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# HOMOGENIZATION IN A PERIODIC AND TIME-DEPENDENT POTENTIAL*

JOSSELIN GARNIER†

**Abstract.** This paper contains a study of the long time behavior of a diffusion process in a periodic potential. The first goal is to determine a suitable rescaling of time and space so that the diffusion process converges to some homogeneous limit. The issue of interest is to characterize the effective evolution equation. The main result is that in some cases large drifts must be removed in order to get a diffusive asymptotic behavior. This is applied to the homogenization of parabolic differential equations.

**Key words.** homogenization, partial differential equations in media with periodic structure, diffusion processes

**AMS subject classifications.** 35B27, 60J60

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## 1. Introduction.

In this paper we discuss homogenization of diffusion equations involving space-time periodic coefficients and present some refinements of the results obtained by Bensoussan, Lions, and Papanicolaou in [2]. We consider the scaling $x \mapsto x/\varepsilon$, $t \mapsto t/\varepsilon^p$, $0 < p < 2$. The martingale method that is used here is not really new (see [5], [8]). Indeed the aim of this paper is not the introduction of a new method but rather a complete and corrected treatment of the problem at hand. The main result is that, in some cases for $1 \le p < 2$, we must remove large drifts in order to get a diffusion limit. This corrects the corresponding results in [2], where the drifts were taken to be zero but are not so in general.

We shall use a probabilistic approach to study the convergence of the solutions of partial differential equations as in Chapter 3 of [2]. Indeed the solution of a parabolic differential equation may be represented as averages of functionals of the process solution of a stochastic differential equation (see [4]). One advantage of the probabilistic approach is to obtain pointwise convergence results. However, the price that has to be paid is some stringent regularity assumptions on the coefficients of the partial differential equations.

First we lay out the situation that will be studied in the paper. Let $b_i$ be periodic and regular enough functions and $a_{ij}$ be symmetric, periodic, strongly elliptic, and regular enough functions. (More precise assumptions will be given later.) Let $\mathcal{O}$ be an open bounded subset whose boundary $\Sigma$ is of class $\mathcal{C}^2$. Let $f$ be defined on $[0,T]\times\bar{\mathcal{O}}$ regular enough and $u_0$ be a continuous function such that $u_0 \mid_\Sigma = 0$. We are particularly interested in proving convergence results for the solutions of the partial differential equations of the type

$$(1) \qquad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + a_{ij}\left(\dfrac{t}{\varepsilon^p}, \dfrac{x}{\varepsilon}\right)\dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \dfrac{1}{\varepsilon} b_i\left(\dfrac{t}{\varepsilon^p}, \dfrac{x}{\varepsilon}\right)\dfrac{\partial u^\varepsilon}{\partial x_i} = f, \\ u^\varepsilon \mid_\Sigma = 0, \quad u^\varepsilon(T,x) = u_0(x), \end{cases}$$

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†Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France (garnier@paris.polytechnique.fr).

where we use here and throughout the paper the convention of summation upon repeated indices. $\varepsilon$ denotes a small parameter which characterizes the periods of the oscillations of the coefficients $a$ and $b$. We shall consider a parameter $p \in (0,2)$. The case $p = 0$ is a straightforward extension of the time-independent case. The case $p = 2$ (and $p > 2$) has been extensively studied in [2]. There exists a unique solution $u^\varepsilon \in \mathcal{C}^{1,2}((0,T) \times \mathcal{O}, \mathbb{R}) \bigcap \mathcal{C}^0([0,T] \times \bar{\mathcal{O}}, \mathbb{R})$ under nice regularity assumptions on the data. Moreover we can give a probabilistic representation for $u^\varepsilon(t,x)$. If $\sigma$ denotes a square root of $a$ (i.e., $a = \frac{1}{2}\sigma\sigma^*$), then there exists a unique continuous process $(Y^\varepsilon_{t,x}(s))_{s\in[t,T]}$ solution of the following stochastic differential equation:

$$dY^\varepsilon_{t,x}(s) = \sigma\left(\frac{s}{\varepsilon^p}, \frac{Y^\varepsilon_{t,x}}{\varepsilon}\right) dW_s + \frac{1}{\varepsilon} b\left(\frac{s}{\varepsilon^p}, \frac{Y^\varepsilon_{t,x}}{\varepsilon}\right) ds,\ Y^\varepsilon_{t,x}(t) = x, \tag{2}$$

where $W$ denotes a standard $d$-dimensional Brownian motion. If we introduce the stopping time $\tau^\varepsilon_{t,x} = \inf\{s \geq t, Y^\varepsilon_{t,x}(s) \notin \mathcal{O}\}$, then we have (see [4], [2], and references therein):

$$u^\varepsilon(t,x) = \mathbb{E}\left[\int_t^{T\wedge\tau^\varepsilon_{t,x}} f(s, Y^\varepsilon_{t,x}(s)) ds\right] + \mathbb{E}\left[u_0(Y^\varepsilon_{t,x}(T)) \mathbb{1}_{T\leq\tau^\varepsilon_{t,x}}\right]. \tag{3}$$

Bensoussan, Lions, and Papanicolaou have shown that $Y^\varepsilon_{t,x}$ converges weakly to a Brownian motion, whose diffusion coefficient may take different values depending on the parameter $p$. We shall show that some terms have been omitted, which may introduce a drift in the limit process. Moreover, using standard Dirichlet-type variational principles, we are able to give variational formulations for the effective coefficients of the limit equations which are not given in [2].

The paper is organized as follows. We consider first a simpler problem than (1). Namely, we state and prove our main results in sections 3–5 in the case of a diffusion in a periodic potential, where $a_{ij}(t,x) = \frac{1}{2}\delta_{ij}$ and $b_i(t,x) = -\frac{\partial V}{\partial x_i}(t,x)$ for some periodic function $V$. Generalizations are given in section 6. Finally in section 7 we consider a problem with random coefficients and prove convergence results for some particular time-space dependence. We shall point out in the proof of Theorem 7.1 that we could obtain the results of section 7 by means of another way which has been developed in a different context by Lebowitz and Rost [7].

**2. Notations and preliminaries.** We begin by introducing some notation. Throughout the paper $(W_t)_{t\geq 0}$ will denote a $d$-dimensional standard Wiener process defined on the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That means that $\Omega$ is the space of all continuous functions in $\mathcal{C}^0([0,\infty), \mathbb{R}^d)$ equipped with the topology generated by the uniform norm on compact subsets in $\mathbb{R}^d$, $\mathcal{F}$ is the Borel $\sigma$-algebra of $\Omega$, and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. We also introduce some function spaces. $\mathcal{C}^{k,l}([0,\infty) \times \mathbb{R}^d, \mathbb{R})$ denotes the set of all the functions $\psi$ of two variables $t$ and $y$ such that the partial derivatives $\frac{\partial^{i+j}\psi}{\partial t^i \partial y^j}$, $0 \leq i \leq k$, $0 \leq j \leq l$, exist and are continuous with respect to $(t,y)$. ($\frac{\partial^j \psi}{\partial y^j}$ is a shorthand for the tensor of the $j$-order derivatives of $\psi$). $\mathcal{P}^{k,l}$ denotes the set of all the functions $\psi$ from $[0,\infty) \times \mathbb{R}^d$ into $\mathbb{R}$ such that

- $\psi \in \mathcal{C}^{k,l}$;
- $y \mapsto \psi(t,y)$ is periodic with period 1 for every $t$;
- $t \mapsto \psi(t,y)$ is periodic with period $T_0$ for every $y$.

For every $\psi \in \mathcal{P}^{k,l}$, we denote $\|\psi\|_0 = \sup_{y\in[0,1],t\in[0,T_0]} |\psi(t,y)|$ and $\|\psi\|_{m,n} = \sum_{i=0}^m \sum_{j=0}^n \|\frac{\partial^{i+j}\psi}{\partial t^i \partial y^j}\|_0$, where $0 \leq m \leq k$ and $0 \leq n \leq l$. $\mathcal{P}^{k,l}$ is equipped with the topology associated with the norm $\|.\|_{k,l}$. We denote by $\nabla\psi$ the partial derivatives with respect to the variable $y$:

$$\nabla\psi = \begin{pmatrix} \dfrac{\partial\psi}{\partial y_1} \\ \vdots \\ \dfrac{\partial\psi}{\partial y_d} \end{pmatrix}.$$

We aim at outlining here the ergodic properties of diffusions in a periodic potential $v$. We consider here the case where $v$ does not depend on time $t$. More exactly, we assume that $v$ is a periodic $\mathcal{C}^2$ function from $\mathbb{R}^d$ into $\mathbb{R}$ of period 1. (We can think of $v$ as a smooth function on the torus $\mathbf{S}^d$ of length 1.) Let $y_x(s)$ be the solution of the stochastic differential equation

$$dy_x(s) = dW_s - \nabla v(y_x(s))\, ds, \qquad y_x(0) = x. \tag{4}$$

The infinitesimal generator of this Markov process is given by $L = \frac{1}{2}\Delta - \nabla v(y).\nabla$. Let us define the projection on the torus $\mathbf{S}^d \simeq \mathbb{R}^d/\mathbb{Z}^d$:

$$x \in \mathbb{R}^d \mapsto \dot{x} = x \bmod 1 \in \mathbf{S}^d.$$

Here and below points in $\mathbb{R}^d$ are denoted by $x$ (or $y$), and points in $\mathbf{S}^d$ by $\dot{x}$ (or $\dot{y}$). The process $\dot{y}_{\dot{x}}(s)$ is Markov. Its generator is $L$ with the domain restricted to the periodic functions with period 1. This process admits therefore a unique invariant probability measure $m$ whose density with respect to the Lebesgue measure over $\mathbf{S}^d$ is

$$m(\dot{y}) = \frac{e^{-2v(\dot{y})}}{\int_{\mathbf{S}^d} e^{-2v(\dot{x})} d\dot{x}}. \tag{5}$$

It has very nice mixing properties, and its generator admits an inverse on the subspace of functions centered with respect to the measure $m$.

PROPOSITION 2.1. *Let $\phi \in L^\infty(\mathbf{S}^d)$ such that $\int_{\mathbf{S}^d} \phi m d\dot{y} = 0$.*

- *There exists a unique (up to a constant) function $\chi$ which belongs to $W^{2,p}(\mathbf{S}^d)$ for any $p \geq 1$ such that $L\chi = \phi$. In addition $\chi$ satisfies*

$$\|\chi - \int_{\mathbf{S}^d} \chi m d\dot{y}\|_\infty \leq C\|\phi\|_\infty, \tag{6}$$

*where $C$ is a constant which depends only on $\|\nabla v\|_\infty$.*

- *The following variational formula holds*:

$$\int_{\mathbf{S}^d} \phi\chi m d\dot{y} = -2 \sup_{\psi \in \mathcal{C}^1(\mathbf{S}^d)} \left\{ 2\int_{\mathbf{S}^d} \phi\psi m d\dot{y} - \int_{\mathbf{S}^d} \|\nabla\psi\|^2 m d\dot{y} \right\}. \tag{7}$$

*Proof.* The first statement is a straightforward corollary of Theorems III-3-2 and 3-3 in [2], whose key argument is the spectral gap of the generator $L$. Let us now establish the variational formula (7). We denote $\frac{1}{4}\int_{\mathbf{S}^d} \|\nabla\chi\|^2 m d\dot{y}$ by $C^2$.

Let $\psi \in \mathcal{C}^1(\mathbf{S}^d)$. Since $\int_{\mathbf{S}^d} (L\chi)\psi m d\dot{y} = -\frac{1}{2}\int_{\mathbf{S}^d} \nabla\chi.\nabla\psi m d\dot{y}$ for any $\psi \in \mathcal{C}^1(\mathbf{S}^d)$, we can deduce from the Schwarz inequality that

$$\left|\int_{\mathbf{S}^d} \phi\psi m d\dot{y}\right| \leq \frac{1}{2}\left(\int_{\mathbf{S}^d} \|\nabla\chi\|^2 m d\dot{y}\right)^{\frac{1}{2}} \left(\int_{\mathbf{S}^d} \|\nabla\psi\|^2 m d\dot{y}\right)^{\frac{1}{2}} = C\left(\int_{\mathbf{S}^d} \|\nabla\psi\|^2 m d\dot{y}\right)^{\frac{1}{2}}.$$

As a consequence,

$$\sup_{\psi\in\mathcal{C}^1(\mathbf{S}^d)}\left\{2\int_{\mathbf{S}^d}\phi\psi m d\dot{y}-\int_{\mathbf{S}^d}\|\nabla\psi\|^2 m d\dot{y}\right\}\le\sup_{a\ge 0}\left\{2C\sqrt{a}-a\right\}=C^2.$$

The conclusion follows, since the fact that $-\frac{1}{2}\chi\in\mathcal{C}^1(\mathbf{S}^d)$ implies

$$\sup_{\psi\in\mathcal{C}^1(\mathbf{S}^d)}\left\{2\int_{\mathbf{S}^d}\phi\psi m d\dot{y}-\int_{\mathbf{S}^d}\|\nabla\psi\|^2 m d\dot{y}\right\}\ge-\int_{\mathbf{S}^d}\phi\chi m d\dot{y}-\frac{1}{4}\int_{\mathbf{S}^d}\|\nabla\chi\|^2 m d\dot{y}$$
$$=\frac{1}{4}\int_{\mathbf{S}^d}\|\nabla\chi\|^2 m d\dot{y}=C^2. \qquad \square$$

**3. Main results.** Let us regard now a time-dependent problem. We consider a time-space periodic potential $V$. Similarly to (5) we introduce the probability measure $m(t,.)$ on the torus $\mathbf{S}^d$ whose density is

$$m(t,\dot{y})=\frac{e^{-2V(t,\dot{y})}}{\int_{\mathbf{S}^d}e^{-2V(t,\dot{x})}d\dot{x}}.$$

In view of Proposition 2.1 we obtain the following result.

PROPOSITION 3.1. *If $V\in\mathcal{P}^{1,2}$ and $\ell$ is a unit vector in $\mathbb{R}^d$, then there exists a unique function $\chi_1^\ell\in\mathcal{P}^{1,2}$ such that, for every $t\in[0,\infty)$, $L_t\chi_1^\ell(t,.)=-\ell.\nabla V(t,.)$ and $\int_{\mathbf{S}^d}\chi_1^\ell(t,\dot{y})m(t,\dot{y})d\dot{y}=0$, where $L_t=\frac{1}{2}\Delta-\nabla V(t,.).\nabla$.*

Let us introduce the effective coefficients (diffusivity and drift):

$$\alpha^\ell(t)=\int_{\mathbf{S}^d}\|\ell-\nabla\chi_1^\ell(t,\dot{y})\|^2 m(t,\dot{y})\,d\dot{y},\quad \alpha^\ell=\frac{1}{T_0}\int_{t=0}^{T_0}\alpha^\ell(t)\,dt, \tag{8}$$

$$\beta_1^\ell(t)=-\int_{\mathbf{S}^d}\frac{\partial\chi_1^\ell}{\partial t}(t,\dot{y})m(t,\dot{y})\,d\dot{y},\quad \beta_1^\ell=\frac{1}{T_0}\int_{t=0}^{T_0}\beta_1^\ell(t)\,dt. \tag{9}$$

Since $\alpha^\ell$ (resp., $\beta_1^\ell$) is a symmetric quadratic form (resp., a linear form) in $\ell$, there exists a unique matrix $\alpha$ (resp., a vector $\beta_1$) such that $\alpha^\ell=\ell.\alpha\ell$ (resp., $\beta_1^\ell=\ell.\beta_1$).

PROPOSITION 3.2. *Let $\ell$ be a unit vector in $\mathbb{R}^d$.*

(1) *$\alpha^\ell(V)$ and $\beta_1^\ell(V)$ defined by* (8) *and* (9) *are continuous mappings from $\mathcal{P}^{1,2}$ into $\mathbb{R}$.*

(2) *The functions $V$ such that $\beta_1(V)\neq 0$ constitute a nonempty open set of $\mathcal{P}^{1,2}$.*

(3) *The following variational formula for the coefficient $\alpha^\ell$ holds*:

$$\alpha^\ell=\inf_{\psi\in\mathcal{P}^{1,2}}\frac{1}{T_0}\int_{t=0}^{T_0}\int_{\mathbf{S}^d}\|\ell-\nabla\psi(t,\dot{y})\|^2 m(t,\dot{y})d\dot{y}dt. \tag{10}$$

(4) *The potential $V$ reduces the diffusion. Indeed,*

$$\frac{1}{T_0}\int_0^{T_0}\frac{1}{\int_{\mathbf{S}^d}e^{-2V(t,\dot{y})}d\dot{y}\int_{\mathbf{S}^d}e^{2V(t,\dot{y})}d\dot{y}}dt\le\alpha^\ell\le 1. \tag{11}$$

The proof of this proposition is deferred to Appendix A. A simple example where $\beta_1\neq 0$ will be given later in this section. We are now able to state the following theorems that will be proven in section 4.

THEOREM 3.3. *If $V\in\mathcal{P}^{1,2}$ and $Y^\varepsilon$ is solution of the stochastic differential equation*

$$dY^\varepsilon(t) = dW_t - \frac{1}{\varepsilon}\nabla V\left(\frac{t}{\varepsilon^p}, \frac{Y^\varepsilon(t)}{\varepsilon}\right)dt, \quad Y^\varepsilon(0) = 0, \tag{12}$$

*then the following convergences hold in* $\mathcal{C}^0([0,\infty), \mathbb{R}^d)$:

- *If* $0 < p < 1$, *then* $Y^\varepsilon$ *converges weakly to the process* $\alpha^{1/2}W_t$.
- *If* $p = 1$, *then* $Y^\varepsilon$ *converges weakly to the process* $\alpha^{1/2}W_t + \beta_1 t$.
- *If* $1 < p < 2$, *then* $\varepsilon^{p-1}Y^\varepsilon$ *converges in probability to the process* $\beta_1 t$.

We refer to section 5 for the study of the process $Y^\varepsilon(t) - \beta_1\varepsilon^{1-p}t$ when $p > 1$.

THEOREM 3.4. *If* $V \in \mathcal{P}^{1,2}$ *and* $u^\varepsilon$ *is the solution of the partial differential equation*

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_i} - \dfrac{1}{\varepsilon}\dfrac{\partial V}{\partial x_i}\left(\dfrac{t}{\varepsilon^p}, \dfrac{x}{\varepsilon}\right)\dfrac{\partial u^\varepsilon}{\partial x_i} = f, \\ u^\varepsilon \mid_\Sigma = 0, \quad u^\varepsilon(T,x) = u_0(x), \end{cases} \tag{13}$$

*where* $f \in L^q((0,T)\times\mathcal{O})$ *and* $u_0 \in W^{2,q}(\mathcal{O}) \cap W_0^{1,q}(\mathcal{O})$, $q > \frac{d}{2}+1$. *Then the following statements hold for every* $(t,x) \in [0,T]\times\bar{\mathcal{O}}$:

- *If* $0 < p < 1$, *then* $u^\varepsilon(t,x)$ *converges to* $u(t,x)$ *solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\alpha_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f, \\ u \mid_\Sigma = 0, \quad u(T,x) = u_0(x). \end{cases}$$

- *If* $p = 1$, *then* $u^\varepsilon(t,x)$ *converges to* $u(t,x)$ *solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\alpha_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \beta_{1i}\dfrac{\partial u}{\partial x_i} = f, \\ u \mid_\Sigma = 0, \quad u(T,x) = u_0(x). \end{cases}$$

- *If* $1 < p < 2$ *and* $\beta_1(V) \neq 0$, *then* $u^\varepsilon$ *admits no limit as* $\varepsilon \to 0$, *except in the trivial case where* $u_0$ *and* $f$ *do not depend on* $x$.

Finally we consider a particular case where the study of the effective drift $\beta_1$ and diffusivity $\alpha$ can be performed more precisely. Let $v$ be a periodic $\mathcal{C}^2$ function from $\mathbb{R}^d$ into $\mathbb{R}$ with period 1 and $\varrho$ be some vector in $\mathbb{R}^d$. We consider the time-dependent periodic potential $V$ given by

$$V(t,x) = v(x - \varrho t). \tag{14}$$

In order to ensure that $V$ is strictly periodic with respect to $t$, we would have to assume that no couple of coordinates of the vector $\varrho$ are incommensurable. In fact it can be easily proven that this is irrelevant for our purpose, and we get the results of Theorems 3.3 and 3.4 with any vector $\varrho$.

PROPOSITION 3.5. *Let* $\alpha$ *be the diffusion matrix and* $\beta_1$ *be the drift associated with the potential* $V$ *given by* (14).

(1) *The matrix* $\alpha$ *is a symmetric and diagonalizable matrix whose eigenvalues belong to* $[\gamma_1, \gamma_2] \subset [0,1]$. *Moreover,*

- $\gamma_1 \geq \dfrac{1}{\int_{\mathbf{S}^d} e^{-2v(\dot{y})}d\dot{y} \int_{\mathbf{S}^d} e^{+2v(\dot{y})}d\dot{y}}$;
- $\gamma_2 \leq 1$ *and* $\gamma_2 = 1$ *if and only if there exists a vector* $\ell$ *such that* $\ell.\nabla v$ *is identically zero.*

(2) *The drift vector* $\beta_1$ *is equal to* $(I_d - \alpha)\varrho$.

In particular the drift $\beta_1$ is different from zero if $\varrho.\nabla v$ is not identically zero. It means that there exists a residual drift as soon as the potential $v$ acts in the direction of $\varrho$. The proof of the proposition is easy and deferred to Appendix B.

By Theorem 3.4 and Proposition 3.5 we are now able to give some new results concerning parabolic partial differential equations. For instance let us consider the evolution problem

$$\left\{\begin{array}{l}\dfrac{\partial X^\varepsilon}{\partial t} + b^\varepsilon(x).\nabla X^\varepsilon(x) = \dfrac{\nu}{2}\Delta X^\varepsilon,\\ X^\varepsilon(0,x) = X_0(x),\end{array}\right. \tag{15}$$

where $b^\varepsilon(x) = \frac{1}{\varepsilon}\nabla v(\frac{x}{\varepsilon}) + \varrho$, $v$ is some periodic function, and $\varrho \in \mathbb{R}^d$. We first regard the behavior of the solution near the place where the "wind" $\varrho$ has led it. Namely, we consider $X^\varepsilon_\varrho(t,x) = X(t, x+\varrho t)$, which is the solution of

$$\left\{\begin{array}{l}\dfrac{\partial X^\varepsilon_\varrho}{\partial t} + \dfrac{1}{\varepsilon}\nabla v\left(\dfrac{x+\varrho t}{\varepsilon}\right).\nabla X^\varepsilon_\varrho(x) = \dfrac{\nu}{2}\Delta X^\varepsilon_\varrho,\\ X^\varepsilon_\varrho(0,x) = X_0(x).\end{array}\right. \tag{16}$$

Applying a slight modification of Theorem 3.4 to $X^\varepsilon_\varrho$ yields the convergence of this function, from which we can deduce that $X^\varepsilon$ pointwise converges to $X$ solution of

$$\left\{\begin{array}{l}\dfrac{\partial X}{\partial t} + (\alpha^\nu\varrho)_i\dfrac{\partial X}{\partial x_i} = \dfrac{\nu}{2}\alpha^\nu_{ij}\dfrac{\partial^2 X}{\partial x_i x_j},\\ X\mid_\Sigma = 0, \quad X(0,x) = X_0(x),\end{array}\right. \tag{17}$$

where $\ell.\alpha^\nu\ell$ is given by

$$\ell.\alpha^\nu\ell = \inf_{\psi\in\mathcal{C}^1(\mathbf{S}^d)} \frac{\displaystyle\int_{\mathbf{S}^d}\|\ell - \nabla\psi(\dot y)\|^2 e^{-\frac{2v}{\nu}(\dot y)}d\dot y}{\displaystyle\int_{\mathbf{S}^d} e^{-\frac{2v}{\nu}(\dot y)}d\dot y}.$$

The striking point is that the drift has been affected. We shall study this problem in a general random context in section 7.

**4. Proof.** This section is devoted to the proofs of Theorems 3.3 and 3.4. We first state a standard tightness criterion in $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ (see Theorem VI-4-1 in [4]).

PROPOSITION 4.1. *Let $x^\varepsilon$ be a family of processes with paths in $\mathcal{C}^0([0,T],\mathbb{R}^d)$ such that $x^\varepsilon(0) = 0$. The family is tight if for any $T > 0$ and $\delta > 0$*

$$\lim_{h\to 0}\sup_{\varepsilon>0}\mathbb{P}\left(\sup_{0\le s<t\le T, |t-s|\le h}\|x^\varepsilon(t) - x^\varepsilon(s)\| > \delta\right) = 0. \tag{18}$$

Now we reduce the problem to the study of the long time behavior of a diffusion process. We claim that, when $p < 2$, the diffusion process $Y^\varepsilon$ given by (12) can be regarded as the long time behavior of the diffusion process $y^\varepsilon$ defined by

$$dy^\varepsilon(t) = dw_t - \nabla V(\varepsilon t, y^\varepsilon(t))dt, \quad y^\varepsilon(0) = 0, \tag{19}$$

where $w_.$ is the auxiliary Brownian motion $\varepsilon^{-\theta/2}W_{\varepsilon^\theta .}$ and $\theta \in (1,\infty)$ is a parameter derived from $p$ through the formula $\theta = (1-p/2)^{-1}$. Indeed, it is easy to check that

$\varepsilon^{\theta/2}y^\varepsilon(\frac{\cdot}{\varepsilon^\theta})$ is equal to $Y^{\varepsilon^{\theta/2}}(.)$. As a consequence, a weak convergence result for the process $\varepsilon^{\theta/2}y^\varepsilon(\frac{\cdot}{\varepsilon^\theta})$ will imply a weak convergence result for the process $Y^\varepsilon(.)$ and, in view of the representation (3), a pointwise convergence result for the function $u^\varepsilon$. We shall then discuss the long time behavior of $y^\varepsilon$. By Itô's formula we can write

$$\chi_1^\ell(\varepsilon t, \dot y^\varepsilon(t)) = \chi_1^\ell(0,0) + \int_0^t \nabla\chi_1^\ell(\varepsilon s, \dot y^\varepsilon(s)).dw_s + \int_0^t \left(\varepsilon\frac{\partial\chi_1^\ell}{\partial t} + L_{\varepsilon s}\chi_1^\ell\right)(\varepsilon s, \dot y^\varepsilon(s))ds.$$

Since $L_t\chi_1^\ell(t,\dot y) = -\ell.\nabla V(t,\dot y)$, we have for any $\theta > 1$

$$-\int_0^{\frac{t}{\varepsilon^\theta}} \ell.\nabla V(\varepsilon s, \dot y^\varepsilon(s))ds = \chi_1^\ell\left(\frac{t}{\varepsilon^{\theta-1}}, \dot y^\varepsilon\left(\frac{t}{\varepsilon^\theta}\right)\right) - \chi_1^\ell(0,0) - \int_0^{\frac{t}{\varepsilon^\theta}} \nabla\chi_1^\ell(\varepsilon s, \dot y^\varepsilon(s)).dw_s - \varepsilon\int_0^{\frac{t}{\varepsilon^\theta}} \frac{\partial\chi_1^\ell}{\partial t}(\varepsilon s, \dot y^\varepsilon(s))ds. \tag{20}$$

The following lemma is a slight modification of Theorem III-10-2 in [2], where the same result is obtained in the case where $\psi$ is assumed to belong to $\mathcal{P}^{1,1}$.

LEMMA 4.2. *If $\psi$ is a function in $\mathcal{P}^{0,1}$ that satisfies $\int_{\mathbf{S}^d}\psi(t,\dot y)m(t,\dot y)d\dot y = 0$ for any $t$, then for any $\theta > 1$ and for every $\varepsilon \in (0,1)$ and $t_1 \geq 0$, $t_0 \geq 0$*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left|\varepsilon^\theta\int_{\frac{t_0}{\varepsilon^\theta}}^{\frac{t_0+t_1}{\varepsilon^\theta}} \psi(\varepsilon s, \dot y^\varepsilon(s))ds\right|^2 / \mathcal{F}_{\frac{t_0}{\varepsilon^2}}\right] = 0. \tag{21}$$

*Proof.* By Proposition 2.1 there exists a unique function $\xi \in \mathcal{P}^{0,2}$ such that, for every $t \in [0,\infty)$, $L_t\xi(t,.) = \psi(t,.)$ and $\int_{\mathbf{S}^d}\xi(t,\dot y)m(t,\dot y)d\dot y = 0$. By Itô's formula we have for any $t' \geq t \geq 0$

$$\xi(\varepsilon t, \dot y^\varepsilon(t')) = \xi(\varepsilon t, \dot y^\varepsilon(t)) + \int_t^{t'} \nabla\xi(\varepsilon t, \dot y^\varepsilon(s)).dw_s + \int_t^{t'} L_{\varepsilon t}\xi(\varepsilon t, \dot y^\varepsilon(s))ds. \tag{22}$$

We shall regard the integral in (21) over intervals which are small compared to the macroscopic scale $O(\varepsilon^{-\theta})$ but large compared to the microscopic scale $O(1)$. Let $M$ be an integer. We consider the partition of the macroscopic interval $[\frac{t_0}{\varepsilon^\theta}, \frac{t_0+t_1}{\varepsilon^\theta})$ which is constituted by the intervals $[\tau_k, \tau_{k+1})$ for $k = 0, \ldots, M-1$, where $\tau_k = \frac{t_0}{\varepsilon^\theta} + \frac{kt_1}{M\varepsilon^\theta}$. From (22) it follows that

$$\int_{\tau_k}^{\tau_{k+1}} \psi(\varepsilon\tau_k, \dot y^\varepsilon(s))ds = \xi(\varepsilon\tau_{k+1}, \dot y^\varepsilon(\tau_{k+1})) - \xi(\varepsilon\tau_k, \dot y^\varepsilon(\tau_k)) - \int_{\tau_k}^{\tau_{k+1}} \nabla\xi(\varepsilon\tau_k, \dot y^\varepsilon(s)).dw_s.$$

Taking the square conditional expectation and using the Burkholder's inequality for the martingale term, then summing over the $M$ steps by Minkowski's inequality, we get

$$\mathbb{E}\left[\left|\int_{\frac{t_0}{\varepsilon^\theta}}^{\frac{t_0+t_1}{\varepsilon^\theta}} \psi(\varepsilon s, \dot y^\varepsilon(s))ds\right|^2 / \mathcal{F}_{\frac{t_0}{\varepsilon^2}}\right]^{\frac{1}{2}} \leq 2M\|\xi\|_0 + \frac{t_1^{\frac{1}{2}}}{\varepsilon^{\frac{\theta}{2}}}\|\xi\|_{0,1} + \frac{t_1}{\varepsilon^\theta}\sup_{|s-t|\leq\frac{t_1}{M\varepsilon^\theta}}\|\psi(\varepsilon s,.) - \psi(\varepsilon t,.)\|_\infty.$$

Optimizing with respect to $M$ and choosing $M \simeq \varepsilon^{1/2-\theta}$, we obtained the desired result. ∎

**4.1. Case $1 < \theta \leq 2$ (i.e., $0 < p \leq 1$).** We denote $\varepsilon^{\theta/2}\left(y^\varepsilon(\frac{t}{\varepsilon^\theta}) - \beta_1 \frac{t}{\varepsilon^{\theta-1}}\right)$ by $\bar{Y}^\varepsilon(t)$. The corrective drift $\varepsilon^{\theta/2}\beta_1 \frac{t}{\varepsilon^{\theta-1}}$ goes to 0 as $\varepsilon \to 0$ when $\theta < 2$ and is equal to $\beta_1 t$ when $\theta = 2$. Using (20) and (19), we get that $\ell.\bar{Y}^\varepsilon$ satisfies

(23)

$$\ell.\bar{Y}^\varepsilon(t) = \varepsilon^{\frac{\theta}{2}}\left(\chi_1^\ell\left(\frac{t}{\varepsilon^{\theta-1}}, \dot{y}^\varepsilon\left(\frac{t}{\varepsilon^\theta}\right)\right) - \chi_1^\ell(0,0)\right) + \varepsilon^{\frac{\theta}{2}}\left(-\beta_1^\ell \frac{t}{\varepsilon^{\theta-1}} + \int_0^{\frac{t}{\varepsilon^{\theta-1}}} \beta_1^\ell(s)ds\right)$$
$$+ \varepsilon^{1-\frac{\theta}{2}}\left(b_\ell^\varepsilon(t) - \varepsilon^{\theta-1}\int_0^{\frac{t}{\varepsilon^{\theta-1}}} \beta_1^\ell(s)ds\right) + M_\ell^\varepsilon(t),$$

where $b_\ell^\varepsilon$ is the drift term given by

$$b_\ell^\varepsilon(t) = -\varepsilon^\theta \int_0^{\frac{t}{\varepsilon^\theta}} \frac{\partial \chi_1^\ell}{\partial t}(\varepsilon s, \dot{y}^\varepsilon(s))ds \tag{24}$$

and $M_\ell^\varepsilon$ is the continuous martingale

$$M_\ell^\varepsilon(t) = \varepsilon^{\frac{\theta}{2}} \int_0^{\frac{t}{\varepsilon^\theta}} \left(\ell - \nabla\chi_1^\ell\right)(\varepsilon s, \dot{y}^\varepsilon(s)).dw_s. \tag{25}$$

The first and second terms in (23) clearly go to 0 uniformly as $\varepsilon \to 0$. Since the family of processes $(b_\ell^\varepsilon(.) - \varepsilon^{\theta-1}\int_0^{\frac{\cdot}{\varepsilon^{\theta-1}}} \beta_1^\ell(s)ds)_{\varepsilon>0}$ fulfills the criterion (18), it is tight in $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$; moreover, by Lemma 4.2 the finite-dimensional distributions converge to 0. Thus the third term in (23) weakly goes to 0 in $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ since $1-\theta/2 \geq 0$. As a consequence $\ell.\bar{Y}^\varepsilon$ is the sum of the continuous martingale $M_\ell^\varepsilon$ whose quadratic variation is

$$\langle\langle M_\ell^\varepsilon\rangle\rangle_t = \varepsilon^\theta \int_0^{\frac{t}{\varepsilon^\theta}} \|\ell - \nabla\chi_1^\ell\|^2(\varepsilon s, \dot{y}^\varepsilon(s))ds$$

and of terms that go to 0 in probability uniformly with respect to $t$ as $\varepsilon \to 0$. Applying the Burkholder's inequality we find immediately that $M_\ell^\varepsilon$ satisfies the tightness criterion (18). Applying Lemma 4.2, we get that the quadratic variation of the martingale $M_\ell^\varepsilon$ converges to the deterministic function $\alpha^\ell t$. Therefore, using a standard martingale central limit theorem (see Theorem III-10-2 in [2]), we find that the continuous martingale $M_\ell^\varepsilon$ converges weakly to $\sqrt{\alpha^\ell}w^1_\cdot$, where $w^1$ is a standard one-dimensional Brownian motion.

**4.2. Case $\theta > 2$ (i.e., $1 < p < 2$).** We denote $\varepsilon^{\theta-1}y^\varepsilon(\frac{t}{\varepsilon^\theta})$ by $\tilde{Y}^\varepsilon(t)$. Using (20) and (19), we get that $\ell.\tilde{Y}^\varepsilon$ satisfies

$$\ell.\tilde{Y}^\varepsilon(t) = \varepsilon^{\theta-1}\left(\chi_1^\ell\left(\frac{t}{\varepsilon^{\theta-1}}, \dot{y}^\varepsilon\left(\frac{t}{\varepsilon^\theta}\right)\right) - \chi_1^\ell(0,0)\right) + b_\ell^\varepsilon(t) + \varepsilon^{\frac{\theta}{2}-1}M_\ell^\varepsilon(t).$$

$\ell.\tilde{Y}^\varepsilon$ is the sum of the drift $b_\ell^\varepsilon$ given by (24) and of terms that go to 0 in probability uniformly with respect to $t$. The process $b_\ell^\varepsilon$ is tight in $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ since it obviously satisfies the criterion (18). On the other hand, by Lemma 4.2 the finite-dimensional distributions of $b_\ell^\varepsilon$ converge to the finite-dimensional distributions of the process $\beta_1 t$. This yields the weak convergence of $b_\ell^\varepsilon$. Besides, since the limit process is a deterministic function, the convergence at hand holds in probability.

**5. Further normalizations.** We can look for a more precise long time behavior of the diffusion process $y^\varepsilon$ defined by (19), provided that the potential $V$ is differentiable enough with respect to $t$.

Let us assume that $V \in \mathcal{P}^{2,2}$. Since $\frac{\partial \chi_1^\ell}{\partial t}(t,.) + \beta_1^\ell(t)$ is centered with respect to the invariant measure $m(t,.)$ of the generator $L_t$ for any $t$, there exists a unique function $\chi_2^\ell \in \mathcal{P}^{1,2}$ such that, for every $t \in [0,\infty)$,

$$L_t\chi_2^\ell(t,.) = -\frac{\partial \chi_1^\ell}{\partial t}(t,.) - \beta_1^\ell(t) \text{ and } \int_{\mathbf{S}^d} \chi_2^\ell(t,\dot{y})m(t,\dot{y})d\dot{y} = 0.$$

Hence we can introduce the coefficients $\beta_2^\ell(t)$, $\beta_2^\ell$, and $\beta_2$ defined by

$$\beta_2^\ell(t) = -\int_{\mathbf{S}^d} \frac{\partial \chi_2^\ell}{\partial t}(t,\dot{y})m(t,\dot{y})d\dot{y}, \quad \beta_2^\ell = \frac{1}{T_0}\int_{t=0}^{T_0} \beta_2^\ell(t)dt, \quad \ell.\beta_2 = \beta_2^\ell. \tag{26}$$

We are now able to state the following proposition.

PROPOSITION 5.1. *If $V \in \mathcal{P}^{2,2}$ and $Y^\varepsilon$ is solution of the stochastic differential equation* (12), *then the following convergences hold in $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$:*

- *If $1 \le p < 3/2$, then the process $Y^\varepsilon(t) - \beta_1\varepsilon^{1-p}t$ converges weakly to $\alpha^{1/2}W_t$.*
- *If $p = 3/2$, then the process $Y^\varepsilon(t) - \beta_1\varepsilon^{-1/2}t$ converges weakly to $\alpha^{1/2}W_t + \beta_2 t$.*

*Proof.* We shall proceed as in section 4. By Itô's formula we can write

$$\begin{aligned}
(\chi_1^\ell + \varepsilon\chi_2^\ell)(\varepsilon t, \dot{y}^\varepsilon(t)) &= (\chi_1^\ell + \varepsilon\chi_2^\ell)(0,0) \\
&\quad + \int_0^t L_{\varepsilon s}\chi_1^\ell(\varepsilon s, \dot{y}^\varepsilon(s))ds + \int_0^t \nabla\chi_1^\ell(\varepsilon s, \dot{y}^\varepsilon(s)).dw_s \\
&\quad + \varepsilon\int_0^t \frac{\partial \chi_1^\ell}{\partial t}(\varepsilon s, \dot{y}^\varepsilon(s))ds + \varepsilon\int_0^t L_{\varepsilon s}\chi_2^\ell(\varepsilon s, \dot{y}^\varepsilon(s))ds \\
&\quad + \varepsilon\int_0^t \nabla\chi_2^\ell(\varepsilon s, \dot{y}^\varepsilon(s)).dw_s + \varepsilon^2\int_0^t \frac{\partial \chi_2^\ell}{\partial t}(\varepsilon s, \dot{y}^\varepsilon(s))ds.
\end{aligned}$$

Since $L_t\chi_1^\ell(t,\dot{y}) = -\ell.\nabla V(t,\dot{y})$ and $\frac{\partial \chi_1^\ell}{\partial t}(t,\dot{y}) + L_t\chi_2^\ell(t,\dot{y}) = -\beta_1^\ell(t)$, we have for any $\theta > 1$

$$\begin{aligned}
-\int_0^{\frac{t}{\varepsilon^\theta}} \ell.\nabla V(\varepsilon s, \dot{y}^\varepsilon(s))ds &= (\chi_1^\ell + \varepsilon\chi_2^\ell)\left(\frac{t}{\varepsilon^{\theta-1}}, \dot{y}^\varepsilon\left(\frac{t}{\varepsilon^\theta}\right)\right) - (\chi_1^\ell + \varepsilon\chi_2^\ell)(0,0) \\
&\quad - \int_0^{\frac{t}{\varepsilon^\theta}} \nabla\chi_1^\ell(\varepsilon s, \dot{y}^\varepsilon(s)).dw_s + \varepsilon\int_0^{\frac{t}{\varepsilon^\theta}} \beta_1^\ell(s)ds \\
&\quad - \varepsilon\int_0^{\frac{t}{\varepsilon^\theta}} \nabla\chi_2^\ell(\varepsilon s, \dot{y}^\varepsilon(s)).dw_s - \varepsilon^2\int_0^{\frac{t}{\varepsilon^\theta}} \frac{\partial \chi_2^\ell}{\partial t}(\varepsilon s, \dot{y}^\varepsilon(s))ds.
\end{aligned}$$

We denote $\varepsilon^{\frac{\theta}{2}}\left(y^\varepsilon(\frac{t}{\varepsilon^\theta}) - \beta_1\frac{t}{\varepsilon^{\theta-1}} - \beta_2\frac{t}{\varepsilon^{\theta-2}}\right)$ by $\hat{Y}^\varepsilon(t)$. Combining (19) with the last relation, we get that $\ell.\hat{Y}^\varepsilon$ satisfies

$$\begin{aligned}
\ell.\hat{Y}^\varepsilon(t) &= \varepsilon^{\frac{\theta}{2}}\left((\chi_1^\ell + \varepsilon\chi_2^\ell)\left(\frac{t}{\varepsilon^{\theta-1}}, \dot{y}^\varepsilon\left(\frac{t}{\varepsilon^\theta}\right)\right) - (\chi_1^\ell + \varepsilon\chi_2^\ell)(0,0)\right) \\
&\quad - \varepsilon\left(\varepsilon^{\frac{\theta}{2}}\int_0^{\frac{t}{\varepsilon^\theta}} \nabla\chi_2^\ell(\varepsilon s, \dot{y}^\varepsilon(s)).dw_s\right) + \varepsilon^{\frac{\theta}{2}+1}\left(-\beta_1^\ell\frac{t}{\varepsilon^\theta} + \int_0^{\frac{t}{\varepsilon^\theta}} \beta_1^\ell(s)ds\right) \\
&\quad - \varepsilon^{2-\frac{\theta}{2}}\left(\varepsilon^\theta\int_0^{\frac{t}{\varepsilon^\theta}} \frac{\partial \chi_2^\ell}{\partial t}(\varepsilon s, \dot{y}^\varepsilon(s))ds + \beta_2^\ell t\right) + M_\ell^\varepsilon(t).
\end{aligned}$$

If $1 < \theta \leq 4$ (i.e., $0 < p \leq 3/2$), then $\ell.\hat{Y}^\varepsilon$ is the sum of the continuous martingale $M_\ell^\varepsilon$ given by (25) and of terms that go in probability to 0 as $\varepsilon \to 0$. The completion of the proof is the same as in section 4. □

Finally, repeating these arguments, if $V$ is smooth enough, we can introduce the functions $\chi_m^\ell$ and the coefficients $\beta_m^\ell$ defined recursively by

$$L_t \chi_m^\ell(t,.) = -\frac{\partial \chi_{m-1}^\ell}{\partial t}(t,.) - \beta_{m-1}^\ell(t), \quad \int_{\mathbf{S}^d} \chi_m^\ell(t,\dot{y}) m(t,\dot{y}) d\dot{y} = 0,$$

$$\beta_m^\ell(t) = -\int_{\mathbf{S}^d} \frac{\partial \chi_m^\ell}{\partial t}(t,\dot{y}) m(t,\dot{y}) d\dot{y}, \quad \beta_m^\ell = \frac{1}{T_0}\int_{t=0}^{T_0} \beta_m^\ell(t) dt, \quad \beta_m^\ell = \ell.\beta_m,$$

and we can state the following theorem.

THEOREM 5.2. *If $V \in \mathcal{P}^{k,2}$, $k \geq 1$, and $Y^\varepsilon$ is solution of the stochastic differential equation* (12), *then the following convergences hold in $\mathcal{C}^0([0,\infty), \mathbb{R}^d)$:*

- *If $1 \leq p < \frac{2k-1}{k}$, then the process $Y^\varepsilon(t) - \beta_1 \varepsilon^{1-p} t - \cdots - \beta_{k-1}\varepsilon^{(2k-3)-(k-1)p} t$ converges weakly to $\alpha^{1/2} W_t$.*
- *If $p = \frac{2k-1}{k}$, then the process $Y^\varepsilon(t) - \beta_1 \varepsilon^{-1+1/k} t - \cdots - \beta_{k-1}\varepsilon^{-1/k} t$ converges weakly to $\alpha^{1/2} W_t + \beta_k t$.*

## 6. Generalizations.

**6.1. Time dependence.** We have assumed in the previous sections that the potential $V$ was periodic with respect to the time variable $t$. This limitation is in fact far too restrictive, and we can easily extend the results of Theorems 3.3 and 3.4 to cases where $V$ is ergodic with respect to the time in the sense that the limits

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbf{S}^d} (I_d - \nabla\chi_1)(I_d - \nabla\chi_1)^*(t,\dot{y}) m(t,\dot{y}) d\dot{y} dt,$$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbf{S}^d} \frac{\partial \chi_1}{\partial t}(t,\dot{y}) m(t,\dot{y}) d\dot{y} dt$$

exist. Then they play the role of $\alpha$ and $-\beta_1$, respectively. Further we can extend our results to processes which are random and ergodic with respect to $t$.

*Example* 6.1. We consider the case where the potential depends on time through a random process $z_t(\tilde{\omega})$, where $z_t$ is an Orstein–Uhlenbeck process defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Namely, $z_t$ is a real-valued Markov process independent of the Brownian motion $W$ whose generator $L$ is given by $\frac{1}{2}\frac{\partial^2}{\partial z^2} - z\frac{\partial}{\partial z}$. $L$ admits therefore a unique invariant probability measure $p$, whose density with respect to the Lebesgue measure over $\mathbb{R}$ is $p(z) = (1/\sqrt{\pi})\exp(-z^2)$. Furthermore $z_t$ is ergodic; i.e., for any continuous and bounded function $F$,

$$\frac{1}{T}\int_0^T F(z_t) dt \underset{T\to\infty}{\longrightarrow} \int_{-\infty}^{+\infty} F(z) p(z) dz \quad \tilde{\mathbb{P}}\text{-almost surely.}$$

We consider here the random potential $V$ defined on $[0,\infty) \times \mathbb{R}^d \times \tilde{\Omega}$ and given by $V(t,y,\tilde{\omega}) = v(z_t(\tilde{\omega}), y)$, where $v$ is a $\mathcal{C}_b^{2,2}(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ function which is periodic with respect to the second variable. Proceeding as in the previous sections, we denote

$$m(z,\dot{y}) = \frac{e^{-2v(z,\dot{y})}}{\int_{\mathbf{S}^d} e^{-2v(z,\dot{x})} d\dot{x}}.$$

There exists a unique function $\chi_1 \in \mathcal{C}_b^{2,2}(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^d)$ such that $y \mapsto \chi_1(z,y)$ is periodic for every $z$ and satisfies

$$\frac{1}{2}\frac{\partial^2 \chi_1^j}{\partial y_i y_i} - \frac{\partial v}{\partial y_i}\frac{\partial \chi_1^j}{\partial y_i} = -\frac{\partial v}{\partial y_j} \quad \text{and} \quad \int_{\mathbf{S}^d} \chi_1^j(z,\dot{y})m(z,\dot{y})d\dot{y} \equiv 0 \text{ for any } j = 1,\dots,d.$$

Then we are able to define the coefficients $\alpha$ and $\beta_1$:

$$\alpha = \int_{-\infty}^{+\infty} \left( \int_{\mathbf{S}^d} (I_d - \nabla\chi_1(z,\dot{y}))\,(I_d - \nabla\chi_1(z,\dot{y}))^*\, m(z,\dot{y})d\dot{y} \right) p(z)dz,$$

$$\beta_1 = -\int_{-\infty}^{+\infty} \left( \int_{\mathbf{S}^d} L\chi_1(z,\dot{y})m(z,\dot{y})d\dot{y} \right) p(z)dz.$$

We get back the results of Theorem 3.3 with the random potential $V$ if we interpret the convergences of the diffusion processes according to the following way: for any $\delta > 0$, for any bounded and continuous functional $F$ from $\mathcal{C}^0([0,\infty),\mathbb{R}^d)$ into $\mathbb{R}$,

$$\lim_{\varepsilon\to 0} \tilde{\mathbb{P}}\left(\tilde{\omega} \text{ such that } |\mathbb{E}[F(Y_{\tilde{\omega}}^{\varepsilon})] - \mathbb{E}[F(Y)]| \geq \delta\right) = 0.$$

As a consequence the conclusions of Theorem 3.4 hold. The convergence of the solution $u^\varepsilon$ of the partial differential equation (13) with the random potential $V(t,y,\tilde{\omega})$ should be understood in the following sense:

$$\text{for any } (t,x), \text{ for any } \delta > 0,\ \lim_{\varepsilon\to 0} \tilde{\mathbb{P}}\left(|u^\varepsilon(t,x,\tilde{\omega}) - u(t,x)| \geq \delta\right) = 0.$$

**6.2. Locally periodic coefficients.** We aim at generalizing the results of section 3 with locally periodic coefficients. Namely, we consider a potential $V(t,x,y)$ which belongs to $\mathcal{C}_b^{1,2,2}([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R})$ and which is periodic with respect to the variable $y$ and the variable $t$. We want to prove a convergence result for the solution of the evolution problem

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_i} - \dfrac{1}{\varepsilon}\dfrac{\partial V}{\partial y_i}\left(\dfrac{t}{\varepsilon^p}, x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial u^\varepsilon}{\partial x_i} = f, \\ u^\varepsilon\ |_{\Sigma} = 0, \quad u^\varepsilon(T,x) = u_0(x), \end{cases} \tag{27}$$

with the same notations as in Theorem 3.4. We begin by introducing the generator $L_{t,x}$ defined on the space of periodic $\mathcal{C}^2$ functions by $\frac{1}{2}\Delta_y - \nabla_y V(t,x,.).\nabla_y$.The invariant probability measure associated to the generator $L_{t,x}$ admits a density with respect to the Lebesgue measure given by

$$m(t,x,\dot{y}) = \frac{e^{-2V(t,x,\dot{y})}}{\int_{\mathbf{S}^d} e^{-2V(t,x,\dot{z})}d\dot{z}}.$$

By Proposition 2.1 there exists a unique periodic function $\chi_1$ in $\mathcal{C}^{1,2,2}([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d)$ which satisfies the equation $L_{t,x}\chi_1(t,x,.) = -\nabla_y V(t,x,.)$ and the centering condition $\int_{\mathbf{S}^d}\chi_1(t,x,\dot{y})m(t,x,\dot{y})d\dot{y} \equiv 0$. Then we are able to define the coefficients $\alpha$, $\beta_1$, and $\gamma$ so that we can state Theorem 6.2:

$$\alpha(x) = \frac{1}{T_0}\int_0^{T_0}\int_{\mathbf{S}^d} (I_d - \nabla_y\chi_1)\,(I_d - \nabla_y\chi_1)^*\,(t,x,\dot{y})m(t,x,\dot{y})d\dot{y}dt,$$

$$\beta_1(x) = -\frac{1}{T_0}\int_0^{T_0}\int_{\mathbf{S}^d} \frac{\partial\chi_1}{\partial t}(t,x,\dot{y})m(t,x,\dot{y})d\dot{y}dt,$$

$$\gamma(x) = \frac{1}{T_0}\int_0^{T_0}\int_{\mathbf{S}^d} \frac{\partial V}{\partial y_i}\frac{\partial\chi_1}{\partial x_i}(t,x,\dot{y})m(t,x,\dot{y})d\dot{y}dt.$$

THEOREM 6.2. *If $u^\varepsilon$ is solution of the partial differential equation* (27), *then the following statements hold for every $(t,x) \in [0,T] \times \bar{\mathcal{O}}$:*

- *If $0 < p < 1$, then $u^\varepsilon(t,x)$ converges to $u(t,x)$ solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\alpha_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \gamma_i(x)\dfrac{\partial u}{\partial x_i} = f, \\ u\mid_\Sigma = 0, \quad u(T,x) = u_0(x). \end{cases}$$

- *If $p = 1$, then $u^\varepsilon(t,x)$ converges to $u(t,x)$ solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\alpha_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} + \left(\beta_{1i}(x) + \gamma_i(x)\right)\dfrac{\partial u}{\partial x_i} = f, \\ u\mid_\Sigma = 0, \quad u(T,x) = u_0(x). \end{cases}$$

We could also consider a reiterated homogenization problem. However, it would not furnish deeper results than the study of the problem with locally periodic coefficients, so we mention only the fact that a reiterated problem with time-dependent coefficients can be dealt with the previous method and gives the same type of results (appearance of a new drift when $p = 1$).

**6.3. Nonconstant diffusive terms.** We have assumed in the previous sections that the diffusive term was constant and that the drift coefficient was the gradient of some periodic function $V$. In fact we can generalize our results to problems (1) under more general assumptions. Let us assume that the coefficients $a_{ij}$ belong to $\mathcal{P}^{1,2}$, are symmetric, and satisfy a strict ellipticity condition, i.e.,

(H1) there exists some $\gamma > 0$ such that, for any $(t,x)$, $\sum_{ij} a_{ij}(t,x)\xi_i\xi_j \geq \gamma \sum_i \xi_i^2$.
Under these conditions, there exists a matrix $\sigma(t,x)$ which is symmetric, positive definite such that $a = \frac{1}{2}\sigma^2$. As a consequence we can express the solution of the partial differential equations (1) according to (2) and (3). We put now

$$L_t = a_{ij}(t,.)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(t,.)\frac{\partial}{\partial x_i}.$$

From the elliptic theory (see [3]), there exists a unique positive function $m$ in $\mathcal{P}^{1,2}$ such that $L_t^* m(t,.) \equiv 0$ and $\int_{\mathbf{S}^d} m(t,\dot{y})d\dot{y} \equiv 1$ for any $t$. If we assume that

(H2) $\int_{\mathbf{S}^d} b_i(t,\dot{y})m(t,\dot{y})d\dot{y} = 0$ for any $t$ and $i$,
then there exists unique functions $\chi_1^i$ in $\mathcal{P}^{1,2}$ which satisfy $L_t\chi_1^i = b_i$ and the centering conditions $\int_{\mathbf{S}^d} \chi_1^i(t,\dot{y})m(t,\dot{y})d\dot{y} \equiv 0$. All the conclusions of Theorems 3.3 and 3.4 hold if we denote

$$\alpha = \frac{2}{T_0}\int_0^{T_0}\int_{\mathbf{S}^d}(I_d - \nabla\chi_1)a(I_d - \nabla\chi_1)^*(t,\dot{y})m(t,\dot{y})d\dot{y}dt,$$

$$\beta_1 = -\frac{1}{T_0}\int_0^{T_0}\int_{\mathbf{S}^d}\frac{\partial \chi_1}{\partial t}(t,\dot{y})m(t,\dot{y})d\dot{y}dt.$$

*Remark* 6.3. Condition (H2) is obviously fulfilled if $a = \frac{1}{2}I_d$ and $b = -\nabla V$ for some $V \in \mathcal{P}^{1,2}$.

*Remark* 6.4. In the self-adjoint case (i.e., $b_i = \frac{\partial a_{ij}}{\partial x_j}$) we have $m \equiv 1$. Consequently (H2) is satisfied and $\beta_1$ is equal to 0 (as are $\beta_m$, $m \geq 1$). We get back the well-known homogenization results in this case.

**7. Random coefficients.** We consider the advection-diffusion of a passive scalar quantity (we may think at temperature) by a velocity field. The problem is particularly interesting and difficult when the field admits a statistical description (see [1]). There exist two space-scales:

• the microscopic scale, which is the typical length scale associated with the random fluctuations of the field;

• the macroscopic scale, which is the typical length scale associated with the variations of the smooth initial data.

To articulate the last feature we introduce a small parameter $\varepsilon$ so that the initial data are $T_0(\varepsilon x)$. We assume also that the deterministic component of the field has a small amplitude compared to that of the fluctuations. More exactly we are considering a velocity field of the type $\varepsilon\varrho + F(x)$, where $\varrho$ is a fixed and homogeneous wind and $F(x)$ is a random, centered, stationary and ergodic process. As a result the quantity $T^\varepsilon$ satisfies

$$\begin{cases} \dfrac{\partial T^\varepsilon}{\partial t} + \left(\varepsilon\varrho + F(x)\right).\nabla T^\varepsilon = \dfrac{1}{2}\nu_0 \Delta T^\varepsilon, \\ T^\varepsilon\mid_{t=0} = T_0(\varepsilon x), \end{cases}$$

We regard the natural large time large distance scaling $t \mapsto t/\varepsilon^2$, $x \mapsto x/\varepsilon$. We aim at determining the effective evolution equation of the macroscopic-scale varying quantity $T^\varepsilon(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ as $\varepsilon \to 0$.

We begin with a precise expression of the random field. We introduce the probability space $(X, \mathcal{G}, \mu)$, with $\eta$ labeling the realization of the field $F$. We assume that on $(X, \mathcal{G}, \mu)$ a group of measure-preserving transformations $\{\tau_x, x \in \mathbb{R}^d\}$ acts ergodically. If $f \in L^2(\mu)$, for almost every $\eta$ we define $T_x f(\eta) = f(\tau_{-x}\eta)$. Assuming that $T_x$ is stochastically continuous, we get that $T_x$ forms a strongly continuous unitary group of operators on $L^2(\mu)$ whose infinitesimal generator $D$ is closed and has a dense domain $\mathcal{D}(D)$ in $L^2(\mu)$:

$$\text{for any } f \in \mathcal{D}(D),\ D_i f(\eta) = \partial_{x_i} f(\tau_{-x}\eta)\mid_{x=0}.$$

We shall assume in the following that the field $F$ belongs to $\mathcal{D}(D)$ and is bounded together with its first derivatives $D_i F$. We denote $F(x,\eta) = F(\tau_{-x}\eta)$.

THEOREM 7.1.

• *If $F$ is a force field which derives from a bounded and stationary potential $V$, then for any $(t,x)$, the random quantity $T^\varepsilon_\eta(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ converges in $\mu$-probability to $T(t,x)$, the solution of*

$$\begin{cases} \dfrac{\partial T}{\partial t} + (\alpha\varrho).\nabla T = \dfrac{1}{2}\nu_0 \alpha_{ij} \dfrac{\partial^2 T}{\partial x_i \partial x_j}, \\ T\mid_{t=0} = T_0(x). \end{cases}$$

*$\alpha$ is a positive matrix whose norm is less than $1$. It means that the effective diffusivity and the effective drift are weaker than the microscopic ones.*

• *If $F$ is an incompressible velocity field (i.e., a free divergence field) and its orthogonal gradient (i.e., a skewsymmetric matrix $H_{ij}$ such that $F_j = D_i H_{ij}$) is bounded, then for any $(t,x)$, the random quantity $T^\varepsilon_\eta(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ converges in $\mu$-probability to $T(t,x)$, the solution of*

$$\begin{cases} \dfrac{\partial T}{\partial t} + \varrho.\nabla T = \dfrac{1}{2}\nu_0 \alpha_{ij} \dfrac{\partial^2 T}{\partial x_i \partial x_j}, \\ T\mid_{t=0} = T_0(x). \end{cases}$$

*$\alpha$ is a positive matrix whose smallest eigenvalue is larger than 1. In this case the diffusivity has been enhanced, while the drift has not been affected by the homogenization.*

The conclusions of the second case are well known. I have stated them in order to underline the differences between the free divergence case and the potential case. Namely, the main point of the first case is the modification of the invariant measure by the exponential of the potential which generates a drift. We shall sketch only the proof. The first step consists of representing the solution $T^\varepsilon$ using a $d$-dimensional Brownian motion $W$:

$$T^\varepsilon_\eta\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) = \mathbf{E}\left[T_0(X^\varepsilon_{\eta,x}(t))\right],$$

where $\mathbf{E}$ stands for the averaging with respect to the Brownian motion and $X^\varepsilon_{\eta,x}$ is the solution of the stochastic differential equation

$$dX^\varepsilon_{\eta,x}(t) = \sqrt{\nu_0}dW_t - \varrho dt - \frac{1}{\varepsilon}F\left(\frac{X^\varepsilon_{\eta,x}(t)}{\varepsilon}\right)dt, \quad X^\varepsilon_{\eta,x}(0) = x.$$

A proof of the weak convergence of $X^\varepsilon_{\eta,x}$ can be performed using Girsanov's formula. This method has been used in another context in [7]. It consists of passing from the problem ($\varrho = 0$) to the problem ($\varrho \neq 0$) through a change of measures. However, the arguments developed in our paper in the periodic case can be successfully adapted to study the convergence of $X^\varepsilon_{\eta,0}$. Indeed, if we denote by $Y^\varepsilon_\eta$ the translated process $Y^\varepsilon_\eta(t) = X^\varepsilon_{\eta,0}(t) + \varrho t$, then $Y^\varepsilon_\eta$ is solution of

$$dY^\varepsilon_\eta(t) = \sqrt{\nu_0}dW_t - \frac{1}{\varepsilon}F\left(\frac{Y^\varepsilon_\eta(t) - \varrho t}{\varepsilon}\right)dt, \quad Y^\varepsilon_\eta(0) = 0.$$

We aim to look at the ergodic properties of the process $\eta^\varepsilon(t) = \tau_{-\varepsilon^{-1}Y^\varepsilon_\eta(\varepsilon^2 t)}\eta$, which is the random environment seen from $\varepsilon^{-1}Y^\varepsilon_\eta(\varepsilon^2 t)$. As in the periodic case the key argument is the resolution of the equation $L\chi = -F$, where $L = \frac{\nu_0}{2}D_iD_i - F_iD_i$. Unfortunately the equation $L\chi = -F$ has no solution in general. Indeed, in the random case we have neither the help of the elliptic theory nor information on the spectral gap of the generator. Then the strategy consists of regarding the solution of the resolvent equation $(\lambda - L)\chi_\lambda = F$ and in studying the limit $\lambda \to 0$. Using the same symmetry properties of the generator as in [6], we obtain the following proposition.

PROPOSITION 7.2. (1) *$L$ admits an invariant probability measure $d\pi(\eta)$, which is the translation-invariant measure $d\mu(\eta)$ in the free divergence case, and $\frac{1}{Z}\exp(-\frac{2V(\eta)}{\nu_0})d\mu(\eta)$ in the potential case, where $Z$ is the normalization constant $\int\exp(-\frac{2V}{\nu_0})d\mu$.*

(2) *For any $\lambda > 0$ and any $j = 1, \dots, d$, let $\chi_{\lambda,j}$ be the solution of the resolvent equation $(\lambda - L)\chi_{\lambda,j} = -F_j$. Then for any $j = 1, \dots, d$,*

$$\lim_{\lambda\to 0}\lambda\langle{\chi_{\lambda,j}}^2\rangle_\pi = 0,$$

*and for any $i, j = 1, \dots, d$, there exists an element $\xi_{i,j}$ in $L^2(\pi)$ such that*

$$\lim_{\lambda\to 0}\langle(D_i\chi_{\lambda,j} - \xi_{i,j})^2\rangle_\pi = 0.$$

As a result, expanding $\chi_{\varepsilon^2}(-\frac{\varrho t}{\varepsilon}, \eta^\varepsilon(\frac{t}{\varepsilon^2}))$ by Itô's formula and applying the relation $\frac{\partial \chi_{\varepsilon^2}}{\partial t}(-\varepsilon\varrho t, \eta) = -\varepsilon(\varrho.D)\chi_{\varepsilon^2}(-\varepsilon\varrho t, \eta)$, we get

$$Y_\eta^\varepsilon(t) = \varepsilon\left(\chi_{\varepsilon^2}\left(-\frac{\varrho t}{\varepsilon}, \eta^\varepsilon\left(\frac{t}{\varepsilon^2}\right)\right) - \chi_{\varepsilon^2}(\eta)\right) + \varepsilon^2 \int_0^{\frac{t}{\varepsilon^2}} (\varrho.D)\chi_{\varepsilon^2}(-\varepsilon\varrho s, \eta^\varepsilon(s))ds \\ + \varepsilon\sqrt{\nu_0}\int_0^{\frac{t}{\varepsilon^2}} (I_d - D\chi_{\varepsilon^2})(-\varepsilon\varrho s, \eta^\varepsilon(s))dw_s, \tag{28}$$

where we have denoted by $w_.$ the auxiliary Brownian motion $\varepsilon^{-1}W_{\varepsilon^2 .}$. We can note that (28) looks like (23) (with $\theta = 2$) in section 4. Namely, taking into account the $L^2$-convergences of $\varepsilon\chi_{\varepsilon^2}$ (resp., $D\chi_{\varepsilon^2}$) to 0 (resp., $\xi$) and combining the ergodic theorem with the martingale central limit theorem we obtain the convergence of the finite-dimensional distributions. In particular the drift converges to $\langle\varrho.\xi\rangle_\pi t$. In the free divergence case this residual drift is equal to 0, because $\pi$ is the translation-invariant measure $\mu$ which satisfies $\langle D\chi\rangle_\mu = 0$ for any $\chi \in L^2(\mu)$; hence $\langle\varrho.\xi\rangle_\mu = \lim_{\lambda\to 0}\langle\varrho.D\chi_\lambda\rangle_\mu = 0$. On the other hand, in the potential case, the invariant measure $\pi$ is not translation invariant. Then we can check that the statements of Proposition 3.5 are still valid if we replace the Lebesgue measure over the torus $\mathbf{S}^d$ with the translation invariant measure $\mu$.

The proof of the tightness in $\mathcal{C}^0([0,\infty), \mathbb{R}^d)$ relies on some uniform estimates (of the type of Nash's estimate) of the transition probability density of the Markov process $Y_\eta^\varepsilon$.

## 8. Appendix.

**8.1. Appendix A.** The aim of this appendix is to prove Proposition 3.2. Let us begin by showing that the mappings $V \mapsto \alpha(V)$ and $V \mapsto \beta_1(V)$ are continuous.

*Proof.* Let $V \in \mathcal{P}^{1,2}$. Since $\chi_1^\ell m$ is periodic with respect to $t$, we can express $\beta_1^\ell$ through $\chi_1^\ell$ by means of an integration by parts:

$$\beta_1^\ell = \frac{1}{T_0}\int_{t=0}^{T_0}\int_{\mathbf{S}^d} \chi_1^\ell \frac{\partial m}{\partial t}(t, \dot{y})d\dot{y}dt. \tag{29}$$

On the other hand, since $\int_{\mathbf{S}^d} \|\nabla\chi_1^\ell\|^2 m d\dot{y} = -2\int_{\mathbf{S}^d} \chi_1^\ell L\chi_1^\ell m d\dot{y} = \int_{\mathbf{S}^d} \ell.\nabla\chi_1^\ell m d\dot{y}$, we have

$$\int_{\mathbf{S}^d} \|\ell - \nabla\chi_1^\ell\|^2 m d\dot{y} = 1 - \int_{\mathbf{S}^d} \ell.\nabla\chi_1^\ell m d\dot{y}, \tag{30}$$

from which we deduce a new expression of the diffusion constant:

$$\alpha^\ell = 1 - \frac{2}{T_0}\int_{t=0}^{T_0}\int_{\mathbf{S}^d} \chi_1^\ell \ell.\nabla V m(t, \dot{y})d\dot{y}dt. \tag{31}$$

Let us fix some positive real $M$. Let $\tilde{V}$ be some function in $\mathcal{P}^{1,2}$ such that $\|\tilde{V}\|_{1,1} \le M$. In the remainder of the proof, $C_i$ will denote constants which depend only on $V$ and $M$. From (29), (31), and analogous expressions for the corresponding coefficients $\tilde{\alpha}^\ell$, $\tilde{\beta}_1^\ell$ associated with $\tilde{V}$, we obtain

$$\begin{aligned}|\alpha^\ell - \tilde{\alpha}^\ell| + |\beta_1^\ell - \tilde{\beta}_1^\ell| &\le \left\|\chi_1^\ell\frac{\partial m}{\partial t} - \tilde{\chi}_1^\ell\frac{\partial \tilde{m}}{\partial t}\right\|_0 + \|\chi_1^\ell \ell.\nabla V m - \tilde{\chi}_1^\ell \ell.\nabla\tilde{V}\tilde{m}\|_0 \\ &\le C_1\|V - \tilde{V}\|_{1,1} + C_2\|\chi_1^\ell - \tilde{\chi}_1^\ell\|_0,\end{aligned}$$

where $C_1$ is derived from the bound of $\|\chi_1^\ell\|_0$ given in Proposition 2.1. It remains to show a suitable estimation of $\|\chi_1^\ell - \tilde{\chi}_1^\ell\|_0$. If we denote $\chi_1^\ell(t,.) - \tilde{\chi}_1^\ell(t,.)$ by $\Psi_t(.)$, then $\Psi_t$ is solution of $\tilde{L}_t\Psi_t = \xi_t$, where $\xi_t(.) = (-\ell + \nabla\chi_1^\ell(t,.)).(\nabla\tilde{V}(t,.) - \nabla V(t,.))$. Using Proposition 2.1, we get that

$$\left\|\Psi_t - \int_{\mathbf{S}^d} \Psi_t(\dot{x})\tilde{m}(t,\dot{x})d\dot{x}\right\|_\infty \leq C(\|\nabla\tilde{V}\|_0)\|\xi_t\|_\infty \leq C_3\|V - \tilde{V}\|_{0,1}. \tag{32}$$

However, $\chi_1^\ell$ (resp., $\tilde{\chi}_1^\ell$) is centered under the measure $m$ (resp., $\tilde{m}$). This implies the relation $\int_{\mathbf{S}^d}(\chi_1^\ell - \tilde{\chi}_1^\ell)\tilde{m}(t,\dot{x})d\dot{x} = \int_{\mathbf{S}^d}\chi_1^\ell(\tilde{m} - m)(t,\dot{x})d\dot{x}$, from which we get that

$$\left|\int_{\mathbf{S}^d} \Psi_t(\dot{x})\tilde{m}(t,\dot{x})d\dot{x}\right| \leq C_4\|V - \tilde{V}\|_0. \tag{33}$$

Combining (32) and (33) gives an estimation of $\|\chi_1^\ell - \tilde{\chi}_1^\ell\|_0$, which yields the result. ◻

Let us prove now the last two statements of Proposition 3.2. Let $\psi \in \mathcal{P}^{1,2}$. We fix some $t$ and forget it in the notation. Expanding the square Euclidian norm $\|\ell - \nabla\psi\|^2$ and integrating by parts yield

$$\int_{\mathbf{S}^d} \|\ell - \nabla\psi\|^2 m d\dot{y} = 1 - 4\int_{\mathbf{S}^d}(\ell.\nabla V)\psi m d\dot{y} + \int_{\mathbf{S}^d}\|\nabla\psi\|^2 m d\dot{y}.$$

Now, applying the variational formula (7) with $\chi = -2\chi_1^\ell$ and $\phi = 2\ell.\nabla V$, which is centered under the invariant measure $m$, we get

$$\inf_{\psi\in\mathcal{C}^1(\mathbf{S}^d)} \int_{\mathbf{S}^d} \|\ell - \nabla\psi\|^2 m d\dot{y} = 1 - 2\int_{\mathbf{S}^d}(\ell.\nabla V)\chi_1^\ell m d\dot{y} = 1 - \int_{\mathbf{S}^d}\ell.\nabla\chi_1^\ell m d\dot{y}. \tag{34}$$

Combining (30) and (34) completes the proof of the variational formula (10). It remains to show (11). On the one hand the upper bound is obvious: it suffices to take $\psi \equiv 0$ in the variational formula. On the other hand, the lower bound can be obtained by a Schwarz inequality. Indeed, since $\chi_1^\ell$ is periodic, we have $\int_{\mathbf{S}^d}\nabla\chi_1^\ell d\dot{y} = 0$. Hence, in view of the expression of the probability measure $m$,

$$\begin{aligned}
1 &= \int_{\mathbf{S}^d} e^{-2V}d\dot{y} \times \left\|\int_{\mathbf{S}^d}(\ell - \nabla\chi_1^\ell)e^{2V}m d\dot{y}\right\| \\
&\leq \int_{\mathbf{S}^d} e^{-2V}d\dot{y} \times \left(\int_{\mathbf{S}^d}\|\ell - \nabla\chi_1^\ell\|^2 m d\dot{y}\right)^{\frac{1}{2}} \times \left(\int_{\mathbf{S}^d} e^{4V} m d\dot{y}\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbf{S}^d} e^{-2V}d\dot{y}\right)^{\frac{1}{2}} \times \left(\int_{\mathbf{S}^d}\|\ell - \nabla\chi_1^\ell\|^2 m d\dot{y}\right)^{\frac{1}{2}} \times \left(\int_{\mathbf{S}^d} e^{2V}d\dot{y}\right)^{\frac{1}{2}}.
\end{aligned}$$

**8.2. Appendix B.** We aim to prove Proposition 3.5. The situation takes place in the case where the time-dependent potential is of the type $V(t,x) = v(x - \varrho t)$. We denote by $L$ the generator associated with $v$ defined by $L = \frac{1}{2}\Delta - \nabla v.\nabla$, and by $m$ its invariant measure given by (5). By Proposition 2.1, for any $i = 1,\dots,d$ there exists a unique periodic function $\chi^i$ which satisfies $L\chi^i = -\frac{\partial v}{\partial x_i}$ and the centering condition $\int_{\mathbf{S}^d}\chi^i(\dot{y})m(\dot{y})d\dot{y}$. Besides we denote by $M(v)$ the $d\times d$ matrix whose coefficients are $M_{ij} = \int_{\mathbf{S}^d}\frac{\partial\chi^i}{\partial x_j}m d\dot{y}$. Then we can note that, for any unit vector $\ell$, the function $\chi_1^\ell$

associated with $V$ and defined as in Proposition 3.1 is given by $\chi_1^\ell(t,y) = \ell_i \chi^i(y - \varrho t)$. As a consequence $\frac{\partial \chi_1^\ell}{\partial t} = -\varrho.\nabla\chi_1^\ell$ and $\beta_1^\ell$ is written

$$\beta_1^\ell = \varrho. \int_{\mathbf{S}^d} \nabla\chi^\ell m d\dot{y} = \ell.M(v)\varrho.$$

It follows that $\beta_1 = M(v)\varrho$. On the other hand, by (30), the coefficient $\alpha^\ell$ can be written as

$$\alpha^\ell = 1 - \int_{\mathbf{S}^d} \ell.\nabla\chi_1^\ell m d\dot{y}. = 1 - \ell.M(v)\ell.$$

Thus $M(v) = I_d - \alpha$ and $\beta_1 = (I_d - \alpha)\varrho$. The other statements of Proposition 3.5 follow readily from Proposition 3.2.

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