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2012 J. Phys. A: Math. Theor. 45 035202

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# Inverse scattering perturbation theory for the nonlinear Schrödinger equation with non-vanishing background

Josselin Garnier and Konstantinos Kalimeris

Laboratoire de Probabilités et Modèles Aléatoires, site Chevaleret, case 7012,  
Université Paris VII, 75251 Paris Cedex 13, France

E-mail: [garnier@math.jussieu.fr](mailto:garnier@math.jussieu.fr) and [kalimeris@math.jussieu.fr](mailto:kalimeris@math.jussieu.fr)

Received 4 August 2011, in final form 30 October 2011

Published 19 December 2011

Online at [stacks.iop.org/JPhysA/45/035202](http://stacks.iop.org/JPhysA/45/035202)

## Abstract

In this paper, a perturbation theory for the nonlinear Schrödinger equation with non-vanishing boundary conditions based on the inverse scattering transform is presented. It is applied to study the stability of the soliton propagation on a continuous-wave background. It is shown that the soliton is rather robust with respect to dispersive perturbations but it can be strongly affected by damping. In particular, it is shown that adiabatic approaches can underestimate the decay of the soliton energy.

PACS numbers: 42.65.–R, 47.20.Ky

(Some figures may appear in colour only in the online journal)

## 1. Introduction

It is well known that the one-dimensional nonlinear Schrödinger (NLS) equation can model the wave dynamics in the deep ocean [20] and pulse propagation in optical fibers [1]. This analogy has recently found an imaginative and insightful development. Indeed, the understanding of the infamous hydrodynamic rogue waves on the surface of the ocean [7] is not straightforward as very few observations are available. The discovery that optical rogue waves can be generated in optical systems [22] has opened the way to new research directions, as it is now possible to produce and study optical rogue waves in laboratories. Furthermore, the analysis of special solutions of the NLS equation that could describe rogue waves has attracted a lot of attention.

The bright soliton solution for the NLS equation with vanishing boundary conditions was exhibited by Zakharov and is now well known [19]. Using the inverse scattering transform, soliton solutions with non-vanishing boundary conditions were first investigated in [16]. It is also possible to find this solution using more direct approaches based on a direct algebraic ansatz [3] or the Bäcklund transform [13]. This solution is nowadays better known under the name of Ma soliton [18]. In the notation of equation (2.1), it is a wave solution periodic in  $t$  and localized in  $x$ . When the background wave goes to zero, one recovers the standard bright

soliton of the NLS equation. The Ma soliton can be seen as a particular class of a more general family of solutions that describes periodic wave solutions in both  $t$  and  $x$  [3]. Note that the Akhmediev breathers are another particular class of this general family of solutions [4]. These breathers are localized in  $t$  and periodic in  $x$  and they can be used to describe a nonlinear stage of the modulational instability. The Peregrine solution [21] is a rational solution of the NLS equation, which can be seen as a limiting case of both a one-parameter family of Ma solitons or a one-parameter family of Akhmediev breathers (when the periods go to infinity). The Peregrine solution was observed in [17] and it is a very good candidate for a ‘wave that appears from nowhere and disappears without a trace’ [2], that is, a rogue wave. The Ma solitons provide one way to study the Peregrine solution.

It is of theoretical and practical interest to understand the stability of the Ma soliton under different types of perturbations [9]. In the hydrodynamic context, one should take into account finite depth, bottom friction, dissipation and other effects present in the ocean [6, 23]. In the nonlinear fiber optics context, the NLS equation is a simplified model of a more general equation that should take into account high-order dispersion and nonlinear effects [8, 24] and damping [6, 14].

The stability of the solutions of the NLS equation with non-vanishing boundary conditions has already been investigated using numerical or analytical tools. Background modulational instability and Raman self-scattering were numerically investigated in [5]. In this paper, we are not concerned with the possible modulational instability of the background but we focus our attention to the stability of the Ma soliton with respect to different types of perturbations. An adiabatic approach was used to study the stability of the soliton in [10]. In this approach, it is assumed that the solution keeps its form and that its parameters slowly evolve. The adiabatic evolutions of the soliton parameters are identified by using the evolution equations of a few conserved integrals, such as the total energy. In our paper, we develop a perturbation theory based on the inverse scattering transform to study the evolution of the soliton. The principle of such a method for a small perturbation of an integrable system was described in [15]. This method was successfully applied to the NLS equation with vanishing boundary conditions in the case of both deterministic and random perturbations [11, 12].

In this paper, we develop the perturbation theory for the solution of the NLS equation with non-vanishing boundary conditions and pay particular attention to the case of the Ma soliton. This theory can be applied to analyze the effect of different dispersive, diffusive (damping), or nonlinear perturbations to the soliton propagation. As we will see, the perturbed inverse scattering transform approach reveals that the evolution of the soliton can be different than that predicted by the adiabatic approach. In particular, radiation can play a significant role in the evolution of the conserved integrals. We will show that the Ma soliton is rather robust with respect to dispersive perturbations, but can be strongly affected by damping terms. Although the total energy is not greatly affected by damping, the soliton energy can decay significantly. These theoretical predictions are confirmed by numerical simulations.

## 2. The nonlinear Schrödinger equation with non-vanishing boundary conditions

In this section, we recall two of the one-soliton solutions of one-dimensional NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (2.1)$$

with non-vanishing boundary conditions. The simple transformation  $v(x, t) = e^{2iv_0^2(t-t_0)}q(x, t)$  in (2.1) gives

$$iq_t + q_{xx} + 2(|q|^2 - v_0^2)q = 0, \quad (2.2)$$

and we consider the non-vanishing boundary conditions  $\lim_{x \rightarrow \pm\infty} q(x, t) = v_0$ .

The general form of the Ma soliton, as it is known in the literature [18] (with shifted time and space), was firstly discovered in [16]<sup>1</sup>, using the inverse scattering transform and is given by the following expression:

$$q(x, t) = v_0 + 2\eta \frac{\eta \cos(4v\eta t + \theta) + iv \sin(4v\eta t + \theta)}{v_0 \cos(4v\eta t + \theta) - v \cosh(2\eta x + \psi)}, \tag{2.3}$$

where  $\psi, \theta$  are real numbers and  $\eta, v$  and  $v_0$  are positive parameters, for which  $\eta = \sqrt{v^2 - v_0^2}$ .

For  $\psi = \theta = 0$ , we obtain the usual form of the Ma soliton

$$q(x, t) = v_0 + 2\eta \frac{\eta \cos(4v\eta t) + iv \sin(4v\eta t)}{v_0 \cos(4v\eta t) - v \cosh(2\eta x)}. \tag{2.4}$$

The limit of (2.4) at  $\eta \rightarrow 0$  gives the Peregrine soliton [21], which is a rational solution of (2.2) and has the following expression:

$$p(x, t) = v_0 \left( 1 - \frac{4 + 16iv_0^2 t}{1 + 4v_0^2 x^2 + 16v_0^4 t^2} \right). \tag{2.5}$$

### 3. Inverse scattering transform

#### 3.1. The spectral problems

Following [16] the generalized eigenvalue problem associated with (2.2) is

$$u_x = D(\lambda; x, t)u, \quad D(\lambda; x, t) = -i\lambda\sigma_3 + Q(x, t), \tag{3.1a}$$

$$u_t = F(\lambda; x, t)u, \quad F(\lambda; x, t) = -i(2\lambda^2 - |q(x, t)|^2 + v_0^2)\sigma_3 + 2\lambda Q(x, t) + i\sigma_3 Q_x(x, t), \tag{3.1b}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Q(x, t) = \begin{pmatrix} 0 & q(x, t) \\ -\bar{q}(x, t) & 0 \end{pmatrix},$$

and the bar stands for the complex conjugate. The compatibility condition  $u_{xt} = u_{tx}$  yields the equation  $D_t - F_x + DF - FD = 0$ , which is equivalent to (2.2).

The Jost functions  $\Phi^\pm(x, t; \lambda, \zeta; \eta, v, v_0)$  are defined as the  $2 \times 2$  matrices which satisfy the ordinary differential equation (ODE) (3.1a) with the boundary conditions

$$\Phi^\pm(x, t; \lambda, \zeta; \eta, v, v_0) \rightarrow T(\lambda, \zeta)J(\zeta x), \quad x \rightarrow \pm\infty, \tag{3.2}$$

where

$$T(\lambda, \zeta) = \begin{pmatrix} -iv_0 & \lambda - \zeta \\ \lambda - \zeta & -iv_0 \end{pmatrix}, \quad J(\zeta x) = \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \quad \text{and} \quad \zeta = \sqrt{\lambda^2 + v_0^2}.$$

From the above problem one can see that

$$\frac{d}{dx}(\text{Det}(\Phi^\pm)) = \text{Tr}(-i\lambda\sigma_3 + Q(x, t)) = -i\lambda \text{Tr}(\sigma_3) = 0.$$

Using conditions (3.2), we obtain

$$d(\lambda, \zeta) := \text{Det}(\Phi^\pm) = 2\zeta(\lambda - \zeta). \tag{3.3}$$

We define the scattering matrix  $S$  by the following relation:

$$\Phi^-(x, t; \lambda, \zeta; \eta, v, v_0) = \Phi^+(x, t; \lambda, \zeta; \eta, v, v_0)S(t; \lambda, \zeta; \eta, v, v_0) \tag{3.4}$$

<sup>1</sup> There is a sign error in equation (6.10) in the original paper [16]. The exact expression is (2.3).

and  $\text{Det}(S) = 1$ , for every  $x$  and  $t$ . Using equations (3.1)–(3.4), one can derive an ODE for the function  $S(t; \lambda, \zeta)$  from which we obtain the following expression:

$$S(t; \lambda, \zeta) = e^{-2i\lambda\zeta\sigma_3 t} S_0(\lambda, \zeta) e^{2i\lambda\zeta\sigma_3 t}, \quad S_0(\lambda, \zeta) = S(t; \lambda, \zeta)|_{t=0}. \quad (3.5)$$

In order to consider the analytic properties in the spectral plane we use the transformed Jost functions  $\Psi^\pm = T^{-1}\Phi^\pm$  which now satisfy the ODE

$$r_x = T^{-1}(DT - T_x)r \quad (r = T^{-1}u)$$

and the conditions

$$\Psi^\pm(x, t; \lambda\zeta) \rightarrow J(\zeta x), \quad \text{as } x \rightarrow \pm\infty.$$

Equation (3.4) remains invariant under this transformation, i.e.  $\Psi^- = \Psi^+ S$ .

For an initial profile that converges sufficiently rapidly to  $v_0$  as  $x \rightarrow \pm\infty$  we can obtain the following theorem, as was stated in [16].

**Theorem 3.1.** *Let  $\Psi_1^\pm$  and  $\Psi_2^\pm$  be the columns of  $\Psi^\pm$  and  $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ . The functions  $\Psi_1^-(\lambda, \zeta, x, t) e^{i\zeta x}$ ,  $\Psi_2^+(\lambda, \zeta, x, t) e^{-i\zeta x}$  and  $S_{11}(\lambda, \zeta)$  are analytic on  $\lambda$ , when  $\text{Im } \zeta > 0$ . The functions  $\Psi_2^-(\lambda, \zeta, x, t) e^{-i\zeta x}$ ,  $\Psi_1^+(\lambda, \zeta, x, t) e^{i\zeta x}$  and  $S_{22}(\lambda, \zeta)$  are analytic on  $\lambda$ , when  $\text{Im } \zeta < 0$ .*

Furthermore, if we assume that the function  $f(x) = q(x, 0) - v_0$  has a compact support, then all of the above functions are analytic in  $\lambda$  if  $\zeta \neq 0$ . Here we consider  $\zeta = \sqrt{\lambda^2 + v_0^2}$  as a single valued function by introducing two Riemann surfaces.

### 3.2. Integrals

On the one hand, asymptotic expansions of the functions listed in theorem 3.1, around  $|\zeta| = \infty$ , in their domain of analyticity, allow us to construct two  $2 \times 2$  systems of integral equations for the Jost functions, (equations (4.4) and (4.5) in [16]). On the other hand, the ODE (3.1a) allows us to construct the following integral representation for the Jost functions:

$$\Phi^\pm(x, t; \lambda, \zeta) = T(\lambda, \zeta)J(\zeta x) - \int_x^{\pm\infty} K^\pm(x, s, t)T(\lambda, \zeta)J(\zeta s) ds. \quad (3.6)$$

Substitution of the integral representation (3.6) into the previously mentioned integral equations yields two  $2 \times 2$  systems of integral equations where Jost functions are eliminated, which give the Gel'fand–Levitan integral equations

$$K^\pm(x, y, t) + H^\pm(x + y, t) = \int_x^{\pm\infty} K^\pm(x, s, t)H^\pm(y + s) ds, \quad y > x, \quad (3.7)$$

where

$$H^\pm = (H_2^\pm, H_1^\pm), \quad H_j^\pm(z) = \frac{1}{4\pi} \int_{\Gamma_j^\pm} \frac{e^{-(-1)^j i\zeta z}}{\zeta} \rho_j(\lambda, \zeta) T(\lambda, \zeta) d\lambda \begin{pmatrix} \delta_{2j} \\ \delta_{1j} \end{pmatrix},$$

where  $\delta_{ij}$  stands for the Kronecker delta.  $\rho_1(\lambda, \zeta) = \frac{S_{21}(\lambda, \zeta)}{S_{11}(\lambda, \zeta)}$  and  $\rho_2(\lambda, \zeta) = \frac{S_{12}(\lambda, \zeta)}{S_{22}(\lambda, \zeta)}$ , for some appropriate contours  $\{\Gamma_j^\pm\}_{j=1}^2$  on the  $\lambda$ -plane, see [16]. From the (+) equations above, one can obtain that  $2K_{12}^+(x, x, t) = q(x, t) - v_0$ .

The following symmetries are valid and will be useful in the following subsection:

$$S(\lambda, \zeta) = \sigma_2 \overline{S(\bar{\lambda}, \bar{\zeta})} \sigma_2, \quad K^\pm(x, y, t) = \sigma_2 \overline{K^\pm(x, y, t)} \sigma_2, \quad \rho_1(\lambda, \zeta) = -\overline{\rho_2(\bar{\lambda}, \bar{\zeta})},$$

where  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ .

Moreover, the fact that the functions  $\Phi^\pm(x, t; \lambda, -\zeta)$  satisfy equation (3.1a), the definition of  $S(t; \lambda, \zeta)$  in (3.4) implies that  $S_{11}(t; \lambda, \zeta) = S_{22}(t; \lambda, -\zeta)$  and  $S_{12}(t; \lambda, \zeta) = S_{21}(t; \lambda, -\zeta)$ . Hence,

$$S_{11}(t; \lambda, \zeta) = \overline{S_{11}(t; \bar{\lambda}, -\bar{\zeta})}. \tag{3.8}$$

### 3.3. Solitons

From (3.5) we find that  $\rho_1(\lambda, \zeta, t) = \rho_1(\lambda, \zeta, 0) e^{4i\lambda\zeta t}$ . So if we consider the discrete part of the functions  $H^\pm(z, t)$  one has to find the zeros of  $\rho_1(\lambda, \zeta, t)$  or equivalently  $S_{11}(\lambda, \zeta, 0)$ —let us note them  $\{(\lambda_j, \zeta_j)\}_{j=1}^n$ . Then, the definition of  $H_1^+(z, t)$  gives the following representation:

$$H_1^+(z, t) = \sum_{j=1}^n \begin{pmatrix} c_j(t) \\ \tilde{c}_j(t) \end{pmatrix} e^{i\zeta_j z}, \quad \text{Im}(\zeta_j) > 0, \tag{3.9}$$

where

$$c_j(t) = \frac{i}{2}(\lambda_j - \zeta_j)b(\lambda_j, \zeta_j) e^{4i\lambda_j\zeta_j t}, \quad \tilde{c}_j(t) = -\frac{1}{2}v_0 b(\lambda_j, \zeta_j) e^{4i\lambda_j\zeta_j t},$$

and

$$b(\lambda, \zeta) = \frac{S_{21}(\lambda, \zeta)}{\zeta \frac{dS_{11}(\lambda, \zeta)}{d\lambda}}.$$

The form of function  $H_1^+(z, t)$  suggests to take the following representation:

$$\begin{pmatrix} K_{12}^+(x, y, t) \\ K_{22}^+(x, y, t) \end{pmatrix} = \sum_{j=1}^n \begin{pmatrix} K_j^+(x, t) \\ \tilde{K}_j^+(x, t) \end{pmatrix} e^{i\zeta_j y}. \tag{3.10}$$

Applying the representations (3.9) and (3.10), of functions  $H^+(z, t)$  and  $K^+(x, y, t)$ , respectively, to the (+) integral equation (3.7), one obtains two  $n \times n$  linear systems of algebraic equations which give the functions  $\{K_j^+(x, t), \tilde{K}_j^+(x, t)\}_{j=1}^n$ .

The symmetry relation (3.8) shows that the zeros of  $S_{11}$  go in pairs, apart from the case that  $\lambda_j \in \mathbb{R}$ . Furthermore, from the definition of  $b(\lambda, \zeta)$  we obtain that  $b(\lambda, \zeta) = \overline{b(\bar{\lambda}, -\bar{\zeta})}$ . Following the previous procedure, we take the pairs of zeros  $\{(\lambda_j, \zeta_j) = ((-1)^j v, \eta)\}_{j=1}^2$  with the constraint  $v = \sqrt{\eta^2 - v_0^2}$  and by letting  $b(\lambda_j, \zeta_j) = \frac{2\eta}{v}$ ,  $j = 1, 2$ , we obtain

$$K^+(x, x, t) = \frac{\eta}{v_0 \cos(4v\eta t) - v \cosh(2\eta x)} \times \begin{pmatrix} v_0 \cos(4v\eta t) - v e^{-2\eta x} & \eta \cos(4v\eta t) + i v \sin(4v\eta t) \\ -\eta \cos(4v\eta t) + i v \sin(4v\eta t) & v_0 \cos(4v\eta t) - v e^{-2\eta x} \end{pmatrix} \tag{3.11}$$

and  $K(x, y, t) = K(x, x, t) e^{\eta(x-y)}$ . Consequently, we obtain the Ma soliton given in (2.4). An arbitrary choice for  $b(\lambda, \zeta)$  will give, in the same way, the more general form (time and space shifted) of the Ma soliton given in equation (2.3).

## 4. The Jost functions

### 4.1. The Jost functions for the Ma soliton

Applying the expression we obtained for  $K^+(x, y, t)$  in (3.11) to the integral representation of the Jost functions in (3.6), we obtain the following expressions for  $\Phi^+ = (\Phi_1^+, \Phi_2^+)$ :

$$\begin{aligned} \Phi_1^+(x, t) = & -\frac{e^{-i\zeta x}}{\zeta - i\eta} \frac{1}{v_0 \cos(4v\eta t) - v \cosh(2\eta x)} \\ & \times \left( \begin{array}{l} i v_0^2 \lambda \cos(4v\eta t) - i v [(\lambda - \zeta)(v \cos(4v\eta t) + i\eta \sin(4v\eta t)) + v_0(\zeta \cosh(2\eta x) - i\eta \sinh(2\eta x))] \\ v_0 \lambda (\lambda - \zeta) \cos(4v\eta t) + v [v_0(v \cos(4v\eta t) - i\eta \sin(4v\eta t)) + (\lambda - \zeta)(\zeta \cosh(2\eta x) - i\eta \sinh(2\eta x))] \end{array} \right) \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \Phi_2^+(x, t) = & -\frac{e^{i\zeta x}}{\zeta + i\eta} \frac{1}{v_0 \cos(4v\eta t) - v \cosh(2\eta x)} \\ & \times \left( \begin{array}{l} v_0 \lambda (\lambda - \zeta) \cos(4v\eta t) + v [v_0(v \cos(4v\eta t) + i\eta \sin(4v\eta t)) + (\lambda - \zeta)(\zeta \cosh(2\eta x) + i\eta \sinh(2\eta x))] \\ i v_0^2 \lambda \cos(4v\eta t) - i v [(\lambda - \zeta)(v \cos(4v\eta t) - i\eta \sin(4v\eta t)) + v_0(\zeta \cosh(2\eta x) + i\eta \sinh(2\eta x))] \end{array} \right). \end{aligned} \quad (4.2)$$

The Jost functions  $\Phi^-$  are given from the  $\Phi^+$  under the substitution  $\eta \rightarrow -\eta$ . Moreover, the equation  $\Phi^- = \Phi^+ S$  yields the following expression:

$$S(\lambda, \zeta, t) = \begin{pmatrix} \frac{\zeta - i\eta}{\zeta + i\eta} & 0 \\ 0 & \frac{\zeta + i\eta}{\zeta - i\eta} \end{pmatrix}. \quad (4.3)$$

Functions  $\Psi^\pm$  are given by equation  $\Psi^\pm = T^{-1} \Phi^\pm$  and we obtain the following expressions:

$$\begin{aligned} \Psi_1^+(x, t) = & \frac{e^{-i\zeta x}}{v_0 \cos(4v\eta t) - v \cosh(2\eta x)} \\ & \times \left( \begin{array}{l} \frac{v_0}{\zeta} (\zeta + i\eta) \cos(4v\eta t) - \frac{v}{\zeta - i\eta} (\zeta \cosh(2\eta x) - i\eta \sinh(2\eta x)) \\ -\frac{i\eta}{\zeta} \frac{1}{\zeta - i\eta} (\eta \lambda \cos(4v\eta t) - i v \zeta \sin(4v\eta t)) \end{array} \right) \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \Psi_2^+(x, t) = & \frac{e^{i\zeta x}}{v_0 \cos(4v\eta t) - v \cosh(2\eta x)} \\ & \times \left( \begin{array}{l} -\frac{i\eta}{\zeta} \frac{1}{\zeta + i\eta} (\eta \lambda \cos(4v\eta t) + i v \zeta \sin(4v\eta t)) \\ \frac{v_0}{\zeta} (\zeta - i\eta) \cos(4v\eta t) - \frac{v}{\zeta + i\eta} (\zeta \cosh(2\eta x) + i\eta \sinh(2\eta x)) \end{array} \right). \end{aligned} \quad (4.5)$$

The functions  $\Psi^-$  are given by the  $\Psi^+$  under the substitution  $\eta \rightarrow -\eta$ . Note that  $\Psi^\pm \rightarrow J(\zeta x)$  as  $x \rightarrow \pm\infty$ .

#### 4.2. The Jost functions for the Peregrine soliton

In the case of the Peregrine soliton (2.5) we can obtain the expressions of the relative spectral functions by taking the limit  $\eta \rightarrow 0$  on the equations of the previous subsection, i.e.

$$K(x, x, t) = -\frac{2v_0}{1 + 4v_0^2 x^2 + 16v_0^4 t^2} \begin{pmatrix} 2v_0 x & 1 + 4iv_0^2 t \\ -1 + 4iv_0^2 t & 2v_0 x \end{pmatrix} \quad (4.6)$$

and  $K(x, y, t) = K(x, x, t)$ . The Jost functions  $\Phi^+ = (\Phi_1^+, \Phi_2^+)$  are given by

$$\Phi_1^+(x, t) = e^{-i\zeta x} \left( \begin{array}{l} -i v_0 \left\{ 1 - \frac{i}{\zeta} \frac{2v_0}{1 + 4v_0^2 x^2 + 16v_0^4 t^2} \left[ 2v_0 x + i \frac{\lambda - \zeta}{v_0} (1 + 4iv_0^2 t) \right] \right\} \\ (\lambda - \zeta) \left\{ 1 - \frac{i}{\zeta} \frac{2v_0}{1 + 4v_0^2 x^2 + 16v_0^4 t^2} \left[ 2v_0 x - i \frac{\lambda + \zeta}{v_0} (1 - 4iv_0^2 t) \right] \right\} \end{array} \right) \quad (4.7)$$

and

$$\Phi_2^+(x, t) = e^{i\zeta x} \begin{pmatrix} (\lambda - \zeta) \left\{ 1 + \frac{i}{\zeta} \frac{2\nu_0}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} \left[ 2\nu_0 x + i \frac{\lambda + \zeta}{\nu_0} (1 + 4i\nu_0^2 t) \right] \right\} \\ -i\nu_0 \left\{ 1 + \frac{i}{\zeta} \frac{2\nu_0}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} \left[ 2\nu_0 x - i \frac{\lambda - \zeta}{\nu_0} (1 - 4i\nu_0^2 t) \right] \right\} \end{pmatrix}. \quad (4.8)$$

Mentioning that  $\Phi^+ \rightarrow TJ$  as  $x \rightarrow \pm\infty$ , we conclude that the Jost functions  $\Phi^-$  are identical to the  $\Phi^+$  functions. This means that  $S(\lambda, \zeta)$  that satisfies the equation  $\Phi^- = \Phi^+ S$  is the identity matrix, which means that the Peregrine soliton is a zero-radiation solution but it does not provide any eigenvalue, i.e.  $S_{11}(\lambda, \zeta) \neq 0$ .

Functions  $\Psi = \Psi^\pm$  are given by equation  $\Psi = T^{-1}\Phi$  and we obtain the following expressions:

$$\Psi(x, t) = \begin{pmatrix} e^{-i\zeta x} \left( 1 - \frac{2\nu_0^2}{\zeta^2} \frac{1 + 2i\zeta x}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} \right) & e^{i\zeta x} \frac{2i\nu_0}{\zeta^2} \frac{\lambda + 4i\nu_0^2 \zeta t}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} \\ e^{-i\zeta x} \frac{2i\nu_0}{\zeta^2} \frac{\lambda - 4i\nu_0^2 \zeta t}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} & e^{i\zeta x} \left( 1 - \frac{2\nu_0^2}{\zeta^2} \frac{1 - 2i\zeta x}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} \right) \end{pmatrix}. \quad (4.9)$$

Alternatively,

$$\Psi(x, t) = J(\zeta x) + \frac{1}{\zeta^2} \frac{2\nu_0}{1 + 4\nu_0^2 x^2 + 16\nu_0^4 t^2} \begin{pmatrix} -e^{-i\zeta x} \nu_0 (1 + 2i\zeta x) & e^{i\zeta x} (i\lambda - 4\nu_0^2 \zeta t) \\ e^{-i\zeta x} (i\lambda + 4\nu_0^2 \zeta t) & -e^{i\zeta x} \nu_0 (1 - 2i\zeta x) \end{pmatrix}. \quad (4.10)$$

## 5. Perturbation theory

In this section, we consider a perturbation theory for the perturbed NLS equation

$$q_t = S[q] + \epsilon R[q], \quad (5.1)$$

where

$$S[q] = iq_{xx} + 2i(|q|^2 - \nu_0^2)q$$

is the right-hand side (rhs) of the unperturbed NLS equation (2.2),  $R[q]$  is a perturbation and  $\epsilon$  is a (small) dimensionless parameter that characterizes the amplitude of the perturbation. We assume that the perturbation  $R[q]$  does not affect the background. We work in a way similar to [15] in order to obtain the evolution of the eigenvalues of the spectral problem in the presence of small perturbations.

### 5.1. Variational derivative

Equations (3.1a) can be rewritten as

$$\left( i\sigma_3 \frac{\partial}{\partial x} + \hat{Q}(x, t) \right) u = \lambda u, \quad \hat{Q} = \begin{pmatrix} 0 & -iq(x, t) \\ -i\bar{q}(x, t) & 0 \end{pmatrix}. \quad (5.2)$$

By taking the variational derivative of (5.2) we obtain

$$\left( i\sigma_3 \frac{\partial}{\partial x} + \hat{Q}(x; t) - \lambda I \right) \frac{\delta u\{q; x, t, \lambda\}}{\delta q(x')} = -\frac{\delta \hat{Q}(x; t)}{\delta q(x')} u\{q; x, t, \lambda\} + \frac{\delta \lambda\{q\}}{\delta q(x')} u\{q; x, t, \lambda\}. \quad (5.3)$$

When  $\lambda$  belongs to the continuous spectrum, we obtain the following problem for the variational derivative of the Jost functions:

$$\left( i\sigma_3 \frac{\partial}{\partial x} + \hat{Q}(x; t) - \lambda I \right) \frac{\delta \Phi^\pm\{q; x, t, \lambda\}}{\delta q(x')} = i\delta(x - x') \Phi^\pm\{q; x, t, \lambda\} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (5.4)$$



with the conditions  $\frac{\delta\Phi^\pm\{q;x,t,\lambda\}}{\delta q(x')} \rightarrow 0$  as  $x \rightarrow \pm\infty$ . These conditions are valid because we consider the problem on the continuous spectrum.

Hence, one has to find Green's function associated with equation (5.2). Using the fact that the columns of the Jost functions are independent solutions of the associated homogeneous problem along with the previous conditions we find that

$$\frac{\delta\Phi^\pm\{q;x,t,\lambda\}}{\delta q(x')} = \frac{\Theta(\pm(x' - x))}{d(\lambda, \zeta)} \Phi^\pm(x) \begin{pmatrix} \phi_{21}^\pm(x')\phi_{22}^\pm(x') & \phi_{22}^\pm(x')^2 \\ -\phi_{21}^\pm(x')^2 & -\phi_{21}^\pm(x')\phi_{22}^\pm(x') \end{pmatrix}, \quad (5.5)$$

where  $d(\lambda, \zeta)$  is given in (3.3) and  $\Theta(y)$  is the Heaviside function  $\Theta(y) = \begin{cases} 0, & y < 0, \\ 1, & y > 0. \end{cases}$  Here we do not write explicitly the time  $t$  on the rhs of the equation, considering it as a parameter. We will keep this notation throughout this section.

Taking the variational derivative of equation (3.4) we obtain

$$\frac{\delta\Phi^-(x)}{\delta q(x')} = \frac{\delta\Phi^+(x)}{\delta q(x')} S + \Phi^+(x) \frac{\delta S}{\delta q(x')},$$

for all  $x \in \mathbb{R}$ . Taking the limit  $x \rightarrow +\infty$ , we obtain

$$\frac{\delta S}{\delta q(x')} = \lim_{x \rightarrow +\infty} \left\{ (\Phi^+(x))^{-1} \frac{\delta\Phi^-(x)}{\delta q(x')} \right\}.$$

Equation (5.5) yields

$$\frac{\delta S}{\delta q(x')} = \frac{1}{d(\lambda, \zeta)} \lim_{x \rightarrow +\infty} \{ (\Phi^+(x))^{-1} \Phi^-(x) \} \begin{pmatrix} \phi_{21}^-(x')\phi_{22}^-(x') & \phi_{22}^-(x')^2 \\ -\phi_{21}^-(x')^2 & -\phi_{21}^-(x')\phi_{22}^-(x') \end{pmatrix}.$$

Using the fact that the quantity  $(\Phi^+(x))^{-1} \Phi^-(x)$  is equal to  $S$  and independent of  $x$  we obtain

$$\frac{\delta S}{\delta q(x)} = \frac{1}{d(\lambda, \zeta)} (\Phi^+(x))^{-1} \Phi^-(x) \begin{pmatrix} \phi_{21}^-(x)\phi_{22}^-(x) & \phi_{22}^-(x)^2 \\ -\phi_{21}^-(x)^2 & -\phi_{21}^-(x)\phi_{22}^-(x) \end{pmatrix}$$

and consequently

$$\frac{\delta S}{\delta q(x)} = \frac{1}{d(\lambda, \zeta)} \begin{pmatrix} \phi_{21}^-(x)\phi_{22}^+(x) & \phi_{22}^-(x)\phi_{22}^+(x) \\ -\phi_{21}^-(x)\phi_{21}^+(x) & -\phi_{22}^-(x)\phi_{21}^+(x) \end{pmatrix}. \quad (5.6)$$

If  $\lambda = i\nu$ ,  $\nu > 0$  is a discrete eigenvalue, i.e.  $S_{11}(i\nu) = 0$ , then we obtain the following identity:

$$\frac{\delta\nu}{\delta q(x)} = \frac{i}{S'_{11}(i\nu)} \frac{\delta S}{\delta q(x)} \Big|_{\lambda=i\nu}, \quad (5.7)$$

where the prime stands for the derivative with respect to  $\lambda$  and consequently

$$\frac{\delta\nu}{\delta q(x)} = \frac{1}{d(i\nu, i\eta)} \frac{i}{S'_{11}(i\nu)} \phi_{21}^-(x)\phi_{22}^+(x) \Big|_{\lambda=i\nu, \zeta=i\eta}. \quad (5.8)$$

In the same way, we obtain the analog of (5.4), (5.5), (5.6) and (5.8) for the function  $\bar{q}$ ,

$$\left( i\sigma_3 \frac{\partial}{\partial x} + \hat{Q}(x; t) - \lambda I \right) \frac{\delta\Phi^\pm\{\bar{q}; x, t, \lambda\}}{\delta \bar{q}(x')} = i\delta(x - x') \Phi^\pm\{\bar{q}; x, t, \lambda\} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (5.9)$$

$$\frac{\delta\Phi^\pm\{q; x, t, \lambda\}}{\delta \bar{q}(x')} = \frac{H(\pm(x' - x))}{d(\lambda, \zeta)} \Phi^\pm(x) \begin{pmatrix} \phi_{11}^\pm(x')\phi_{12}^\pm(x') & \phi_{12}^\pm(x')^2 \\ -\phi_{11}^\pm(x')^2 & -\phi_{11}^\pm(x')\phi_{12}^\pm(x') \end{pmatrix}, \quad (5.10)$$

$$\frac{\delta S}{\delta \bar{q}(x)} = \frac{1}{d(\lambda, \zeta)} \begin{pmatrix} \phi_{11}^-(x)\phi_{12}^+(x) & \phi_{12}^-(x)\phi_{12}^+(x) \\ -\phi_{11}^-(x)\phi_{11}^+(x) & -\phi_{12}^-(x)\phi_{11}^+(x) \end{pmatrix}, \quad (5.11)$$

and, if  $S_{11}(i\nu) = 0$ , then we obtain

$$\frac{\delta\nu}{\delta \bar{q}(x)} = \frac{1}{d(i\nu, i\eta)} \frac{i}{S'_{11}(i\nu)} \phi_{11}^-(x)\phi_{12}^+(x) \Big|_{\lambda=i\nu, \zeta=i\eta}. \quad (5.12)$$

### 5.2. Evolution of the discrete spectrum

Let  $F\{q\}$  be a functional that depends on  $q(x, t)$ ,  $x, t \in \mathbb{R}$ . Its time derivative is given by

$$\frac{dF\{q\}}{dt} = \int_{-\infty}^{+\infty} \left[ \frac{\delta F}{\delta q(x)} \frac{dq}{dt} + \frac{\delta F}{\delta \bar{q}(x)} \frac{d\bar{q}}{dt} \right] dx. \quad (5.13)$$

Applying this to the perturbed NLS equation (5.1) we obtain that

$$\frac{dF\{q\}}{dt} = \int_{-\infty}^{+\infty} \left( \frac{\delta F}{\delta q(x)} S[q] + \frac{\delta F}{\delta \bar{q}(x)} \bar{S}[q] \right) dx + \epsilon \int_{-\infty}^{+\infty} \left( \frac{\delta F}{\delta q(x)} R[q] + \frac{\delta F}{\delta \bar{q}(x)} \bar{R}[q] \right) dx. \quad (5.14)$$

If  $\lambda = i\nu$ ,  $\nu > 0$  is a discrete eigenvalue, then using the fact that the eigenvalues of the unperturbed NLS are time independent we obtain the following identity:

$$\int_{-\infty}^{+\infty} \left( \frac{\delta \nu}{\delta q(x)} S[q] + \frac{\delta \nu}{\delta \bar{q}(x)} \bar{S}[q] \right) dx = \left. \frac{d\nu}{dt} \right|_{\epsilon=0} = 0$$

and consequently the formula that describes the evolution of these eigenvalues is

$$\frac{d\nu}{dt} = \epsilon \int_{-\infty}^{+\infty} \left( \frac{\delta \nu}{\delta q(x)} R[q] + \frac{\delta \nu}{\delta \bar{q}(x)} \bar{R}[q] \right) dx, \quad (5.15)$$

where  $\frac{\delta \nu}{\delta q(x)}$  and  $\frac{\delta \nu}{\delta \bar{q}(x)}$  are given by (5.8) and (5.12).

From now on we develop a first-order perturbation theory. In this case, we can substitute into the rhs of (5.15) the corresponding expression with the unperturbed solution  $q(x, t)$  and this gives the evolution equation of the parameter  $\nu$  to first order in  $\epsilon$ .

## 6. Evolution of the Ma soliton under small perturbations

In what follows we will explicitly write the evolution of the eigenvalue, under the following perturbations, which do not affect the background:

- (i)  $R[q] = q_{xx}$
- (ii)  $R[q] = q_{xxx}$
- (iii)  $R[q] = iq_{xxxx}$ .

By making use of equations (4.1), (4.2), (4.3), (3.3) and the fact that  $\Phi^-$  is given by  $\Phi^+$  under the substitution  $\eta \rightarrow -\eta$ , equations (5.8) and (5.12) become

$$\frac{\delta \nu}{\delta q(x)} = \frac{\eta \nu \nu_0 - \nu \cosh(2\eta x - i4\nu\eta t) - \eta \sinh(2\eta x - i4\nu\eta t)}{2 [ \nu_0 \cos(4\nu\eta t) - \nu \cosh(2\eta x) ]^2} \quad (6.1)$$

and

$$\frac{\delta \nu}{\delta \bar{q}(x)} = \frac{\eta \nu \nu_0 - \nu \cosh(2\eta x - i4\nu\eta t) + \eta \sinh(2\eta x - i4\nu\eta t)}{2 [ \nu_0 \cos(4\nu\eta t) - \nu \cosh(2\eta x) ]^2}. \quad (6.2)$$

### 6.1. Diffusive perturbations

We compute the evolution of the parameter  $\nu$ , given by (5.15), when  $R[q] = q_{xx}$ . Equations (6.1), (6.2) and the solution (2.4) give

$$\frac{d\nu}{dt} = 4\eta^5 \nu^2 \epsilon \int_{-\infty}^{+\infty} \frac{-3\nu + 2\nu_0 \cos(4\nu\eta t) \cosh(2\eta x) + \nu \cosh(4\eta x)}{[ \nu_0 \cos(4\nu\eta t) - \nu \cosh(2\eta x) ]^4} dx. \quad (6.3)$$

Consequently,

$$\frac{d\nu}{dt} = - \frac{4\eta^4 \nu}{3 [ \nu^2 - \nu_0^2 \cos^2(4\nu\eta t) ]} \epsilon D(\eta, t), \quad (6.4)$$

where

$$D(\eta, t) = \frac{2v^2 + v_0^2 \cos^2(4v\eta t)}{v^2 - v_0^2 \cos^2(4v\eta t)} + \frac{6v^2 v_0 \tan^{-1} \left( \sqrt{\frac{v+v_0 \cos(4v\eta t)}{v-v_0 \cos(4v\eta t)}} \right) \cos(4v\eta t)}{[v^2 - v_0^2 \cos^2(4v\eta t)]^{3/2}}$$

and we recall that  $v = \sqrt{\eta^2 + v_0^2}$ . Moreover using that  $\frac{d\eta}{dt} = \frac{v}{\eta} \frac{dv}{dt}$ , we obtain

$$\frac{d\eta}{dt} = -\frac{4\eta^3 v^2}{3[v^2 - v_0^2 \cos^2(4v\eta t)]} \epsilon D(\eta, t). \quad (6.5)$$

The rhs is negative valued which shows that damping induces a decay of the soliton parameter.

One can compute the order  $\epsilon$  evolution of the total energy  $\mathcal{E}_{\text{tot}}$ , using the following formula:

$$\mathcal{E}_{\text{tot}} = \int_{-\infty}^{+\infty} (|q(x, t)|^2 - v_0^2) dx, \quad \frac{\partial \mathcal{E}_{\text{tot}}}{\partial t} = -2\epsilon \int_{-\infty}^{+\infty} |q_x(x, t)|^2 dx. \quad (6.6)$$

Straightforward computations give, to leading order  $\epsilon$ ,

$$\int_{-\infty}^{+\infty} |q_x(x, t)|^2 dx = \frac{8\eta^3}{3} D(\eta, t). \quad (6.7)$$

Using the fact that the energy of the soliton is

$$\mathcal{E}_{\text{sol}} = \int_{-\infty}^{+\infty} (|q(x, t)|^2 - v_0^2) dx = 4\eta,$$

we find

$$\frac{\partial \mathcal{E}_{\text{tot}}}{\partial t} = \frac{\partial \mathcal{E}_{\text{sol}}}{\partial t} \frac{v^2 - v_0^2 \cos^2(4v\eta t)}{v^2}. \quad (6.8)$$

This shows that the decay of the soliton energy is larger than the decay of the total energy. Therefore, an adiabatic approach based on the identification of the soliton parameter via the decay of the total energy and the hypothesis that the wave solution has the form of soliton would underestimate the decay of the solution [10]. Note, however, that the adiabatic approach gives the right prediction when  $v_0 \rightarrow 0$ , that is the classical case of the bright soliton solution of the NLS equation with vanishing boundary condition.

### 6.2. Dispersive perturbations

When  $R[q] = q_{xxx}$ , straightforward calculations yield

$$\begin{aligned} \frac{dv}{dt} &= 6\eta^6 v^2 \epsilon \int_{-\infty}^{+\infty} \\ &\times \frac{[-11v^2 + 2v_0^2 \cos^2(4v\eta t) + 8v v_0 \cos(4v\eta t) \cosh(2\eta x) + v^2 \cosh(4\eta x)] \sinh(2\eta x)}{[v_0 \cos(4v\eta t) - v \cosh(2\eta x)]^5} dx, \end{aligned} \quad (6.9)$$

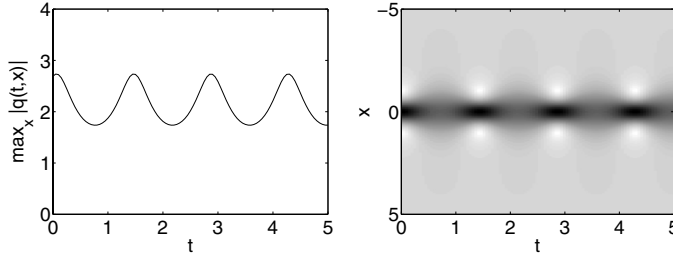
which gives

$$\frac{dv}{dt} = \frac{d\eta}{dt} = 0.$$

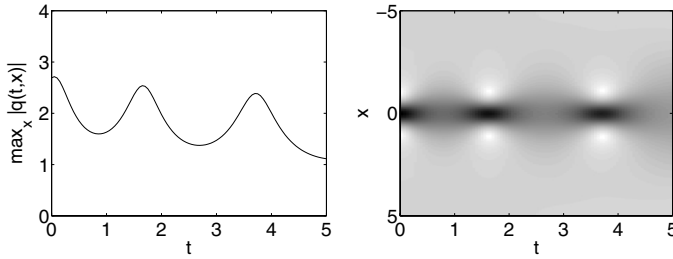
This shows that the Ma soliton is stable with respect to third-order dispersion.

When  $R[q] = iq_{xxx}$  we obtain

$$\begin{aligned} \frac{dv}{dt} &= 4\eta^6 v^2 \epsilon \int_{-\infty}^{+\infty} \frac{v_0 [v - v_0 \cos(4v\eta t) \cosh(2\eta x)] \sin(4v\eta t) + i [v^2 - v_0^2 \cos^2(4v\eta t)] \sinh(2\eta x)}{[v_0 \cos(4v\eta t) - v \cosh(2\eta x)]^7} \\ &\{4v v_0^2 \cos^2(4v\eta t)[-5 + 11 \cosh(4\eta x)] + v^3 [115 - 76 \cosh(4\eta x) + \cosh(8\eta x)] \\ &+ 4v_0 \cos(4v\eta t) \cosh(2\eta x) [2v_0^2 \cos^2(4v\eta t) - 29v^2 + 11v^2 \cosh(2\eta x)]\} dx. \end{aligned} \quad (6.10)$$



**Figure 1.** Propagation of the Ma soliton without perturbation. The left picture plots the maximum of the spatial profile as a function of time, the right picture plots the spatial profiles as a function of time. Here  $v_0 = 0.5$ ,  $\eta = 1$ .



**Figure 2.** Propagation of the Ma soliton with a second-order diffusive perturbation. Here  $v_0 = 0.5$ ,  $\eta = 1$  and  $\epsilon = 0.02$ .

From the first line of the above integral we can see that the imaginary part has no contribution to the above expression. Consequently,

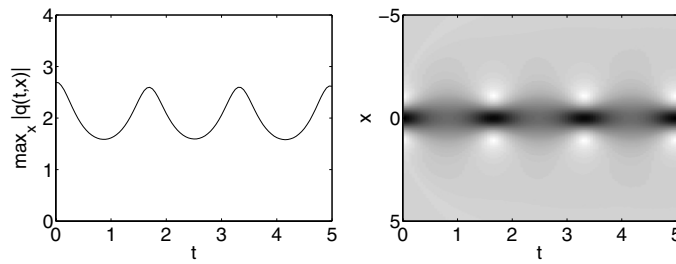
$$\begin{aligned} \frac{dv}{dt} = & 8\eta^5 v^2 v_0 \sin(4v\eta t) \epsilon \left\{ -\frac{1}{[v^2 - v_0^2 \cos^2(4v\eta t)]^2} + \frac{15v^2 v_0}{[v^2 - v_0^2 \cos^2(4v\eta t)]^3} \right. \\ & \left. + \left[ \frac{4 + 30v_0^2}{[v^2 - v_0^2 \cos^2(4v\eta t)]^{5/2}} - \frac{30v_0^4}{[v^2 - v_0^2 \cos^2(4v\eta t)]^{7/2}} \right] \tan^{-1} \left( \sqrt{\frac{v + v_0 \cos(4v\eta t)}{v - v_0 \cos(4v\eta t)}} \right) \right\}. \end{aligned} \tag{6.11}$$

Moreover, the above rhs expression is an odd and periodic function; hence, its integral over a period is equal to zero. This shows that the Ma soliton is rather stable with respect to fourth-order dispersion, as it only experiences breathing (to first order in  $\epsilon$ ).

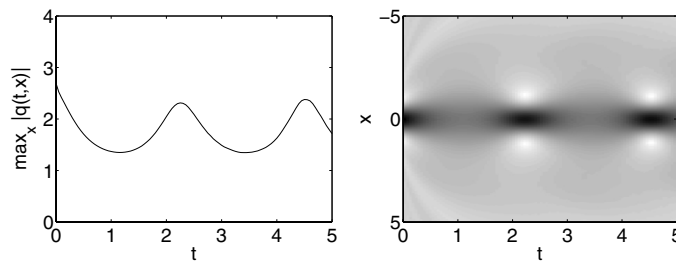
### 7. Numerical simulations

In this section, we carry out direct numerical simulations of the NLS equation (2.2) to illustrate our theoretical predictions. We consider an initial soliton of the form (2.4). The  $t$ -period of the unperturbed soliton is  $\pi/(2\eta v)$ , with  $v = \sqrt{v_0^2 + \eta^2}$  and its maximum at time 0 is  $\max_x |q(0, x)| = 2v + v_0$ .

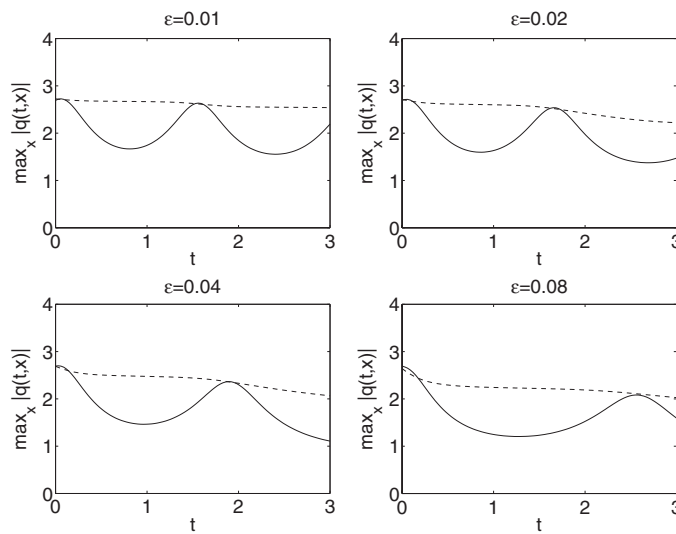
We use a second-order split-step Fourier method to numerically solve the NLS equation [11]. In the numerical simulations the background is  $v_0 = 0.5$  and the initial parameter of the soliton is  $\eta = 1$ . In figures 1–4 we plot the maximum of the solution  $\max_x |q(t, x)|$  as a function of  $t$  and the profiles  $|q(t, x)|$  as a function of  $(t, x)$  in the absence of perturbations



**Figure 3.** Propagation of the Ma soliton with a third-order dispersive perturbation. Here  $v_0 = 0.5$ ,  $\eta = 1$  and  $\epsilon = 0.02$ .



**Figure 4.** Propagation of the Ma soliton with a fourth-order dispersive perturbation. Here  $v_0 = 0.5$ ,  $\eta = 1$  and  $\epsilon = 0.02$ .



**Figure 5.** Propagation of the Ma soliton with a second-order diffusive perturbation. Here  $v_0 = 0.5$ ,  $\eta = 1$  and  $\epsilon$  takes different values from 0.01 to 0.08. The solid lines stand for the numerical maxima as functions of time, the dashed lines plot the functions  $m_{\text{theo}}(t) = 2\sqrt{v_0^2 + \eta_{\text{theo}}^2(t)} + v_0$ , where  $\eta_{\text{theo}}(t)$  is the solution of (6.5).

(figure 1) and in the presence of the three perturbations addressed in section 6 (figures 2–4). Here the amplitude of the perturbation is  $\epsilon = 0.02$ . We can observe that the soliton is more robust with respect to dispersive perturbations than with respect to diffusive perturbations. In

particular, we can observe a nice periodic behavior in figure 3 in the case of a third-order dispersion, as predicted by the theory. The same holds true in the case of a fourth-order dispersion (figure 4) although the period has changed compared to the unperturbed case. Finally, we can observe a decay of the soliton and a continuous increase of its period in the case of a diffusive perturbation, which is also in agreement with the theoretical predictions. To be complete, we must add that the stability of the soliton also becomes affected by dispersive perturbations when  $\epsilon \geq 0.1$ . Our theory is therefore valid only for weak perturbations.

More quantitatively, we can numerically integrate the ODE (6.5) to find  $\eta_{\text{theo}}(t)$ , take care of  $m_{\text{theo}}(t) = 2\nu_{\text{theo}}(t) + \nu_0 = 2\sqrt{\nu_0^2 + \eta_{\text{theo}}^2(t)} + \nu_0$  and compare with the maximum of the solution after one oscillation. The results are reported in figure 5 for different values of  $\epsilon$ , which again shows good agreement.

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