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THE MEAN ESCAPE TIME FOR A NARROW ESCAPE PROBLEM WITH MULTIPLE SWITCHING GATES*

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Abstract. This article deals with the narrow escape problem when there are two gates which open alternatively in a random way. We set up the problem and carry out a rigorous asymptotic analysis to derive the mean escape time (MET) for a Brownian particle inside a domain to exit the domain through the switching gates. We show that the MET decreases as the switching rate between the gates increases, and we give upper and lower bounds for the decay rate. We then consider the case when there are multiple switching gates and derive the leading-order term of the asymptotic expansion of the MET.

Key words. mean escape time, narrow escape problem, asymptotic expansions

AMS subject classifications. 35J25, 60H30, 60J60

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1. Introduction.

Recently the narrow escape problem has attracted a lot of attention in connection with cellular and molecular biology. The narrow escape problem is the problem of computing the mean first passage time (MFPT) for a Brownian particle that moves in a domain before it exits through a narrow gate on the boundary of the domain. Here the gate is an absorbing spot on an otherwise reflecting boundary, which is why the MFPT coincides with the mean escape time (MET). A main concern for this problem is to derive an asymptotic expansion of the MET when the size of the gate tends to zero. We refer readers to a review paper of Holcman [9] for an overview of the narrow escape problem.

There have been several significant works deriving the leading-order and higher-order terms of the asymptotic expansions of MET in one, two, and three dimensions [3], [6], [10], [11], [12], [14], [17], [18], [19], [22], [20], [21]. These works deal with the narrow escape problem when there are one or several gates and these gates are not fluctuating in time; in other words, the gates are open all the time. However, there are some cases when the reactivities of the gates fluctuate in time, or, in an equivalent way, when the gates are time-independent, but the diffusing particle stochastically switches between several states and can exit in only one state (see [15], [16], and references therein). This is an important problem because a diffusing particle may randomly switch between different states due to chemical interactions or conformational changes, and when the exit through a gate is possible in a specific state only, switching can significantly alter the process [7], [23].

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In this article we study the narrow escape problem when there are several switching gates. First, we deal with the case when there are two gates which open alternatively according to a telegraph process. This is an interesting stochastic system, as the dynamics of the diffusing particle and that of the gates interact in a nontrivial way. We formulate the problem and rigorously derive an asymptotic expansion for the MET in this case. The asymptotic expansion is expressed in terms of the sizes of the gates. The nontrivial interaction manifests itself in that the MET depends on the switching rate of the gates, and one of the major objectives of this article is to investigate how the MET depends on the switching rate. We then consider the problem when there are more than two gates which open alternatively according to Markovian dynamics. We formulate the problem in this case and derive the leading order term of the MET. In our case the switching corresponds to a switching of boundary conditions, while in [4], for instance, the Markov switching corresponds to a switching of the dynamics itself, which has applications in econometric and financial modeling [5]. Here the focus is on the case when the gates are relatively small and on a situation motivated by the biological applications.

The main findings of this article are twofold. We show that the leading-order term of the asymptotic expansion of the MET is twice that of the MET when there are two small gates which are open all the time. It means, in particular, that the switching rate does not affect the leading-order MET (this is the case even if there are multiple switching gates). On the other hand, we show that the next term in the asymptotic expansion of the MET decreases as the switching rate increases. In other words, the more rapid the two gates switch, the faster the particle exits the domain.

This article is organized as follows. In the next section we formulate the escape problem when there are two switching gates. In section 3, we derive the relevant integral equations using the Neumann function. In section 4, we derive the leading-order term and the first-order correction of the asymptotic expansion of the MET when there are two switching gates. Section 5 deals with the case when there are more than two switching gates. This article ends with a brief conclusion.

2. Formulation of the problem. We denote by Ω a smooth, bounded, and simply connected domain in $\mathbb{R}^2$. We consider the reflected Brownian particle $(\boldsymbol{X}_t)_{t\geq 0}$ confined to Ω (in fact, it is not the standard Brownian motion, but the one with the generator Δ instead of $(1/2)\Delta$). The reflected process $(\boldsymbol{X}_t)_{t\geq 0}$ can be rigorously defined as the unique solution to the following equation [13]:

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \sqrt{2}\boldsymbol{B}_t - \int_0^t \boldsymbol{n}(\boldsymbol{X}_s)\, dL_s, \tag{2.1}$$

where $(\boldsymbol{B}_t)_{t\geq 0}$ is a standard two-dimensional Brownian motion, $\boldsymbol{n}$ is the outward normal vector on the boundary $\partial\Omega$, and $(L_t)_{t\geq 0}$ is a continuous nondecreasing process with $L_0 = 0$ which increases only when $\boldsymbol{X}_t$ is on the boundary $\partial\Omega$:

$$L_t = \int_0^t \mathbf{1}_{\boldsymbol{X}_s \in \partial\Omega}\, dL_s.$$

Here we consider the situation that the boundary $\partial\Omega$ is partitioned into three parts, $\partial\Omega_1$, $\partial\Omega_2$, and the complementary $\partial\Omega_c$. We model an escape problem in which there are two gates at the arcs $\partial\Omega_1$ and $\partial\Omega_2$ that open alternatively according to a telegraph process $(N_t)_{t\geq 0}$ with parameter a (a is the rate of switching between the gates). When the particle hits the open gate it escapes rather than being reflected. The telegraph process

$(N_t)_{t\geq 0}$ with parameter $a>0$ is a memoryless continuous-time stochastic process that takes on two distinct values, say, 1 and 2 [8]. Given that $N_0=1$, its random dynamics can be described by

$$N_t = \begin{cases} 1 & \text{if } T_{2j} \leq t < T_{2j+1} \text{ for some } j, \\ 2 & \text{if } T_{2j+1} \leq t < T_{2j+2} \text{ for some } j, \end{cases}$$

where $T_0=0$, $T_j=\sum_{l=1}^{j}\tau_l$, and $(\tau_l)_{l\geq 1}$ is a sequence of independent and identically distributed exponential random variables with parameter $a>0$ ($\mathbb{E}[\tau_l]=a^{-1}$). The dynamics are Markovian and the transition probabilities $p(t,j|s,k)=\mathbb{P}(N_t=j|N_s=k)$ satisfy the master equations:

$$\frac{\partial p(t,1|s,k)}{\partial t} = -ap(t,1|s,k)+ap(t,2|s,k), \quad t\geq s, \quad p(s,1|s,k)=\delta_{1k},$$
$$\frac{\partial p(t,2|s,k)}{\partial t} = ap(t,1|s,k)-ap(t,2|s,k), \quad t\geq s, \quad p(s,2|s,k)=\delta_{2k}.$$

Note that $(N_t)_{t\geq 0}$ is reversible and that its stationary distribution is the uniform distribution over $\{1,2\}$.

The goal is to compute the expectation of the hitting time for the process $(\boldsymbol{X}_t)_{t\geq 0}$ to the time-dependent boundary segment $\partial\Omega_{N_t}$ (i.e., $\partial\Omega_1$ when $N_t=1$ and $\partial\Omega_2$ when $N_t=2$). We denote by $\mathcal{T}$ the stopping time which corresponds to the "escape" of the particle:

$$\mathcal{T}=\mathcal{T}_1\wedge\mathcal{T}_2, \quad \mathcal{T}_j=\inf\{t\geq 0, N_t=j \text{ and } \boldsymbol{X}_t\in\partial\Omega_j\}, \quad j=1,2.$$

The MET is the expectation $\mathbb{E}_{k,\boldsymbol{x}}[\mathcal{T}]$ of the stopping time when the initial states are $N_0=k$ and $\boldsymbol{X}_0=\boldsymbol{x}$ with $k\in\{1,2\}$ and $\boldsymbol{x}\in\Omega$. If N_0 follows the uniform distribution over $\{1,2\}$, then the MET is given by $\mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}]$, where

$$\mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}]=\frac{1}{2}(\mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]+\mathbb{E}_{2,\boldsymbol{x}}[\mathcal{T}]). \tag{2.2}$$

We have the following proposition for the MET.

PROPOSITION 2.1. *For $\boldsymbol{x}\in\Omega$ and $k=1,2$, we have*

$$\mathbb{E}_{k,\boldsymbol{x}}[\mathcal{T}]=u_k(\boldsymbol{x}), \tag{2.3}$$

where the pair of functions $(u_1(\boldsymbol{x}),u_2(\boldsymbol{x}))$ is the solution of the coupled system

$$-\begin{pmatrix}1\\1\end{pmatrix}=a\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}u_1(\boldsymbol{x})\\u_2(\boldsymbol{x})\end{pmatrix}+\Delta\begin{pmatrix}u_1(\boldsymbol{x})\\u_2(\boldsymbol{x})\end{pmatrix} \tag{2.4}$$

with the boundary conditions

$$\partial_\nu u_j|_{\partial\Omega_c}=0, \qquad j=1,2, \tag{2.5}$$

$$u_1|_{\partial\Omega_1}=0, \qquad u_2|_{\partial\Omega_2}=0, \tag{2.6}$$

$$\partial_\nu u_1|_{\partial\Omega_2}=0, \qquad \partial_\nu u_2|_{\partial\Omega_1}=0. \tag{2.7}$$

Here and throughout this article ∂_ν stands for the normal derivative.

Proof. Note that the right-hand side of the system (2.4) is the infinitesimal generator $\mathcal{A}$ of the process $(\sqrt{2}\boldsymbol{B}_t, N_t)_{t\geq 0}$. Denoting $u(\boldsymbol{x}, j) = u_j(\boldsymbol{x})$, we can check that, for any $s < t$,

$$\begin{aligned}
&\mathbb{E}\bigg[u(\boldsymbol{X}_{t\wedge\mathcal{T}}, N_{t\wedge\mathcal{T}}) - u(\boldsymbol{X}_{s\wedge\mathcal{T}}, N_{s\wedge\mathcal{T}}) - \int_{s\wedge\mathcal{T}}^{t\wedge\mathcal{T}} \mathcal{A}u(\boldsymbol{X}_r, N_r)\, dr|\mathcal{F}_{s\wedge\mathcal{T}}\bigg] \\
&\quad = -\mathbb{E}\bigg[\int_{s\wedge\mathcal{T}}^{t\wedge\mathcal{T}} \partial_\nu u(\boldsymbol{X}_r, N_r)\, dL_r|\mathcal{F}_{s\wedge\mathcal{T}}\bigg],
\end{aligned}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(\sqrt{2}\boldsymbol{B}_t, N_t)_{t\geq 0}$ and L_t is defined as above. The Neumann boundary conditions (2.5) and (2.7) imposed on u show that when the particle reaches the boundary $\partial_\nu u(\boldsymbol{X}_r, N_r) = 0$ if $r < \mathcal{T}$. We also have $\mathcal{A}u(\boldsymbol{X}_r, N_r) = -1$ by the partial differential equation satisfied by u. This shows that

$$\mathbb{E}[u(\boldsymbol{X}_{t\wedge\mathcal{T}}, N_{t\wedge\mathcal{T}}) - u(\boldsymbol{X}_{s\wedge\mathcal{T}}, N_{s\wedge\mathcal{T}}) + (t\wedge\mathcal{T} - s\wedge\mathcal{T})|\mathcal{F}_{s\wedge\mathcal{T}}] = 0;$$

i.e., the process $(u(\boldsymbol{X}_{t\wedge\mathcal{T}}, N_{t\wedge\mathcal{T}}) + t\wedge\mathcal{T})_{t\geq 0}$ is a martingale. Applying this equality with $s = 0$ and letting $t \to \infty$, we obtain

$$\mathbb{E}[u(\boldsymbol{X}_{\mathcal{T}}, N_{\mathcal{T}}) - u(\boldsymbol{X}_0, N_0) + \mathcal{T}|\mathcal{F}_0] = 0.$$

The Dirichlet boundary conditions (2.6) imposed on u show that $u(\boldsymbol{X}_{\mathcal{T}}, N_{\mathcal{T}}) = 0$. Therefore, we have

$$\mathbb{E}[\mathcal{T}|\mathcal{F}_0] = \mathbb{E}[u(\boldsymbol{X}_0, N_0)|\mathcal{F}_0].$$

This gives the desired result when the initial distribution is such that $N_0 = k$ and $\boldsymbol{X}_0 = \boldsymbol{x}$. ◻

3. The Neumann functions and integral equations.

We denote by $N(\boldsymbol{x}, \boldsymbol{z})$ the Neumann function for $-\Delta$ in Ω with a Dirac mass at $\boldsymbol{z} \in \Omega$:

$$\begin{cases} \Delta_{\boldsymbol{x}} N = -\delta_{\boldsymbol{z}}, & \boldsymbol{x} \in \Omega, \\ \partial_\nu N|_{\partial\Omega} = -\frac{1}{|\partial\Omega|}, & \int_{\partial\Omega} N(\boldsymbol{x}, \boldsymbol{z})\, d\sigma(\boldsymbol{x}) = 0. \end{cases} \tag{3.1}$$

If $\boldsymbol{z} \in \partial\Omega$, then the Neumann function, which we denote by $N_{\partial\Omega}(\boldsymbol{x}, \boldsymbol{z})$, can be expanded as

$$N_{\partial\Omega}(\boldsymbol{x}, \boldsymbol{z}) = -\frac{1}{\pi} \ln|\boldsymbol{x} - \boldsymbol{z}| + R(\boldsymbol{x}, \boldsymbol{z}), \tag{3.2}$$

where $R(\cdot, \boldsymbol{z})$ belongs to $H^{3/2}(\Omega)$, the standard Sobolev space of order $3/2$, uniformly in $\boldsymbol{z} \in \partial\Omega$; see [1].

For a positive constant a, we denote by $M^a(\boldsymbol{x}, \boldsymbol{z})$ the Neumann function for $-\Delta + 2a$ in Ω with a Dirac mass at $\boldsymbol{z} \in \Omega$:

$$\begin{cases} \Delta_{\boldsymbol{x}} M^a - 2aM^a = -\delta_{\boldsymbol{z}}, & \boldsymbol{x} \in \Omega, \\ \partial_\nu M^a|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

Note that we can write $M^a(\boldsymbol{x}, \boldsymbol{z})$ as

$$M^a(\boldsymbol{x},\boldsymbol{z}) = \sum_{l=0}^{\infty} \frac{\psi_l(\boldsymbol{x})\psi_l(\boldsymbol{z})}{\lambda_l + 2a}, \tag{3.4}$$

where $(\lambda_l, \psi_l)_{l\geq 0}$ are pairs of eigenvalues and corresponding (normalized) eigenvectors of $-\boldsymbol{\Delta}$ over $\boldsymbol{\Omega}$ with Neumann boundary conditions on $\partial\boldsymbol{\Omega}$. In particular, $\lambda_0 = 0$ and $\psi_0 = 1/\sqrt{|\Omega|}$. It then follows that [2]

$$\int_\Omega M^a(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x} = \frac{1}{2a}. \tag{3.5}$$

If $\boldsymbol{z} \in \partial\boldsymbol{\Omega}$, then $M^a_{\partial\Omega}(\boldsymbol{x},\boldsymbol{z})$ can be expanded as

$$M^a_{\partial\Omega}(\boldsymbol{x},\boldsymbol{z}) = -\frac{1}{\pi}\,\ln|\boldsymbol{x}-\boldsymbol{z}| + R_a(\boldsymbol{x},\boldsymbol{z}), \tag{3.6}$$

where $R_a(\cdot,\boldsymbol{z})$ also belongs to $H^{3/2}(\Omega)$ uniformly in $\boldsymbol{z} \in \partial\boldsymbol{\Omega}$. We emphasize that R and R_a are symmetric in their arguments.

Using Green's formula, (2.5), and (2.7), one can show that the solutions $(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ to (2.4) have the following representations for all $\boldsymbol{z} \in \boldsymbol{\Omega}$ (see Appendix A):

$$\begin{aligned} u_j(\boldsymbol{z}) = g(\boldsymbol{z}) + \frac{C}{2} &+ \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x}) \frac{N_{\partial\Omega} + M^a_{\partial\Omega}}{2}(\boldsymbol{z},\boldsymbol{x})\,d\sigma(\boldsymbol{x}) \\ &+ \int_{\partial\Omega_{3-j}} \partial_\nu u_{3-j}(\boldsymbol{x}) \frac{N_{\partial\Omega} - M^a_{\partial\Omega}}{2}(\boldsymbol{z},\boldsymbol{x})\,d\sigma(\boldsymbol{x}), \qquad j = 1, 2, \end{aligned} \tag{3.7}$$

where $g(\boldsymbol{z})$ is the function defined by

$$g(\boldsymbol{z}) = \int_\Omega N(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x}, \qquad \boldsymbol{z} \in \boldsymbol{\Omega}, \tag{3.8}$$

and C is the constant defined by

$$C = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} (u_1(\boldsymbol{x}) + u_2(\boldsymbol{x}))\,d\sigma(\boldsymbol{x}). \tag{3.9}$$

In view of (2.6), we have, for all $j = 1$, 2 and for all $\boldsymbol{z} \in \partial\boldsymbol{\Omega}_j$,

$$\begin{aligned} g(\boldsymbol{z}) + \frac{C}{2} - \frac{1}{\pi} &\int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x})\,\ln|\boldsymbol{x}-\boldsymbol{z}|\,d\sigma(\boldsymbol{x}) \\ + \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x}) \frac{R + R_a}{2}(\boldsymbol{z},\boldsymbol{x})\,d\sigma(\boldsymbol{x}) &+ \int_{\partial\Omega_{3-j}} \partial_\nu u_{3-j}(\boldsymbol{x}) \frac{R - R_a}{2}(\boldsymbol{z},\boldsymbol{x})\,d\sigma(\boldsymbol{x}) = 0. \end{aligned} \tag{3.10}$$

We solve the integral equations (3.10) asymptotically for $\partial_\nu u_1$ and $\partial_\nu u_2$ in the next section. The method follows the arguments found in [12], but here we pay special attention to justify the asymptotic expansions which require control of the Sobolev norms of the Neumann functions R and R_a.

4. Asymptotic expansions. We parameterize $\partial\boldsymbol{\Omega}_j$ by arclength:

$$\partial\boldsymbol{\Omega}_j = \{\boldsymbol{X}_j(t), t \in [-\varepsilon_j, \varepsilon_j]\} \tag{4.1}$$

with $\varepsilon_j \ll 1$, $j = 1, 2$. The goal of the article is to study the asymptotic behavior of the MET when $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ goes to zero. The domain $\mathbf{\Omega}$ is fixed in the analysis, which means, in particular, that there are no bottlenecks (at the scale ε) in $\mathbf{\Omega}$. We denote by $\boldsymbol{x}_j^* = \boldsymbol{X}_j(0)$ the centers of the arcs, and we define, for $j = 1, 2$,

$$\phi_j(t) = \varepsilon_j \partial_\nu u_j(\boldsymbol{X}_j(\varepsilon_j t)), \qquad t \in [-1, 1].$$

We then introduce the operator $\mathcal{L}$ from $\mathbf{X} = \{\phi, \int_{-1}^1 \sqrt{1-s^2}\phi(s)^2 ds < \infty\}$ to $\mathbf{Y} = \{\phi \in \mathcal{C}(-1,1), \phi' \in \mathbf{X}\}$ by

$$\mathcal{L}[\phi](t) = \int_{-1}^1 \ln|t-s|\phi(s)\,ds. \tag{4.2}$$

The important theoretical result derived in [2] is that the operator $\mathcal{L}\colon \mathbf{X} \to \mathbf{Y}$ is invertible and

$$\mathcal{L}^{-1}[1](t) = -\frac{1}{\pi \ln 2}\frac{1}{\sqrt{1-t^2}}. \tag{4.3}$$

After scaling, (3.10) takes the forms

$$\begin{aligned}\frac{1}{\pi}\mathcal{L}[\phi_j](t) + \varepsilon\mathcal{K}_{j,j}[\phi_j](t) + \varepsilon\mathcal{K}_{j,3-j2}[\phi_{3-j}](t) &= \left(-\frac{\ln \varepsilon_j}{\pi} + \frac{R+R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*)\right)\int_{-1}^1 \phi_j(s)\,ds \\ &\quad + \frac{R-R_a}{2}(\boldsymbol{x}_{3-j}^*, \boldsymbol{x}_j^*)\int_{-1}^1 \phi_{3-j}(s)\,ds + g(\boldsymbol{X}_j(\varepsilon_j t)) + \frac{C}{2}\end{aligned}$$

for $j = 1, 2$ and $t \in [-1, 1]$, where the operators $\mathcal{K}_{j,k}$, j, $k = 1, 2$, are defined by

$$\begin{aligned}\mathcal{K}_{j,j}[\phi](t) &= \frac{1}{\pi\varepsilon}\int_{-1}^1 \ln\left(\frac{|\boldsymbol{X}_j(\varepsilon_j t) - \boldsymbol{X}_j(\varepsilon_j s)|}{\varepsilon_j|t-s|}\right)\phi(s)\,ds \\ &\quad - \frac{1}{\varepsilon}\int_{-1}^1 \left(\frac{R+R_a}{2}(\boldsymbol{X}_j(\varepsilon_j t), \boldsymbol{X}_j(\varepsilon_j s)) - \frac{R+R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*)\right)\phi(s)\,ds, \\ \mathcal{K}_{j,3-j}[\phi](t) &= -\frac{1}{\varepsilon}\int_{-1}^1 \left(\frac{R-R_a}{2}(\boldsymbol{X}_j(\varepsilon_j t), \boldsymbol{X}_{3-j}(\varepsilon_{3-j} s)) - \frac{R-R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_{3-j}^*)\right)\phi(s)\,ds\end{aligned}$$

for $j = 1, 2$. We emphasize that $\mathcal{K}_{j,k}$ is a bounded operator from $\mathbf{X}$ into $\mathbf{Y}$ independently of ε since $\boldsymbol{X}_j(t)$ is $\mathcal{C}^2$ and R, R_a are in $H^{3/2}$. Denoting

$$A_j = \int_{-1}^1 \phi_j(s)\,ds = \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x}), \qquad j = 1, 2,$$

we can rewrite the equations in the form

$$\frac{1}{\pi}\begin{pmatrix}\mathcal{L} & 0 \\ 0 & \mathcal{L}\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}(t) + \varepsilon\begin{pmatrix}\mathcal{K}_{1,1} & \mathcal{K}_{1,2} \\ \mathcal{K}_{2,1} & \mathcal{K}_{2,2}\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}(t) = \begin{pmatrix}q_1 \\ q_2\end{pmatrix} + O(\varepsilon), \qquad t \in [-1, 1],$$

where

$$q_j = g(\boldsymbol{x}_j^*) + \frac{C}{2} + \left(-\frac{\ln \varepsilon_j}{\pi} + \frac{R+R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*)\right)A_j + \frac{R-R_a}{2}(\boldsymbol{x}_{3-j}^*, \boldsymbol{x}_j^*)A_{3-j}, \qquad j = 1, 2.$$

It then follows from (4.3) that, for $t \in [-1, 1]$,

$$\phi_j(t) = \pi q_j \mathcal{L}^{-1}[1](t) + O(\varepsilon) = -\frac{q_j}{\ln 2} \frac{1}{\sqrt{1-t^2}} + O(\varepsilon), \qquad j = 1, 2.$$

Integrating this identity in t we find

$$g(\boldsymbol{x}_j^*) + \frac{C}{2} + \left(-\frac{\ln \varepsilon_j/2}{\pi} + \frac{R+R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*) \right) A_j + \frac{R-R_a}{2}(\boldsymbol{x}_{3-j}^*, \boldsymbol{x}_j^*) A_{3-j} = O(\varepsilon) \tag{4.4}$$

for $j = 1, 2$. Moreover, we have the compatibility condition

$$A_1 + A_2 = -2|\boldsymbol{\Omega}|, \tag{4.5}$$

which can be obtained by integrating $\Delta(u_1 + u_2)$ over $\boldsymbol{\Omega}$.

Equations (4.4)–(4.5) determine the value of C up to terms of order ε:

$$C = \frac{C_n}{C_d}, \tag{4.6}$$

where

$$\begin{aligned} C_n = {} & \frac{2|\boldsymbol{\Omega}|}{\pi^2} \left(\ln \frac{2}{\varepsilon_1} \right) \left(\ln \frac{2}{\varepsilon_2} \right) + \sum_{j=1}^{2} \frac{1}{\pi} \left(\ln \frac{2}{\varepsilon_j} \right) (|\boldsymbol{\Omega}|(R+R_a)(\boldsymbol{x}_{3-j}^*, \boldsymbol{x}_{3-j}^*) - g(\boldsymbol{x}_{3-j}^*)) \\ & - \sum_{j=1}^{2} g(\boldsymbol{x}_j^*) \left(\frac{R+R_a}{2}(\boldsymbol{x}_{3-j}^*, \boldsymbol{x}_{3-j}^*) - \frac{R-R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_{3-j}^*) \right) \\ & + \frac{R+R_a}{2}(\boldsymbol{x}_2^*, \boldsymbol{x}_2^*) \frac{R+R_a}{2}(\boldsymbol{x}_1^*, \boldsymbol{x}_1^*) - \frac{R-R_a}{2}(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)^2, \end{aligned} \tag{4.7}$$

$$C_d = \frac{1}{2\pi} \left(\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2} \right) + \sum_{j=1}^{2} \frac{R+R_a}{4}(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*) - \frac{R-R_a}{2}(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*). \tag{4.8}$$

They also determine the value of A_j up to terms of order ε:

$$A_j = -\frac{g(\boldsymbol{x}_j^*) + \frac{C}{2} - |\boldsymbol{\Omega}|(R-R_a)(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)}{\frac{1}{\pi} \ln \frac{2}{\varepsilon_j} + \frac{R+R_a}{2}(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*) - \frac{R-R_a}{2}(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)}, \qquad j = 1, 2. \tag{4.9}$$

In view of (3.7) we have

$$\mathbb{E}_{j,\boldsymbol{x}}[\mathcal{T}] = g(\boldsymbol{x}) + \frac{C}{2} + \frac{A_j}{2}(N_{\partial\Omega} + M_{\partial\Omega}^a)(\boldsymbol{x}, \boldsymbol{x}_j^*) + \frac{A_{3-j}}{2}(N_{\partial\Omega} - M_{\partial\Omega}^a)(\boldsymbol{x}, \boldsymbol{x}_{3-j}^*) \tag{4.10}$$

for $j = 1, 2$, up to terms of order ε as long as $\boldsymbol{x}$ is away from $\partial\boldsymbol{\Omega}_1$ and $\partial\boldsymbol{\Omega}_2$. If the initial distribution of N_0 is uniform over $\{1, 2\}$, then we have

$$\mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}] = g(\boldsymbol{x}) + \frac{C}{2} + \frac{A_1}{2} N_{\partial\Omega}(\boldsymbol{x}, \boldsymbol{x}_1^*) + \frac{A_2}{2} N_{\partial\Omega}(\boldsymbol{x}, \boldsymbol{x}_2^*). \tag{4.11}$$

Asymptotic expansions of the MET can be derived using (4.6)–(4.9), which we do in the next section.

5. Role of the switching rate. In this section we study the expressions (4.10)–(4.11) and show that the leading-order term (of order $\ln(2/\varepsilon_j)$) of the MET is independent of the switching rate a, while the first-order correction term (of order one) depends on a and is a decaying function of a.

5.1. Leading-order term. We find from (4.6)–(4.8) that the leading-order term of C is independent of a and $\boldsymbol{x}$:

$$C = \frac{4|\Omega|}{\pi} \frac{\ln \frac{2}{\varepsilon_1} \ln \frac{2}{\varepsilon_2}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} + O(1). \tag{5.1}$$

Since

$$A_j = -\frac{\pi C}{2 \ln \frac{2}{\varepsilon_j}} + O(1), \qquad j = 1, 2,$$

which can be seen from (4.9), one can see that the leading-order term of the MET is also independent of a:

$$\mathbb{E}_{k,\boldsymbol{x}}[\mathcal{T}] = \frac{2|\Omega|}{\pi} \frac{\ln \frac{2}{\varepsilon_1} \ln \frac{2}{\varepsilon_2}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} + O(1). \tag{5.2}$$

It is worth emphasizing that it is also independent of the initial state N_0 and position $\boldsymbol{x}$.

It is quite interesting to notice that the number

$$\frac{|\Omega|}{\pi} \frac{\ln \frac{2}{\varepsilon_1} \ln \frac{2}{\varepsilon_2}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}}$$

is the leading-order term of the MET when there are two (well-separated) gates of half-length ε_1 and ε_2 which are open all the time (see [12] or [3, Theorem 3.3]). Therefore, (5.2) shows that the leading order term in the MET with two switching gates is twice the one with two time-independent gates. In this scaling regime with small gates that are open only half the time, the time to find a gate doubles to leading order.

Another interpretation follows from the fact that the MET for a single gate problem with a gate that is always open at $\partial\Omega_1$ is, to leading order,

$$\frac{|\Omega|}{\pi} \frac{1}{\ln \frac{2}{\varepsilon_1}}.$$

A similar result holds for the MET for a single gate problem with a gate that is always open at $\partial\Omega_2$. Therefore, to leading order, the MET for the switching gates problem is the harmonic average of the MET of the single gate problems.

5.2. First-order correction. We can expand the expressions for C, A_1, and A_2 in powers of $\ln(2/\varepsilon_j)$. Taking into account the terms of order $\ln(2/\varepsilon_j)$ and of order one, we find that

$$C = \frac{4|\boldsymbol{\Omega}|}{\pi} \frac{\ln \frac{2}{\varepsilon_1} \ln \frac{2}{\varepsilon_2}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} + 2|\boldsymbol{\Omega}| \sum_{j=1}^{2} \frac{\left(\ln \frac{2}{\varepsilon_{3-j}}\right)^2}{\left(\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}\right)^2} (R + R_a)(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*)$$
$$+ 4|\boldsymbol{\Omega}| \frac{\left(\ln \frac{2}{\varepsilon_1}\right)\left(\ln \frac{2}{\varepsilon_2}\right)}{\left(\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}\right)^2} (R - R_a)(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*) + 2 \sum_{j=1}^{2} \frac{\ln \frac{2}{\varepsilon_{3-j}}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} g(\boldsymbol{x}_j^*)$$
$$+ O\left(\frac{1}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}}\right)$$

and

$$A_j = -2|\boldsymbol{\Omega}| \frac{\ln \frac{2}{\varepsilon_{3-j}}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} + O\left(\frac{1}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}}\right), \qquad j = 1, 2. \tag{5.3}$$

This gives for the MET

$$\mathbb{E}_{j,\boldsymbol{x}}[\mathcal{T}] = g(\boldsymbol{x}) + \frac{C}{2} - |\boldsymbol{\Omega}| \frac{\ln \frac{2}{\varepsilon_{3-j}}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} (N_{\partial\Omega} + M_{\partial\Omega}^a)(\boldsymbol{x}, \boldsymbol{x}_j^*)$$
$$- |\boldsymbol{\Omega}| \frac{\ln \frac{2}{\varepsilon_j}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} (N_{\partial\Omega} - M_{\partial\Omega}^a)(\boldsymbol{x}, \boldsymbol{x}_{3-j}^*) + O\left(\frac{1}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}}\right) \tag{5.4}$$

for $j = 1, 2$, and finally

$$\mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}] = g(\boldsymbol{x}) + \frac{C}{2} - \sum_{j=1}^{2} |\boldsymbol{\Omega}| \frac{\ln \frac{2}{\varepsilon_j}}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}} N_{\partial\Omega}(\boldsymbol{x}, \boldsymbol{x}_{3-j}^*) + O\left(\frac{1}{\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}}\right). \tag{5.5}$$

The terms of order one depend on a through the terms C and $M_{\partial\Omega}$. We have, up to terms of order $O((\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2})^{-1})$,

$$\frac{\partial C}{\partial a} = 2|\boldsymbol{\Omega}| \sum_{j=1}^{2} \frac{\left(\ln \frac{2}{\varepsilon_{3-j}}\right)^2}{\left(\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}\right)^2} \frac{\partial R_a(\boldsymbol{x}_j^*, \boldsymbol{x}_j^*)}{\partial a} - 4|\boldsymbol{\Omega}| \frac{\left(\ln \frac{2}{\varepsilon_1}\right)\left(\ln \frac{2}{\varepsilon_2}\right)}{\left(\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2}\right)^2} \frac{\partial R_a(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)}{\partial a}. \tag{5.6}$$

The following lemma applied with $p = \ln(2/\varepsilon_2)/[\ln(2/\varepsilon_2) + \ln(2/\varepsilon_1)]$ shows that

$$\frac{\partial C}{\partial a} < 0, \tag{5.7}$$

provided that ε_j is small enough.

Lemma 5.1. *We have the equality*

$$\frac{\partial R_a(\boldsymbol{x}_j^*, \boldsymbol{x}_k^*)}{\partial a} = -2 \int_{\Omega} M_{\partial\Omega}^a(\boldsymbol{x}, \boldsymbol{x}_j^*) M_{\partial\Omega}^a(\boldsymbol{x}, \boldsymbol{x}_k^*) d\boldsymbol{x}, \qquad j, k = 1, 2. \tag{5.8}$$

Moreover, for any $p \in (0,1)$, we have the inequality

$$\frac{p^2}{2}\frac{\partial R_a(\boldsymbol{x}_1^*,\boldsymbol{x}_1^*)}{\partial a}+\frac{(1-p)^2}{2}\frac{\partial R_a(\boldsymbol{x}_2^*,\boldsymbol{x}_2^*)}{\partial a}-p(1-p)\frac{\partial R_a(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*)}{\partial a}<0. \tag{5.9}$$

Proof. We denote $M'_a(\boldsymbol{x},\boldsymbol{z}) := \partial_a M^a(\boldsymbol{x},\boldsymbol{z})$ for $\boldsymbol{z}\in\boldsymbol{\Omega}$. It is the solution to the problem

$$\Delta_{\boldsymbol{x}} M'_a(\boldsymbol{x},\boldsymbol{z}) - 2aM'_a(\boldsymbol{x},\boldsymbol{z}) = 2M^a(\boldsymbol{x},\boldsymbol{z}), \qquad \partial_\nu M'_a(\cdot,\boldsymbol{z})|_{\partial\boldsymbol{\Omega}} = 0.$$

Thus we obtain by Green's formula

$$M'_a(\boldsymbol{x},\boldsymbol{z}) = -2\int_{\boldsymbol{\Omega}} M^a(\boldsymbol{x},\boldsymbol{y})M^a(\boldsymbol{y},\boldsymbol{z})\,d\boldsymbol{y}.$$

Since $\partial_a R_a(\boldsymbol{x},\boldsymbol{x}_j^*) = \partial_a M^a_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_j^*)$, we obtain (5.8).

From the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|p(1-p)\frac{\partial R_a(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*)}{\partial a}\right| &= \left|2\int_{\boldsymbol{\Omega}} pM^a_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_1^*)(1-p)M^a_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_2^*)\,d\boldsymbol{x}\right|\\ &< \left[2p^2\int_{\boldsymbol{\Omega}} M^a_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_1^*)^2\,d\boldsymbol{x}\right]^{1/2}\left[2(1-p)^2\int_{\boldsymbol{\Omega}} M^a_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_2^*)^2\,d\boldsymbol{x}\right]^{1/2}.\end{aligned}$$

The inequality is strict since $\boldsymbol{x}_1^* \neq \boldsymbol{x}_2^*$, and therefore we cannot have $M_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_2^*) = M_{\partial\boldsymbol{\Omega}}(\boldsymbol{x},\boldsymbol{x}_1^*)$ for almost every $\boldsymbol{x}\in\boldsymbol{\Omega}$. Using the inequality $\alpha\beta \le (1/2)\alpha^2 + (1/2)\beta^2$, we obtain (5.9). ∎

By substituting (3.4) into (5.6) we can obtain a representation formula of the quantity $\partial C/\partial a$ in terms of the pairs of eigenvalues and eigenvectors $(\lambda_l,\psi_l)_{l\ge 0}$ of $-\Delta$ over $\boldsymbol{\Omega}$ with Neumann boundary conditions on $\partial\boldsymbol{\Omega}$:

$$\frac{\partial C}{\partial a} = -4|\boldsymbol{\Omega}|\sum_{l\ge 0}\left[\frac{\ln\frac{2}{\varepsilon_2}\psi_l(\boldsymbol{x}_1^*)+\ln\frac{2}{\varepsilon_1}\psi_l(\boldsymbol{x}_2^*)}{(\lambda_l+2a)(\ln\frac{2}{\varepsilon_1}+\ln\frac{2}{\varepsilon_2})}\right]^2. \tag{5.10}$$

Since $\lambda_0 = 0$ and $\psi_0 = 1/\sqrt{|\boldsymbol{\Omega}|}$, we obtain the straightforward upper bound

$$\frac{\partial C}{\partial a} \le -\frac{1}{a^2}. \tag{5.11}$$

This inequality is close to an equality when a is small since we have

$$-\frac{1}{a^2} - C_1 \le \frac{\partial C}{\partial a} \le -\frac{1}{a^2}$$

with

$$C_1 = 4|\boldsymbol{\Omega}|\sum_{l\ge 1}\left[\frac{\ln\frac{2}{\varepsilon_2}\psi_l(\boldsymbol{x}_1^*)+\ln\frac{2}{\varepsilon_1}\psi_l(\boldsymbol{x}_2^*)}{\lambda_l(\ln\frac{2}{\varepsilon_1}+\ln\frac{2}{\varepsilon_2})}\right]^2. \tag{5.12}$$

We have, up to terms of order $O((\ln(2/\varepsilon_1)+\ln(2/\varepsilon_2))^{-1})$,

$$\frac{\partial \mathbb{E}_{j,\boldsymbol{x}}[\mathcal{T}]}{\partial a}=\frac{1}{2}\frac{\partial C}{\partial a}-|\Omega|\frac{\ln\frac{2}{\varepsilon_{3-j}}}{\ln\frac{2}{\varepsilon_1}+\ln\frac{2}{\varepsilon_2}}\frac{\partial R_a(\boldsymbol{x},\boldsymbol{x}_j^*)}{\partial a}+|\Omega|\frac{\ln\frac{2}{\varepsilon_j}}{\ln\frac{2}{\varepsilon_1}+\ln\frac{2}{\varepsilon_2}}\frac{\partial R_a(\boldsymbol{x},\boldsymbol{x}_{3-j}^*)}{\partial a}$$

for $j=1,2$, and finally

$$\frac{\partial \mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}]}{\partial a}=\frac{1}{2}\frac{\partial C}{\partial a}.$$

The last result allows us to conclude that the MET decays with the switching rate whatever the initial starting point $\boldsymbol{x}$, since we have, for ε_j small enough,

$$\frac{\partial \mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}]}{\partial a}<0, \tag{5.13}$$

with the hypothesis that the initial open gate is chosen randomly between the two possible ones. This is the main practical result of this article. It answers in a positive way the conjecture that the MET decreases with the switching rate. More quantitatively, we have

$$-\frac{1}{2a^2}-\frac{C_1}{2}\leq\frac{\partial \mathbb{E}_{u,\boldsymbol{x}}[\mathcal{T}]}{\partial a}\leq-\frac{1}{2a^2}, \tag{5.14}$$

where C_1 is defined by (5.12).

If the first open gate is $\partial\Omega_1$ (i.e., if $N_0=1$), then the sign of $\partial_a\mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]$ is not guaranteed by the estimates of this article for all $\boldsymbol{x}\in\Omega$. However, one can show that the average quantity $(1/|\Omega|)\int_\Omega\mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}$ actually decays in a. Indeed, we have from (3.7)

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega\mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}=\frac{1}{|\Omega|}\int_\Omega g(\boldsymbol{x})\,d\boldsymbol{x}+\frac{C}{2}+\frac{A_1}{2}\frac{1}{|\Omega|}\int_\Omega(N_{\partial\Omega}+M^a_{\partial\Omega})(\boldsymbol{x},\boldsymbol{x}_1^*)\,d\boldsymbol{x}\\+\frac{A_2}{2}\frac{1}{|\Omega|}\int_\Omega(N_{\partial\Omega}-M^a_{\partial\Omega})(\boldsymbol{x},\boldsymbol{x}_2^*)\,d\boldsymbol{x}.\end{aligned}$$

Since A_j does not depend on a to this order (see (5.3)), we obtain

$$\begin{aligned}\frac{\partial}{\partial a}\left(\frac{1}{|\Omega|}\int_\Omega\mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}\right)=\frac{1}{2}\frac{\partial C}{\partial a}+\frac{A_1}{2}\frac{\partial}{\partial a}\left(\frac{1}{|\Omega|}\int_\Omega R_a(\boldsymbol{x},\boldsymbol{x}_1^*)\,d\boldsymbol{x}\right)\\-\frac{A_2}{2}\frac{\partial}{\partial a}\left(\frac{1}{|\Omega|}\int_\Omega R_a(\boldsymbol{x},\boldsymbol{x}_2^*)\,d\boldsymbol{x}\right).\end{aligned}$$

By differentiating (3.5) we get

$$\frac{\partial}{\partial a}\int R_a(\boldsymbol{x},\boldsymbol{x}_j^*)\,d\boldsymbol{x}=-\frac{1}{2a^2}.$$

Thus we have

$$\frac{\partial}{\partial a}\left(\frac{1}{|\Omega|}\int_\Omega\mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}\right)=\frac{1}{2}\frac{\partial C}{\partial a}+\frac{1}{2a^2}\frac{\ln\frac{2}{\varepsilon_2}-\ln\frac{2}{\varepsilon_1}}{\ln\frac{2}{\varepsilon_1}+\ln\frac{2}{\varepsilon_2}},$$

up to the term of order $O((\ln \frac{2}{\varepsilon_1} + \ln \frac{2}{\varepsilon_2})^{-1})$. Using (5.11) we get

$$\frac{\partial}{\partial a}\left(\frac{1}{|\Omega|}\int_\Omega \mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}\right) \le -\frac{1}{a^2}\frac{\ln\frac{2}{\varepsilon_1}}{\ln\frac{2}{\varepsilon_1}+\ln\frac{2}{\varepsilon_2}}, \tag{5.15}$$

which shows that the quantity $(1/|\Omega|)\int_\Omega \mathbb{E}_{1,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}$ is decreasing in a. As noted above this inequality is close to an equality when a is small. A similar reasoning shows the same for $(1/|\Omega|)\int_\Omega \mathbb{E}_{2,\boldsymbol{x}}[\mathcal{T}]\,d\boldsymbol{x}$.

6. Generalization to an arbitrary number of gates.

6.1. The Markovian dynamics of the gates. Suppose that the boundary $\partial\Omega$ is partitioned into $n+1$ parts, $\partial\Omega_1, \dots, \partial\Omega_n$, and the complementary $\partial\Omega_c$. We model an escape problem in which there are n gates at the arcs $\partial\Omega_j$, $j = 1, \dots, n$, that open alternatively: at time t the open gate has index N_t and $(N_t)_{t\ge 0}$ is a jump Markov process that is stepwise constant and takes values in $\{1, \dots, n\}$. The Markovian dynamics can be described as follows: if at time t the process is in state $k \in \{1, \dots, n\}$, then during the small time interval $[t, t+h]$ the process can either jump to a new state or stay in k. The probability that it jumps to state $j \in \{1, \dots, n\} \setminus \{k\}$ is

$$\mathbb{P}(N_{t+h} = j | N_t = k) = Q_{jk}h + o(h),$$

and the probability that it stays in k is

$$\mathbb{P}(N_{t+h} = k | N_t = k) = 1 + Q_{kk}h + o(h).$$

We have $Q_{jk} \ge 0$ for $j \ne k$, $Q_{kk} \le 0$, and $\sum_{j=1}^n Q_{jk} = 0$. The $n \times n$ matrix $\mathbf{Q} = (Q_{jk})_{j,k=1}^n$ characterizes the distribution of the Markov process $(N_t)_{t\ge0}$, and it is called the infinitesimal generator. The transition probabilities $P(t, j|s, k) = \mathbb{P}(N_t = j|N_s = k)$ satisfy the master equations:

$$\frac{\partial P(t,j|s,k)}{\partial t} = \sum_{l=1}^n Q_{jl}P(t,l|s,k), \quad t \ge s, \quad P(s,j|s,k) = \delta_{jk}, \quad j,k = 1,\dots,n.$$

Example 1. The telegraph process addressed in section 2 is a particular example with $n = 2$ and

$$\mathbf{Q} = a\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Example 2. Here $n \ge 2$. Assume that the Markov process is stepwise constant during time intervals whose durations follow independent and identically distributed exponential random variables with mean $1/a$ and that the Markov process chooses another gate with equiprobability amongst the $n-1$ available gates when it jumps. Then

$$\mathbf{Q} = \frac{a}{n-1}\begin{pmatrix} 1-n & 1 & \dots & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & \dots & & 1-n & 1 \\ 1 & \dots & & 1 & 1-n \end{pmatrix}.$$

Example 3. Here $n \geq 3$. Assume that the Markov process is stepwise constant during time intervals whose durations follow independent and identically distributed exponential random variables with mean $1/a$ and that the Markov process chooses either the right or the left gate with equiprobability when it jumps. Then

$$\mathbf{Q} = \frac{a}{2} \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & 0 & 1 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix}.$$

We assume that the Markov process is irreducible; i.e., there is a nonzero probability of transitioning (even if in more than one step) from any state to any other state. It means that for any $j \neq k$, $Q_{jk} > 0$ or there exists $j_1, \dots, j_p$ such that $Q_{jj_1} Q_{j_1 j_2} \cdots Q_{j_p k} > 0$.

We also assume that the matrix $\mathbf{Q}$ is symmetric, and therefore the process is reversible (with respect to the uniform measure). Note that the process is irreducible and reversible in the three examples listed above.

The matrix $\mathbf{Q}$ is therefore diagonalizable:

$$\mathbf{Q} = \mathbf{P}\mathbf{D}\mathbf{P}^T, \tag{6.1}$$

and, by the Perron–Frobenius theorem, the matrix $\mathbf{Q}$ has one zero eigenvalue and the other eigenvalues are negative and denoted by

$$(D_{11}, \dots, D_{nn}) = (-a_1, -a_2, \dots, -a_n) \tag{6.2}$$

with $a_1 = 0$ and $a_j > 0$, $j \geq 2$. The eigenvector associated with the zero eigenvalue is the normalized uniform vector:

$$P_{j1} = \frac{1}{\sqrt{n}}, \qquad j = 1, \dots, n. \tag{6.3}$$

The process $(N_t)_{t \geq 0}$ is therefore ergodic, and the transition probabilities $P(t, j|s, k)$ converge as $t \to \infty$ to the unique stationary distribution of the process which is the uniform distribution over $\{1, \dots, n\}$.

The goal is to compute the expectation of the hitting time by the process $(\boldsymbol{X}_t)_{t \geq 0}$ of the time-dependent domain $\partial\Omega_{N_t}$. We denote by $\mathcal{T}$ the stopping time which corresponds to the "escape" of the particle:

$$\mathcal{T} = \wedge_{j=1}^{n} \mathcal{T}_j, \quad \mathcal{T}_j = \inf\{t \geq 0, N_t = j \,\text{and}\, \boldsymbol{X}_t \in \partial\Omega_j\}, \quad j = 1, \dots, n.$$

The MET is the expectation $\mathbb{E}_{k,\boldsymbol{x}}[\mathcal{T}]$ of the stopping time when the initial states are $N_0 = k$ and $\boldsymbol{X}_0 = \boldsymbol{x}$. Proceeding as in section 2, we have the following proposition for the MET.

PROPOSITION 6.1. *For $\boldsymbol{x} \in \Omega$ and $k = 1, \dots, n$, we have*

$$\mathbb{E}_{k,\boldsymbol{x}}[\mathcal{T}] = u_k(\boldsymbol{x}), \tag{6.4}$$

where the vector of functions $\boldsymbol{u}(\boldsymbol{x}) = (u_j(\boldsymbol{x}))^T_{j=1,\dots,n}$ *is the solution of the coupled linear system*

$$-\boldsymbol{j} = \mathbf{Q}\boldsymbol{u}(\boldsymbol{x}) + \Delta\boldsymbol{u}(\boldsymbol{x}) \tag{6.5}$$

with $\boldsymbol{j} = (1, \dots, 1)^T$ *and with the boundary conditions*

$$\partial_\nu u_j|_{\partial\Omega_c} = 0, \qquad j = 1, \dots, n, \tag{6.6}$$

$$u_j|_{\partial\Omega_j} = 0, \qquad j = 1, \dots, n, \tag{6.7}$$

$$\partial_\nu u_j|_{\partial\Omega_k} = 0, \qquad j \neq k. \tag{6.8}$$

6.2. Integral equations. We introduce $\boldsymbol{v} = \mathbf{P}^T\boldsymbol{u}$. For $k = 1, \dots, n$ the function $v_k = \sum_{j=1}^n P_{jk}u_j$, $k = 1, \dots, n$, satisfies

$$\Delta v_k - a_k v_k = -\sum_{j=1}^n P_{jk}.$$

Using (6.3) and the orthogonality of the eigenvectors of $\mathbf{Q}$ we have

$$\sum_{j=1}^n P_{jk} = \begin{cases} \sqrt{n} & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Therefore, Green's formula gives us

$$v_k(\boldsymbol{z}) = \sqrt{n}\delta_{k1}g(\boldsymbol{z}) + \delta_{k1}\frac{1}{|\partial\Omega|}\int_{\partial\Omega} v_1(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) + \int_{\partial\Omega} M_{\partial\Omega}^{a_k/2}(\boldsymbol{x},\boldsymbol{z})\partial_\nu v_k(\boldsymbol{x})\,d\sigma(\boldsymbol{x})$$

for $\boldsymbol{z} \in \Omega$. Here $M^a_{\partial\Omega}$ is defined by (3.3)–(3.6), and we have set $M^0_{\partial\Omega} = N_{\partial\Omega}$, the Neumann function (3.1)–(3.2). Since $\boldsymbol{u} = \mathbf{P}\boldsymbol{v}$, we can show that the following representation holds for all $l = 1, \dots, n$ and for all $\boldsymbol{z} \in \Omega$:

$$\begin{aligned} u_l(\boldsymbol{z}) &= g(\boldsymbol{z}) + \frac{C}{n} + \sum_{j=1}^n \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x}) \sum_{k=1}^n P_{lk}P_{jk}M_{\partial\Omega}^{a_k/2}(\boldsymbol{x},\boldsymbol{z})\,d\sigma(\boldsymbol{x}) \\ &= g(\boldsymbol{z}) + \frac{C}{n} - \frac{1}{\pi}\int_{\partial\Omega_l} \ln|\boldsymbol{x}-\boldsymbol{z}|\partial_\nu u_l(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) \\ &\quad + \sum_{j=1}^n \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x}) \sum_{k=1}^n P_{lk}P_{jk}S_k(\boldsymbol{x},\boldsymbol{z})\,d\sigma(\boldsymbol{x}), \end{aligned} \tag{6.9}$$

where C is the constant defined by

$$C = \frac{1}{|\partial\Omega|}\sum_{j=1}^n \int_{\partial\Omega} u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) \tag{6.10}$$

and

$$S_1(\boldsymbol{x},\boldsymbol{z}) = R(\boldsymbol{x},\boldsymbol{z}), \quad S_k(\boldsymbol{x},\boldsymbol{z}) = R_{a_k/2}(\boldsymbol{x},\boldsymbol{z}), \quad k = 2, \dots, n. \tag{6.11}$$

Here R and R_a are functions defined by (3.2) and (3.6), respectively. We then find that, for all $l = 1, \ldots, n$ and for all $\boldsymbol{z} \in \partial\boldsymbol{\Omega}_l$,

$$
\begin{aligned}
&g(\boldsymbol{z}) + \frac{C}{n} - \frac{1}{\pi}\int_{\partial\Omega_l} \ln|\boldsymbol{x} - \boldsymbol{z}|\partial_\nu u_l(\boldsymbol{x})\, d\sigma(\boldsymbol{x}) \\
&+ \sum_{j=1}^{n} \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x}) \sum_{k=1}^{n} P_{lk}P_{jk}S_k(\boldsymbol{x}, \boldsymbol{z})\, d\sigma(\boldsymbol{x}) = 0.
\end{aligned} \tag{6.12}
$$

6.3. Asymptotic expansions. Denoting by $\boldsymbol{x}_j^*$ the center of the arc $\partial\boldsymbol{\Omega}_j$ and by $\varepsilon_j \ll 1$ its half-length, we find that the numbers A_j, $j = 1, \ldots, n$, defined by

$$
A_j = \int_{\partial\Omega_j} \partial_\nu u_j(\boldsymbol{x})\, d\sigma(\boldsymbol{x}), \qquad j = 1, \ldots, n, \tag{6.13}
$$

satisfy the n equations

$$
g(\boldsymbol{x}_l^*) + \frac{C}{n} - \frac{\ln \varepsilon_l/2}{\pi} A_l + \sum_{j=1}^{n}\left(\sum_{k=1}^{n} P_{lk}P_{jk}S_k(\boldsymbol{x}_j^*, \boldsymbol{x}_l^*)\right) A_j = O(\varepsilon), \qquad l = 1, \ldots, n, \tag{6.14}
$$

where $\varepsilon = \max(\varepsilon_1, \ldots, \varepsilon_n)$, together with the compatibility condition

$$
\sum_{j=1}^{n} A_j = -n|\boldsymbol{\Omega}|. \tag{6.15}
$$

Therefore, the $n+1$ linear equations (6.14) and (6.15) determine the constant C and A_j, $j = 1, \ldots, n$ (up to terms of order ε), and they can then be substituted in (6.9) to get the expansion of the MET:

$$
u_l(\boldsymbol{z}) = g(\boldsymbol{z}) + \frac{C}{n} + \sum_{j=1}^{n}\sum_{k=1}^{n} A_j P_{lk} P_{jk} M_{\partial\boldsymbol{\Omega}}^{a_k/2}(\boldsymbol{x}_j^*, \boldsymbol{z}) + O(\epsilon) \tag{6.16}
$$

for $l = 1, \ldots, n$ and $\boldsymbol{z} \in \boldsymbol{\Omega}$ away from $\boldsymbol{x}_j^*$, $j = 1, \ldots, n$.

To leading order in ε we find that

$$
\frac{C}{n} \simeq \frac{|\boldsymbol{\Omega}|}{\pi}\left(\frac{1}{n}\sum_{l=1}^{n}\frac{1}{\ln \frac{2}{\varepsilon_l}}\right)^{-1} + O(1), \tag{6.17}
$$

and one can see that the leading-order term of the MET is independent of the initial state N_0 and of the detailed dynamics of the Markov process:

$$
\mathbb{E}_{k,\boldsymbol{x}}[\mathcal{T}] = \frac{|\boldsymbol{\Omega}|}{\pi}\left(\frac{1}{n}\sum_{l=1}^{n}\frac{1}{\ln \frac{2}{\varepsilon_l}}\right)^{-1} + O(1). \tag{6.18}
$$

This shows that, to leading order, the MET for the switching gates problem is the harmonic average of the MET of the single gate problems. It is also quite interesting to notice that the leading-order term of the MET for the n switching gates problem is n times the leading-order term of the MET when the gates are time-independent and open.

7. Conclusion. In this article we have considered the narrow escape problem and derived the asymptotic expansion of the MET when there are multiple gates which open alternatively according to Markovian dynamics. The major findings of this article are (i) the leading-order term of the asymptotic of the MET does not depend on the initial state or the particular dynamics of the Markov process; it only depends on the number of gates and their sizes; (ii) when there are two switching gates, the leading-order term of the MET is twice that which occurs when there are two gates which are open all the time; (iii) the higher the switching rate, the shorter the MET in its first-order correction, and we give lower and upper bounds for the decay rate of the MET as a function of the switching rate. The first-order correction of the MET also depends on the starting point and on the distribution of which gate is initially open. We finally remark that it would be interesting to extend the result of this article to three dimensions. The analysis is more tedious in the three-dimensional case since Green's function has a Coulomb singularity and an additional weaker logarithmic singularity, but it should be possible to carry out a full analysis in the case when the gates have the forms of small circular disks as was done in [19].

Appendix A. Proof of (3.7). First we apply Green's second identity with the functions u_j and M^a:

$$\int_\Omega u_j(\boldsymbol{x})\Delta_{\boldsymbol{x}} M^a(\boldsymbol{x},\boldsymbol{z}) - M^a(\boldsymbol{x},\boldsymbol{z})\Delta u_j(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\partial\Omega} u_j(\boldsymbol{x})\partial_\nu M^a(\boldsymbol{x},\boldsymbol{z}) - M^a(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x}).$$

Using (2.4), (3.3), and the boundary conditions (2.5) and (2.7), we find

$$u_j(\boldsymbol{z}) = a\int_\Omega (u_1+u_2)(\boldsymbol{x})M^a(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x} + \int_\Omega M^a(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x} + \int_{\partial\Omega_j} M^a(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x})$$

for $j=1,2$. Taking the difference of these two equations gives

$$u_1(\boldsymbol{z}) - u_2(\boldsymbol{z}) = \int_{\partial\Omega_1} M^a(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_1(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) - \int_{\partial\Omega_2} M^a(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_2(\boldsymbol{x})\,d\sigma(\boldsymbol{x}). \tag{A.1}$$

Second we apply Green's second identity with the functions u_j and N:

$$\int_\Omega u_j(\boldsymbol{x})\Delta_{\boldsymbol{x}} N(\boldsymbol{x},\boldsymbol{z}) - N(\boldsymbol{x},\boldsymbol{z})\Delta u_j(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\partial\Omega} u_j(\boldsymbol{x})\partial_\nu N(\boldsymbol{x},\boldsymbol{z}) - N(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x}).$$

Using (2.4), (3.1), and the boundary conditions (2.5) and (2.7), we find

$$u_j(\boldsymbol{z}) = \int_\Omega (1+a(u_1+u_2))(\boldsymbol{x})N(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x} + \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) + \int_{\partial\Omega_j} N(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_j(\boldsymbol{x})\,d\sigma(\boldsymbol{x})$$

for $j=1,2$. Taking the sum of these two equations gives

$$
\begin{aligned}
u_1(\boldsymbol{z}) + u_2(\boldsymbol{z}) = {} & 2\int_\Omega N(\boldsymbol{x},\boldsymbol{z})\,d\boldsymbol{x} + \frac{1}{|\partial\Omega|}\int_{\partial\Omega}(u_1+u_2)(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) \\
& + \int_{\partial\Omega_1} N(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_1(\boldsymbol{x})\,d\sigma(\boldsymbol{x}) + \int_{\partial\Omega_2} N(\boldsymbol{x},\boldsymbol{z})\partial_\nu u_2(\boldsymbol{x})\,d\sigma(\boldsymbol{x}).
\end{aligned}
\tag{A.2}
$$

The two formulas (A.1) and (A.2) give (3.7).

REFERENCES

[1] H. Ammari and H. Kang, *Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory*, Appl. Math. Sci. 162, Springer-Verlag, New York, 2007.

[2] H. Ammari, H. Kang, and H. Lee, *Layer Potential Techniques in Spectral Analysis*, Math. Surveys Monogr. 153, AMS, Providence, RI, 2009.

[3] H. Ammari, H. Kang, K. Kalimeris, and H. Lee, *Layer potential techniques for the narrow escape problem*, submitted.

[4] J.-B. Bardet, H. Guérin, and F. Malrieu, *Long time behavior of diffusions with Markov switching*, ALEA Lat. Am. J. Probab. Math. Stat., 7 (2010), pp. 151–170.

[5] W. S. Chan, L. X. Zhang, and S. Hun, *Temporal aggregation of Markov-switching financial return models*, Appl. Stoch. Models Bus. Ind., 25 (2009), pp. 359–383.

[6] A. F. Cheviakov, M. J. Ward, and R. Straube, *An asymptotic analysis of the mean first passage time for narrow escape problems: Part* II*: The sphere*, Multiscale Model. Simul., 8 (2010), pp. 836–868.

[7] J. Elf, G. Li, and X. Xie, *Probing transcription factor dynamics at the single-molecule level in a living cell*, Science, 316 (2007), pp. 1191–1194.

[8] S. Goldstein, *On diffusion by discontinuous movements and the telegraph equation*, Quart. J. Mech. Appl. Math., 4 (1951), pp. 129–156.

[9] D. Holcman, *Diffusion in cellular microdomains: Application to synapses*, Contemp. Math. 494, H. Ammari and H. Kang, eds., AMS, Providence, RI, 2009, pp. 1–26.

[10] D. Holcman and Z. Schuss, *Escape through a small opening: Receptor trafficking in a synaptic membrane*, J. Stat. Phys., 117 (2004), pp. 975–1014.

[11] D. Holcman and Z. Schuss, *Diffusion escape through a cluster of small absorbing windows*, J. Phys. A, 41 (2008), 155001.

[12] D. Holcman and Z. Schuss, *Diffusion through a cluster of small windows and flux regulation in microdomains*, Phys. Lett. A, 372 (2008), pp. 3768–3772.

[13] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math., 37 (1984), pp. 511–537.

[14] S. Pillay, M. J. Ward, A. Peirce, and T. Kolokolnikov, *An asymptotic analysis of the mean first passage time for the narrow escape problems: Part* I*: Two-dimensional domain*, Multiscale Model. Simul., 8 (2010), pp. 803–835.

[15] J. Reingruber and D. Holcman, *Gated narrow escape time for molecular signaling*, Phys. Rev. Lett., 103 (2009), 148102.

[16] J. Reingruber and D. Holcman, *Narrow escape for a stochastically gated Brownian ligand*, J. Phys. Condens. Matter, 22 (2010), 065103.

[17] Z. Schuss, A. Singer, and D. Holcman, *The narrow escape problem for diffusion in cellular microdomains*, Proc. Natl. Acad. Sci. USA, 104 (2007), pp. 16098–16103.

[18] A. Singer and Z. Schuss, *Activation through a narrow opening*, SIAM J. Appl. Math., 68 (2007), pp. 98–108.

[19] A. Singer, Z. Schuss, and D. Holcman, *Narrow escape and leakage of Brownian particles*, Phys. Rev. E, 78 (2008), 051111.

[20] A. Singer, Z. Schuss, and D. Holcman, *Narrow escape, Part* II*: The circular disk*, J. Stat. Phys., 122 (2006), pp. 465–489.

[21] A. Singer, Z. Schuss, and D. Holcman, *Narrow escape, Part* III*. Non-smooth domains and Riemann surfaces*, J. Stat. Phys., 122 (2006), pp. 491–509.

[22] A. Singer, Z. Schuss, D. Holcman, and R. S. Eisenberg, *Narrow escape, Part* I, J. Stat. Phys., 122 (2006), pp. 437–463.

[23] Y. M. Wang, R. H. Austin, and E. C. Cox, *Single molecule measurements of repressor protein* 1*D diffusion on DNA*, Phys. Rev. Lett., 97 (2006), 048302.