power transmission coefficient $|T(\omega, L)|^2$ are given in Proposition 2. Moreover, the coherence properties of the transmission coefficient with respect to the frequency ω are known [4, Chapter 9]. In particular the coherence frequency ω_c is of the order of $\sigma^2 l_c \omega_0^2 / c_0$. We assume in this section that the bandwidth B is much larger than the coherence frequency ω_c of the transmission coefficient, which will ensure the statistical stability of the quantities of interest.

7.1. **Optimization of the output intensity by time-reversal.** In order to maximize the output intensity at time t_0, Cauchy-Schwarz inequality indicates that we need to take:

$$\hat{\psi}_{\rm in}^{\rm opt}(\omega) = \sqrt{2\pi} \frac{\overline{T(\omega, L)} e^{i\omega t_0}}{\big(\int_{\omega_0 - B/2}^{\omega_0 + B/2} |T(\omega', L)|^2 d\omega'\big)^{1/2}}. \tag{39}$$

This optimal field has a simple interpretation. It is proportional to the time-reversed field received at the input of the channel when a Dirac pulse is emitted from the output of the channel at time t_0. By reciprocity this is also the time-reversed field received at the output of the channel when a Dirac pulse is emitted from the input of the channel at time t_0. This result shows that time-reversal is the optimal strategy to get the maximal intensity at some given time.

Let us first briefly address the case of a homogeneous channel, for which the transmission coefficient is (38) and the optimal illumination is the uniform illumination

$$\hat{\psi}_{\rm in}^{\rm unif}(\omega) \equiv \frac{\sqrt{2\pi}}{\sqrt{B}} e^{i\omega t_0 - i\beta_1(\omega)L}. \tag{40}$$

Proposition 6. *In the case of a homogeneous channel, using a uniform illumination, the transmitted field is*

$$\psi_{\rm out}^{\rm unif}(t) = \frac{\sqrt{B}}{\sqrt{2\pi}} e^{i\omega_0(t_0-t)} {\rm sinc}\Big(\frac{B(t-t_0)}{2}\Big),$$

which shows that
1) the transmitted intensity is maximal at $t = t_0$ and it is equal to $I_{t_0} = B/(2\pi)$,
2) the total transmitted energy is equal to the input energy.

In the case of a random channel we compare the transmitted intensity and energy using the optimal illumination $\hat{\psi}_{\rm in}^{\rm opt}$ and the ones obtained with uniform illumination $\hat{\psi}_{\rm in}^{\rm unif}$ (which is the illumination that we would use if the channel was homogeneous).

Proposition 7. *1. Using a uniform illumination, the transmitted intensity at time t_0 is a random quantity whose expectation is given by*

$$\mathbb{E}\big[I_{t_0}^{\rm unif}\big] = \frac{1}{2\pi B} \int_{\omega_0-B/2}^{\omega_0+B/2} \int_{\omega_0-B/2}^{\omega_0+B/2} \exp\Big(\frac{L}{L_{\rm loc}^{1/2}(\omega) L_{\rm loc}^{1/2}(\omega')} - \frac{L}{L_{\rm loc}(\omega)} - \frac{L}{L_{\rm loc}(\omega')}\Big) d\omega d\omega' \tag{41}$$

The total transmitted energy is a statistically stable quantity given by

$$\mathcal{P}_{\rm out}^{\rm unif} = \frac{1}{B} \int_{\omega_0-B/2}^{\omega_0+B/2} \xi_1(L/L_{\rm loc}(\omega)) d\omega. \tag{42}$$

2. Using the optimal illumination calibrated with the target time t_0, the transmitted intensity at time t_0 and the total transmitted energy are self-averaging quantities given by

$$I_{t_0}^{\rm opt} = \frac{1}{2\pi} \int_{\omega_0-B/2}^{\omega_0+B/2} \xi_1(L/L_{\rm loc}(\omega)) d\omega, \qquad \mathcal{P}_{\rm out}^{\rm opt} = \frac{\int_{\omega_0-B/2}^{\omega_0+B/2} \xi_2(L/L_{\rm loc}(\omega)) d\omega}{\int_{\omega_0-B/2}^{\omega_0+B/2} \xi_1(L/L_{\rm loc}(\omega)) d\omega}, \tag{43}$$

where the functions ξ_1 and ξ_2 are defined by (18).

The first point of the proposition follows from the O'Doherty-Anstey theory [4, Section 8.2.1] for the transmitted intensity and from an energy conservation property [4, Section 7.2.2] for the total transmitted energy. The transmitted intensity at time t_0 for a uniform illumination is random because the front pulse shape is deterministic but the arrival time is randomly shifted.

The second point of the proposition can be proved using the same arguments as in the proof of Proposition 3.

In the case in which $B \ll \omega_0$ and we can neglect the ω-dependence of $L_{\rm loc}(\omega)$ for $\omega \in [\omega_0 - B/2, \omega_0 + B/2]$, we can simplify the expressions and write

$$\mathbb{E}\big[I_{t_0}^{\rm unif}\big] \simeq \frac{B}{2\pi} \exp(-L/L_{\rm loc}(\omega_0)), \qquad \mathcal{P}_{\rm out}^{\rm unif} \simeq \xi_1(L/L_{\rm loc}(\omega_0)),$$

$$I_{t_0}^{\rm opt} \simeq \frac{B}{2\pi} \xi_1(L/L_{\rm loc}(\omega_0)), \qquad \mathcal{P}_{\rm out}^{\rm opt} \simeq \frac{\xi_2(L/L_{\rm loc}(\omega_0))}{\xi_1(L/L_{\rm loc}(\omega_0))}.$$

We can see that the transmitted intensity and energy enhancements using the optimal illumination have the same properties as the ones discussed in the previous sections. In particular, the main mechanism for transmission enhancement is the selection of the small frequency bands for which transmittivity is anomalously large. The main effect is that, in the strong localization regime $L \gg L_{\rm loc}(\omega_0)$, the total transmitted energy using the optimal illumination is $1/4$, while it is exponentially small $\sim \exp[-L/(4L_{\rm loc}(\omega_0))]$ if uniform illumination is used.

7.2. **Optimization of the transmitted energy by iterated time-reversal.** In this subsection we consider the problem of the optimization of the total transmitted energy. That means that we consider the optimization problem (for the input profile $\hat{\psi}_{\rm in}$):

$$\text{maximize} \quad \mathcal{E}_{\rm out} = \int |\psi_{\rm out}(t)|^2 dt = \frac{1}{2\pi} \int |\hat{\psi}_{\rm in}(\omega)|^2 |T(\omega,L)|^2 d\omega,$$

with the constraint

$$\int |\hat{\psi}_{\rm in}(\omega)|^2 d\omega = 2\pi.$$

The -obvious- answer is that the energy of the input field should be concentrated on a frequency $\omega^{\rm opt}$ such that

$$|T(\omega^{\rm opt},L)|^2 = \max_{\omega \in [\omega_0 - B/2, \omega_0 + B/2]} |T(\omega,L)|^2,$$

and then

$$\mathcal{E}_{\rm out}^{\rm opt} = |T(\omega^{\rm opt},L)|^2.$$

The situation is therefore similar to the time-harmonic multi-mode problem, in which the optimal illumination consists in selecting the mode whose power transmission coefficient is maximal. The optimal illumination can be nearly achieved by an iterated time-reversal procedure as we now explain:
step 0: consider the signal $\psi_0(t) = \sqrt{B/(2\pi)} \mathrm{sinc}(Bt/2) \exp(i\omega_0 t)$, whose spectrum is flat in the band $[\omega_0 - B/2, \omega_0 + B/2]$.
step n: emit from the input of the channel the signal ψ_{n-1}, record at the output of the channel and time-reverse the recorded signal, which gives

$$\hat{\psi}_n(\omega) = \overline{T(\omega,L)} \hat{\psi}_{n-1}(\omega).$$

The first $n = 1$ time-reversed signal is proportional to (39) with the choice $t_0 = 0$. The n-th iterated time-reversed signal is given by

$$\hat{\psi}_n(\omega) = \begin{cases} |T(\omega,L)|^n \sqrt{2\pi/B} \mathbf{1}_{[\omega_0 - B/2, \omega_0 + B/2]}(\omega) & \text{if } n \text{ is even,} \\ |T(\omega,L)|^{n-1} \overline{T(\omega,L)} \sqrt{2\pi/B} \mathbf{1}_{[\omega_0 - B/2, \omega_0 + B/2]}(\omega) & \text{if } n \text{ is odd.} \end{cases}$$

We now use the input signal

$$\psi_{\rm in}^{{\rm opt},n}(\omega) = \sqrt{2\pi} \frac{\hat{\psi}_n(\omega)}{\big(\int_{\omega_0 - B/2}^{\omega_0 + B/2} |\hat{\psi}_n(\omega')|^2 d\omega' \big)^{1/2}},$$

where the denominator has been introduced to get an input field with unit energy. For any n, the total transmitted energy obtained with this input signal is self-averaging and given by

$$\mathcal{E}_{\rm out}^{{\rm opt},n} = \frac{\int_{\omega_0 - B/2}^{\omega_0 + B/2} \xi_{n+1}(L/L_{\rm loc}(\omega)) d\omega}{\int_{\omega_0 - B/2}^{\omega_0 + B/2} \xi_n(L/L_{\rm loc}(\omega)) d\omega} \overset{B \ll \omega_0}{\simeq} \frac{\xi_{n+1}(L/L_{\rm loc}(\omega_0))}{\xi_n(L/L_{\rm loc}(\omega_0))}.$$

When $L \gg L_{\rm loc}(\omega_0)$ we find from (20)

$$\mathcal{E}_{\rm out}^{{\rm opt},n} \stackrel{B\ll\omega_0,\, L\gg L_{\rm loc}(\omega_0)}{\simeq} \Big(1-\frac{1}{2n}\Big)^2 .$$

This result shows that iterated time-reversal is the optimal scheme to achieve total energy transmission through a one-dimensional randomly perturbed system in the localization regime.

8. **Conclusion.**

In this paper we have first studied the optimal illumination problem for a randomly perturbed multi-mode system in the localization regime and in the time-harmonic domain. This problem consists in identifying the optimal spatial wave front in the input plane of the channel that maximizes the intensity transmitted through the random system at a target point in the output plane. The physical mechanism that allows enhanced transmission using an optimized wave front is based on the selection of the open channels of the system by the optimized illumination. More quantitatively, if we denote by L the length of the system and by $L_{\rm loc}$ the localization length, most of the mode power transmission coefficients of the system are exponentially small in L, of the order of $\exp[-L/L_{\rm loc}]$, but a very small fraction of them are of order one. The proportion of these "open channels" is $\exp[-L/(4L_{\rm loc})]$. By optimizing the wave front one can select and excite open channels.

We have shown that the optimal illumination can be obtained by a time-reversal operation. This explains why the intensity transmitted at the target point and the total transmitted power are found to be statistically stable quantities, since statistical stability has been shown to be one of the key features of time reversal for waves in random media [14, 15, 4]. The optimal illumination process increases by a factor N (i.e. the number of modes) the intensity at the target point and at very neighboring points. By doing so, the optimal illumination process also decreases the background intensity (i.e. the intensities at the other points) when $L < 2.06 L_{\rm loc}$, but increases the background intensity when $L > 2.06 L_{\rm loc}$. The overall result is that the total transmitted power is always significantly enhanced. More quantitatively the total transmitted power is a decaying function of the ratio $L/L_{\rm loc}$ that converges to $1/4$ as $L/L_{\rm loc} \to \infty$. This represents a dramatic power transmission enhancement since the total transmitted power decays as $\exp[-L/(4L_{\rm loc})]$ in the case of a uniform illumination. We have also shown that the total transmitted power can be optimized using an iterated time-reversal operation. After $2q+1$ iterations the total transmitted power is found to be equal to $(1-1/(4q+2))^2$.

Finally, in the single-mode and time-dependent problem studied in Section 7 we have shown that time reversal is optimal for the maximization of the transmitted intensity at some prescribed time while iterated time reversal is optimal for the maximization of the total transmitted energy. Quantitative results are obtained that describe precisely the enhanced transmission.

Appendix A. **Proofs of Propositions 3 and 4.**

We first give a short lemma used for the derivation of the main result.

Lemma A.1. *For any input and output wave fields $\psi_{\rm in}(\boldsymbol{x})$ and $\psi_{\rm out}(\boldsymbol{y})$ we have*

$$\int_\Omega \big|\psi_{\rm in}(\boldsymbol{x})\big|^2 d\boldsymbol{x} = \frac{|\Omega|}{n}\sum_{k=1}^n \big|\psi_{\rm in}(\boldsymbol{x}_k)\big|^2, \qquad \int_\Omega \big|\psi_{\rm out}(\boldsymbol{y})\big|^2 d\boldsymbol{y} = \frac{|\Omega|}{p}\sum_{l=1}^p \big|\psi_{\rm out}(\boldsymbol{y}_l)\big|^2 .$$

Proof. We have

$$\psi_{\rm in}(\boldsymbol{x}) = \sum_{j=1}^N \alpha_j \phi_j(\boldsymbol{x}), \qquad \alpha_j = \int_\Omega \overline{\phi_j(\boldsymbol{x})} \psi_{\rm in}(\boldsymbol{x}) d\boldsymbol{x}.$$

On the one hand the continuous orthonormality property gives

$$\int_\Omega \big|\psi_{\rm in}(\boldsymbol{x})\big|^2 d\boldsymbol{x} = \sum_{j,j'=1}^N \alpha_j \overline{\alpha_{j'}} \int_\Omega \phi_j(\boldsymbol{x}) \overline{\phi_{j'}(\boldsymbol{x})} d\boldsymbol{x} = \sum_{j=1}^N |\alpha_j|^2.$$

On the other hand the discrete orthonormality property gives

$$\sum_{k=1}^n \big|\psi_{\rm in}(\boldsymbol{x}_k)\big|^2 = \sum_{j,j'=1}^N \alpha_j \overline{\alpha_{j'}} \sum_{k=1}^n \phi_j(\boldsymbol{x}_k) \overline{\phi_{j'}(\boldsymbol{x}_k)} = \frac{n}{|\Omega|} \sum_{j=1}^N |\alpha_j|^2.$$

Combining these two relations gives the desired result. The proof is the same for the output wave field. ◻

We can now give the proof of Proposition 3. In the case of a uniform illumination the coefficients $(\alpha_j)_{j=1,\dots,N}$ are uniformly distributed over the unit sphere in $\mathbb{C}^N$. If we denote by $\langle \cdot \rangle$ the expectation with respect to this distribution, then we have

$$\langle \psi_{\rm in}(\boldsymbol{x}) \rangle = 0, \qquad \big\langle |\psi_{\rm in}(\boldsymbol{x})|^2 \big\rangle = \frac{1}{N} \sum_{j=1}^N |\phi_j(\boldsymbol{x})|^2 = \frac{1}{|\Omega|},$$

where we have used the uniformity property (2). The intensity at $\boldsymbol{y}_{l_0}$ and total transmitted power are:

$$I_{l_0}^{\rm unif} = \sum_{j,j'=1}^N \alpha_j \overline{\alpha_{j'}} T_j \overline{T_{j'}} \phi_j(\boldsymbol{y}_{l_0}) \overline{\phi_{j'}(\boldsymbol{y}_{l_0})}, \qquad \mathcal{P}_{\rm out}^{\rm unif} = \sum_{j=1}^N |\alpha_j|^2 |T_j|^2.$$

We can compute the expectations of these quantities:

$$\begin{aligned}
\big\langle \mathbb{E}\big[I_{l_0}^{\rm unif}\big] \big\rangle &= \sum_{j=1}^N \big\langle |\alpha_j|^2 \big\rangle |\phi_j(\boldsymbol{y}_{l_0})|^2 \xi_1(L/L_{\rm loc}) = \frac{1}{N} \sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^2 \xi_1(L/L_{\rm loc}) \\
&= \frac{\xi_1(L/L_{\rm loc})}{|\Omega|}, \\
\big\langle \mathbb{E}\big[\mathcal{P}_{\rm out}^{\rm unif}\big] \big\rangle &= \sum_{j=1}^N \big\langle |\alpha_j|^2 \big\rangle \xi_1(L/L_{\rm loc}) = \xi_1(L/L_{\rm loc}),
\end{aligned}$$

and their variances:

$$\begin{aligned}
\mathrm{Var}\big(\mathcal{P}_{\mathrm{out}}^{\mathrm{unif}}\big) &= \sum_{j\neq j'} \big\langle |\alpha_j|^2|\alpha_{j'}|^2\big\rangle\, \xi_1^2(L/L_{\mathrm{loc}}) + \sum_{j=1}^N \big\langle |\alpha_j|^4\big\rangle\, \xi_2(L/L_{\mathrm{loc}}) - \xi_1^2(L/L_{\mathrm{loc}})\\
&= \frac{2}{N+1}\big[\xi_2(L/L_{\mathrm{loc}}) - \xi_1^2(L/L_{\mathrm{loc}})\big],\\
\mathrm{Var}\big(I_{l_0}^{\mathrm{unif}}\big) &= 2\sum_{j\neq j'} \big\langle |\alpha_j|^2|\alpha_{j'}|^2\big\rangle\, |\phi_j(\boldsymbol{y}_{l_0})|^2|\phi_{j'}(\boldsymbol{y}_{l_0})|^2\xi_1^2(L/L_{\mathrm{loc}})\\
&\quad + \sum_{j=1}^N \big\langle |\alpha_j|^4\big\rangle\, |\phi_j(\boldsymbol{y}_{l_0})|^4\xi_2(L/L_{\mathrm{loc}}) - \frac{\xi_1^2(L/L_{\mathrm{loc}})}{|\Omega|^2}\\
&= \frac{N-3}{N+1}\frac{\xi_1^2(L/L_{\mathrm{loc}})}{|\Omega|^2} + 2\xi_2(L/L_{\mathrm{loc}})\Big[\frac{1}{N(N+1)}\sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^4\Big],
\end{aligned}$$

where we have used the uniformity property (2) and the following results which come from the fact that $(|\alpha_j|^2)_{j=1\ldots,N}$ is uniformly distributed in $\sum_{j=1}^N |\alpha_j|^2 = 1$:

$$\big\langle|\alpha_j|^2\big\rangle = \frac{1}{N}, \qquad \big\langle|\alpha_j|^2|\alpha_{j'}|^2\big\rangle = \frac{1}{N^2} + \begin{cases} \frac{-1}{N^2(N+1)} & \text{if } j\neq j',\\ \frac{N-1}{N^2(N+1)} & \text{if } j=j'.\end{cases}$$

Taking the limit $N\to\infty$ and using (3) gives that $\mathrm{Var}(\mathcal{P}_{\mathrm{out}}^{\mathrm{unif}})\to 0$ and $\mathrm{Var}(I_{l_0}^{\mathrm{unif}})\to \xi_1^2/|\Omega|^2$.

From now on we consider the case in which we use the optimal illumination (28). The intensity transmitted to $\boldsymbol{y}_{l_0}$ is

$$\begin{aligned}
I_{l_0}^{\mathrm{opt}} &= \frac{|\Omega|}{n}\sum_{k=1}^n \Big|\sum_{j=1}^N \phi_j(\boldsymbol{y}_{l_0})T_j\overline{\phi_j(\boldsymbol{x}_k)}\Big|^2\\
&= \frac{|\Omega|}{n}\sum_{j,j'=1}^N \phi_j(\boldsymbol{y}_{l_0})\overline{\phi_{j'}(\boldsymbol{y}_{l_0})}T_j\overline{T_{j'}}\sum_{k=1}^n \overline{\phi_j(\boldsymbol{x}_k)}\phi_{j'}(\boldsymbol{x}_k).
\end{aligned}$$

Using the discrete orthonormality property we obtain

$$I_{l_0}^{\mathrm{opt}} = \sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^2|T_j|^2. \tag{44}$$

We compute the expectation and the variance of $I_{l_0}^{\mathrm{opt}}$:

$$\begin{aligned}
\mathbb{E}\Big[I_{l_0}^{\mathrm{opt}}\Big] &= \sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^2\mathbb{E}[|T_j|^2] = \sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^2\xi_1(L/L_{\mathrm{loc}}),\\
\mathrm{Var}\Big(I_{l_0}^{\mathrm{opt}}\Big) &= \sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^4\mathrm{Var}(|T_j|^2) = \sum_{j=1}^N |\phi_j(\boldsymbol{y}_{l_0})|^4\big[\xi_2(L/L_{\mathrm{loc}}) - \xi_1^2(L/L_{\mathrm{loc}})\big],
\end{aligned}$$

where we have used the independence of the mode power transmission coefficients. Using the uniformity property (2-3) gives

$$\mathbb{E}\Big[\frac{1}{N}I_{l_0}^{\mathrm{opt}}\Big] = \frac{\xi_1(L/L_{\mathrm{loc}})}{|\Omega|}, \qquad \mathrm{Var}\Big(\frac{1}{N}I_{l_0}^{\mathrm{opt}}\Big) = o(1),$$

which shows that the quantity $I_{l_0}^{\rm opt}$ is self-averaging in the regime $N \gg 1$ and

$$I_{l_0}^{\rm opt} = N \frac{\xi_1(L/L_{\rm loc})}{|\Omega|}. \tag{45}$$

Using Lemma A.1 we have

$$\mathcal{P}_{\rm out}^{\rm opt} = \frac{|\Omega|}{p} \sum_{l=1}^{p} \big|\psi_{\rm out}^{\rm opt}(\boldsymbol{y}_l)\big|^2,$$

so we can write the total transmitted power in the form

$$\begin{aligned}
\mathcal{P}_{\rm out}^{\rm opt} &= \frac{|\Omega|}{p} \sum_{l=1}^{p} \Big| \sum_{k=1}^{n} M_{lk} E_k^{\rm opt} \Big|^2 = \frac{|\Omega|}{p} \frac{|\Omega|}{n} \sum_{l=1}^{p} \Big| \frac{\sum_{k=1}^{n} M_{lk} \overline{M_{l_0 k}}}{\big[\sum_{m=1}^{n} |M_{l_0 m}|^2\big]^{1/2}} \Big|^2 \\
&= \frac{|\Omega|}{p} \frac{|\Omega|}{n} \frac{\sum_{l=1}^{p} \big| \sum_{k=1}^{n} M_{lk} \overline{M_{l_0 k}} \big|^2}{\sum_{k=1}^{n} |M_{l_0 k}|^2}.
\end{aligned} \tag{46}$$

We have

$$\sum_{k=1}^{n} M_{lk} \overline{M_{l_0 k}} = \sum_{j,j'=1}^{N} \phi_j(\boldsymbol{y}_l) \overline{\phi_{j'}(\boldsymbol{y}_{l_0})} T_j \overline{T_{j'}} \sum_{k=1}^{n} \phi_j(\boldsymbol{x}_k) \phi_{j'}(\boldsymbol{x}_k).$$

Using the discrete orthonormality property we get

$$\sum_{k=1}^{n} M_{lk} \overline{M_{l_0 k}} = \frac{n}{|\Omega|} \sum_{j=1}^{N} \phi_j(\boldsymbol{y}_l) \overline{\phi_j(\boldsymbol{y}_{l_0})} |T_j|^2. \tag{47}$$

For $l = l_0$ we have $\sum_{k=1}^{n} |M_{l_0 k}|^2 = (n/|\Omega|) I_{l_0}^{\rm opt}$ given by (44), which is self-averaging by (45), so that

$$\frac{1}{N} \sum_{k=1}^{n} |M_{l_0 k}|^2 = \frac{n}{N|\Omega|} \sum_{j=1}^{N} |\phi_j(\boldsymbol{y}_{l_0})|^2 |T_j|^2 = \frac{n}{|\Omega|^2} \xi_1(L/L_{\rm loc}). \tag{48}$$

Substituting (47-48) into (46) gives

$$\mathcal{P}_{\rm out}^{\rm opt} = \frac{|\Omega|}{p} \frac{|\Omega|}{\xi_1(L/L_{\rm loc})} \frac{1}{N} \sum_{l=1}^{p} \Big| \sum_{j=1}^{N} \phi_j(\boldsymbol{y}_l) \overline{\phi_j(\boldsymbol{y}_{l_0})} |T_j|^2 \Big|^2. \tag{49}$$

We have

$$\sum_{l=1}^{p} \Big| \sum_{j=1}^{N} \phi_j(\boldsymbol{y}_l) \overline{\phi_j(\boldsymbol{y}_{l_0})} |T_j|^2 \Big|^2 = \sum_{j,j'=1}^{N} \overline{\phi_j(\boldsymbol{y}_{l_0})} \phi_{j'}(\boldsymbol{y}_{l_0}) |T_j|^2 |T_{j'}|^2 \sum_{l=1}^{p} \phi_j(\boldsymbol{y}_l) \overline{\phi_{j'}(\boldsymbol{y}_l)}.$$

Using once again the discrete orthonormal property gives

$$\sum_{l=1}^{p} \Big| \sum_{j=1}^{N} \phi_j(\boldsymbol{y}_l) \overline{\phi_j(\boldsymbol{y}_{l_0})} |T_j|^2 \Big|^2 = \frac{p}{|\Omega|} \sum_{j=1}^{N} |\phi_j(\boldsymbol{y}_{l_0})|^2 |T_j|^4.$$

Using the independence of the $|T_j|^4$ and the uniformity property (2) gives

$$\frac{1}{N} \sum_{l=1}^{p} \Big| \sum_{j=1}^{N} \phi_j(\boldsymbol{y}_l) \overline{\phi_j(\boldsymbol{y}_{l_0})} |T_j|^2 \Big|^2 = \frac{p}{|\Omega|^2} \xi_2(L/L_{\rm loc}).$$

Substituting into (49) completes the proof of Proposition 3.

We finally give the proof of Proposition 4. Using the computations carried out in the proof of Proposition 3 we find that

$$I_l^{\rm opt} = \frac{|\Omega|}{N\xi_1(L/L_{\rm loc})} \Big| \sum_{j=1}^N \phi_j(\boldsymbol{y}_l)\overline{\phi_j(\boldsymbol{y}_{l_0})} |T_j|^2 \Big|^2 .$$

Taking the expectation we obtain:

$$\mathbb{E}\big[I_l^{\rm opt}\big] = \frac{N\xi_1(L/L_{\rm loc})}{|\Omega|} C(\boldsymbol{y}_l,\boldsymbol{y}_{l_0}) + \frac{|\Omega|}{N\xi_1(L/L_{\rm loc})} \sum_{j=1}^N |\phi_j(\boldsymbol{y}_l)|^2 |\phi_j(\boldsymbol{y}_{l_0})|^2 (\xi_2 - \xi_1^2),$$

which gives the desired result.

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