Toward a wave turbulence formulation of statistical nonlinear optics

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Received March 27, 2012; accepted June 5, 2012;
posted June 11, 2012 (Doc. ID 165489); published August 1, 2012

During this last decade, several remarkable phenomena inherent to the nonlinear propagation of incoherent optical waves have been reported in the literature. This article is aimed at providing a generalized wave turbulence kinetic formulation of random nonlinear waves governed by the nonlinear Schrödinger equation in the presence of a nonlocal or a noninstantaneous nonlinear response function. Depending on the amount of nonlocal (noninstantaneous) nonlinear interaction and the amount of inhomogeneous (nonstationary) statistics of the incoherent wave, different types of kinetic equations are obtained. In the spatial domain, when the incoherent wave exhibits fluctuations that are statistically homogeneous in space, the relevant kinetic equation is the wave turbulence (Hasselmann) kinetic equation. It describes, in particular, the process of optical wave thermalization to thermodynamic equilibrium, which slows down significantly as the interaction becomes highly nonlocal. When the incoherent wave is characterized by inhomogeneous statistical fluctuations, different forms of the Vlasov equation are derived, which depend on the amount of nonlocality in the system. This Vlasov approach describes, in particular, the processes of incoherent modulational instability and the formation of localized incoherent soliton structures. In the temporal domain, the noninstantaneous nonlinear response function is constrained by the causality condition. It turns out that the relevant kinetic equation has a form analogous to the weak Langmuir turbulence equation, which describes, in particular, the formation of nonlocalized spectral incoherent solitons. In the regime of a highly noninstantaneous nonlinear response and a stationary statistics of the incoherent wave, the weak Langmuir turbulence equation reduces to the Korteweg–de Vries equation. Conversely, in the regime of a highly noninstantaneous response in the presence of a nonstationary statistics, we derive a long-range Vlasov-like kinetic equation in the temporal domain, whose self-consistent potential is constrained by the causality condition. From a broader perspective, this work indicates that the wave turbulence theory may constitute the appropriate theoretical framework to formulate statistical nonlinear optics.

OCIS codes: 190.0190, 030.6600.

1. INTRODUCTION

The nonlinear propagation of coherent optical fields has been explored in the framework of nonlinear optics [1,2], while the linear propagation of incoherent fields has been studied in the framework of statistical optics [3]. However, these two fundamental fields of optics have been mostly developed independently, so that a complete and satisfactory understanding of *statistical nonlinear optics* is still lacking. This article is aimed at providing a generalized wave turbulence (WT) description of partially coherent optical waves propagating in Kerr media with a nonlocal or a noninstantaneous nonlinear response.

The dynamics of partially coherent nonlinear optical beams has received a renewed interest since the first experimental demonstration of incoherent solitons (ISs) in photorefractive crystals [4,5]. The IS formation results from the spatial self-trapping of incoherent light that propagates in a noninstantaneous response nonlinear medium [6–15]. The remarkable simplicity of experiments realized in photorefractive crystals has led to a fruitful investigation of the dynamics of incoherent nonlinear waves. Several fundamental phenomena have been predicted theoretically and confirmed experimentally, such as, e.g., the modulational instability of incoherent optical waves [11,12], the existence of incoherent dark solitons [14,15], or ISs in optical lattices [16,17] (for a review, see [2]). More specifically, several different theoretical approaches have been developed to describe the evolution of incoherent nonlinear waves in slowly responding nonlinear materials, namely the mutual coherence function approach [18], the self-consistent multimode theory [19], the coherent density method [20], and the Wigner–Moyal transform approach [21,22]. Subsequently, these four different theoretical methods have been shown to be formally equivalent [23,24]. We note in particular that the Wigner–Moyal equation has been recently shown to be integrable by finding a recursive relation generating an infinite number of invariants [25,26].

It is important to underline that the mechanism underlying the formation of these IS states finds its origin in the existence of a *self-consistent potential*, which is responsible for a spatial self-trapping of the incoherent optical beam. From this point of view, these ISs are of the same nature as the ISs predicted in plasma physics a long time ago in the framework of the Vlasov equation [27–29]. This analogy with nonlinear plasma waves has been also exploited in optics in different circumstances [30–32], in particular to interpret the existence of a threshold in the incoherent modulational instability as a consequence of the phenomenon of Landau damping [21].

ISs can be also supported by a nonlocal nonlinearity in the spatial domain, instead of the conventional noninstantaneous nonlinearity discussed here above [33–38]. A nonlocal wave interaction means that the response of the nonlinearity at a particular point is not determined solely by the wave intensity at that point, but also depends on the wave intensity in the neighborhood of this point. Nonlocality thus constitutes a generic property of a large number of nonlinear wave systems [39–46], and the dynamics of nonlocal nonlinear waves has been widely investigated in this last decade [47–51]. In particular, in the highly nonlocal limit, i.e., in the limit where the range of the nonlocal response is much larger than the size of the beam, the propagation equation reduces to a linear and local equation with an effective guiding potential given by the nonlocal response function. The optical beam can thus be guided by the nonlocal response of the material, a process originally termed "accessible solitons" in [51,52]. In this highly nonlocal limit, it has been shown theoretically and experimentally that a speckled beam can be guided and trapped by the effective waveguide induced by the nonlocal response [33,34].

More recently, the long-term evolution of a modulationally unstable homogeneous wave has been studied in the presence of a nonlocal response [35]. Contrarily to the expected soliton turbulence process where a coherent soliton is eventually generated in the midst of small-scale fluctuations [53–57], a highly nonlocal response is responsible for an *IS turbulence* process. It is characterized by the spontaneous formation of an IS structure starting from an initially homogeneous plane-wave. A WT approach of the problem revealed that this type of IS can be described in detail in the framework of a Vlasov-like kinetic equation, which is shown to provide an "exact" statistical description of the highly nonlocal random wave system. We note that this Vlasov equation differs from the traditional Vlasov equation considered for the study of incoherent modulational instability and ISs in plasmas [28,29,58], hydrodynamics [59], and optics [21,30–32], while its structure is analogous to that recently used to describe systems of particles with long-range interactions [60].

From a different perspective, another type of IS has been recently identified in the temporal domain, by exploiting the noninstantaneous property of the nonlinear Raman response in optical fibers [61–63]. This IS is of a fundamentally different nature than the ISs discussed above. In particular, it does not exhibit a confinement in the spatio-temporal domain, but exclusively in the frequency domain. For this reason, it has been termed "spectral IS." Indeed, the optical field exhibits a stationary statistics (i.e., the field exhibits random fluctuations that are statistically stationary in time), so that the soliton behavior only manifests in the spectral domain. A WT analysis of the system has revealed that the kinetic equation that describes spectral ISs has a structure analogous to that considered in plasma physics to study weak Langmuir turbulence [64–68]. For this reason, we will term this kinetic equation "weak Langmuir turbulence" equation.

It turns out that the generation of ISs requires either a noninstantaneous or a nonlocal response of the nonlinear medium in which the wave propagates. Conversely, when a statistically homogeneous (stationary) incoherent wave propagates in a medium whose nonlinear response can be considered as local (instantaneous), the wave is expected to exhibit a process of thermalization. This process is characterized by an irreversible evolution toward the thermodynamic equilibrium state, i.e., the Rayleigh–Jeans spectrum that realizes the maximum entropy. The essential properties of wave thermalization are described by the WT theory on the basis of an irreversible kinetic equation [69–74], which was originally derived by Hasselmann [75,76]. The irreversible behavior to thermal equilibrium is expressed by an H-theorem of entropy growth, in analogy with the Boltzmann kinetic equation relevant to kinetic gas theory. The process of optical wave thermalization [77–80], and its breakdown [81–84], have been studied in various circumstances, such as, e.g., wave condensation [85–88] and supercontinuum generation [89–91], as well as in various optical media characterized by different nonlinearities [92–94].

Our aim in this article is to provide a generalized kinetic description of incoherent waves governed by the nonlinear Schrödinger (NLS) equation in the presence of either a nonlocal (in the spatial domain) or a noninstantaneous (in the temporal domain) response function. For the sake of clarity, we consider separately the physical situations where the Kerr nonlinearity exhibits a nonlocal and a noninstantantaneous response. Depending on the amount of nonlocal (noninstantaneous) nonlinear interaction and the amount of inhomogeneous (nonstationary) statistics of the incoherent wave, different types of kinetic equations are obtained. This is illustrated schematically in Figs. 1 and 2, respectively, for a spatially nonlocal and a temporally noninstantaneous response of the nonlinearity. In substance, when the statistics of the wave are homogeneous in space, the relevant kinetic equation is the Hasselmann WT kinetic equation. In the limit of a conservative wave interaction (unforced system), this kinetic equation describes the irreversible process of thermalization to the Rayleigh–Jeans equilibrium distribution. Interestingly, we show here that this thermalization process slows down in a

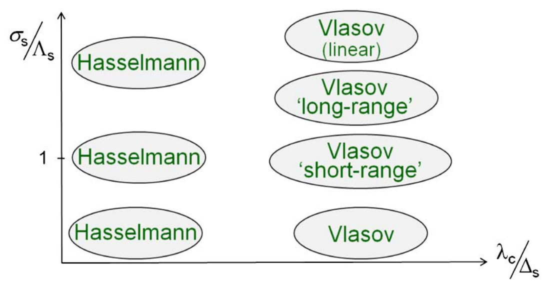

Fig. 1. (Color online) Schematic illustration of the validity of the fundamental kinetic equations in the framework of a spatially nonlocal nonlinear response: σ_s/Λ_s represents the amount of nonlocality of the nonlinear interaction, while λ_c/Δ_s represents the amount of inhomogeneous statistics of the incoherent wave. When the incoherent wave is characterized by fluctuations that are statistically homogeneous in space, the relevant kinetic description is provided by the WT (Hasselmann) kinetic equation, which describes in particular the irreversible process of thermalization to the thermodynamic Rayleigh–Jeans equilibrium spectrum. When the incoherent wave exhibits an inhomogeneous statistics, the relevant kinetic description is provided by different variants of the Vlasov equation, whose self-consistent potential depends on the amount of nonlocality in the system. The Vlasov kinetic equation describes, in particular, incoherent modulational instability and localized IS structures. (σ_s is the range of the nonlocal interaction, Λ_s is the healing length, λ_c is the correlation length, and Δ_s is the length scale of inhomogeneous statistics).

significant way as the nonlinear interaction becomes highly nonlocal. Conversely, in the presence of inhomogeneous statistics, the relevant kinetic equation is the Vlasov equation, whose self-consistent nonlinear potential is shown to depend on the amount of nonlocality in the system (see Fig. 1). The "short-range" Vlasov equation was not considered before in the literature, while the "long-range" Vlasov equation was recently considered in [35] to describe nonlocal IS solitons. A similar diagram of the kinetic description of the wave interaction is obtained in the temporal domain in the presence of a noninstantaneous nonlinear response, as illustrated in Fig. 2. There is, however, an essential difference with the spatial case because the response function of the material is constrained by the causality condition in the temporal domain. As a result of the causality property, a kinetic equation similar to the weak Langmuir turbulence equation turns out to be the relevant kinetic equation, irrespective of the nature of the statistics of the incoherent wave, which may be either stationary or nonstationary. When the incoherent wave exhibits a stationary statistics in the presence of a highly noninstantaneous nonlinear response of the material, we show that the weak Langmuir turbulence equation reduces to the Korteweg–de Vries equation, which thus describes the evolution of the averaged spectrum of the incoherent wave. Conversely, when the wave exhibits a nonstationary statistics in the presence of a highly noninstantaneous response, we derive a long-range Vlasov-like kinetic equation in the temporal domain, whose self-consistent potential is constrained by the causality condition. To our knowledge, it is the first time that this long-range Vlasov-like equation is derived, even beyond the context of optics. The self-consistent potential of this kinetic equation

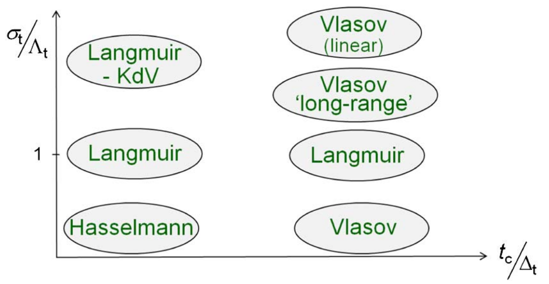

Fig. 2. (Color online) Schematic illustration of the validity of the fundamental kinetic equations in the framework of a temporally noninstantaneous nonlinear response: σ_t/Λ_t represents the amount of noninstantaneous response of the nonlinearity, while t_c/Δ_t represents the amount of nonstationary statistics of the incoherent wave. The diagram for the temporal domain reported here is similar to that reported in the spatial domain in Fig. 1. The essential difference between the spatial and the temporal domain relies on the fact that in the temporal domain, the response function is constrained by the causality condition. It turns out that when the finite response time of the nonlinearity cannot be neglected, the relevant kinetic description is provided by an equation analogous to the weak Langmuir turbulence equation, irrespective of the nature of the fluctuations that may be either stationary or nonstationary. This equation has been shown to describe nonlocalized spectral ISs. In the presence of a highly noninstantaneous nonlinear response and a stationary statistics of the incoherent wave, the weak Langmuir turbulence reduces to the Korteweg–de Vries equation. Conversely, when the wave exhibits a nonstationary statistics still in the presence of a highly noninstantaneous response, we derive a "temporal long-range" Vlasov equation, whose self-consistent potential is constrained by the causality condition of the noninstantaneous response function. (σ_t is the response time of the nonlinearity, Λ_t is the "healing time," t_c is the correlation time, and Δ_t is the time scale of nonstationary statistics).

can be decomposed into an even contribution that leads to the conventional conservative Vlasov dynamics and an odd contribution that leads to a spectral red-shift in frequency space. We remark that the dynamics of optical waves in slowly responding Kerr materials is also attracting a growing interest thanks to recent advances in the fabrication of photonic crystal fibers filled with molecular liquids displaying highly noninstantaneous Kerr responses (see, e.g., [95]). Finally, to summarize the analogy between the kinetic wave approach and the kinetic theory relevant to a gas system, we schematically report in Fig. 3 a qualitative and intuitive physical insight into the meaning of the fundamental three kinetic equations discussed in this article.

This work reports a significant progress in the understanding of the dynamics of partially coherent optical waves propagating in nonlinear media. We note that, besides ISs and optical wave thermalization, the study of statistical nonlinear optics is a subject of growing interest in various fields of investigations, including, e.g., optical wave condensation [85–88,96–100], wave propagation in periodic media [94], nonlinear imaging [101], polarization effects [79,102], cavity systems [97–99, 103–107], or nonlinear interferometry [108]. From a more general perspective, this work reports a unified kinetic formulation of random nonlinear waves governed by the NLS equation. Given the universality of this equation in nonlinear science, this work finds applicability in many areas of physics including hydrodynamics, plasma physics, and low-temperature condensed matter.

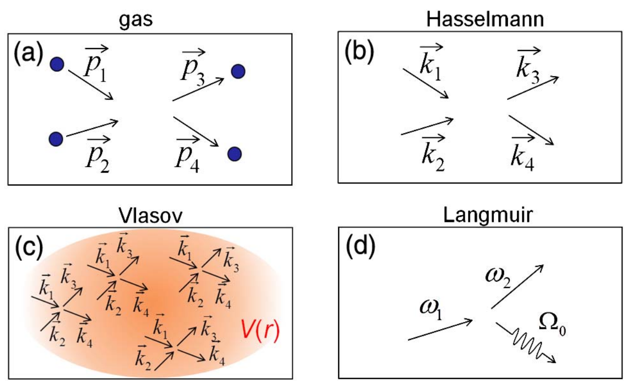

Fig. 3. (Color online) Analogy between a system of classical particles and the propagation of an incoherent optical wave in a cubic nonlinear medium. (a) As described by the kinetic gas theory (Boltzmann kinetic equation), collisions between particles are responsible for an irreversible evolution of the gas towards thermodynamic equilibrium. (b) In complete analogy, the WT (Hasselmann) kinetic equation and the underlying four-wave mixing describe an irreversible evolution of the incoherent optical wave toward the thermodynamic Rayleigh–Jeans equilibrium state. (c) When the incoherent optical wave exhibits an inhomogeneous statistics, the four-wave interaction no longer takes place locally; i.e., the quasi-particles feel the presence of an effective self-consistent potential, $V(r)$, which prevents them from relaxing to thermal equilibrium. The dynamics of the incoherent optical wave turns out to be described by a Vlasov-like kinetic equation. (d) In the presence of a noninstantaneous nonlinear interaction, the causality condition inherent to the response function changes the physical picture: the nonlinear interaction involves a material excitation (e.g., molecular vibration in the example of Raman scattering). The dynamics of the incoherent optical wave turns out to be described by a kinetic equation analogous to the weak Langmuir turbulence equation. Note, however, that a highly noninstantaneous nonlinear response is no longer described by the weak Langmuir turbulence equation, but instead by the "long-range" Vlasov-like equation (see Fig. 2).

We underline that this article is also aimed to render the mathematical tools of the WT theory accessible to a broad audience in the nonlinear optics community. Indeed, several detailed calculations underlying the derivation of the kinetic equations are reported in Appendix A. In this respect, this paper can also be considered as a pedagogical introduction to the WT kinetic theory in the context of statistical nonlinear optics.

2. NONLOCAL RESPONSE

In this section, we study the transverse spatial evolution of a partially coherent wave that propagates in a nonlocal nonlinear medium. A nonlocal nonlinear response is found in several systems such as, e.g., dipolar Bose–Einstein condensates [39], atomic vapors [40], nematic liquid crystals [41,42], photorefractive media [43], thermal susceptibilities [44,45], and plasmas physics [46]. For this reason, the dynamics of nonlocal nonlinear waves has been widely investigated [47,48,50,51]. We consider here the standard nonlocal NLS model equation describing a nonlocal nonlinear wave interaction

$$i\partial_z\psi = -\beta_s\nabla^2\psi + \gamma\psi\int U(\boldsymbol{x}-\boldsymbol{x}')|\psi|^2(z,\boldsymbol{x}')\mathrm{d}\boldsymbol{x}', \quad (1)$$

where $\boldsymbol{x}$ denotes the position in the transverse plane of dimension d and ∇^2 denotes the corresponding transverse Laplacian ($\nabla^2 = \partial_x^2$ for $d = 1$, $\nabla^2 = \partial_x^2 + \partial_y^2$ for $d = 2$). The nonlocal response function $U(\boldsymbol{x})$ is a real and even function normalized in such a way that $\int U(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$, so that in the limit of a local response [$U(\boldsymbol{x}) = \delta(\boldsymbol{x})$, $\delta(\boldsymbol{x})$ being the Dirac function], Eq. (1) recovers the standard local NLS equation. The parameters $\beta_s(> 0)$ and γ refer to the linear and nonlinear coefficients, respectively. Note that a positive (negative) value of γ corresponds to a defocusing (focusing) nonlinear interaction. Besides the momentum, Eq. (1) conserves the power (or number of particles) $\mathcal{N} = \int|\psi(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}$, and the Hamiltonian $\mathcal{H} = \mathcal{E} + \mathcal{U}$, where $\mathcal{E} = \beta_s\int|\nabla\psi(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}$ and $\mathcal{U} = \frac{\gamma}{2}\iint U(\boldsymbol{x}-\boldsymbol{x}')|\psi(\boldsymbol{x})|^2|\psi(\boldsymbol{x}')|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'$ denote the linear (kinetic) and nonlinear contributions to the energy $\mathcal{H}$.

We denote by σ_s the spatial extension of $U(\boldsymbol{x})$, which characterizes the amount of nonlocality in the system. This length scale has to be compared with the healing length $\Lambda_s = \sqrt{\beta_s/(\gamma n)}$, where n is the typical density of power (particles) $|\psi|^2$. We recall that Λ_s denotes the typical wavelength excited by the modulational instability of a homogeneous background in the limit of a local nonlinearity, $\sigma_s \to 0$. Another important length is the typical length Δ_s that characterizes the homogeneity of the statistics. It reflects the typical length scale over which the fluctuations of the incoherent wave can be considered as homogeneous in space.

A. Short-Range Vlasov Equation

1. Nonlocal Case

Following the standard procedure, we derive an equation for the evolution of the autocorrelation function of the field, $B(\boldsymbol{x},\boldsymbol{\xi},z) = \langle\psi(\boldsymbol{x}+\boldsymbol{\xi}/2,z)\psi^*(\boldsymbol{x}-\boldsymbol{\xi}/2,z)\rangle$, where $\langle\cdot\rangle$ denotes an averaging over the realizations of the initial noise of the field $\psi(\boldsymbol{x}, z = 0)$. Because of the nonlinear character of the NLS equation, the evolution of the second-order moment of the wave depends on the fourth-order moment. In the same way, the equation for the fourth-order moment depends on the sixth-order moment, and so on. A simple way to achieve a closure of the infinite hierarchy of moment equations is to assume that the field has Gaussian statistics. This approximation is justified in the weakly nonlinear regime, $\rho = L_d/L_{\mathrm{nl}} \ll 1$, where $L_d = \lambda_c^2/\beta_s$ is the diffraction length, λ_c is the coherence length, and $L_{\mathrm{nl}} = 1/(\gamma n)$ is the characteristic length of nonlinear interaction [58,64,69,70]. Exploiting the property of factorizability of moments of Gaussian fields, one obtains the following closed equation for the evolution of the autocorrelation function:

$$i\partial_z B(\boldsymbol{x},\boldsymbol{\xi},z) = -2\beta_s\nabla_{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{\xi}}B(\boldsymbol{x},\boldsymbol{\xi},z) + \gamma P(\boldsymbol{x},\boldsymbol{\xi},z) + \gamma Q(\boldsymbol{x},\boldsymbol{\xi},z), \quad (2)$$

where

$$P(\boldsymbol{x},\boldsymbol{\xi}) = B(\boldsymbol{x},\boldsymbol{\xi})\int U(\boldsymbol{y})[N(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{\xi}/2) - N(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{\xi}/2)]\mathrm{d}\boldsymbol{y}, \quad (3)$$

$$Q(\boldsymbol{x},\boldsymbol{\xi}) = \int U(\boldsymbol{y})[B(\boldsymbol{x}-\boldsymbol{y}/2+\boldsymbol{\xi}/2,\boldsymbol{y})B(\boldsymbol{x}-\boldsymbol{y}/2,\boldsymbol{\xi}-\boldsymbol{y}) - B(\boldsymbol{x}-\boldsymbol{y}/2,\boldsymbol{\xi}+\boldsymbol{y})B(\boldsymbol{x}-\boldsymbol{y}/2-\boldsymbol{\xi}/2,-\boldsymbol{y})]\mathrm{d}\boldsymbol{y}, \quad (4)$$

and

$$N(\boldsymbol{x},z) \equiv B(\boldsymbol{x},\boldsymbol{\xi}=0,z) = \langle|\psi(\boldsymbol{x},z)|^2\rangle \quad (5)$$

denotes the averaged power of the field, which depends on the spatial variable $\boldsymbol{x}$ because the statistics of the field is *a priori* inhomogeneous. Note that we have omitted the z-label in Eqs. (3) and (4).

Equations (2)–(4) are quite involved. To provide an insight into their physics, we assume that the incoherent wave exhibits a quasi-homogeneous statistics. We introduce the small parameter ε, which is the ratio between the coherence length of the field λ_c (i.e., the length scale of the random fluctuations) and the length scale of homogeneous statistics Δ_s (i.e., typically the size of the incoherent beam), $\varepsilon = \lambda_c/\Delta_s$. We assume that the range of the response function is of the same order as the healing length, $\sigma_s \sim \Lambda_s$. Defining the local spectrum of the wave as the Wigner-like transform of the autocorrelation function,

$$n_{\boldsymbol{k}}(\boldsymbol{x},z) = \int B(\boldsymbol{x},\boldsymbol{\xi},z)\exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},$$

and performing a multiscale expansion of the solution

$$B(\boldsymbol{x},\boldsymbol{\xi},z) = B^{(0)}(\varepsilon\boldsymbol{x},\boldsymbol{\xi},\varepsilon z) + O(\varepsilon), \quad (6)$$

we obtain in the first-order in ε the following Vlasov-like kinetic equation (see Appendix A.1):

$$\partial_z n_{\boldsymbol{k}}(\boldsymbol{x},z) + \partial_{\boldsymbol{k}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z)\cdot\partial_{\boldsymbol{x}}n_{\boldsymbol{k}}(\boldsymbol{x},z) - \partial_{\boldsymbol{x}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z)\cdot\partial_{\boldsymbol{k}}n_{\boldsymbol{k}}(\boldsymbol{x},z) = 0. \quad (7)$$

The generalized dispersion relation reads

$$\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z) = \omega(\boldsymbol{k}) + V_{\boldsymbol{k}}(\boldsymbol{x},z), \quad (8)$$

where $\omega(\boldsymbol{k}) = \beta_s k^2$ is the linear dispersion relation of the NLS Eq. (1), and the self-consistent potential reads

$$V_k(x,z) = \frac{\gamma}{(2\pi)^d} \int (1 + \tilde{U}_{k-k'}) n_{k'}(x,z) dk', \quad (9)$$

where $\tilde{U}(k) = \int U(x) \exp(-ik \cdot x) dx$ is the Fourier transform of $U(x)$ and

$$N(x,z) = \frac{1}{(2\pi)^d} \int n_k(x,z) dk \quad (10)$$

is the averaged spatial intensity profile of the wave [see Eq. (5)]. Note that the effective potential (9) of this Vlasov equation also depends on the spatial frequency k, which considerably complicates the study of the Vlasov equation. To our knowledge, it is the first time that this Vlasov equation with the self-consistent potential (9) is derived in the context of a wave system that exhibits a nonlocal interaction. The dependence of the potential (9) on k is expected to introduce new dynamical behaviors, which will be the subject of future investigations.

Several important properties of the Vlasov Eq. (7) result from its Poisson bracket structure. First of all, the Vlasov equation is a formally reversible equation; i.e., it is invariant under the transformation $(z,k) \to (-z,-k)$. Moreover, it conserves the number of particles, $\mathcal{N} = (2\pi)^{-d} \iint n_k(x,z) dx dk$, and the Hamiltonian $\mathcal{H}_{Vl} = \iint \omega(k) n_k(x,z) dx dk + \frac{1}{2(2\pi)^{2d}} \iiint n_{k_1}(x,z) \tilde{U}_{k_1-k_2} n_{k_2}(x,z) dx dk_1 dk_2$. In addition, the Vlasov Eqs. (7)–(9) also conserve the so-called Casimirs, $\mathcal{M} = \iint f[n] dx dk$, where $f[n]$ is an arbitrary functional of the distribution $n_k(x,z)$.

2. Local Limit

We remark that in the limit of a local interaction, $U(x) \to \delta(x)$, the Vlasov equation derived here above recovers the traditional Vlasov equation, whose self-consistent potential (9) becomes k-independent and reduces to

$$V(x,z) = 2\gamma N(x,z). \quad (11)$$

This type of Vlasov equation was considered in various different fields to study incoherent modulational instability and ISs in plasmas [28,29,58], hydrodynamics [59], and optics [21,30–32].

B. Long-Range Vlasov Equation

1. Long-Range Response

Let us now consider a long-range nonlocal nonlinear response, $\sigma_s/\Lambda_s \gg 1$. In this case, the random field exhibits fluctuations whose spatial inhomogeneities are of the same order as the range of the nonlocal potential, $\sigma_s \sim \Delta_s$. This aspect is well illustrated, e.g., by the IS recently discussed in [35], whose typical size is determined by σ_s. Accordingly, the derivation of the long-range Vlasov equation is obtained by following a procedure similar to that for the short-range case ($\sigma_s \sim \Lambda_s$), except that we have to introduce the following scaling for the nonlocal potential:

$$U(x) = \varepsilon U^{(0)}(\varepsilon x). \quad (12)$$

Note that the prefactor ε is required by the normalization condition, $\int U(x) dx = \int U^{(0)}(\varepsilon x) d(\varepsilon x) = 1$. Following the multiscale expansion technique, we derive in Appendix A.2 the Vlasov-like kinetic Eq. (7), with the effective dispersion relation (8), and the long-range self-consistent potential

$$V(x,z) = \gamma \int U(x - x') N(x',z) dx'. \quad (13)$$

This effective potential then appears as a simple convolution of the nonlocal response with the intensity profile of the incoherent wave. Contrarily to the short-range potential, it does not depend on the spatial frequency k. The long-range Vlasov equation conserves the number of particles, $\mathcal{N} = (2\pi)^{-d} \iint n_k(x,z) dx dk$, the Hamiltonian $\mathcal{H}_{Vl} = \iint \omega(k) n_k(x,z) dx dk + \frac{1}{2} \int V(x) N(x,z) dx$, and $\mathcal{M} = \iint f[n] dx dk$, where $f[n]$ is an arbitrary functional of $n_k(x,z)$. This Vlasov equation was considered for the first time in [35] to describe highly nonlocal spatial ISs. In this previous work, we also generalized to a nonlocal potential the soliton solution obtained in the limit of a local nonlinear response, $U(x) \to \delta(x)$ [28,29].

It is important to underline here that, thanks to the long-range nonlocal response, the system exhibits a self-averaging property of the nonlinear response, $\int U(x - x') |\psi(x',z)|^2 dx' \simeq \int U(x - x') N(x',z) dx'$. Substitution of this property into the nonlocal NLS Eq. (1) thus leads to a closure of the hierarchy of the moment equations. More specifically, using statistical arguments similar as those in [31], one can show that, owing to the highly nonlocal response, the statistics of the incoherent wave turns out to be Gaussian. Then, contrarily to a conventional Vlasov equation, whose validity is constrained by the assumptions of (i) weakly nonlinear interaction and (ii) quasi-homogeneous statistics, the long-range Vlasov equation provides an *exact* statistical description of the random wave $\psi(x,z)$ in the highly nonlocal regime, $\varepsilon \ll 1$. This property is corroborated by the fact that the Vlasov equation considered here is formally analogous to the Vlasov equation considered to study long-range interacting systems [60]. In this context, it has been rigorously proven that, in the limit of an infinite number of particles, the dynamics of *mean-field Hamiltonian systems* is governed by the long-range Vlasov equation [60]. Note, however, that the term "long-range" used in [60] refers to a response function whose integral diverges, $\int U(x) dx = +\infty$, while the response functions considered here refer to exponential or Gaussian shaped functions typically encountered in optical materials (see, e.g., [51]).

2. Highly Nonlocal Response: Linear Limit

Note that in the limit of a highly nonlocal nonlinear interaction, the range of the response function can be much larger than the scale of inhomogeneous statistics, $\sigma_s \gg \Delta_s$. In this limit, the response function can be extracted from the convolution integral in the effective potential (13), which thus leads to

$$V(x) = \gamma \mathcal{N} U(x). \quad (14)$$

It is interesting to note that in this limit, the response function plays the role of the effective potential. Accordingly, the Vlasov equation loses its self-consistent nonlinear character and thus reduces to a *linear* kinetic equation.

As a matter of fact, this highly nonlocal limit was originally explored by Snyder and Mitchell in [52], and has then been the subject of a detailed investigation in optics in the framework of the so-called "accessible solitons" [41,42]. Indeed, the above approximation ($\sigma_s \gg \Delta_s$) can be done directly into the original NLS Eq. (1), which is thus reduced to a *local and linear Schrödinger wave equation*,

$$i\partial_z\psi = -\beta_s\nabla^2\psi + \gamma\mathcal{N}U(\boldsymbol{x})\psi. \quad (15)$$

This equation describes the evolution of an optical beam trapped in an effective waveguide structure whose profile is given by the nonlocal response function $U(\boldsymbol{x})$. Because this equation is linear, it does not describe modulational instability, or the generation of new frequency components. It is in this highly nonlocal limit that the ISs reported in [33,34] were studied. These ISs may thus be viewed as a random superposition of the linear eigenmodes of the potential $U(\boldsymbol{x})$, which are preserved during the linear propagation of the incoherent beam. We finally note that the Vlasov equation with the effective potential (14) can also be readily derived from the linear Schrödinger equation (15).

3. NONINSTANTANEOUS RESPONSE

In this section, we study the longitudinal temporal evolution of a partially coherent wave that propagates in a nonlinear medium characterized by a noninstantaneous response. We note that a noninstantaneous response of the material arises in almost any radiation–matter interaction. A typical example in one-dimensional systems is provided by the Raman effect in optical fibers, which finds its origin in the delayed molecular response of the material [2]. We consider the standard one-dimensional NLS equation accounting for a noninstantaneous nonlinear response function

$$i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi\int_{-\infty}^{+\infty} R(t-t')|\psi|^2(z,t')\mathrm{d}t', \quad (16)$$

where the response function $R(t)$ is constrained by the causality condition, $R(t) = 0$ for $t < 0$. Because of this property, the real and imaginary parts of the Fourier transform of the response function,

$$\tilde{R}(\omega) = \tilde{U}(\omega) + ig(\omega), \quad (17)$$

are related by the Kramers–Krönig relations, $\tilde{U}(\omega) = -\frac{1}{\pi}\mathcal{P}\int\frac{g(\omega')}{\omega'-\omega}\mathrm{d}\omega'$, and $g(\omega) = \frac{1}{\pi}\mathcal{P}\int\frac{\tilde{U}(\omega')}{\omega'-\omega}\mathrm{d}\omega'$, where $\mathcal{P}$ denotes the principal Cauchy value. We recall in particular that the real part $\tilde{U}(\omega)$ is an even function, which will be shown to lead to a conservative dynamics, in a way similar to the nonlocal potential $U(\boldsymbol{x})$ in the spatial domain. On the other hand, the imaginary part $g(\omega)$ is an odd function, which is known to play the role of a gain spectrum and will be shown to lead to a spectral red-shift, a well-known feature in the example of the Raman effect. The causality condition breaks the Hamiltonian structure of the NLS equation, so that Eq. (16) only conserves the total power ("number of particles") of the wave, $\mathcal{N} = \int|\psi|^2\mathrm{d}t$. To avoid cumbersome notations in Eq. (16), we denote by β_t the (second-order) dispersion coefficient of the material and by σ_t the spatial extension of the response function $R(t)$, i.e., the response time. We recall that $\beta_t > 0$ ($\beta_t < 0$) denotes the regime of normal (anomalous) dispersion. The dynamics is ruled by the comparison of the response time and the "healing time," $\Lambda_t = \sqrt{\beta_t/(\gamma n)}$. As in the spatial case, the weakly nonlinear regime of interaction refers to the regime in which linear dispersive effects dominate nonlinear effects, i.e., $\rho = L_d/L_{\mathrm{nl}} \ll 1$ where $L_d = t_c^2/\beta_t$ and $L_{\mathrm{nl}} = 1/(\gamma n)$ refer to the dispersive and nonlinear characteristic lengths, respectively. Here t_c is the correlation time of the field and n is the typical density of power (particles) $|\psi|^2$.

A kinetic equation can be derived by following a procedure similar to that exposed in the spatial case in Section 2. One looks for a kinetic equation describing the evolution of the local spectrum of the wave, defined by

$$n_\omega(t,z) = \int_{-\infty}^{+\infty} B(t,\tau,z)\exp(-i\omega\tau)\mathrm{d}\tau, \quad (18)$$

where the autocorrelation function reads $B(t,\tau,z) = \langle\psi(t+\tau/2)\psi^*(t-\tau/2)\rangle$. We refer the reader to [32] for details regarding the derivation of a kinetic equation in the temporal domain. There is, however, an important difference with respect to this previous work. In [32], the response function was assumed to exhibit both instantaneous and noninstantaneous contributions. More precisely, it was implicitly assumed that the *noninstantaneous contribution is perturbative with respect to the instantaneous contribution*, which led to the derivation of a generalized Vlasov–Langmuir kinetic description of the system. Conversely, we consider here a nonlinear medium characterized by a purely noninstantaneous response function, which thus constitutes a nonperturbative contribution. As we will see below, this leads to a substantial change in the kinetic description of the system.

A. Short-Range Response Time: Weak Langmuir Turbulence Equation

We consider here the case of a noninstantaneous nonlinearity characterized by a short-range response time, i.e., the regime where the response time is of the same order as the "healing time," $\sigma_t \sim \Lambda_t$. We proceed as in the spatial case discussed above in 2.A and perform a multiscale expansion with

$$B(t,\tau,z) = B^{(0)}(\varepsilon t,\tau,\varepsilon z) + O(\varepsilon),$$

where $\varepsilon = t_c/\Delta_t$ is the ratio of the time correlation and the characteristic time of nonstationary fluctuations. However, contrarily to the spatial case, the response function is constrained by the causality condition in the temporal domain, an important property that completely changes the picture. It turns out that the relevant kinetic equation describing the evolution of the averaged spectrum of the wave is the WT Langmuir equation (see Appendix A.3)

$$\partial_z n_\omega(t,z) = \frac{\gamma}{\pi}n_\omega(t,z)\int_{-\infty}^{+\infty} g(\omega-\omega')n_{\omega'}(t,z)\mathrm{d}\omega', \quad (19)$$

where we recall that $g(\omega) = \Im[\tilde{R}(\omega)]$ is an odd function that refers to the imaginary part of the Fourier transform of the response function. In the spatial case, this function vanishes simply because the response function $U(\boldsymbol{x})$ is a real and even function, and thus $\tilde{U}(\boldsymbol{k})$ is real and even too. Accordingly, the derivation of the Vlasov kinetic Eqs. (7)–(9) in the spatial case requires a first-order perturbation expansion in ε, whereas the WT Langmuir Eq. (19) is obtained at the zeroth order (see Appendix A.3 for details). Then the important point to underline is that the existence of the WT Langmuir equation originates in the causality property of the noninstantaneous response function $R(t)$.

The fact that the WT Langmuir Eq. (19) is relevant for an incoherent wave whose fluctuations are statistically stationary in time has been pointed out recently in the context of optics [61–63], as well as in previous works in the context of Langmuir turbulence and stimulated Compton scattering in plasmas [64–68]. One would have expected that the description of a nonstationary statistics would naturally involve a Vlasov-like kinetic equation, as discussed above in the spatial case. However, it turns out that the WT Langmuir Eq. (19) is also relevant for the description of a statistically nonstationary random wave. In this respect, an initial condition, $n_\omega(t, z = 0)$, localized in both the spectral and temporal domains, will exhibit a nontrivial deformation in the plane (ω, t)—the temporal regions characterized by a high spectral amplitude will exhibit a fast spectral shift as compared to the regions with a lower spectral amplitude. This aspect was studied qualitatively in the optical experiment reported in [91]. However, to our knowledge, the WT Langmuir equation (19) has not been the subject of a detailed study for statistically nonstationary random waves.

We briefly summarize here the essential properties of the kinetic Eq. (19) in the limit of a stationary statistics; i.e., the local spectrum does not depend on the time t. We first note that the WT Langmuir equation does not account for dispersion effects [Eq. (19) does not depend on β_t], although the role of dispersion in its derivation is essential in order to verify the criterion of weakly nonlinear interaction, $\rho \ll 1$. The fact that the dynamics ruled by the WT Langmuir equation does not depend on the sign of the dispersion coefficient has been verified by direct numerical simulations of the NLS Eq. (16) in [62]. In this previous work, a quantitative agreement between the simulations of the stochastic NLS Eq. (16) and the WT Langmuit Eq. (19) has been obtained, *without using any adjustable parameter*. The kinetic Eq. (19) conserves the power of the field $N = (2\pi)^{-1} \int n_\omega(z) d\omega$. Moreover, as discussed above for the Vlasov equation, the WT Langmuir Eq. (19) is a formally reversible equation [it is invariant under the transformation $(z, \omega) \to (-z, -\omega)$], a feature that is consistent with the fact that it also conserves the nonequilibrium entropy, $S = \int \ln[n_\omega(z)] d\omega$.

The WT Langmuir equation also admits solitary wave solutions [64–68]. This may be anticipated by remarking that, as a result of the convolution product in Eq. (19), the odd spectral gain curve $g(\omega)$ amplifies the low-frequency components of the wave at the expense of the high-frequency components, thus leading to a global red-shift of the spectrum. In the general case where the spectral bandwidth of the wave, $\Delta\omega$, is of the same order as the bandwidth of the gain spectrum, $\Delta\omega_g \sim \Delta\omega$, an analytical soliton solution was derived in the particular case where $g(\omega)$ is the derivative of a Gaussian [68]. This solution has been subsequently generalized for a generic gain spectrum in [32]. Note that the WT Langmuir equation also admits discrete soliton solutions, a feature that was originally studied in the framework of a completely integrable discrete δ-peak model [64]. These discrete spectral ISs have been recently studied in the context of stimulated Raman scattering in optics [62] and have been observed experimentally through the process of supercontinuum generation in photonic crystal fibers [63].

B. Short-Range Response Time: Korteweg–de Vries Limit

Let us consider the case of a highly noninstantaneous nonlinear response of the material. This case is interesting because, as the response time σ_t increases, the typical bandwidth of the gain spectrum $g(\omega)$ decreases, and can thus become much smaller than the spectral bandwidth of the incoherent wave. Assuming furthermore that the wave spectrum evolves in the presence of a strong noise background, it can be shown that the WT Langmuir equation reduces to the Korteweg–de Vries equation. This aspect was already remarked in [67]. Here we provide a rigorous derivation of the Korteweg–de Vries equation from the WT Langmuir equation.

We start from the WT Langmuir equation governing the evolution of a statistically stationary incoherent wave

$$\partial_z n_\omega(z) = \frac{\gamma}{\pi} n_\omega(z) \int_{-\infty}^{+\infty} g(\omega - \omega') n_{\omega'}(z) d\omega'. \quad (20)$$

We introduce the small parameter $\varepsilon = \Delta\omega_g / \Delta\omega \ll 1$, where we recall that $\Delta\omega_g$ is the bandwidth of the spectral gain curve $g(\omega)$ and $\Delta\omega$ is the bandwidth of the incoherent wave. The incoherent wave is assumed to evolve in the presence of a high level of constant spectral noise background of amplitude n_0. In this way, the gain curve can be written $g(\omega) = g^{(0)}(\frac{\omega}{\varepsilon})$ and we look for the spectrum in the form

$$n_\omega(z) = n_0 + \tilde{n}_\omega(z),$$

with $\tilde{n}_\omega(z) = \varepsilon^2 \tilde{n}_\omega^{(0)}(\varepsilon^2 z) + O(\varepsilon^4)$. In these conditions, a multiscale expansion shows that $\tilde{n}_\omega$ satisfies the Korteweg–de Vries equation (see Appendix A.4)

$$\partial_z \tilde{n}_\omega(z) - \frac{\gamma n_0 g_1}{\pi} \partial_\omega \tilde{n}_\omega(z) = \frac{\gamma g_1}{\pi} \tilde{n}_\omega(z) \partial_\omega \tilde{n}_\omega(z) + \frac{\gamma n_0 g_3}{6\pi} \partial_\omega^3 \tilde{n}_\omega(z), \quad (21)$$

where

$$g_1 = -\int_{-\infty}^{+\infty} \omega g(\omega) d\omega, \qquad g_3 = -\int_{-\infty}^{+\infty} \omega^3 g(\omega) d\omega.$$

The soliton solutions of this integrable equation have been used to interpret the formation of jets in the frequency space in the study of weak Langmuir turbulence [67]. Note, however, that the Korteweg–de Vries equation considered in [67] differs substantially from Eq. (21). To our knowledge, this Korteweg–de Vries equation governing the evolution of the averaged spectrum of the incoherent wave has not yet been exploited in a context different from plasma physics.

C. Long-Range Response Time: Temporal Long-Range Vlasov Equation

1. Long-Range Noninstantaneous Response

In analogy with the study of long-range response functions in the spatial domain, let us now consider the case of long-range noninstantaneous response function $R(t)$, i.e., the regime $\sigma_t / \Lambda_t \gg 1$. We assume that the response function has the form $R(t) = \varepsilon R^{(0)}(\varepsilon t)$ and we look for the autocorrelation function in the multiscale form $B(t, \tau, z) = B^{(0)}(\varepsilon t, \tau, \varepsilon z) + O(\varepsilon)$, with $B^{(0)}(\varepsilon t, \tau, \varepsilon z) = (2\pi)^{-1} \int n_\omega^{(0)}(\varepsilon t, \varepsilon z) \exp(i\omega\tau) d\omega$. In these conditions, we derive in Appendix A.5 the temporal version of the long-range Vlasov equation

$$\partial_z n_\omega(t,z) + \partial_\omega \tilde{k}_\omega(t,z) \partial_t n_\omega(t,z) - \partial_t \tilde{k}_\omega(t,z) \partial_\omega n_\omega(t,z) = 0, \quad (22)$$

where the generalized dispersion relation reads

$$\tilde{k}_\omega(t,z) = k(\omega) + V(t,z), \quad (23)$$

with $k(\omega) = \beta_t\omega^2$ and the effective potential

$$V(t,z) = \gamma \int R(t-t')N(t',z)\mathrm{d}t'. \quad (24)$$

The intensity profile of the incoherent wave is $N(t,z) = B(t,\tau=0,z) = (2\pi)^{-1}\int n_\omega(t,z)\mathrm{d}\omega$. Equation (22) conserves $\mathcal{N} = (2\pi)^{-1}\iint n_\omega(t,z)\mathrm{d}\omega\mathrm{d}t$, and more generally $\mathcal{M} = \iint f[n]\mathrm{d}\omega\mathrm{d}t$, where $f[n]$ is an arbitrary functional of n. However, because of the causality property of $R(t)$, Eq. (22) is no longer Hamiltonian.

Actually, one can decompose the Vlasov Eqs. (22)–(24) into a Hamiltonian contribution and a nonconservative contribution. Indeed, according to the decomposition given in Eq. (17), $\tilde{R}(\omega) = \tilde{U}(\omega) + ig(\omega)$, the response function can be split into the sum of an even contribution, $U(t)$, and an odd contribution, $G(t)$; i.e., $R(t) = U(t) + G(t)$, where

$$U(t) = \frac{1}{2\pi}\int \tilde{U}(\omega)\exp(i\omega t)\mathrm{d}\omega, \quad (25)$$

$$G(t) = \frac{i}{2\pi}\int g(\omega)\exp(i\omega t)\mathrm{d}\omega. \quad (26)$$

Accordingly, the Vlasov Eq. (22) can be written in the following form:

$$\partial_z n_\omega(t,z) + \partial_\omega k_\omega(t,z)\partial_t n_\omega(t,z) - \partial_t V_U(t,z)\partial_\omega n_\omega(t,z) = \partial_t V_G(t,z)\partial_\omega n_\omega(t,z), \quad (27)$$

where $V_U(t,z) = \gamma\int U(t-t')N(t',z)\mathrm{d}t'$ and $V_G(t,z) = \gamma\int G(t-t')N(t',z)\mathrm{d}t'$. The lhs of Eq. (27) thus refers to a Hamiltonian Vlasov equation, which conserves $\mathcal{H}_{\mathrm{Vl}} = \iint k(\omega)n_\omega(t,z)\mathrm{d}t\mathrm{d}\omega + \frac{1}{2}\int V_U(t,z)N(t,z)\mathrm{d}t$, while the rhs of Eq. (27) may be viewed as a "collision" term, which may eventually lead to a spectral red-shift of the wave. We remark that, in spite of such expected red-shift, the Vlasov Eqs. (22)–(24) predict the existence of a genuine incoherent MI in the temporal domain, which has been found in quantitative agreement with the numerical simulations of the corresponding NLS equation, without using adjustable parameters [109]. This confirms that the long-range Vlasov, either in its spatial or temporal version, provides provide an "exact" statistical description of the random nonlinear wave.

2. Instantaneous Limit

For the sake of completeness, we briefly comment here on the limit of an instantaneous response function. Making use of the assumptions that the incoherent wave exhibits a quasi-stationary statistics and that it evolves into the weakly nonlinear regime, one obtains the traditional form of the Vlasov Eq. (22) with the self-consistent potential

$$V(t,z) = 2\gamma N(t,z). \quad (28)$$

This self-consistent potential is nothing but the temporal counterpart of the spatial potential discussed above in 2.A.2 in the limit of a purely local nonlinear response. In particular, the factor 2 in the effective potential (28) has the same origin as in Eq. (11).

3. Highly Noninstantaneous Response: Linear Limit

The limit of a highly noninstantaneous response function corresponds to the temporal counterpart of the highly nonlocal limit discussed above in the spatial case. Indeed, in the limit $\sigma_t \gg \Delta_t$, the response function can be extracted from the convolution integral in the effective potential (24), which thus leads to the temporal Vlasov Eq. (22) with the potential

$$V(t) = \gamma\mathcal{N}U(t). \quad (29)$$

As for the highly nonlocal limit, the temporal response function $U(t)$ plays the role of the effective potential, so that the Vlasov equation recovers a linear kinetic equation. We remark that the dynamics of *coherent* optical waves in a highly noninstantaneous response nonlinear medium has been recently explored theoretically in [95], in the framework of gas- or liquid-filled photonic crystal fibers. By deriving a linear Schrödinger wave equation analogous to that discussed above [Eq. (15)] in the spatial domain, the authors of [95] predicted that a highly noninstantaneous responding medium can support the existence of coherent soliton solutions.

4. DISCUSSION AND CONCLUSION

We have provided a generalized kinetic formulation of random nonlinear waves governed by the NLS equation in the presence of a nonlocal or a noninstantaneous response of the nonlinear medium. In the spatial domain, the general picture is that a statistically homogeneous incoherent wave is governed by the WT (Hasselmann) kinetic equation, while it is governed by different kinds of Vlasov equations when it exhibits an inhomogeneous statistics. Besides the traditional forms of the Vlasov equation, we have derived the short-range and the long-range Vlasov equations, the latter providing an "exact" statistical description of the random wave. In the temporal domain, the causality property inherent to the noninstantaneous response function changes the physical picture. Whenever the noninstantaneous response of the material cannot be neglected, the WT Langmuir equation turns out to be the relevant kinetic equation, for both a stationary and a nonstationary statistics of the incoherent wave. When the incoherent wave exhibits a stationary statistics in the presence of a highly noninstantaneous nonlinear response of the material, the weak Langmuir turbulence equation reduces to the Korteweg–de Vries equation. Conversely, when the wave exhibits a nonstationary statistics still in the presence of a highly noninstantaneous response, we have derived a long-range Vlasov kinetic equation in the temporal domain. Its self-consistent potential depends on the response function of the material and is thus constrained by the causality condition, a key feature that is expected to describe new phenomenologies, which will be the subject of future investigations.

Although we considered here the case of the $\chi^{(3)}$ nonlinearity in the framework of generalized NLS equations, we underline that the WT kinetic approach can also be applied to the $\chi^{(2)}$ nonlinearity in the framework of the three-wave interaction (TWI). In particular, in the presence of a homogeneous (stationary) statistics and local (instantaneous) nonlinear response, one obtains three coupled Hasselmann-like kinetic equations governing the coupled evolutions of the spectra of the waves [92]. Work is in progress in order to extend the Vlasov and WT Langmuir equations to wave propagation in a quadratic nonlinear material. Also, the WT approach discussed here in either the spatial or temporal domains may be extended to describe the *spatio-temporal* coherence properties of the incoherent wave within a unified theoretical framework (see, e.g., [92,110]). We remark that the kinetic approach can also be used to treat problems in which phase correlations spontaneously emerge in a system of incoherent optical waves [36,77,111,112]. The spontaneous emergence of a mutual coherence between incoherent waves is the essential mechanism underlying the existence of ISs in instantaneous response nonlinear media [36,37]. Another interesting problem concerns the study of incoherent nonlinear wave interactions in the counterpropagating configuration (see, e.g., [113]), in which the physical significance of the irreversible process of thermalization needs to be analyzed with special care.

A. Inertial Nonlinearity

It is important to comment on the limit of the so-called "inertial nonlinearity" that has been widely explored in the framework of optical experiments realized in slowly responding nonlinear media, such as photorefractive crystals or liquid crystals [6–10]. These experiments were aimed at studying the *transverse spatial dynamics* of a speckle beam, whose nonlinear evolution is averaged out by the *highly noninstantaneous response time* of the nonlinear material—the time correlation of the incoherent wave is assumed much smaller than the response time of the nonlinearity. As a result, the spatial dynamics of the incoherent wave is governed by an *averaged NLS equation*, $i\partial_z\psi = -\beta_s\nabla^2\psi + \gamma\langle|\psi|^2\rangle\psi$, where the brackets $\langle\cdot\rangle$ denote a temporal averaging over a time of the same order as the inertial response time of the material. Because of the underlying averaged nonlinearity, this NLS equation does not lead to an infinite hierarchy of moment equations; i.e., the derivation of a closed equation for the second-order moment of the wave does not require any additional assumption on the nature of the statistics of the field. In this way, the Wigner–Moyal equation governing the evolution of the autocorrelation function of the incoherent wave, $B(\boldsymbol{x},\boldsymbol{\xi},z)$, can be derived without any approximations [21]. In the limit of a quasi-homogeneous statistics, the Wigner–Moyal equation reduces to the Vlasov kinetic Eq. (7), with the effective potential $V(\boldsymbol{x},z) = \gamma N(\boldsymbol{x},z)$.

B. Hasselmann Equation

We recall that the Vlasov and WT Langmuir equations are *quadratic nonlinear equations*, whose derivations refer to a first-order closure of the hierarchy of moments equations. These kinetic equations are formally reversible and describe, in particular, the spontaneous formation of IS structures. Let us now consider the following two limits. (i) In the spatial domain, the limit of homogeneous statistics of the incoherent wave: in this case the Vlasov equation becomes irrelevant, i.e., $\partial_z n_{\boldsymbol{k}}(\boldsymbol{x},z) = 0$ (see Fig. 1). (ii) In the temporal domain, the limit of stationary statistics and instantaneous response of the nonlinearity: in this case the WT Langmuir equation becomes irrelevant, i.e., $\partial_z n_\omega(z) = 0$ (see Fig. 2). In both limits, we thus need to close the hierarchy of the moments equations to the second-order, and one obtains the Hasselmann equation, which is thus a *cubic nonlinear equation*.

In the temporal domain, the Hasselmann equation proved efficient in describing certain properties of the process of supercontinuum generation in photonic crystal fibers [89–91]. It has been shown that, under certain conditions, the dramatic spectral broadening inherent to supercontinuum generation can be described as a consequence of the natural thermalization of the optical wave toward the Rayleigh–Jeans equilibrium distribution. We discuss here the spatial Hasselmann equation and in particular the role of a nonlocal interaction in the process of thermalization.

Starting from the nonlocal NLS Eq. (1) and following the procedure outlined in [69,70], one obtains the following Hasselmann equation governing the evolution of the averaged spectrum of the wave, $\langle\tilde{\psi}(\boldsymbol{k}_1,z)\tilde{\psi}^*(\boldsymbol{k}_2,z)\rangle = n_{\boldsymbol{k}_1}(z)\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)$ [with $\tilde{\psi}(\boldsymbol{k},z) = (2\pi)^{-d/2}\int\psi(\boldsymbol{x},z)\exp(-i\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}$]:

$$\partial_z n_{\boldsymbol{k}} = \frac{4\pi\gamma^2}{(2\pi)^{2d}}\iiint \mathcal{Q}(n_{\boldsymbol{k}},n_{\boldsymbol{k}_1},n_{\boldsymbol{k}_2},n_{\boldsymbol{k}_3})T^2_{k123}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}) \times\delta(\beta_s(k_1^2+k_2^2-k_3^2-k^2))\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2\mathrm{d}\boldsymbol{k}_3, \tag{30}$$

where $\mathcal{Q}(n_{\boldsymbol{k}},n_{\boldsymbol{k}_1},n_{\boldsymbol{k}_2},n_{\boldsymbol{k}_3}) = n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}n_{\boldsymbol{k}}(n_{\boldsymbol{k}}^{-1}+n_{\boldsymbol{k}_3}^{-1}-n_{\boldsymbol{k}_2}^{-1}-n_{\boldsymbol{k}_1}^{-1})$, and the tensor may be written in its symmetric form, $T_{k123} = \frac{1}{4}(\tilde{U}_{12}+\tilde{U}_{13}+\tilde{U}_{k3}+\tilde{U}_{k2})$, with $\tilde{U}_{ij} = \tilde{U}(\boldsymbol{k}_i-\boldsymbol{k}_j)$. Note that in the limit of a local interaction $[U(\boldsymbol{x})\to\delta(\boldsymbol{x})]$, we have $T_{k123} = 1$ and Eq. (30) recovers the standard local Hasselmann equation. Equation (30) conserves the power ("number of quasi-particles") of the wave $\mathcal{N} = \int n_{\boldsymbol{k}}\mathrm{d}\boldsymbol{k}$ and the kinetic energy $\mathcal{E} = \int\beta_s k^2 n_{\boldsymbol{k}}\mathrm{d}\boldsymbol{k}$. Contrary to the Vlasov and WT Langmuir equations, the Hasselmann Eq. (30) is formally irreversible, a feature expressed by an H-theorem of entropy growth, $\partial_z\mathcal{S}(z)\geq 0$, where the nonequilibrium entropy reads $\mathcal{S}(z) = \int\ln[n_{\boldsymbol{k}}(z)]\mathrm{d}\boldsymbol{k}$. Accordingly, Eq. (30) describes an irreversible evolution of the wave spectrum toward the Rayleigh–Jeans thermodynamic equilibrium distribution,

$$n_k^{\mathrm{RJ}} = \frac{T}{\beta_s k^2-\mu}. \tag{31}$$

Note that the divergence of this equilibrium distribution as $-\mu\to 0$ is responsible for a phenomenon of condensation of nonlinear waves, which has a purely classical physical origin [85–88]. This effect is characterized by an irreversible evolution of the wave toward an equilibrium state characterized by a plane-wave ("condensate") that remains immersed in a sea of thermalized small-scale fluctuations ("uncondensed particles"). The Hasselmann equation and the process of wave condensation have recently been extended by considering the propagation of the optical wave in a waveguide potential [88].

Recent numerical simulations reveal that the process of thermalization of a nonlocal system slows down in a significant way as the nonlocal response length σ_s increases. This numerical observation can be interpreted through a qualitative analysis of the kinetic Eq. (30). Indeed, the functions $\tilde{U}(\boldsymbol{k})$, i.e., the tensor T_{k123}, get all the more narrower as the nonlocal range of the response function increases, which thus quenches the efficiency of the four-wave resonances involved in the collision term of Eq. (30). More precisely, in the highly nonlocal limit, we use the same scaling for the nonlocal response as that used to derive the long-range Vlasov equation [see Eq. (12), $U(\boldsymbol{x}) = \varepsilon U^{(0)}(\varepsilon\boldsymbol{x})$]. The Fourier transform of the response function thus reads $\tilde{U}(\boldsymbol{k}) = \tilde{U}^{(0)}(\boldsymbol{k}/\varepsilon)$. Using the change of variables $\boldsymbol{k}_j = \boldsymbol{k} + \varepsilon\boldsymbol{\kappa}_j$ $(j = 1, 2, 3)$, we find after integration in $\boldsymbol{\kappa}_3$ that Eq. (30) is equivalent to

$$\partial_z n_{\boldsymbol{k}} = \frac{4\pi\gamma^2\varepsilon^{2d-2}}{(2\pi)^{2d}} \iint \mathcal{Q}(n_{\boldsymbol{k}}, n_{\boldsymbol{k}+\varepsilon\boldsymbol{\kappa}_1}, n_{\boldsymbol{k}+\varepsilon\boldsymbol{\kappa}_2}, n_{\boldsymbol{k}+\varepsilon\boldsymbol{\kappa}_1+\varepsilon\boldsymbol{\kappa}_2}) \times [\tilde{U}^{(0)}(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) + \tilde{U}^{(0)}(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2) + 2\tilde{U}^{(0)}(\boldsymbol{\kappa}_2)] \times \delta(2\beta_s \boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2)\mathrm{d}\boldsymbol{\kappa}_1 \mathrm{d}\boldsymbol{\kappa}_2.$$

To interpret this expression, let us define the characteristic length of thermalization of the mode k, say λ_k, as $\partial_z n_k / n_k \sim 1/\lambda_k$. Since the width of the function $\tilde{U}^{(0)}(\boldsymbol{\kappa})$ is of order 1, we see in this expression that a highly nonlocal interaction slows down the thermalization process by a factor of order ε^{2d-2},

$$\lambda_k^{\mathrm{nonloc}} \sim \lambda_k^{\mathrm{loc}} (\sigma_s/\Lambda_s)^{2d-2}. \tag{32}$$

We remark that these arguments have no physical meaning in one spatial dimension, because the collision term of the Hasselmann equation vanishes identically for $d = 1$; i.e., Eq. (30) is only relevant for $d > 1$. The slowing down of the thermalization process due to a highly nonlocal response is an important phenomenon that will be the subject of future investigations, in relation with the "exact" statistical description of the random nonlinear wave provided by the long-range Vlasov-like kinetic equation.

APPENDIX A

1. Derivation of the Short-Range Spatial Vlasov Equation

We use the multiscale expansion, $B(\boldsymbol{x}, \boldsymbol{\xi}, z) = B^{(0)}(\varepsilon\boldsymbol{x}, \boldsymbol{\xi}, \varepsilon z)$, where $n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z) = \int B^{(0)}(\boldsymbol{X}, \boldsymbol{\xi}, Z) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$, with $\boldsymbol{X} = \varepsilon\boldsymbol{x}$ and $Z = \varepsilon z$. In this way, the second term in the equation for the autocorrelation function (2), $\int P(\boldsymbol{x}, \boldsymbol{\xi}, z) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \int P(\boldsymbol{X}/\varepsilon, \boldsymbol{\xi}, Z/\varepsilon) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$, can be calculated following the method described in [32]. One obtains

$$\int P(\boldsymbol{x}, \boldsymbol{\xi}, z) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \frac{1}{(2\pi)^{2d}} \times \int U(\boldsymbol{y})[n_{\boldsymbol{k}_2}^{(0)}(\boldsymbol{X} - \varepsilon\boldsymbol{y} + \varepsilon\boldsymbol{\xi}/2, Z) - n_{\boldsymbol{k}_2}^{(0)}(\boldsymbol{X} - \varepsilon\boldsymbol{y} - \varepsilon\boldsymbol{\xi}/2, Z)] \times n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z) \exp[i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{\xi}]\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{y}. \tag{A1}$$

Expanding the integrand to first-order in ε and integrating by parts with respect to $\boldsymbol{k}_1$ gives

$$\int P(\boldsymbol{x}, \boldsymbol{\xi}, z) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = i\varepsilon\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z) \cdot \partial_{\boldsymbol{X}}\left(\frac{1}{(2\pi)^d}\int n_{\boldsymbol{k}'}^{(0)}(\boldsymbol{X}, Z)\mathrm{d}\boldsymbol{k}'\right). \tag{A2}$$

The third term in the equation for the autocorrelation function (2) can be calculated in a similar way. Expanding in powers of ε, one obtains $\int Q(\boldsymbol{x}, \boldsymbol{\xi}, z) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \tilde{Q}_0 + \varepsilon\tilde{Q}_1$, where

$$\tilde{Q}_0 = \frac{1}{(2\pi)^d}\int U(\boldsymbol{y}) n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z) n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z)[\exp[i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{y}] - \exp[-i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{y}]]\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{y}. \tag{A3}$$

Defining the Fourier transform of the response function, $\tilde{U}_{\boldsymbol{k}} = \int U(\boldsymbol{x}) \exp(-i\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, one readily obtains

$$\tilde{Q}_0 = \frac{2i}{(2\pi)^d} n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z) \int \mathfrak{I}(\tilde{U}_{\boldsymbol{k}-\boldsymbol{k}'}) n_{\boldsymbol{k}'}^{(0)}(\boldsymbol{X}, Z)\mathrm{d}\boldsymbol{k}', \tag{A4}$$

where $\mathfrak{I}(\tilde{U}_{\boldsymbol{k}})$ denotes the imaginary part of $\tilde{U}_{\boldsymbol{k}}$. The contribution $\tilde{Q}_0$ vanishes because the response function $U(\boldsymbol{x})$ is real-valued and even, which thus leads to $\mathfrak{I}(\tilde{U}_{\boldsymbol{k}}) = 0$. Note that this will not be the case in the temporal domain, because of the causality condition of the response function, as discussed here below. The contribution of order ε can be written

$$\tilde{Q}_1 = \frac{1}{(2\pi)^{2d}}\int U(\boldsymbol{y})[(u_+ + v)\exp[-i(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{y}] + (u_- - v)\exp[i(\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{y}]]\exp[-i(\boldsymbol{k} - \boldsymbol{k}_2)\cdot\boldsymbol{\xi}]\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2 \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\xi}, \tag{A5}$$

where $u_\pm = \frac{1}{2} n_{\boldsymbol{k}_2}^{(0)}(\boldsymbol{X}, Z)(\boldsymbol{\xi} \pm \boldsymbol{y})\cdot\partial_{\boldsymbol{X}} n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z)$, $v = \frac{1}{2} n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z)\boldsymbol{y} \cdot\partial_{\boldsymbol{X}} n_{\boldsymbol{k}_2}^{(0)}(\boldsymbol{X}, Z)$. By means of some algebraic manipulations, this expression reads

$$\tilde{Q}_1 = -\frac{i}{2(2\pi)^d}\partial_{\boldsymbol{X}}\cdot\int U(\boldsymbol{y})\partial_{\boldsymbol{k}_1}(n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z))n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z) \times [\exp[i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{y}] + \exp[-i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{y}]]\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{y} + \frac{i}{2(2\pi)^d}\partial_{\boldsymbol{k}}\cdot\int U(\boldsymbol{y}) n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z)\partial_{\boldsymbol{X}}(n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z)) \times [\exp[i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{y}] + \exp[-i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{y}]]\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{y} \tag{A6}$$

and can finally be written in the following compact form:

$$\tilde{Q}_1 = \frac{i}{(2\pi)^d}\partial_{\boldsymbol{X}}\left(\int \tilde{U}_{\boldsymbol{k}-\boldsymbol{k}_1} n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z)\mathrm{d}\boldsymbol{k}_1\right)\cdot\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z) - \frac{i}{(2\pi)^d}\partial_{\boldsymbol{k}}\left(\int \tilde{U}_{\boldsymbol{k}-\boldsymbol{k}_1} n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z)\mathrm{d}\boldsymbol{k}_1\right)\cdot\partial_{\boldsymbol{X}} n_{\boldsymbol{k}}^{(0)}(\boldsymbol{X}, Z). \tag{A7}$$

Then collecting all terms in Eqs. (A7) and (A2), and coming back to the original variables, $z = Z/\varepsilon$ and $\boldsymbol{x} = \boldsymbol{X}/\varepsilon$, one obtains the short-range Vlasov-like kinetic Eq. (7), with the effective potential given in Eq. (9).

2. Derivation of the Long-Range Spatial Vlasov Equation

We proceed as in Appendix A.1, but we use the following scaling for the highly nonlocal potential $U(\boldsymbol{x}) = \varepsilon U^{(0)}(\varepsilon\boldsymbol{x})$. We thus have

$$\int P(\boldsymbol{x}, \boldsymbol{\xi}, z) \exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \frac{1}{(2\pi)^d} \times \int U^{(0)}(\boldsymbol{Y})[n_{\boldsymbol{k}_2}^{(0)}(\boldsymbol{X} - \boldsymbol{Y} + \varepsilon\boldsymbol{\xi}/2, Z) - n_{\boldsymbol{k}_2}^{(0)}(\boldsymbol{X} - \boldsymbol{Y} - \varepsilon\boldsymbol{\xi}/2, Z)]n_{\boldsymbol{k}_1}^{(0)}(\boldsymbol{X}, Z) \times \exp[i(\boldsymbol{k}_1 - \boldsymbol{k})\cdot\boldsymbol{\xi}]\mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2 \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{Y}. \tag{A8}$$

Expanding the difference in the brackets to the first order in ε, one obtains

$$\int P(\boldsymbol{x},\boldsymbol{\xi},z)\exp(-i\boldsymbol{k}\cdot\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = i\varepsilon\partial_k n_k^{(0)}(\boldsymbol{X},Z) \cdot \partial_X \int U^{(0)}(\boldsymbol{X}-\boldsymbol{X}')\left(\frac{1}{(2\pi)^d}\int n_{k'}^{(0)}(\boldsymbol{X}',Z)\mathrm{d}\boldsymbol{k}'\right)\mathrm{d}\boldsymbol{X}'. \quad \text{(A9)}$$

The same procedure applied to the term Q in the equation for the autocorrelation function reveals that the expansion in the first order in ε vanishes—the first nonvanishing term is of second order, ε^2. Coming back to the original variables, one thus obtains the Vlasov Eq. (7) with the effective potential (13).

3. Derivation of the Weak Langmuir Turbulence Equation

The method for the derivation of the WT Langmuir equation follows the procedure reported in Appendix A.1 for the derivation of the short-range Vlasov equation in the spatial domain. The difference is that the calculation is carried out in the temporal domain; i.e., the variables are transformed as follows $\boldsymbol{x}\to t$, $\boldsymbol{\xi}\to\tau$, while the nonlocal response function is substituted by the noninstantaneous response, $U(\boldsymbol{x})\to R(t)$. Accordingly, the multiscale expansion reads $B(t,\tau,z) = B^{(0)}(\varepsilon t,\tau,\varepsilon z)+O(\varepsilon)$, where $n_\omega^{(0)}(T,Z)=\int B^{(0)}(T,\tau,Z)\exp(-i\omega\tau)\mathrm{d}\tau$, with $T=\varepsilon t$ and $Z=\varepsilon z$. Expanding the term P in first-order in ε gives

$$\int_{-\infty}^{+\infty} P(t,\tau,z)\exp(-i\omega\tau)\mathrm{d}\tau = i\varepsilon\partial_\omega n_\omega^{(0)}(T,Z) \times\partial_T\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} n_\omega^{(0)}(T,Z)\mathrm{d}\omega'\right). \quad \text{(A10)}$$

The term Q can be expanded in the same way, $\int Q(t,\tau,z)\exp(-i\omega\tau)\mathrm{d}\tau = \tilde{Q}_0+\varepsilon\tilde{Q}_1$, where

$$\tilde{Q}_0 = \frac{i}{\pi}n_\omega^{(0)}(T,Z)\int_{-\infty}^{+\infty}\mathfrak{I}(\tilde{R}_{\omega-\omega'})n_{\omega'}^{(0)}(T,Z)\mathrm{d}\omega', \quad \text{(A11)}$$

where $\mathfrak{I}(\tilde{R}_\omega)$ denotes the imaginary part of the Fourier transform of the response function $R(t)$. Contrarily to the spatial case where this function vanishes, in the temporal domain it refers to the gain spectrum of the nonlinearity, $\mathfrak{I}(\tilde{R}_\omega) = g(\omega)$, as discussed through Eq. (17). It turns out that, thanks to the causality condition, the zeroth order expansion in ε no longer vanishes in Eq. (A11). In this way the first term P in Eq. (A10) of order ε is negligible with respect to the term (A11). Coming back to the original $z=Z/\varepsilon$ and $t=T/\varepsilon$, one obtains the WT Langmuir kinetic Eq. (19).

4. Derivation of the Korteweg–de Vries Equation

By substituting the form of the spectral gain curve, $g(\omega)=g^{(0)}(\frac{\omega}{\varepsilon})$, and of the spectrum, $n_\omega(z)=n_0+\varepsilon^2\tilde{n}_\omega^{(0)}(\varepsilon^2 z)+O(\varepsilon^4)$, into the rhs of Eq. (20), we obtain

$$\begin{aligned}\frac{\gamma}{\pi}n_\omega\int_{-\infty}^{+\infty}g(\omega-\omega')n_{\omega'}\mathrm{d}\omega' &= \frac{\gamma\varepsilon}{\pi}[n_0+\varepsilon^2\tilde{n}_\omega^{(0)}] \times\int_{-\infty}^{+\infty}g^{(0)}(\omega')[n_0+\varepsilon^2\tilde{n}_{\omega-\varepsilon\omega'}^{(0)}]\mathrm{d}\omega' = \frac{\gamma\varepsilon^4}{\pi}[n_0+\varepsilon^2\tilde{n}_\omega^{(0)}] \\ &\quad\times\int_{-\infty}^{+\infty}g^{(0)}(\omega')\left[-\omega'\partial_\omega\tilde{n}_\omega^{(0)}-\omega'^3\frac{\varepsilon^2}{6}\partial_\omega^3\tilde{n}_\omega^{(0)}\right]\mathrm{d}\omega' \\ &= \frac{\gamma\varepsilon^4 n_0 g_1^{(0)}}{\pi}\partial_\omega\tilde{n}_\omega^{(0)}+\frac{\gamma\varepsilon^6 g_1^{(0)}}{\pi}\tilde{n}_\omega^{(0)}\partial_\omega\tilde{n}_\omega^{(0)}+\frac{\gamma\varepsilon^6 n_0 g_3^{(0)}}{6\pi}\partial_\omega^3\tilde{n}_\omega^{(0)},\end{aligned}$$

up to terms of order ε^8, where we have used the fact that $g^{(0)}$ is an odd function and we have defined

$$g_1^{(0)} = -\int_{-\infty}^{+\infty}\omega g^{(0)}(\omega)\mathrm{d}\omega, \qquad g_3^{(0)} = -\int_{-\infty}^{+\infty}\omega^3 g^{(0)}(\omega)\mathrm{d}\omega.$$

Of course the lhs of Eq. (20) reads

$$\partial_z n_\omega(z) = \varepsilon^4\partial_Z\tilde{n}_\omega^\varepsilon(\varepsilon^2 z),$$

with $Z=\varepsilon^2 z$. By dividing Eq. (20) by ε^4 and by collecting the terms of order up to $O(\varepsilon^2)$, we find

$$\partial_Z\tilde{n}_\omega^{(0)}(Z)-\frac{\gamma n_0 g_1^{(0)}}{\pi}\partial_\omega\tilde{n}_\omega^{(0)}(Z) = \frac{\gamma\varepsilon^2 g_1^{(0)}}{\pi}\tilde{n}_\omega^{(0)}(Z) \times\partial_\omega\tilde{n}_\omega^{(0)}(Z)+\frac{\gamma\varepsilon^2 n_0 g_3^{(0)}}{6\pi}\partial_\omega^3\tilde{n}_\omega^{(0)}(Z).$$

By coming back to the original variables, we obtain the Korteweg–de Vries Eq. (21).

5. Derivation of the Long-Range Temporal Vlasov Equation

The derivation of the long-range Vlasov equation in the temporal domain follows the lines of the corresponding derivation in the spatial domain outlined in Appendix A.2. In particular, the scaling for the long-range response function reads $R(t)=\varepsilon R^{(0)}(\varepsilon t)$. For the term P, one thus obtains

$$\int_{-\infty}^{+\infty}P(t,\tau,z)\exp(-i\omega\tau)\mathrm{d}\tau = i\varepsilon\partial_\omega n_\omega^{(0)}(T,Z) \times\partial_T\int_{-\infty}^{+\infty}R^{(0)}(T-T')\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}n_{\omega'}^{(0)}(T',Z)\mathrm{d}\omega'\right)\mathrm{d}T'. \quad \text{(A12)}$$

As in the spatial case, the same procedure applied to the term Q in the equation for the autocorrelation function reveals that the first nonvanishing term is of second order, ε^2. Coming back to the original variables, one thus obtains the Vlasov Eq. (22) with the effective potential (24).

ACKNOWLEDGMENTS

A.P. acknowledges S. Ruffo, T. Dauxois, and A. Campa for discussions on long range systems during the early stages of the work. The authors also thank S. Rica, H. R. Jauslin, G. Millot, C. Michel, and B. Kibler for valuable comments.

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