Incoherent Dispersive Shocks in the Spectral Evolution of Random Waves

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We predict theoretically and numerically the existence of incoherent dispersive shock waves. They manifest themselves as an unstable singular behavior of the spectrum of incoherent waves that evolve in a noninstantaneous nonlinear environment. This phenomenon of ''spectral wave breaking'' develops in the weakly nonlinear regime of the random wave. We elaborate a general theoretical formulation of these incoherent objects on the basis of a weakly nonlinear statistical approach: a family of singular integro-differential kinetic equations is derived, which provides a detailed deterministic description of the incoherent dispersive shock wave phenomenon.

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Introduction.—Shock waves play an important role in many different branches of physics [1]. When dissipative effects are negligible, the shock wave formation is regularized, owing to dispersion, through the onset of rapidly oscillating nonstationary structures, the so-called dispersive shock waves (DSWs) or undular bores [2]. DSWs have been constructed mathematically [3] long ago, after pioneering investigations in the fields of tidal waves [4] and collisionless (extremely rarefied) plasma [5–7]. It is only recently, however, that DSWs have emerged as a general signature of singular fluid-type behavior in areas as different as Bose-Einstein condensed atoms [8], nonlinear optics [9,10], oceanography [11], quantum liquids [12], nonlinear chains or granular materials [13], and electrons [14]. Although DSWs are, generally speaking, nonsolitonic slow modulations of fast periodic waves emerging from a gradient catastrophe [3,15], in specific cases they can exhibit pure multisoliton content [6,11,14,16], yet with solitons emerging only after the breaking time (distance).

A natural question concerns the effect of disorder on DSWs. So far, only the role of structural disorder of the medium has been investigated [17], while perfectly deterministic (coherent) wave amplitudes are assumed as in the rest of the literature body. Our aim in this Letter is to address the opposite problem, namely, the evolution of incoherent (random) waves in deterministic (homogeneous) media, showing that they can exhibit DSWs of a fundamental different nature than their coherent counterpart. Specifically, we show that incoherent DSWs manifest themselves as a wave-breaking process (''gradient catastrophe'') only in the spectral dynamics of the incoherent field that evolves in a noninstantaneous nonlinear environment. These incoherent DSWs develop in the highly incoherent regime of the random wave, in which linear dispersive effects dominate nonlinear effects. This allows us to develop a general theoretical formulation of incoherent DSWs on the basis of a weakly nonlinear statistical approach. The theory reveals that that these incoherent objects are described, as a general rule, by singular integro-differential kinetic equations (SIDKE), which provide a detailed description of the mechanism underlying the formation, or viceversa the inhibition, of spectral incoherent shocks. This theoretical approach also reveals unexpected links with the 3D vorticity equation in incompressible fluids [18], or the integrable Benjamin-Ono (BO) equation [19] originally derived in hydrodynamics for stratified fluids and investigated in the semiclassical (coherent breaking) limit recently [20].

We present the theory in the context of nonlinear optics because fibers [21,22] and waveguides [23] turn out to be ideal experimental test beds for our predictions, thanks to the easily tailorable noninstantaneous response via the well-known Raman effect, as well as other recently investigated mechanisms involving liquid cores or photoionizable noble gases and surface plasmon polaritons [24]. Nevertheless, given the universality of the nonlinear Schrödinger equation (NLSE), incoherent DSWs shed new light on singular nonequilibrium behaviors of a large variety of turbulent wave systems [25,26]. Furthermore, our kinetic approach finds applications in biological systems in the framework of the Lotka-Volterra equation [27], which is known as a key model for the coupled dynamics of competing biological species.

NLSE simulations.—The starting point is the NLSE accounting for a noninstantaneous nonlinearity

$$i\partial_z \psi = -\sigma \partial_{tt} \psi + \psi \int R(t-t')|\psi|^2(t')dt', \qquad (1)$$

where the response function $R(t)$ satisfies the causality condition $R(t) = 0$ for $t < 0$, and the typical width of $R(t)$ denotes the nonlinear response time, τ_R. The problem has been normalized with respect to the ''healing time''$\tau_0 = \sqrt{|\alpha| L_{\rm nl}}$, where α is the dispersion coefficient [$\sigma = {\rm sign}(\alpha)$], $L_{\rm nl} = 1/(\gamma\rho)$ the nonlinear length, γ the nonlinear coefficient, and ρ the wave intensity. The dimensional variables can be recovered through the substitution

$\psi \to \psi\sqrt{\rho}$, $t \to t\tau_0$, $z \to zL_{\rm nl}$. In the following, we consider the highly incoherent (i.e., weakly nonlinear) regime where $L_d \ll L_{\rm nl}$ (or equivalently $t_c \ll \tau_0$), $L_d = t_c^2/|\alpha|$ being the dispersion length and t_c the time correlation of the random field $\psi(z, t)$. It is in this regime that incoherent DSWs develop, at variance with conventional coherent DSWs, which occur in the opposite nonlinear regime, $L_d \gg L_{\rm nl}$. According to linear response theory, the real and imaginary parts of the Fourier transform of $R(t)$ satisfy the Kramers-Kronig relations. The imaginary part is an odd function known as the nonlinear spectral gain, $g_\omega = \mathrm{Im}[\int_0^\infty R(t)e^{-i\omega t}dt]$ [inset of Fig. 1(b)]. Its typical width denotes the natural spectral scale of the problem, $\Delta\omega_g \sim \tau_R^{-1}$.

In the Supplemental Material [28], we report the theory of incoherent DSWs developed for a general form of the response function, $R(t) = H(t)\bar{R}(t)$, where $\bar{R}(t)$ is a smooth function while the Heaviside function $H(t)$ ensures the causality property. We illustrate the theory by considering two physically relevant examples of response functions, which, respectively, induce and inhibit the formation of incoherent shock waves. We first present numerical simulations of the NLSE with the example of the damped harmonic oscillator response, which is known to model a great variety of nonlinear systems (e.g., the Raman effect), $\bar{R}(t) = [(1+\beta^2)/(\beta\tau_R)]\sin(\beta t/\tau_R)\exp(-t/\tau_R)$. Figure 1 reports a typical evolution of a broad initial spectrum of the incoherent wave, $\Delta\omega \gg \Delta\omega_g$ [$t_c \ll \tau_R$, see the inset in Fig. 1(b)]. The initial condition is a Gaussian-shaped spectrum with random spectral phases; i.e., $\psi(z = 0, t)$ is of zero mean and characterized by fluctuations that are

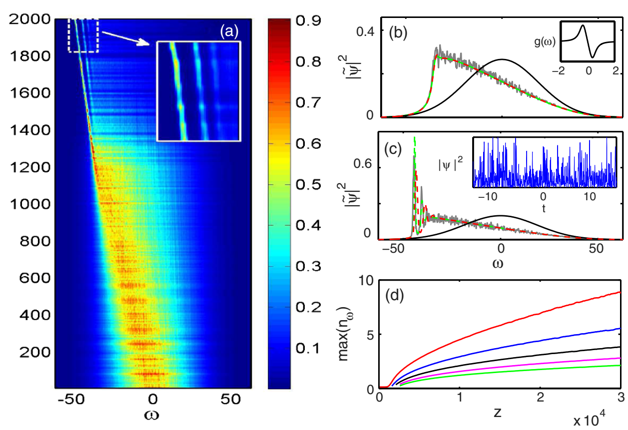

FIG. 1 (color online). (a) Numerical simulation of the NLSE (1): the stochastic spectrum $|\tilde{\psi}|^2(\omega, z)$ develops a DSW at $z \simeq 1200$ ($\tau_R = 3$, $\beta = 1$, $\sigma = 1$). Snapshots at $z = 1040$ (b), $z = 1400$ (c): NLSE outcome (tiny gray) is compared with KE [Eq. (2)] (green), SIDKE [Eq. (3)] (dashed red), and input (solid black). (d) First five maxima of n_ω vs z in the long-term postshock dynamics: the spectral peaks keep evolving, revealing the nonsolitonic nature of the incoherent DSW. Insets: (b) gain spectrum g_ω; note that $\Delta\omega_g \ll \Delta\omega$; (c) corresponding temporal profile $|\psi(t)|^2$ showing the incoherent wave with stationary statistics.

stationary in time. The broad spectrum exhibits a global collective deformation on a spectral scale much larger than $\Delta\omega_g$, which means that the system exhibits a kind of "long-range interaction in frequency space"[29]. This evolution features a pronounced self-steepening of the spectrum at low frequencies, with spectral wave breaking being ultimately regularized by the onset of fast large-amplitude spectral oscillations ("incoherent dispersive shock"). This behavior is reminiscent of the conventional phenomenon of DSWs studied for coherent wave amplitudes. Here, the DSW takes place within a genuine incoherent wave and manifests itself solely in the spectral domain, while in the temporal domain, the random field exhibits stationary fluctuations [see inset in Fig. 1(c)]. Importantly, the incoherent DSWs develop irrespective of the sign of $\sigma = \pm 1$ (i.e., in the focusing or defocusing regime), a feature which is consistent with the weakly nonlinear regime and the theory developed below. Note that the rapid oscillatory spectral wave train induced by the shock cannot be interpreted in this case as a soliton train. Indeed, contrary to the dynamics ruled by integrable models where the solitons quickly stabilize as they emerge [16,20], here the spectral peaks continue to exhibit an adiabatic growth and temporal narrowing even over long-term evolution after the catastrophe [see Fig. 1(d)].

Incoherent DSWs develop also in the presence of a spectral background and even when the spectrum is a hole, as shown in Fig. 2. In this case, features analogous to those widely studied for coherent DSWs [30,31] are exhibited by the incoherent wave in the spectral domain. The evolving spectrum gives rise indeed to an expansion (rarefaction) wave on the leading edge and a gradient catastrophe on the trailing edge, which is resolved by an expanding dispersive wave train, both features being fully captured by our theory based on SIDKEs.

In general, the experimental observation of these incoherent DSWs is readily accessible by exploiting the natural Raman effect in photonic crystal fibers pumped, e.g., by supercontinuum sources [32] (see the Supplemental Material [28]).

Singular integro-differential kinetic equations.—The incoherent wave evolves in the weakly nonlinear regime ($L_d \ll L_{\rm nl}$) and thus preserves a Gaussian statistics during

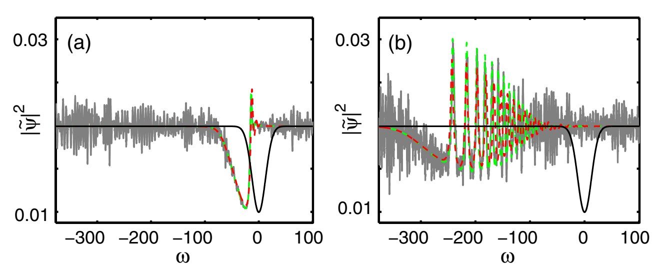

FIG. 2 (color online). Incoherent DSW from a darklike input spectrum with background noise (solid black) at (a) $z = 6 \times 10^3$ and (b) $z = 40 \times 10^3$. NLSE (1), tiny gray; KE (2), green; BO Eq. (4), dashed red. Here $\tau_R = 2$, $\beta = 1$, $\sigma = 1$.

the whole shock (a property that we have verified through the analysis of the kurtosis and the probability density function of the random wave). To get physical insight into incoherent DSWs, we thus resort to a weakly nonlinear statistical approach. On the basis of the random phase approximation, one obtains a closure of the hierarchy of moments equations [22,33]: the averaged spectrum of the wave, $n_\omega(z) = \int B(z,\tau)e^{-i\omega\tau}d\tau$, where $B(z,\tau) = \langle\psi(z, t-\tau/2)\psi^*(z, t+\tau/2)\rangle$ is the correlation function, evolves according to the kinetic equation (KE)

$$\partial_z n_\omega = \frac{1}{\pi} n_\omega \int_{-\infty}^{+\infty} g_{\omega-\omega'} n_{\omega'} d\omega'. \tag{2}$$

This equation conserves $N = \int n_\omega(z)d\omega$ and $S = \int \log[n_\omega(z)]d\omega$. It has been recently considered to describe spectral incoherent solitons in optics [22]. A similar equation was considered in plasma physics to describe weak Langmuir turbulence and stimulated Compton scattering [26,34,35].

To grasp the properties of incoherent DSWs, we now look for a reduction of the KE (2) in the regime $\tau_R \gg 1$ (i.e., $\tau_R \gg \tau_0$ in dimensional units). For this purpose, we stress the fact that, because of the causality condition of the response function $R(t)$, its Fourier transform necessarily decays algebraically at infinity [35,37]. In the example of the harmonic oscillator response, the spectral gain decays as $g_\omega \sim 1/\omega^3$, while for the exponential response discussed below $g_\omega \sim 1/\omega$. This algebraic behavior leads to divergent integrals in the mathematical derivation. We solve this tricky problem by accurately addressing the singularities involved in the convolution integral of the KE (2). The rigorous mathematical proof underlying the derivation of the SIDKEs is reported in detail in the Supplemental Material [28] for a general form of the response function. The theory reveals that, as a rule, a singular integro-differential operator arises systematically in the derivation of the reduced KE (SIDKE), though under different forms describing linear or nonlinear dispersive effects, as well as purely nonlinear effects. The theory is validated by the excellent agreement exhibited by the numerical simulations of the NLSE, KE, and SIDKEs obtained without using any adjustable parameters in all cases.

We first consider the example of the damped harmonic oscillator response. In this case, the KE (2) can be written without any approximations in the form of a SIDKE

$$\tau_R^2 \partial_z n_\omega = (1+\beta^2)\left(n_\omega \partial_\omega n_\omega - \frac{1}{\tau_R} n_\omega \mathcal{H}\partial_\omega^2 n_\omega + \frac{1}{\tau_R^2} I[n_\omega]\right), \tag{3}$$

where $I = [n_\omega/(\pi\beta)]\int_0^\infty [\partial_\omega^3 n_{\omega+u/\tau_R} + \partial_\omega^3 n_{\omega-u/\tau_R}]\times G(u)du$, with $G(u) = \int_u^\infty (F(v) - \beta/v)dv$, $F(u) = \pi\beta/2 - (1/2)[v\arctan(v) - (1/2)\log(1+v^2)]_{u-\beta}^{u+\beta}$, and the singular operator $\mathcal{H}$ refers to the Hilbert transform, $\mathcal{H}f(\omega) = \pi^{-1}\mathcal{P}\int_{-\infty}^{+\infty}(f(\omega-u)/u)du$, where $\mathcal{P}$ denotes the Cauchy principal value. In the regime $\tau_R \gg 1$, the SIDKE (3) describes the essence of incoherent DSWs shown in Fig. 1. The leading-order term is reminiscent of the inviscid Burgers equation and thus drives the formation of the shock. The singularity is subsequently regularized by the second nonlinear dispersive term involving the Hilbert operator. The last $(1/\tau_R^2)$ term plays a negligible role in the development of the incoherent shock, while it becomes comparable to the other terms in the long-term postshock dynamics [see Fig. 1(d) and the Supplemental Material [28]].

BO kinetic equation.—When the incoherent wave evolves in the presence of a significant background spectral noise, $n_\omega(z) = n_0 + \tilde{n}_\omega(z)$, a multiscale expansion with $\tilde{n}_\omega(z) \sim n_0/\tau_R$ leads to the SIDKE (see the Supplemental Material [28])

$$\tau_R^2 \partial_z \tilde{n}_\omega - (1+\beta^2)n_0\partial_\omega\tilde{n}_\omega = (1+\beta^2)\left(\tilde{n}_\omega\partial_\omega\tilde{n}_\omega - \frac{1}{\tau_R}n_0\mathcal{H}\partial_\omega^2\tilde{n}_\omega\right). \tag{4}$$

Equation (4) has the form of the integrable BO equation, which was originally derived to model internal waves in stratified fluids [19]. Here, it unexpectedly provides the deterministic description of the spectral dynamics of incoherent shocks illustrated in Figs. 2 and 3. In particular, the case illustrated in Fig. 3 is reminiscent of the "solitonic" DSW obtained for the BO equation with positive initial data vanishing at infinity [20] (as it is the case for $\tilde{n}_\omega = n_\omega - n_0$ at $z = 0$, in Fig. 3). Indeed, the peaks of the wave train evolve into genuine BO solitons (see the Supplemental Material [28]), which, at variance with those of the "nonsolitonic" DSW described by the nonintegrable SIDKE (3) and discussed in Fig. 1, are stabilized by the presence of the spectral background. From a more general perspective, the derivation of the deterministic SIDKEs paves the way for the analysis of spectral incoherent DSWs in a way analogous to conventional coherent DSWs. For instance, one can consider the Bohr-Sommerfeld limit of the inverse scattering [1,38] to study the distribution function for soliton amplitudes in the wave train by using the semiclassical quantization rule from the BO equation or by means of a modulation theory.

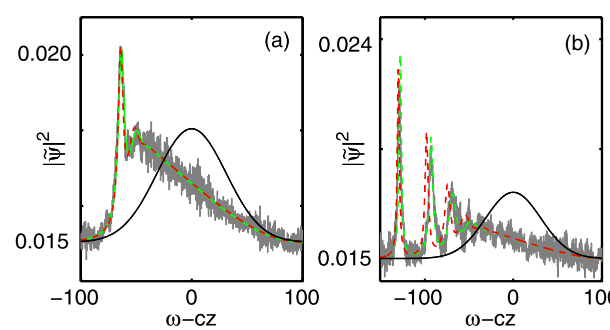

FIG. 3 (color online). (a), (b) Solitonic incoherent DSW in the presence of a background noise (harmonic oscillator response): the shock is regularized by the emission of incoherent BO solitons. NLSE (1), tiny gray; KE (2), green; BO Eq. (4), dashed red; input spectrum (solid black). [$\tau_R = 5$, $\beta = 1$, $\sigma = 1$, (a) $z = 3\times10^5$, (b) $z = 7\times10^5$].

We remark that, in the regime $\tau_R \gg 1$ and assuming that the wave spectrum evolves in the presence of a strong background noise, a heuristic derivation of the Korteweg–de Vries (KdV) equation was considered in the context of plasma physics [26,35,36], although the possible existence of shocks was not discussed there. A derivation taking into account the algebraic decay of g_ω reveals that the validity of the KdV equation is limited to a gain spectrum which decays faster than $g_\omega \sim 1/\omega^5$ as $\omega \to \pm\infty$ (see Ref. [33] and the Supplemental Material [28]). As discussed above, usual physical response functions do not lead to such rapidly decaying spectral gains. A decay faster than $g(\omega) \sim 1/\omega^5$ would require, e.g., an artificial response function of the form, $\bar{R}(t) \sim t^\nu \exp(-t/\tau_R)$ with $\nu \geq 3$ [37]. This reveals that the KdV equation is not appropriate to describe incoherent DSWs.

Inhibition of incoherent shocks.—The KdV equation would also lead to the erroneous conclusion that incoherent shocks occur unconditionally, irrespective of the form of the response function $R(t)$. In contrast, we show here that DSWs can be inhibited. We illustrate this by considering the widespread example of a purely exponential response function, $\bar{R}(t) = \exp(-t/\tau_R)/\tau_R$. In the regime $\tau_R \gg 1$, we obtain the SIDKE (see the Supplemental Material [28])

$$\tau_R \partial_z n_\omega = -n_\omega \mathcal{H} n_\omega - \frac{1}{\tau_R} n_\omega \partial_\omega n_\omega + \frac{1}{2\tau_R^2} n_\omega \mathcal{H} \partial_\omega^2 n_\omega. \quad (5)$$

Note that the second-order Burgers term produces a shock toward the high-frequency components ($\omega > 0$), so that the leading-order term is the only one liable to produce a shock.

The first term of the rhs of Eq. (5) was considered as a one-dimensional model of the vorticity formulation of the 3D Euler equation of incompressible fluid flows [18]. A detailed analysis of this first term reveals that it does not produce a shock (see Ref. [18] and the Supplemental Material [28]). If the initial condition is Lorentzian, $n_\omega^0 = N\omega_0/[\pi(\omega_0^2 + \omega^2)]$, then the spectrum propagates as a solitary-wave solution, $n_\omega(z) = N\omega_0/[\pi(\omega_0^2 + (\omega - \tilde{c}z)^2)]$ with $\tilde{c} = -N/(2\pi\tau_R)$. If the initial condition decays faster than a Lorentzian and vanishes at $\omega = \omega_c$, the spectrum exhibits a genuine collapse at $z_c = -2\tau_R/(\mathcal{H} n^0_{\omega=\omega_c})$ [18]. In this case, the last term in Eq. (5) will ultimately regularize the singularity. If the initial condition decays without ever vanishing, then the spectrum moves at velocity $\tilde{c}$, while its peak amplitude increases according to $\sim 4\tau_R^2/[z^2 n^0(\omega = \tilde{c}z)]$, a property remarkably confirmed by the simulations of the NLSE [see Fig. 4(a)].

Periodic behavior.—This collapselike behavior changes in a dramatic way when the incoherent wave evolves in the presence of a significant spectral background. A multiscale expansion with $n_\omega(z) = n_0 + \tilde{n}_\omega(z)$ and $\tilde{n}_\omega(z) \sim n_0/\tau_R$ (see the Supplemental Material [28]) leads to the SIDKE

$$\tau_R \partial_z \tilde{n}_\omega = -(n_0 + \tilde{n}_\omega)\mathcal{H}\tilde{n}_\omega - \frac{1}{\tau_R}(n_0 + \tilde{n}_\omega)\partial_\omega \tilde{n}_\omega + \frac{n_0}{2\tau_R^2}\mathcal{H}\partial_\omega^2 \tilde{n}_\omega. \quad (6)$$

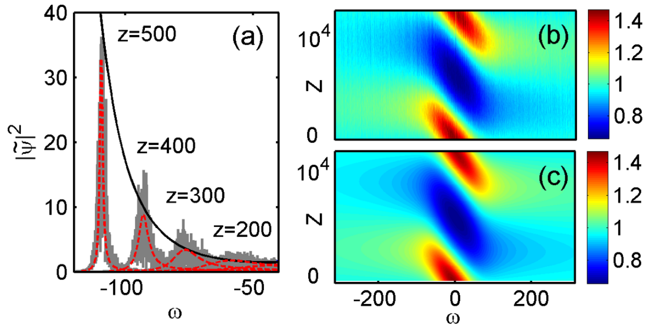

FIG. 4 (color online). Inhibition of an incoherent DSW with an exponential response function. (a) Without background, the spectrum exhibits a collapselike behavior: NLSE (1), tiny gray; SIDKE (5), dashed red ($\tau_R = 5$). The black line stands for the law $\sim 1/[z^2 n^0(\omega = \tilde{c}z)]$ predicted from the first term in Eq. (5), see text. (b), (c) A spectral background turns the evolution into periodic: (b) NLSE simulation; (c) plot of the analytical solution [Eq. (7)], $\tau_R = 2$.

Since $n_0 \gg \tilde{n}_\omega(z)$, the spectral dynamics of the incoherent wave is dominated by the first linear term in the rhs of Eq. (6), which admits the following analytical solution:

$$\tilde{n}_\omega(z) = \cos(n_0 z/\tau_R)\tilde{n}_\omega^0 - \sin(n_0 z/\tau_R)\mathcal{H}\tilde{n}_\omega^0, \quad (7)$$

where $\tilde{n}_\omega^0 = \tilde{n}_\omega(z = 0)$ refers to the initial condition. This unexpected periodic behavior of the incoherent spectrum has been found in quantitative agreement with the simulations of the whole SIDKE (6), as well as with those of the KE (2) and the NLSE (1), as remarkably illustrated in Figs. 4(b) and 4(c).

Conclusion.—We have reported the existence of incoherent DSWs, as well as collapselike and periodic behaviors in the spectral dynamics of incoherent waves. The great generality of our mathematical treatment suggests that these phenomenons could be found in a disparate area of nonlinear science. For instance, the discrete version of the KE (2) recovers a form of the Lotka-Volterra equation, $\partial_t n_j = n_j \sum_i g_{ji} n_i$ with $g_{ji} = -g_{ij}$, where $n_j(t)$ denotes the temporal evolution of the population of the jth biological species (or chemical reacting component) [27]. For a nearest-neighbor predator-prey interaction (which maps the limit $\tau_R \gg 1$), the numerical simulations indicate the existence of a "discrete shock" effect whose "wave breaking" is regularized by the formation of discrete oscillations (see the Supplemental Material [28]).

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[2] We warn the reader that we use the term DSW in this commonly accepted broad sense, though a more restrictive definition relevant, in particular, for the asymptotic study of the shock can be found in the literature, in which the

DSW refers to a dispersive wave train strictly linking two different fixed states at infinity.

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