# Propagation of matter wave solitons in periodic and random nonlinear potentials

Fatkhulla Kh. Abdullaev† and Josselin Garnier‡††

† *Dipartimento di Fisica "E.R. Caianiello", Universitá di Salerno, 84081 Baronissi (SA), Italy*
‡ *Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions,*
*Université Paris VII, 2 Place Jussieu, 75251 Paris Cedex 5, France*

We study the motion of bright matter wave solitons in nonlinear potentials, produced by periodic or random spatial variations of the atomic scattering length. We obtain analytical results for the soliton motion, the radiation of matter wave, and the radiative soliton decay in such configurations of the Bose-Einstein condensate. The stable regimes of propagation are analyzed. The results are in remarkable agreement with the numerical simulations of the Gross-Pitaevskii equation with periodic or random spatial variations of the mean field interactions.

*Introduction.* Nonlinear excitations in Bose-Einstein condensates (BEC) have attracted a lot of attention recently. In particular matter wave solitons are interesting from the fundamental point of view [1]. The discovery of matter waves solitons in BEC [2–5] has opened the possibility to develop new methods for generating and controlling solitons. The investigation of the soliton dynamics in inhomogeneous BEC is of interest, in particular with inhomogeneities periodic in time or space. Time variations can be achieved with the Feshbach resonance (FR) management technique and it has been studied in [6–9]. New type of solitons can be generated by this way, and stabilization of higher-dimensional solitons in attractive condensate has been shown. Spatial variations have been investigated in the form of a periodic or random linear potential. In particular, a periodic optical lattice can be produced by counter propagating laser beams. Such potentials can be used to control the soliton parameters [10, 11] or to generate gap bright solitons [5]. Propagation in a random linear potential was considered from the point of view of the Anderson localization in BEC and the observation of the crossover between the Anderson localization regime and the nonlinear regime [12]. Dark soliton propagation in a linear random potential was studied in [13]. In the recent work [14] the properties of stationary localized states in the nonlinear optical lattice has been investigated. The problems of matter wave soliton propagation, when the mean field nonlinearity varies periodically or randomly in space remain open.

In this Rapid Communication we consider the propagation of nonlinear matter wavepackets and waves emission in the presence of a new type of inhomogeneities, namely under *nonlinear periodic or random potential*, produced by periodic or random variations of the atomic scattering length in space. The strength and the sign of the interatomic interactions, i.e. the value and sign of the

atomic scattering length $a_s$, can be varied using the FR method. Small variations of an external magnetic field near the FR can lead to large variations of $a_s$. For example if we consider the one-dimensional Bose gas close to the magnetic wire, then by small variations of the current one can induce spatially random magnetic field fluctuations. This in turn generates random spatial fluctuations of the strength of the interatomic interactions [15, 16]. Such variations can be achieved also by the optically induced FR [17, 18]. The Gross-Pitaevskii (GP) equation describing such configurations has a periodically or randomly varying in space mean field nonlinear coefficient.

The dynamics of quasi-1D nonlinear matter waves is described by the GP equation [7, 9]

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + g_{1D}|\psi|^2\psi. \quad (1)$$

Here $\psi$ is the mean field wavefunction, with $\int |\psi|^2 dx = N$, $N$ is the number of atoms, $g_{1D} = 2\hbar\omega_\perp a_s$, where $\omega_\perp$ is the transverse oscillator frequency, $a_s(x) = a_{s0} + a_{s1}f(x)$ is the spatially dependent atomic scattering length. The spatial dependence will be assumed to be periodic or random. In dimensionless variables where the distance $x$ is measured in units of the healing length $\xi = \hbar/\sqrt{n_0 g_{1D} m}$, with $n_0$ the peak density, and the time $t$ is measured in $t_0 = \xi/(2c)$, where $c = \sqrt{n_0 g_{1D}/m}$, we obtain the equation

$$iu_t + u_{xx} + 2|u|^2 u = -V(x)|u|^2 u. \quad (2)$$

*Matter wave soliton motion in a nonlinear periodic potential.* We consider the case when the incident wave is the soliton incoming from the left:

$$u^{(s)}(x,t) = 2\nu_0 \frac{\exp i(2\mu_0(x - 4\mu_0 t) + 4(\nu_0^2 + \mu_0^2)t)}{\cosh(2\nu_0(x - 4\mu_0 t))}, \quad (3)$$

where $2\nu_0$ and $4\mu_0$ are the soliton amplitude and velocity, respectively. The scattering length has spatial periodic modulations, so we have $V(x) = V_0 \cos(Kx)$, $V_0 = 2a_{s1}/a_{s0}$. For small values of $V_0$, the solution resembles the unperturbed soliton with modulated parameters in the early step of the propagation. This is true

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as long as the radiative emission of matter wave is negligible. Using the collective coordinate ansatz, we get that the soliton mass is preserved, while the soliton center $\zeta$ obeys the effective particle equation $\zeta_{tt} = -\partial V/\partial\zeta$ starting from $\zeta(0) = \zeta_0, \zeta_t(0) = 4\mu_0$, where the effective potential is

$$V(\zeta) = -A_{nl}\cos(K\zeta), \ A_{nl} = \frac{2\pi}{3}\frac{V_0\nu K}{\sinh(\frac{\pi K}{4\nu})}\left[1 + \frac{K^2}{16\nu^2}\right].$$

Here $A_{nl}$ is the barrier for the soliton moving in the nonlinear periodic potential. Note that in comparison with the linear periodic potential $V(x)u(x,t)$ in Eq. (2), the influence of the nonlinear periodic potential on the soliton is enhanced. For the broad soliton case $K/\nu \gg 1$, the enhancement factor $\alpha = A_{nl}/A_l$ is $K^2/12, \alpha > 1$. For the narrow soliton case $K/\nu \ll 1$, the enhancement factor is $\alpha = 4\nu^2/3$. The soliton is moving as a classical particle, and it can be trapped at $\zeta = 2\pi n/K, n = 0, 1, 2....$ In the trapped regime the soliton performs the oscillatory motion with the small oscillation frequency $\Omega = \sqrt{A_{nl}K}$. The critical velocity for depinning of the soliton starting from the minimum of the potential is $v_{dp} = \sqrt{2A_{nl}}$.

When the soliton width is much larger than the period of the nonlinear potential, i.e. $K/\nu \gg 1$, the radiation emission phenomenon can be divided into two time steps. First, the soliton emits a small but quick burst of radiation which is trapped in the form of soliton shape modulation. Second, the soliton continues to radiate slowly on long time scales $\sim V_0^{-2}$, as we shall see below. The first step can be described in the "renormalized particle limit" [10]. The dressed solution of Eq. (2) can be searched in the form $u = u_s(x,t)(1 + \chi(x,t))$. For $v < K$, where $v = 4\mu_0$ is the soliton velocity, the solution for $\chi$ is

$$\chi(x,t) = V_0\left(\frac{\cos(Kx)}{K^2 - v^2} - i\frac{v\sin(Kx)}{K(K^2 - v^2)}\right)|u^{(s)}|^2.$$

At $v \sim K$ the approximation used for the derivation is violated, and the radiation emission should be studied more carefully.

*Emission of waves by soliton in a nonlinear periodic potential.* The soliton propagating under action of periodic nonlinear potential emits matter wave radiation. When the modulations are weak we can use the perturbation theory based on the Inverse Scattering Transform to calculate the radiation emission [19]. By conservation of the total mass $N$ and energy $H$,

$$N = \int_{-\infty}^{\infty}|u|^2dx, \ H = \int_{-\infty}^{\infty}|u_x|^2 - (1 + \frac{V(x)}{2})|u|^4dx,$$

the soliton parameters during the time interval $\Delta T$ are modified according to $\Delta\nu = -F(\nu,\mu,\Delta T), \Delta\mu = -G(\nu,\mu,\Delta T)$, where

$$F(\nu,\mu,\Delta T) = \frac{1}{4}\int n(\lambda,\Delta T)d\lambda,$$

$$G(\nu,\mu,\Delta T) = \frac{1}{8}\int(\frac{\lambda^2}{\mu\nu} + \frac{\nu}{\mu} - \frac{\mu}{\nu})n(\lambda,\Delta T)d\lambda.$$

Here $n(\lambda,\Delta T)$ is the emitted mass density during the time interval $\Delta T$, and $k = 2\lambda$ (resp. $\omega = 4\lambda^2$) is the wavenumber (resp. the frequency) of the emitted radiation. The mass (number of atoms) emitted by the soliton is $N_{rad} = \int n(\lambda)d\lambda$. When $V_0 \ll 1$ and the propagation times are of order $V_0^{-2}$ we can calculate the emitted mass density and the evolution of the soliton parameters. Different regimes are possible.

*Regime 1.* If the modulation wavenumber $K$ is smaller that $\nu_0^2/\mu_0$, then the radiative emission is negligible for times of order $V_0^{-2}$. The soliton parameters are almost constant in this regime.

*Regime 2.* If the modulation wavenumber $K$ is larger than $\nu_0^2/\mu_0$, then the soliton emits a significant amount of radiation. The soliton amplitude and velocity satisfy the system of ordinary differential equations

$$\nu_t = -F(\nu,\mu), \quad \mu_t = -G(\nu,\mu), \tag{4}$$

starting from $\nu(0) = \nu_0, \mu(0) = \mu_0$. The functions $F$ and $G$ are given by

$$F(\nu,\mu) = \frac{V_0^2}{16K\mu\lambda_+}\left[\psi(\lambda_+)^2 + \psi(\lambda_-)^2\right],$$

$$\begin{aligned}G(\nu,\mu) = \frac{V_0^2}{16K\mu\nu\lambda_+}\Big[&\psi(\lambda_+)^2(\frac{K}{2} + \lambda_+)\\ &+\psi(\lambda_-)^2(\frac{K}{2} + \lambda_-)\Big],\end{aligned}$$

where $\lambda_\pm = \pm\sqrt{K\mu - \nu^2}$ and

$$\begin{aligned}\psi(\lambda) = &\frac{\pi}{6\cosh(\pi(\lambda + K/2)/(2\nu))}[\nu^2 + (\lambda + \frac{K}{2})^2]\\ &\times[\nu^2 + 2K\mu + (\lambda - \frac{K}{2})^2].\end{aligned}$$

The maximal radiative decay is obtained for $K$ close to $\nu^2/\mu$. By integrating Eq. (4), we can put into evidence that there are two subcases: *a).* The soliton mass $4\nu$ decays, while the velocity increases or decays slowly, so that the soliton stays in the regime $K\mu > \nu^2$. Eq. (4) can be used to describe the long time behavior of the soliton, whose mass decays to zero (Figure 1). *b).* The soliton mass $4\nu$ decays, but the velocity $4\mu$ decays faster, so that the soliton parameters reach the condition $K\mu = \nu^2$ at the critical time. This state is a stable equilibrium, we recover the regime 1. This shows that we can have a stable soliton even in the case $K > \mu_0/\nu_0^2$, at the expense of the emission of a small amount of radiation to allow the soliton to reach a stable state (Figures 2-3).

*Regime 3.* If we start from critical initial conditions such that $\mu_0 K = \nu_0^2$, then there are three sub-cases: *a).* If $K < 4\mu_0$, then the soliton is attracted by the regime 2 and its mass decays to 0. *b).* If $K > 4\mu_0$, then the soliton is attracted by the regime 1 and its parameters remain constant. *c).* If $K = 4\mu_0$ (and thus $\nu_0 = 2\mu_0$),
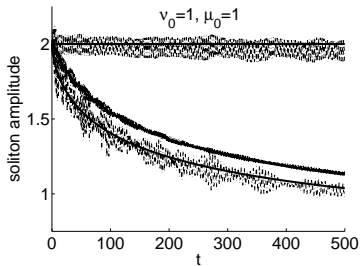
FIG. 1: Soliton amplitude. The initial soliton parameters are $\nu_0 = \mu_0 = 1$. A periodic modulation $0.1\cos(Kx)$ is applied to the nonlinear parameter. We compare the results from full numerical simulations of the perturbed GP equation (thin dashed lines) with the theoretical predictions of (4) (thick solid lines). From top to bottom: $K = 1.5$ (regime 1), $K = 0.5$ (regime 3a), $K = 1$ (regime 2a). The strongest decay is achieved for the critical case $K = 1 = \nu_0^2/\mu_0$, as predicted by the theory.
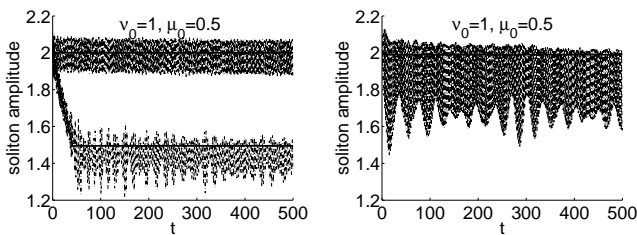


FIG. 2: Soliton amplitude. The initial soliton parameters are $\nu_0 = 1$, $\mu_0 = 0.5$. Left picture, from top to bottom: $K = 1.5$ (regime 1), $K = 2.5$ (regime 2b). Right picture: $K = 2$ (regime 3c). As predicted by the theory, the critical case experiences oscillations but is stable.

the soliton experiences strong oscillations but its mass does not decay (Figure 2).

*Emission of waves by soliton in a nonlinear random potential.* Let us assume that the function $V$ is the realization of a random zero-mean stationary process. The correlation function is $B(x, l_c) = \langle V(x)V(0)\rangle$, where $l_c$ is the correlation length. For times of order $B(0, l_c)^{-1}$ the soliton parameters satisfy the *deterministic* system
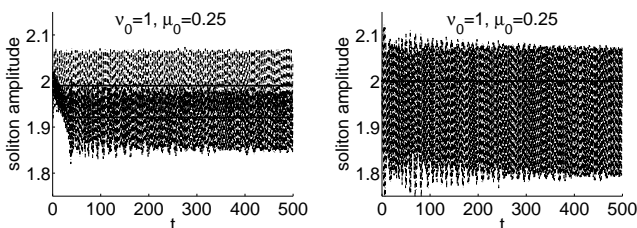


FIG. 3: Soliton amplitude. The initial soliton parameters are $\nu_0 = 1$, $\mu_0 = 0.25$. Left picture, from top to bottom: $K = 2$ (regime 1), $K = 8$ (regime 2b). Right picture: $K = 4$ (regime 3b). As predicted by the theory, the critical case is stable.

(4) where the functions $F, G$ are defined by

$$F(\nu, \mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} c^2(\nu, \mu, \lambda) d(k(\nu, \mu, \lambda)) d\lambda,$$

$$G(\nu, \mu) = \int_{-\infty}^{\infty} \frac{\lambda^2 + \nu^2 - \mu^2}{8\pi\mu\nu} c^2(\nu, \mu, \lambda) d(k(\nu, \mu, \lambda)) d\lambda.$$

Here the function $c$ is given by

$$c(\nu, \mu, \lambda) = \frac{\pi}{96\mu^{9/2}}(\nu^2 + 17\mu^2 - 6\lambda\mu + \lambda^2)$$
$$\times \frac{((\lambda + \mu)^2 + \nu^2)((\lambda - \mu)^2 + \nu^2)}{\cosh(\pi(\nu^2 + \lambda^2 - \mu^2)/(4\mu\nu))},$$

and the coefficients $d$ and $k$ by:

$$k = \frac{(\lambda - \mu)^2 + \nu^2}{\mu}, \quad d(k) = 2\int_0^{\infty} B(x)\cos(kx)dx.$$

Note that $k(\lambda) \geq \nu^2/\mu$ for all $\lambda$. Thus, the interaction between the soliton and the nonlinear random potential depends only the tail of the power spectral density $d(k)$ of $V$ for $k > \nu^2/\mu$. There are two regimes of propagation:

*Regime 1.* If $\mu_0 \gg \nu_0$, then the emitted radiation density is concentrated around the wavenumbers $\pm 2\mu_0$. Besides the system (4) can be simplified, and we obtain that the velocity of the soliton is almost constant, while the mass decays as a power law:

$$\nu(t) \simeq \nu_0 \left(1 + \frac{t}{T_c}\right)^{-1/4}, \quad T_c = \frac{3\mu_0}{32d(4\mu_0)\nu_0^4}. \quad (5)$$

In this regime the radiative decay prevents from transmitting nonlinear wavepackets. The decay time is inversely proportional to the correlation function of the disorder and the forth power of the soliton amplitude. This means that this type of disorder intensively destroys heavy solitons.

*Regime 2.* If $\mu_0 \ll \nu_0$, then the soliton emits a very small amount of broadband radiation, its mass is almost constant, while the velocity decays very slowly, typically as a logarithm [20]. The amount of emitted radiation is proportional to $d(\nu^2/\mu)$. In this regime the soliton can be transmitted.

The analysis of the system (4) shows that these two regimes are attractive, in the sense that, after a transient regime, one observes the regime 1 (resp. 2) if $\mu_0/\nu_0$ is above (resp. below) a critical value. We have checked these predictions by numerical simulations of the randomly perturbed GP equation (2). We consider the case of a stepwise constant process $V$. The constant step is equal to $l_c$, and the process $V$ takes random independent values over each elementary interval that are uniformly distributed in $[-\sigma, \sigma]$. The power spectral density is $d(k) = 2\sigma^2[1 - \cos(kl_c)]/[k^2 l_c]$. We compare in Figure 4 the numerical results and the theoretical predictions in a case close to the regime 1 (top figure) and close to
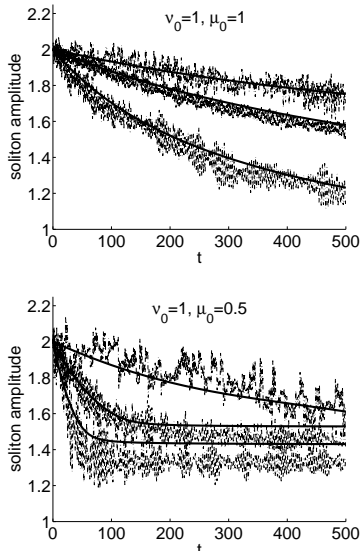
FIG. 4: Soliton amplitude. A random modulation with amplitude $\sigma$ and correlation length $l_c$ is applied to the nonlinear parameter. The initial soliton parameters and potential amplitude are $\nu_0 = \mu_0 = 1$, $\sigma = 0.1$ (top) and $\nu_0 = 1$, $\mu_0 = 0.5$ $\sigma = 0.2$ (bottom). We compare the results from full numerical simulations of the perturbed GP equation (thin dashed lines) with the theoretical predictions of (4) (thick solid lines). For each figure, we have from top to bottom: $l_c = 8$, $l_c = 0.125$, $l_c = 1$.

the regime 2 (bottom figure). The power law radiative decay is noticeable in the top figure, with a decay rate $T_c^{-1}$ that is maximal for $l_c \simeq 0.58$. The full transmission regime (up to a transient regime where the soliton emits radiation) can be observed in the bottom figure.

To estimate the typical values of the parameters, let us consider the case of $^7$Li condensate with the FR at $B = 720$G. The typical values of the scattering length used in the experiments are: at $B = 352$G is $a_s = -0.23$nm and $a_s(300G) \approx -0.2$nm [3]. Thus by changing periodically in space the magnetic field between these values we can obtain the nonlinear periodic potential with the dimensionless amplitude of modulations $V_0 \approx 0.26$. In the trap with $\omega_\perp = 2\pi \cdot 10^3$Hz and $n_0 = 10^6$cm$^{-1}$, we have $\xi \approx 2\mu$m, $c \approx 2$ mm/s, so $t_0 \approx 1$ ms. A soliton with velocity $v \sim c$ travels through the region with modulated scattering length with $L \sim 0.25$mm in the dimensionless time $t \sim 125$. The soliton width is of the order of $1.5\mu$m, and if the grating period varies in the interval $(1, 15)\mu$m, then both limits $K\nu \gg 1$ and $K\nu \ll 1$ are covered. The optically induced FR method gives a grating period $\sim 1\mu$m. The critical case $K = 1 = \nu_0^2/\mu_0$ (Fig. 1) corresponds for these parameters to the grating period $\approx 6\mu$m and the soliton velocity $\approx 8$mm/s. The random modulations with the deviation strength is $a_s/a_{s1} = 0.1$ can be achieved by the random distribution of the current in wire along the atom chip[15]. For the soliton velocity $v_s = 2c$, and the initial number of atoms

in soliton is $\sim 10^3$, we find then the soliton decay time $T_c \sim 50$, which is equal to $\sim 0.1$s in physical units.

*In conclusion* we have investigated the transmission of matter wave bright solitons through periodic or random nonlinear potential, generated by periodic or random spatial variations of the atomic scattering length. The condition for the emission of matter waves and radiative soliton decay are obtained. We show that critical cases support the stable propagation of bright solitons.

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