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WAVE PROPAGATION AND IMAGING IN MOVING RANDOM MEDIA*

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Abstract. We present a study of sound wave propagation in a time dependent random medium and an application to imaging. The medium is modeled by small temporal and spatial random fluctuations in the wave speed and density, and it moves due to an ambient flow. We develop a transport theory for the energy density of the waves, in a forward scattering regime, within a cone (beam) of propagation with small opening angle. We apply the transport theory to the inverse problem of estimating a stationary wave source from measurements at a remote array of receivers. The estimation requires knowledge of the mean velocity of the ambient flow and the second-order statistics of the random medium. If these are not known, we show how they may be estimated from additional measurements gathered at the array, using a few known sources. We also show how the transport theory can be used to estimate the mean velocity of the medium. If the array has large aperture and the scattering in the random medium is strong, this estimate does not depend on the knowledge of the statistics of the random medium.

Key words. time dependent random medium, Wigner transform, transport, imaging

AMS subject classifications. 76B15, 35Q99, 60F05

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1. Introduction.

We study sound wave propagation in a time dependent medium modeled by the wave speed $c(t,\vec{\boldsymbol{x}})$ and density $\rho(t,\vec{\boldsymbol{x}})$ that are random perturbations of the constant values c_o and ρ_o. The medium is moving due to an ambient flow, with velocity $\vec{\boldsymbol{v}}(t,\vec{\boldsymbol{x}})$ that has a constant mean $\vec{\boldsymbol{v}}_o$ and small random fluctuations. The source is at a stationary location and emits a signal in the range direction denoted henceforth by the coordinate z, as illustrated in Figure 1.1. The signal is typically a pulse defined by an envelope function of compact support, modulated at frequency ω_o. It generates a wave that undergoes scattering as it propagates through the random medium. The goal of the paper is to analyze from first principles the net scattering at long range and to apply the results to the inverse problem of estimating the source location and medium velocity from measurements of the wave at a remote, stationary array of receivers.

Various models of sound waves in moving media are described in [16, Chapter 2] using the linearization of the fluid dynamics equations about an ambient flow, followed by simplifications motivated by scaling assumptions. Here we consider Pierce's equations [16, section 2.4.6] derived in [19] for media that vary at longer scales than the

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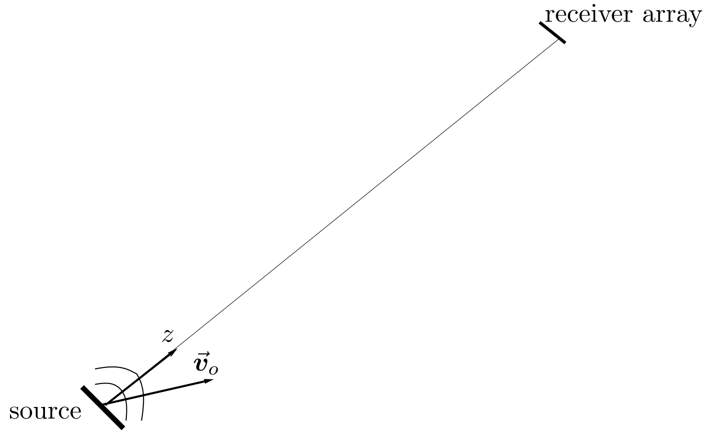

FIG. 1.1. *Illustration of the setup. A stationary source emits a wave in the range direction z, in a moving medium with velocity $\vec{\boldsymbol{v}}(t,\vec{\boldsymbol{x}})$ that has small random fluctuations about the constant mean $\vec{\boldsymbol{v}}_o$. The orientation of $\vec{\boldsymbol{v}}_o$ with respect to the range direction is arbitrary. The wave is recorded by a stationary, remote array of receivers.*

central wavelength $\lambda_o = 2\pi c_o/\omega_o$ of the wave generated by the source. Pierce's model gives the acoustic pressure

$$p(t,\vec{\boldsymbol{x}}) = -\rho(t,\vec{\boldsymbol{x}}) D_t \phi(t,\vec{\boldsymbol{x}}) \tag{1.1}$$

in terms of the velocity quasi-potential $\phi(t,\vec{\boldsymbol{x}})$, which satisfies the equation

$$D_t\Big[\frac{1}{c^2(t,\vec{\boldsymbol{x}})} D_t\phi(t,\vec{\boldsymbol{x}})\Big] - \frac{1}{\rho(t,\vec{\boldsymbol{x}})}\nabla_{\vec{\boldsymbol{x}}}\cdot\Big[\rho(t,\vec{\boldsymbol{x}})\nabla_{\vec{\boldsymbol{x}}}\phi(t,\vec{\boldsymbol{x}})\Big] = \mathrm{s}(t,\vec{\boldsymbol{x}}) \tag{1.2}$$

for spatial variable $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^{d+1}$ and time $t \in \mathbb{R}$, with natural number $d \geq 1$. Here $\boldsymbol{x} \in \mathbb{R}^d$ lies in the cross-range plane, orthogonal to the range axis z. Moreover, $\nabla_{\vec{\boldsymbol{x}}}$ and $\nabla_{\vec{\boldsymbol{x}}}\cdot$ are the gradient and divergence operators in the variable $\vec{\boldsymbol{x}}$ and

$$D_t = \partial_t + \vec{\boldsymbol{v}}(t,\vec{\boldsymbol{x}})\cdot\nabla_{\vec{\boldsymbol{x}}}$$

is the material (Lagrangian) derivative, with ∂_t denoting the partial derivative with respect to time. The source is modeled by the function $\mathrm{s}(t,\vec{\boldsymbol{x}})$ localized at the origin of range and with compact support. Prior to the source excitation there is no wave

$$\phi(t,\vec{\boldsymbol{x}}) \equiv 0, \quad t \ll 0, \tag{1.3}$$

but the medium is in motion due to the ambient flow.

Sound wave propagation in ambient flows due to wind in the atmosphere or ocean currents arises in applications like the quantification of the effects of temperature fluctuations and wind on the rise time and shape of sonic booms [4] or on radio-acoustic sounding [12], monitoring noise near airports [21], acoustic tomography [14], and so on.

Moving media also arise in optics, for example, in Doppler velocimetry or anemometry [6, 7], which uses lasers to determine the flow velocity $\vec{\boldsymbol{v}}_o$. This has applications in wind tunnel experiments for testing aircraft [10], in velocity analysis of water flow for ship hull design [13], in navigation and landing [1], and in medicine and bioengineering [15]. A description of light propagation models used in this context can be found in [8, Chapter 8].

Much of the applied literature on waves in moving random media considers either discrete models with Rayleigh or Mie scattering by moving particles [8] or continuum models described by the classic wave equation with wave speed $c(0, \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}t)$. These use Taylor's hypothesis [11, Chapter 19], where the medium is "frozen" over the duration of the experiment and simply shifted by the uniform ambient flow. A transport theory in such frozen-in media is obtained, for example, in [11, Chapter 20] and [16, Chapter 8], in the paraxial regime where the waves propagate in a narrow angle cone around the range direction. The formal derivation of this theory assumes that the random fluctuations of the wave speed are Gaussian and uses the Markov approximation, where the fluctuations are δ-correlated in range, i.e., at any two distinct ranges, no matter how close, the fluctuations are assumed uncorrelated.

In this paper we study the wave equation (1.2) with coefficients $c(t, \vec{\boldsymbol{x}}), \rho(t, \vec{\boldsymbol{x}})$, and $\vec{\boldsymbol{v}}(t, \vec{\boldsymbol{x}})$ that have random correlated fluctuations at spatial scale ℓ and temporal scale T. These fluctuations are not necessarily Gaussian. We analyze the solution $\phi(t, \vec{\boldsymbol{x}})$ and therefore the acoustic pressure $p(t, \vec{\boldsymbol{x}})$ in a forward scattering regime, where the propagation is within a cone (beam) with axis along the range direction z. The analysis uses asymptotics in the small parameter $\varepsilon = \lambda_o/L \ll 1$, where L is the range scale that quantifies the distance between the source and the array of receivers. Pierce's equations (1.1)–(1.2) are justified for small $\lambda_o/\ell \ll 1$. By fixing λ_o/ℓ or letting it tend to zero, independent of ε, and by appropriate scaling of the spatial support of the source $\mathrm{s}(t, \vec{\boldsymbol{x}})$, we obtain two wave propagation regimes. The first is called the wide beam regime because the cone of propagation has finite opening angle. The second is the paraxial regime, where the cone has very small opening angle. We use the diffusion approximation theory given in [9, Chapter 6] and [17, 18] to study both regimes and obtain transport equations that describe the propagation of energy. These equations are simpler in the paraxial case and we use them to study the inverse problem of locating the source. Because the inversion requires knowledge of the mean velocity $\vec{\boldsymbol{v}}_o$ of the ambient flow and the second-order statistics of the random medium, we also discuss their estimation from additional measurements of waves generated by known sources.

The paper is organized as follows. We begin in section 2 with the mathematical formulation of the problem. Then we state in section 3 the transport equations. These equations are derived in section 5 and we use them for the inverse problem in section 4. We end with a summary in section 6.

2. Formulation of the problem.

We study the sound wave modeled by the acoustic pressure $p(t, \vec{\boldsymbol{x}})$ defined in (1.1) in terms of the velocity quasi potential $\phi(t, \vec{\boldsymbol{x}})$, the solution of the initial value problem (1.2)–(1.3). The problem is to characterize the acoustic pressure $p(t, \vec{\boldsymbol{x}})$ in the scaling regime described in section 2.2 and then use the results for localizing the source and estimating the mean medium velocity $\vec{\boldsymbol{v}}_o$.

2.1. Medium and source.

The coefficients in (1.2) are random fields, defined by

$$\vec{\boldsymbol{v}}(t, \vec{\boldsymbol{x}}) = \vec{\boldsymbol{v}}_o + V\sigma_v \vec{\boldsymbol{\nu}}\Big(\frac{t}{T}, \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_o t}{\ell}\Big), \tag{2.1}$$

$$\rho(t, \vec{\boldsymbol{x}}) = \rho_o \exp\Big[\sigma_\rho \nu_\rho\Big(\frac{t}{T}, \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_o t}{\ell}\Big)\Big], \tag{2.2}$$

$$c(t, \vec{\boldsymbol{x}}) = c_o\Big[1 + \sigma_c \nu_c\Big(\frac{t}{T}, \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_o t}{\ell}\Big)\Big]^{-1/2}, \tag{2.3}$$

where c_o, ρ_o are the constant background wave speed and density, $\vec{\boldsymbol{v}}_o$ is the constant mean velocity of the ambient flow, and V is a velocity scale (of the order of $|\vec{\boldsymbol{v}}_o|$) that will be specified later. Equations (2.1)–(2.3) describe a randomly perturbed medium, with typical spatial structures of the size of the order of ℓ and lifetimes of the order of T, advected with the ambient flow with mean velocity $\vec{\boldsymbol{v}}_o$. It is possible to consider that the perturbations are steady, i.e., ν_ρ and ν_c do not depend explicitly on time and $\vec{\boldsymbol{\nu}} = \mathbf{0}$, as discussed briefly in section 3.3. However, we assume a more general model where the perturbations are unsteady and correlated, to be consistent with the basic equations satisfied by the perturbations [16, Chapter 2], moreover, to capture typical physical regimes when the medium decorrelates in time. The fluctuations in (2.1)–(2.3) are given by the random stationary processes $\vec{\boldsymbol{\nu}}$, ν_ρ, and ν_c of dimensionless arguments and mean zero

$$\mathbb{E}\big[\vec{\boldsymbol{\nu}}(\tau,\vec{\boldsymbol{r}})\big] = 0, \quad \mathbb{E}[\nu_\rho(\tau,\vec{\boldsymbol{r}})] = 0, \quad \mathbb{E}[\nu_c(\tau,\vec{\boldsymbol{r}})] = 0. \tag{2.4}$$

We assume that $\vec{\boldsymbol{\nu}} = (\nu_j)_{j=1}^{d+1}$, ν_ρ, and ν_c are twice differentiable, with bounded derivatives almost surely, have ergodic properties in the z direction, and are correlated, with covariance entries

$$\mathbb{E}\big[\nu_\alpha(\tau,\vec{\boldsymbol{r}})\nu_\beta(\tau',\vec{\boldsymbol{r}}')\big] = \mathscr{R}_{\alpha\beta}(\tau-\tau',\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'). \tag{2.5}$$

Here the indices α and β are either $1,\dots,d+1$, or ρ, or c. The covariance is an even and integrable symmetric matrix valued function, which is four times differentiable and satisfies the normalization conditions

$$\mathscr{R}_{\alpha\alpha}(0,\mathbf{0}) = 1 \text{ or } O(1), \qquad \int_{\mathbb{R}} \mathrm{d}\tau \int_{\mathbb{R}^{d+1}} \mathrm{d}\vec{\boldsymbol{r}}\, \mathscr{R}_{\alpha\alpha}(\tau,\vec{\boldsymbol{r}}) = 1 \ \text{ or } O(1). \tag{2.6}$$

The scale T in definitions (2.1)–(2.3) is the correlation time, the typical lifespan of a spatial realization of the fluctuations, and ℓ is the correlation length, the typical length scale of the fluctuations. The dimensionless positive numbers σ_v, σ_ρ, and σ_c quantify the standard deviation of the fluctuations. They are of the same order and small, so definitions (2.2) and (2.3) can be approximated by

$$\rho(t,\vec{\boldsymbol{x}}) \approx \rho_o\Big[1 + \sigma_\rho \nu_\rho\Big(\frac{t}{T}, \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_o t}{\ell}\Big)\Big], \quad c(t,\vec{\boldsymbol{x}}) \approx c_o\Big[1 - \frac{\sigma_c}{2}\nu_c\Big(\frac{t}{T}, \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_o t}{\ell}\Big)\Big],$$

with c_o and ρ_o close to the mean wave speed and density. The exponential in (2.2) and the inverse of the square root in (2.3) are used for convenience because some important effective properties of the medium are defined in terms of $\mathbb{E}[\log \rho]$ and $\mathbb{E}[c^{-2}]$, which are equal to $\log \rho_o$ and c_o^{-2}.

The origin of the coordinates is at the center of the source location, modeled by

$$\mathrm{s}(t,\vec{\boldsymbol{x}}) = \sigma_s e^{-i\omega_o t} S\Big(\frac{t}{T_s}, \frac{\boldsymbol{x}}{\ell_s}\Big)\delta(z), \tag{2.7}$$

for $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$, using the continuous function S of dimensionless arguments and compact support. The length scale ℓ_s is the radius of the support of $\mathrm{s}(t,\vec{\boldsymbol{x}})$ in cross-range and the time scale T_s is the duration of the emitted signal. Note that $\mathrm{s}(t,\vec{\boldsymbol{x}})$ is modulated by the oscillatory exponential at the frequency ω_o. We call it the central frequency because the Fourier transform of $\mathrm{s}(t,\vec{\boldsymbol{x}})$ with respect to time is supported in the frequency interval $|\omega - \omega_o| \le O(1/T_s)$. The solution $\phi(t,\vec{\boldsymbol{x}})$ of (2.2) depends linearly on the source, so we use σ_s to control its amplitude.

To be able to set radiation conditions for the wave field resolved over frequencies, we make the mathematical assumption that the random fluctuations of $\vec{\boldsymbol{v}}(t, \vec{\boldsymbol{x}})$, $\rho(t, \vec{\boldsymbol{x}})$, and $c(t, \vec{\boldsymbol{x}})$ are supported in a domain of finite range that is much larger than L. In practice this assumption does not hold, but the wave equation is causal and with finite speed of propagation, so the truncation of the support of the fluctuations does not affect the wave measured at the array up to time $O(L/c_o)$.

2.2. Scaling regime. Because the fluctuations of the coefficients (2.1)–(2.2) are small, they have negligible effect on the wave at short range, meaning that $\phi(t, \vec{\boldsymbol{x}}) \approx \phi_o(t, \vec{\boldsymbol{x}})$, the solution of (1.2)–(1.3) with constant wave speed c_o, density ρ_o, and velocity $\vec{\boldsymbol{v}}_o$. We are interested in a long range L, where the wave undergoes many scattering events in the random medium and $\phi(t, \vec{\boldsymbol{x}})$ is quite different from $\phi_o(t, \vec{\boldsymbol{x}})$. We model this long range regime with the small and positive, dimensionless parameter

$$\varepsilon = \frac{\lambda_o}{L} \ll 1 \tag{2.8}$$

and use asymptotics in the limit $\varepsilon \to 0$ to study the random field $\phi(t, \vec{\boldsymbol{x}})$.

The relation between the wavelength, the correlation length, and the cross-range support of the source is described by the positive, dimensionless parameters

$$\gamma = \frac{\lambda_o}{\ell}, \quad \gamma_s = \frac{\lambda_o}{\ell_s}, \tag{2.9}$$

which are small but independent of ε. The positive, dimensionless parameter

$$\eta = \frac{T}{T_L} \tag{2.10}$$

determines how fast the medium changes on the scale of the travel time $T_L = L/c_o$.

The duration of the source signal is modeled by the positive, dimensionless parameter

$$\eta_s = \frac{T_s}{T_L}, \tag{2.11}$$

which is independent of ε. The Fourier transform of this signal is supported in the frequency interval centered at ω_o and of length (bandwidth) $O(1/T_s)$, where

$$\frac{1}{T_s} = \frac{1}{\eta_s T_L} \ll \frac{1}{\varepsilon T_L} = \frac{c_o}{\varepsilon L} = \frac{c_o}{\lambda_o} = O(\omega_o). \tag{2.12}$$

Thus, the source has a small bandwidth in the $\varepsilon \to 0$ limit.

Our asymptotic analysis assumes the order relation

$$\varepsilon \ll \min\{\gamma, \gamma_s, \eta, \eta_s\}, \tag{2.13}$$

meaning that we take the limit $\varepsilon \to 0$ for fixed $\gamma, \gamma_s, \eta, \eta_s$. The standard deviations of the fluctuations are scaled as

$$\sigma_c = \sqrt{\varepsilon\gamma}\bar{\sigma}_c, \quad \sigma_\rho = \sqrt{\varepsilon\gamma}\bar{\sigma}_\rho, \quad \sigma_v = \sqrt{\varepsilon\gamma}\bar{\sigma}_v, \tag{2.14}$$

with $\bar{\sigma}_c, \bar{\sigma}_\rho, \bar{\sigma}_v = O(1)$ to obtain a $O(1)$ net scattering effect.

The ambient flow, due, for example, to wind, has much smaller velocity than the reference sound speed c_o. We model this assumption with the scaling relation

$$|\vec{\boldsymbol{v}}_o|/V = O(1), \quad \text{where } V = \varepsilon c_o. \tag{2.15}$$

Although $V \ll c_o$, the medium moves on the scale of the wavelength over the duration of the propagation

$$VT_L = V\frac{L}{c_o} = \varepsilon L = \lambda_o < \ell, \tag{2.16}$$

so the motion has a $O(1)$ net scattering effect. Slower motion is negligible, whereas faster motion gives different phenomena than those analyzed in this paper.

We scale the amplitude of the source as

$$\sigma_s = \frac{1}{\varepsilon \eta_s L}\Big(\frac{\gamma_s}{\varepsilon}\Big)^d \tag{2.17}$$

to obtain $\phi(t,\vec{\boldsymbol{x}}) = O(1)$ in the limit $\varepsilon \to 0$. Since (2.2) is linear, any other source amplitude can be taken into account by multiplication of our wave field with that given amplitude.

Note that in section 3.2.2 we consider the secondary scaling relation

$$\gamma \sim \gamma_s \ll 1, \tag{2.18}$$

corresponding to the paraxial regime, where the symbol "$\sim$" means of the same order. Moreover, in section 4.3 we assume $\eta/\eta_s \ll 1$ corresponding to a regime of statistical stability. In this secondary scaling regime we let

$$|\vec{\boldsymbol{v}}_o| = O\left(\frac{\varepsilon c_o}{\eta\gamma}\right) \tag{2.19}$$

to obtain the distinguished limit in which the medium velocity impacts the quantities of interest.

3. Results of the analysis of the wave field.

We show in section 5 that in the scaling regime described in (2.8)–(2.17), the pressure is given by

$$p(t,\vec{\boldsymbol{x}}) \approx i\omega_o\rho_o \int_{\mathcal{O}} \frac{\mathrm{d}\omega \mathrm{d}\boldsymbol{k}}{(2\pi)^{d+1}} \frac{a(\omega,\boldsymbol{k},z)}{\sqrt{\beta(\boldsymbol{k})}} e^{-i(\omega_o+\omega)t + i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}} \tag{3.1}$$

for $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$ and $\mathcal{O} = \{\omega \in \mathbb{R}\} \times \{\boldsymbol{k} \in \mathbb{R}^d,\ |\boldsymbol{k}| < k_o\}$, where the approximation error vanishes in the limit $\varepsilon \to 0$. This expression is a Fourier synthesis of forward propagating time-harmonic plane waves (modes) at frequency $\omega_o + \omega$, with wave vectors $\vec{\boldsymbol{k}}$ defined by

$$\vec{\boldsymbol{k}} = \big(\boldsymbol{k}, \beta(\boldsymbol{k})\big), \quad \beta(\boldsymbol{k}) = \sqrt{k_o^2 - |\boldsymbol{k}|^2}, \quad k_o = 2\pi/\lambda_o. \tag{3.2}$$

The scattering effects in the random medium are captured by the mode amplitudes, which form a Markov process $\big(a(\omega,\boldsymbol{k},z)\big)_{(\omega,\boldsymbol{k})\in\mathcal{O}}$ that evolves in z, starting from

$$a(\omega,\boldsymbol{k},0) = a_o(\omega,\boldsymbol{k}) = \frac{i\sigma_s T_s \ell_s^d}{2\sqrt{\beta(\boldsymbol{k})}}\widehat{S}(\omega T_s, \ell_s \boldsymbol{k}). \tag{3.3}$$

This process satisfies the conservation relation

$$\int_{\mathcal{O}} \mathrm{d}\omega \mathrm{d}\boldsymbol{k} \left|a(\omega,\boldsymbol{k},z)\right|^2 = \int_{\mathcal{O}} \mathrm{d}\omega \mathrm{d}\boldsymbol{k} \left|a_o(\omega,\boldsymbol{k})\right|^2 \quad \forall z>0. \tag{3.4}$$

The statistical moments of $\big(a(\omega,\boldsymbol{k},z)\big)_{(\omega,\boldsymbol{k})\in\mathcal{O}}$ are characterized explicitly in the limit $\varepsilon \to 0$, as explained in section 5.7 and Appendix A. Here we describe the expectation of the amplitudes, which defines the coherent wave, and the second moments that define the mean Wigner transform of the wave, i.e., the energy resolved over frequencies and direction of propagation.

3.1. The coherent wave. The expectation of the acoustic pressure (the coherent wave) is obtained from (3.1) using the mean amplitudes

$$\mathbb{E}[a(\omega,\boldsymbol{k},z)] = a_o(\omega,\boldsymbol{k}) \exp\left[i\theta(\omega,\boldsymbol{k})z + D(\boldsymbol{k})z\right]. \tag{3.5}$$

These are derived in section 5.7.1, with $a_o(\omega,\boldsymbol{k})$ given in (3.3). The exponential describes the effect of the random medium, as follows.

The first term in the exponent is the phase

$$\theta(\omega,\boldsymbol{k}) = \frac{k_o}{\beta(\boldsymbol{k})}\Big(\frac{\omega}{c_o} - \frac{\boldsymbol{v}_o}{c_o}\cdot\boldsymbol{k}\Big) + \frac{\sigma_\rho^2}{8\beta(\boldsymbol{k})\ell^2}\Delta_{\vec{\boldsymbol{r}}}\mathscr{R}_{\rho\rho}(0,\vec{\boldsymbol{r}})|_{\vec{\boldsymbol{r}}=0} \tag{3.6}$$

and consists of two parts. The first part models the Doppler frequency shift and depends on the cross-range component $\boldsymbol{v}_o$ of the mean velocity $\vec{\boldsymbol{v}}_o = (\boldsymbol{v}_o, v_{oz})$. It comes from the expansion of the mode wavenumber

$$\sqrt{\Big(k_o + \frac{\omega - \boldsymbol{v}_o\cdot\boldsymbol{k}}{c_o}\Big)^2 - |\boldsymbol{k}|^2} \approx \beta(\boldsymbol{k}) + \frac{k_o(\omega - \boldsymbol{v}_o\cdot\boldsymbol{k})}{c_o\beta(\boldsymbol{k})},$$

in the limit $\varepsilon \to 0$, using the scaling relation (2.15) and $\omega \ll \omega_o$ obtained from (2.12). The second part is due to the random medium and it is small when $\gamma \ll 1$, i.e., $\lambda_o \ll \ell$.

The second term in the exponent in (3.5) is

$$\begin{aligned} D(\boldsymbol{k}) = -\frac{k_o^4\ell^{d+1}}{4}\int_{|\boldsymbol{k}'|<k_o}\frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^d}\,\frac{1}{\beta(\boldsymbol{k})\beta(\boldsymbol{k}')}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{r}\int_0^\infty \mathrm{d}r_z\, e^{-i\ell(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{r}}} \\ \times\Big[\sigma_c^2\mathscr{R}_{cc}(0,\vec{\boldsymbol{r}}) + \frac{\sigma_\rho^2}{4(k_o\ell)^4}\Delta^2_{\vec{\boldsymbol{r}}}\mathscr{R}_{\rho\rho}(0,\vec{\boldsymbol{r}}) - \frac{\sigma_\rho\sigma_c}{(k_o\ell)^2}\Delta_{\vec{\boldsymbol{r}}}\mathscr{R}_{c\rho}(0,\vec{\boldsymbol{r}})\Big], \end{aligned} \tag{3.7}$$

where we used the notation $\vec{\boldsymbol{r}} = (\boldsymbol{r}, r_z)$ and definition (3.2). This complex exponent accounts for the significant effect of the random medium, seen especially in the term proportional to $\mathscr{R}_{cc}$ which dominates the other ones in the $\gamma \ll 1$ regime. Because the covariance is even, the real part of $D(\boldsymbol{k})$ derives from

$$\int_{\mathbb{R}^{d+1}} \mathrm{d}\vec{\boldsymbol{r}}\,\mathscr{R}_{cc}(0,\vec{\boldsymbol{r}})e^{-i\ell(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{r}}} = \int_{\mathbb{R}}\frac{\mathrm{d}\Omega}{2\pi}\,\widetilde{\mathscr{R}}_{cc}\big(\Omega, \ell(\vec{\boldsymbol{k}}-\vec{\boldsymbol{k}}')\big), \tag{3.8}$$

where

$$\widetilde{\mathscr{R}}_{cc}(\Omega,\vec{\boldsymbol{q}}) = \int_{\mathbb{R}}\mathrm{d}\tau\int_{\mathbb{R}^{d+1}}\mathrm{d}\vec{\boldsymbol{r}}\,\mathscr{R}_{cc}(\tau,\vec{\boldsymbol{r}})e^{i\Omega\tau - i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} \geq 0 \tag{3.9}$$

is the power spectral density of ν_c. This is nonnegative by Bochner's theorem, so $\mathrm{Re}\big[D(\boldsymbol{k})\big] < 0$ and the mean amplitudes decay exponentially in z, on the length scale

$$\mathscr{S}(\boldsymbol{k}) = -\frac{1}{\mathrm{Re}\big[D(\boldsymbol{k})\big]}, \tag{3.10}$$

called the scattering mean free path. Note that $|\boldsymbol{k}|, |\boldsymbol{k}'| = O(1/\ell)$ in the support of $\widetilde{\mathscr{R}}_{cc}$ in (3.8) and that by choosing the standard deviation σ_c as in (2.14), we obtain from (3.7)–(3.10) that $\mathscr{S}(\boldsymbol{k}) = O(L)$ in the $\varepsilon \to 0$ followed by the $\gamma \to 0$ limit. This shows that the decay of the mean amplitudes in z is significant in our regime. It is the manifestation of the randomization of the wave due to scattering in the medium.

3.2. The Wigner transform. The strength of the random fluctuations of the mode amplitudes is described by the Wigner transform (energy density)

$$W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}\, e^{i\boldsymbol{q}\cdot(\nabla\beta(\boldsymbol{k})z+\boldsymbol{x})}\mathbb{E}\Big[a\Big(\omega, \boldsymbol{k}+\frac{\boldsymbol{q}}{2}, z\Big)\overline{a\Big(\omega, \boldsymbol{k}-\frac{\boldsymbol{q}}{2}, z\Big)}\Big], \tag{3.11}$$

where the bar denotes complex conjugate and the integral is over all $\boldsymbol{q} \in \mathbb{R}^d$ such that $|\boldsymbol{k} \pm \boldsymbol{q}/2| < k_o$. The Wigner transform satisfies the equation

$$\begin{aligned}\big[\partial_z - \nabla\beta(\boldsymbol{k})\cdot\nabla_{\boldsymbol{x}}\big]W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) = \int_{\mathcal{O}} \frac{\mathrm{d}\omega'\mathrm{d}\boldsymbol{k}'}{(2\pi)^{d+1}}\, \mathcal{Q}(\omega, \omega', \boldsymbol{k}, \boldsymbol{k}')\big[W(\omega', \boldsymbol{k}', \boldsymbol{x}, z)\\ - W(\omega, \boldsymbol{k}, \boldsymbol{x}, z)\big],\end{aligned} \tag{3.12}$$

for $z > 0$, with initial condition

$$W(\omega, \boldsymbol{k}, \boldsymbol{x}, 0) = |a_o(\omega, \boldsymbol{k})|^2\delta(\boldsymbol{x}). \tag{3.13}$$

The integral kernel in (3.12) is called the differential scattering cross section. It is defined by

$$\mathcal{Q}(\omega, \omega', \boldsymbol{k}, \boldsymbol{k}') = \frac{k_o^4\ell^{d+1}T}{4\beta(\boldsymbol{k})\beta(\boldsymbol{k}')}\Big[\sigma_c^2\widetilde{\mathscr{R}}_{cc} + \frac{\sigma_\rho^2}{4(k_o\ell)^4}\widetilde{\Delta_{\vec{\boldsymbol{r}}}^2\mathscr{R}_{\rho\rho}} - \frac{\sigma_c\sigma_\rho}{(k_o\ell)^2}\widetilde{\Delta_{\vec{\boldsymbol{r}}}\mathscr{R}_{c\rho}}\Big], \tag{3.14}$$

where the power spectral densities in the square bracket are evaluated as

$$\widetilde{\mathscr{R}}_{cc} = \widetilde{\mathscr{R}}_{cc}\big(T(\omega - \omega' - (\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{v}}_o), \ell(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}')\big), \tag{3.15}$$

and similarly for the other two terms, which are proportional to the Fourier transform of $\Delta_{\vec{\boldsymbol{r}}}^2\mathscr{R}_{\rho\rho}$ and $\Delta_{\vec{\boldsymbol{r}}}\mathscr{R}_{c\rho}$. The total scattering cross section is defined by the integral of (3.14) and satisfies

$$\Sigma(\boldsymbol{k}) = \int_{\mathcal{O}} \frac{\mathrm{d}\omega'\mathrm{d}\boldsymbol{k}'}{(2\pi)^{d+1}}\, \mathcal{Q}(\omega, \omega', \boldsymbol{k}, \boldsymbol{k}') = \frac{2}{\mathscr{S}(\boldsymbol{k})}. \tag{3.16}$$

Note that the last two terms in the square brackets in (3.14) are small in the $\gamma \ll 1$ regime, because $1/(k_o\ell) = \gamma/(2\pi) \ll 1$ and $\sigma_\rho/\sigma_c = O(1)$. If σ_ρ/σ_c were large, of the order γ^{-2}, then these terms would contribute. However, this would change only the interpretation of the differential scattering cross section and not its qualitative form.

3.2.1. The radiative transfer equation. The evolution equation (3.12) for the Wigner transform is related to the radiative transfer equation [5, 20]. Indeed, we show in Appendix D that $W(\omega, \boldsymbol{k}, \boldsymbol{x}, z)$ is the solution of (3.12)–(3.13) if and only if

$$V(\omega, \vec{\boldsymbol{k}}, \vec{\boldsymbol{x}}) = \frac{1}{\beta(\boldsymbol{k})}W(\omega, \boldsymbol{k}, \boldsymbol{x}, z)\delta\big(k_z - \beta(\boldsymbol{k})\big), \quad \vec{\boldsymbol{k}} = (\boldsymbol{k}, k_z), \tag{3.17}$$

solves the radiative transfer equation

$$\nabla_{\vec{\boldsymbol{k}}}\Omega(\vec{\boldsymbol{k}})\cdot\nabla_{\vec{\boldsymbol{x}}}V(\omega,\vec{\boldsymbol{k}},\vec{\boldsymbol{x}}) = \int_{\mathbb{R}^{d+1}}\frac{d\vec{\boldsymbol{k}}'}{(2\pi)^{d+1}}\int\frac{d\omega'}{2\pi}\mathfrak{S}\big(\omega,\omega',\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}'\big)\big[V(\omega',\vec{\boldsymbol{k}}',\vec{\boldsymbol{x}}) - V(\omega,\vec{\boldsymbol{k}},\vec{\boldsymbol{x}})\big] \tag{3.18}$$

with $\Omega(\vec{\boldsymbol{k}}) = c_o|\vec{\boldsymbol{k}}|$ and the scattering kernel

$$\mathfrak{S}\big(\omega,\omega',\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}'\big) = \frac{2\pi c_o^2}{k_o^2}\beta(\boldsymbol{k})\beta(\boldsymbol{k}')\mathcal{Q}(\omega,\omega',\boldsymbol{k},\boldsymbol{k}')\delta\big(\Omega(\vec{\boldsymbol{k}})-\Omega(\vec{\boldsymbol{k}}')\big). \tag{3.19}$$

The initial condition is specified at $\vec{\boldsymbol{x}} = (\boldsymbol{x},0)$ by

$$V(\omega,\vec{\boldsymbol{k}},(\boldsymbol{x},0)) = |a_o(\omega,\boldsymbol{k})|^2\delta(\boldsymbol{x})\delta(k_z-\beta(\boldsymbol{k})) \tag{3.20}$$

with $a_o(\omega,\boldsymbol{k})$ defined in (3.3).

This result shows that the generalized (singular) phase space energy (3.17) evolves as in the standard three-dimensional radiative transfer equation, but it is supported on the phase vectors with range component

$$k_z = \beta(\boldsymbol{k}), \quad |\boldsymbol{k}| < k_o. \tag{3.21}$$

Indeed, if $\vec{\boldsymbol{k}}' = (\boldsymbol{k}',\beta(\boldsymbol{k}'))$ and $\vec{\boldsymbol{k}} = (\boldsymbol{k},k_z)$, then

$$\delta\big(\Omega(\vec{\boldsymbol{k}})-\Omega(\vec{\boldsymbol{k}}')\big) = \frac{1}{c_o}\delta\big(|\vec{\boldsymbol{k}}|-k_o\big) = \frac{k_o}{c_o\beta(\boldsymbol{k})}\delta\big(k_z-\beta(\boldsymbol{k})\big),$$

so the evolution of $V(\omega,\vec{\boldsymbol{k}},\vec{\boldsymbol{x}})$ is confined to the hypersurface in (3.21). Physically, this means that the wave energy is traveling with constant speed in a cone of directions centered at the range axis z.

3.2.2. Paraxial approximation. The paraxial approximation of the Wigner transform is obtained from (3.12)–(3.13) in the limit

$$\gamma = \lambda_o/\ell \to 0 \quad \text{so that } \gamma/\gamma_s = \text{finite},$$

as explained in section 5.8. In this case the phase space decomposition of the initial wave energy given by (3.3) and (3.13) is supported in a narrow cone around the range axis z, with opening angle scaling as

$$\frac{\lambda_o}{\ell_s} = \gamma_s \ll 1.$$

Moreover, from the expression (3.14) of the differential scattering cross section and (3.15) we see that the energy coupling takes place in a small cone of differential directions whose opening angle is

$$\frac{\lambda_o}{\ell} = \gamma \ll 1.$$

In the paraxial regime (3.12) simplifies to

$$\left[\partial_z + \frac{\boldsymbol{k}}{k_o}\cdot\nabla_{\boldsymbol{x}}\right]W(\omega,\boldsymbol{k},\boldsymbol{x},z) = \int_{\mathbb{R}^d}\frac{d\boldsymbol{k}'}{(2\pi)^d}\int_{\mathbb{R}}\frac{d\omega'}{2\pi}\mathcal{Q}_{\text{par}}(\omega',\boldsymbol{k}') \times W(\omega-\omega'-\boldsymbol{k}'\cdot\boldsymbol{v}_o,\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{x},z) - \Sigma_{\text{par}}W(\omega,\boldsymbol{k},\boldsymbol{x},z), \tag{3.22}$$

where we obtained from definition (3.2) and the scaling relations (2.10), (2.15) that in the limit $\gamma \to 0$,

$$\beta(\boldsymbol{k}) \to k_o, \quad \ell\big|\beta(\boldsymbol{k}) - \beta(\boldsymbol{k}')\big| \to 0, \quad T\big|v_{oz}(\beta(\boldsymbol{k}) - \beta(\boldsymbol{k}'))\big| \to 0.$$

The differential scattering cross section becomes

$$\mathcal{Q}_{\rm par}(\omega, \boldsymbol{k}) = \frac{k_o^2 \sigma_c^2 \ell^{d+1} T}{4} \widetilde{\mathscr{R}}_{cc}\big(T\omega, \ell\boldsymbol{k}, 0\big), \tag{3.23}$$

and the total scattering cross section is

$$\Sigma_{\rm par} = \int_{\mathbb{R}^d} \frac{{\rm d}\boldsymbol{k}'}{(2\pi)^d} \int_{\mathbb{R}} \frac{{\rm d}\omega'}{2\pi}\, \mathcal{Q}_{\rm par}(\omega', \boldsymbol{k}') = \frac{\sigma_c^2 \ell k_o^2}{4} \mathscr{R}(0, \boldsymbol{0}) = \frac{2}{\mathscr{S}_{\rm par}}, \tag{3.24}$$

where $\mathscr{S}_{\rm par}$ is the scattering mean free path in the paraxial regime and

$$\mathscr{R}(\tau, \boldsymbol{r}) = \int_{\mathbb{R}} {\rm d}r_z\, \mathscr{R}_{cc}(\tau, \vec{\boldsymbol{r}}), \quad \vec{\boldsymbol{r}} = (\boldsymbol{r}, r_z). \tag{3.25}$$

The initial condition is as in (3.13), with a_o defined in (3.3),

$$W(\omega, \boldsymbol{k}, \boldsymbol{x}, 0) = \frac{\sigma_s^2 T_s^2 \ell_s^{2d}}{4k_o} \big|\widehat{S}(T_s\omega, \ell_s\boldsymbol{k})\big|^2 \delta(\boldsymbol{x}). \tag{3.26}$$

Note that the right-hand side of (3.22) is a convolution, so we can write the Wigner transform explicitly using Fourier transforms, as explained in Appendix C. The result is

$$\begin{aligned} W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) = &\frac{\sigma_s^2 T_s \ell_s^d}{4k_o} \int_{\mathbb{R}} \frac{{\rm d}\Omega}{2\pi} \int_{\mathbb{R}^d} \frac{{\rm d}\boldsymbol{K}}{(2\pi)^d} \,|\widehat{S}(\Omega, \boldsymbol{K})|^2 \int_{\mathbb{R}} {\rm d}t \int_{\mathbb{R}^d} {\rm d}\boldsymbol{y} \\ &\times \int_{\mathbb{R}^d} \frac{{\rm d}\boldsymbol{q}}{(2\pi)^d} \exp\Big\{ i\Big(\omega - \frac{\Omega}{T_s}\Big)t - i\boldsymbol{y}\cdot\Big(\boldsymbol{k} - \frac{\boldsymbol{K}}{\ell_s}\Big) + i\boldsymbol{q}\cdot\Big(\boldsymbol{x} - \frac{\boldsymbol{K}}{k_o}\frac{z}{\ell_s}\Big) \\ &+ \frac{\sigma_c^2 \ell k_o^2}{4} \int_0^z {\rm d}z' \Big[\mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{y} - \frac{\boldsymbol{q}}{k_o}(z - z') - \boldsymbol{v}_o t}{\ell}\Big) - \mathscr{R}(0, \boldsymbol{0})\Big]\Big\}, \end{aligned} \tag{3.27}$$

and we use it next in the inverse problem of estimating the source location and the mean flow velocity $\vec{\boldsymbol{v}}_o$.

3.3. Steady perturbations. Here we discuss briefly the case of steady medium perturbations (2.1)–(2.3), where $\vec{\boldsymbol{\nu}} = \boldsymbol{0}$ and ν_ρ and ν_c are advected by $\vec{\boldsymbol{v}}_o$ but do not depend explicitly on time. This case can be studied directly or it can be obtained as the limit $T \to +\infty$ of the previous results:

- The mean amplitudes decay as (3.5) and the expressions of the phase $\theta(\omega, \boldsymbol{k})$ and complex damping term $D(\boldsymbol{k})$ are given by (3.6) and (3.7), respectively.
- The Wigner transform (3.11) satisfies (3.12) with the singular differential scattering cross section

$$\begin{aligned} \mathcal{Q}(\omega, \omega', \boldsymbol{k}, \boldsymbol{k}') = &\frac{2\pi k_o^4 \ell^{d+1}}{4\beta(\boldsymbol{k})\beta(\boldsymbol{k}')} \delta\big(\omega - \omega' - (\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}')\cdot\vec{\boldsymbol{v}}_o\big) \\ &\times \Big[\sigma_c^2 \widehat{\mathscr{R}}_{cc} + \frac{\sigma_\rho^2}{4(k_o\ell)^4} \widehat{\Delta^2_{\vec{\boldsymbol{r}}} \mathscr{R}_{\rho\rho}} - \frac{\sigma_c\sigma_\rho}{(k_o\ell)^2} \widehat{\Delta_{\vec{\boldsymbol{r}}} \mathscr{R}_{c\rho}}\Big]\big(\ell(\vec{\boldsymbol{k}} - \vec{\boldsymbol{k}}')\big), \end{aligned} \tag{3.28}$$

  where $\vec{\boldsymbol{k}} = (\boldsymbol{k}, \beta(\boldsymbol{k}))$ and

$$\widehat{\mathscr{R}}_{cc}(\vec{\boldsymbol{q}}) = \int_{\mathbb{R}^{d+1}} {\rm d}\vec{\boldsymbol{r}}\, \mathscr{R}_{cc}(0, \vec{\boldsymbol{r}}) e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}.$$

- In particular, in the paraxial regime, if we introduce

$$\widehat{W}(\omega, \boldsymbol{k}, \boldsymbol{x}, z) = W(\omega + \boldsymbol{k}\cdot \boldsymbol{v}_o, \boldsymbol{k}, \boldsymbol{x}, z),$$

then we find that it satisfies the following equation in which ω is frozen:

$$\big[\partial_z + \frac{\boldsymbol{k}}{k_o}\cdot \nabla_{\boldsymbol{x}}\big]\widehat{W}(\omega, \boldsymbol{k}, \boldsymbol{x}, z) = \int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^d}\, \widehat{\mathcal{Q}}_{\rm par}(\boldsymbol{k}')\big[\widehat{W}(\omega, \boldsymbol{k}-\boldsymbol{k}', \boldsymbol{x}, z) - \widehat{W}(\omega, \boldsymbol{k}, \boldsymbol{x}, z)\big],$$

for $z > 0$, with the initial condition $\widehat{W}(\omega, \boldsymbol{k}, \boldsymbol{x}, 0) = |a_o(\omega + \boldsymbol{k}\cdot \boldsymbol{v}_o, \boldsymbol{k})|^2\delta(\boldsymbol{x})$ and with the reduced differential scattering cross section

$$\widehat{\mathcal{Q}}_{\rm par}(\boldsymbol{k}) = \frac{k_o^2\sigma_c^2\ell^{d+1}}{4}\widehat{\mathscr{R}}_{cc}(\ell\boldsymbol{k}, 0).$$

This is the transport equation derived in [2, section III.C] in the absence of ambient flow.

4. Application to imaging. In this section we use the transport theory in the paraxial regime, stated in section 3.2.2, to localize a stationary in space time-harmonic source in a moving random medium with smooth and isotropic random fluctuations, from measurements at a stationary array of receivers. The case of a time-harmonic source is interesting because it shows the beneficial effect of the motion of the random medium for imaging. In the absence of this motion, the wave received at the array is time-harmonic, it oscillates at the frequency ω_o, and it is impossible to determine from it the range of the source. The random motion of the medium causes broadening of the frequency support of the wave field, which makes the range estimation possible.

We consider a strongly scattering regime, where the wave received at the array is incoherent. This means explicitly that the range L is much larger than the scattering mean free path $\mathscr{S}_{\rm par}$ or, equivalently, from (3.24),

$$\frac{\sigma_c^2\ell k_o^2 L}{4}\mathscr{R}(0, \mathbf{0}) \gg 1. \tag{4.1}$$

We also suppose that

$$\frac{\eta}{\eta_s} = \frac{T}{T_s} \ll 1 \tag{4.2}$$

to ensure that the imaging functions are statistically stable with respect to the realizations of the random medium. We begin in section 4.1 with the approximation of the Wigner transform (3.27) for a time-harmonic source, in the strongly scattering regime. This Wigner transform quantifies the time-space coherence properties of the wave, as described in section 4.2. Then, we explain in section 4.3 how we can estimate the Wigner transform from the measurements at the array. The source localization problem is discussed in section 4.4 and the estimation of the mean medium velocity is discussed in section 4.5.

4.1. Wigner transform for time-harmonic source and strong scattering. To derive the Wigner transform for a time-harmonic source, we take the limit $T_s \to \infty$ in (3.27), after rescaling the source amplitude as

$$\sigma_s = \sigma/\sqrt{T_s}, \quad \sigma = O(1). \tag{4.3}$$

We assume for convenience[1] that the source has a Gaussian profile,

$$\int_{\mathbb{R}} \mathrm{d}\Omega \, \big|\widehat{S}(\Omega, \boldsymbol{K})\big|^2 = (2\pi)^d e^{-|\boldsymbol{K}|^2}, \tag{4.4}$$

so we can calculate explicitly the integral over $\boldsymbol{K}$ in (3.27). We obtain after the change of variables $\boldsymbol{y} = \boldsymbol{\xi} + (\boldsymbol{q}/k_o)z$ that

$$\begin{aligned} W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) =& \frac{\sigma^2 \ell_s^d \pi^{d/2}}{4k_o(2\pi)^d} \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\xi} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{q} \, \exp\Big\{ i\omega t - \frac{|\boldsymbol{\xi}|^2}{4\ell_s^2} - i\boldsymbol{\xi}\cdot\boldsymbol{k} + i\boldsymbol{q}\cdot\Big(\boldsymbol{x} - \frac{\boldsymbol{k}}{k_o}z\Big) \\ &+ \frac{\sigma_c^2 \ell k_o^2}{4} \int_0^z \mathrm{d}z' \Big[\mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{\xi} + \frac{\boldsymbol{q}}{k_o}z' - \boldsymbol{v}_o t}{\ell}\Big) - \mathscr{R}(0, \boldsymbol{0}) \Big] \Big\}. \end{aligned} \tag{4.5}$$

Note that the last term in the exponent in (4.5) is negative, because $\mathscr{R}$ is maximal at the origin. Moreover, the relation (4.1) that defines the strongly scattering regime implies that the integrand in (4.5) is negligible for $t/T \geq 1$ and $|\boldsymbol{\xi} + \boldsymbol{q}/k_o z' - \boldsymbol{v}_o t|/\ell \geq 1$. Thus, we can restrict the integral in (4.5) to the set

$$\Big\{ (t, \boldsymbol{\xi}, \boldsymbol{q}) \in \mathbb{R}^{2d+1} : \quad |t| \ll T, \quad \big|\boldsymbol{\xi} + \frac{\boldsymbol{q}}{k_o}z' - \boldsymbol{v}_o t\big| \ll \ell \Big\}$$

and approximate

$$\mathscr{R}(\tau, \boldsymbol{r}) \approx \mathscr{R}(0, \boldsymbol{0}) - \frac{\alpha_o}{2}\tau^2 - \frac{\vartheta_o}{2}|\boldsymbol{r}|^2 \tag{4.6}$$

with $\alpha_o, \vartheta_o > 0$. Here we used that the Hessian of $\mathscr{R}$ evaluated at the origin is negative definite and because the medium is statistically isotropic, it is also diagonal, with the entries $-\alpha_o$ and $-\vartheta_o$. We obtain that

$$\frac{\sigma_c^2 \ell k_o^2}{4} \Big[\mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{\xi} + \frac{\boldsymbol{q}}{k_o}z' - \boldsymbol{v}_o t}{\ell}\Big) - \mathscr{R}(0, \boldsymbol{0}) \Big] \approx -\frac{\alpha}{2}\Big(\frac{t}{T}\Big)^2 - \frac{\vartheta}{2}\Big(\frac{|\boldsymbol{\xi} + \frac{\boldsymbol{q}}{k_o}z' - \boldsymbol{v}_o t|}{\ell} \Big)^2$$

with the positive parameters

$$\alpha = \alpha_o \frac{\sigma_c^2 \ell k_o^2}{4}, \qquad \vartheta = \vartheta_o \frac{\sigma_c^2 \ell k_o^2}{4}. \tag{4.7}$$

Substituting in (4.5) and integrating in z' we obtain

$$\begin{aligned} W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) \approx& \frac{\sigma^2 \ell_s^d \pi^{d/2}}{4k_o(2\pi)^d} \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\xi} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{q} \, \exp\Big\{ i\omega t - \frac{\alpha z}{2}\Big(\frac{t}{T}\Big)^2 - \frac{\vartheta z}{2\ell^2}\big|\boldsymbol{\xi} - \boldsymbol{v}_o t\big|^2 \\ &- \frac{|\boldsymbol{\xi}|^2}{4\ell_s^2} - i\boldsymbol{\xi}\cdot\boldsymbol{k} - \frac{\vartheta z^2}{2\ell^2}(\boldsymbol{\xi} - \boldsymbol{v}_o t)\cdot\frac{\boldsymbol{q}}{k_o} - \frac{\vartheta z^3}{6\ell^2}\Big|\frac{\boldsymbol{q}}{k_o}\Big|^2 + i\boldsymbol{q}\cdot\Big(\boldsymbol{x} - \frac{\boldsymbol{k}}{k_o}z\Big) \Big\}. \end{aligned} \tag{4.8}$$

The imaging results are based on this expression. Before we present them, we study the coherence properties of the transmitted wave and define the coherence parameters which affect the performance of the imaging techniques.

[1]The results extend qualitatively to other profiles but the formulas are no longer explicit.

4.2. Time-space coherence. Let us define the time-space coherence function

$$C(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) = \frac{\lambda_o}{2\pi (c_o\rho_o)^2} \int_{\mathbb{R}} \mathrm{d}t\, p(t+\Delta t, \boldsymbol{x}+\Delta\boldsymbol{x}/2, z)\overline{p(t, \boldsymbol{x}-\Delta\boldsymbol{x}/2, z)} e^{i\omega_o \Delta t} \tag{4.9}$$

and obtain from (3.1) that in the paraxial regime

$$\begin{aligned} C(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) \approx & \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}\, a(\omega, \boldsymbol{k}+\boldsymbol{q}/2, z)\overline{a(\omega, \boldsymbol{k}-\boldsymbol{q}/2, z)} \\ & \times \exp\Big\{ i\boldsymbol{q}\cdot[z\nabla\beta(\boldsymbol{k}) + \boldsymbol{x}] + i\Delta\boldsymbol{x}\cdot\boldsymbol{k} - i\omega\Delta t \Big\}. \end{aligned} \tag{4.10}$$

Moreover, in view of (3.11) and the fact that we average in time so that the statistical fluctuations of C are small (see Remark 4.1) we have

$$\begin{aligned} C(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) &\approx \mathbb{E}\Big[C(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) \Big] \\ &\approx \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d} W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) e^{-i\omega\Delta t + i\Delta\boldsymbol{x}\cdot\boldsymbol{k}}. \end{aligned} \tag{4.11}$$

This shows formally that we can characterize the Wigner transform as the Fourier transform of the coherence function

$$W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) \approx \int_{\mathbb{R}} \mathrm{d}\Delta t \int_{\mathbb{R}^d} \mathrm{d}\Delta\boldsymbol{x}\, C(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) e^{i\omega\Delta t - i\Delta\boldsymbol{x}\cdot\boldsymbol{k}}. \tag{4.12}$$

Using the expression (4.5) of the Wigner transform in (4.11) we find after evaluating the integrals that

$$\begin{aligned} C(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) \approx & \frac{\sigma^2 \ell_s^d}{2^{2+d/2} k_o R_z^d} \exp\big[i\varphi(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) \big] \\ & \times \exp\left[-\frac{\Delta t^2}{2\mathcal{T}_z^2} - \frac{|\boldsymbol{x}|^2}{2R_z^2} - \frac{|\Delta\boldsymbol{x}|^2}{2\mathcal{D}_{1z}^2} - \frac{|H_z \Delta\boldsymbol{x} - \boldsymbol{v}_o\Delta t|^2}{2\mathcal{D}_{2z}^2} \right] \end{aligned} \tag{4.13}$$

with phase

$$\varphi(\Delta t, \Delta \boldsymbol{x}, \boldsymbol{x}, z) = \frac{k_o \boldsymbol{x}\cdot\Big[\Big(1+\vartheta z\Big(\frac{\ell_s}{\ell}\Big)^2\Big)\Delta\boldsymbol{x} - \vartheta z \Big(\frac{\ell_s}{\ell}\Big)^2 \boldsymbol{v}_o \Delta t \Big]}{z\Big[1+\frac{2}{3}\vartheta z\Big(\frac{\ell_s}{\ell}\Big)^2\Big]} \tag{4.14}$$

and coefficients

$$\mathcal{T}_z = \frac{T}{\sqrt{\alpha z}}, \qquad R_z = \frac{z}{\sqrt{2}\ell_s k_o}\left(1+\frac{2\ell_s^2}{3\mathcal{D}_z^2}\right)^{1/2}, \qquad \mathcal{D}_z = \frac{\ell}{\sqrt{\vartheta z}}, \tag{4.15}$$

$$\mathcal{D}_{1z} = 2\mathcal{D}_z \left[3\Big(1+\frac{\ell_s^2}{6\mathcal{D}_z^2}\Big) \right]^{1/2}, \quad \mathcal{D}_{2z} = \mathcal{D}_z \left(\frac{1+\frac{2\ell_s^2}{3\mathcal{D}_z^2}}{1+\frac{\ell_s^2}{6\mathcal{D}_z^2}} \right)^{1/2}, \quad H_z = 1 - \frac{1}{2\Big(1+\frac{\ell_s^2}{6\mathcal{D}_z^2}\Big)}. \tag{4.16}$$

The decay of the coherence function in Δx models the spatial decorrelation of the wave on the length scale corresponding to the characteristic speckle size. This is quantified by the length scales $\mathcal{D}_{1z}$ and $\mathcal{D}_{2z}$, which are of the order of $\mathcal{D}_z$. We call $\mathcal{D}_z$ the decoherence length and obtain from (2.14) and (4.7) that it is of the order of the typical size ℓ of the random fluctuations of the medium,

$$\mathcal{D}_z = \frac{\ell}{\sqrt{\vartheta z}} = \frac{\ell}{\pi\sqrt{\vartheta_o}}\sqrt{\frac{L}{z}} = O(\ell). \tag{4.17}$$

The decay of the coherence function in Δt models the temporal decorrelation of the wave, on the time scale

$$\mathcal{T}_z = \frac{T}{\sqrt{\alpha z}} = \frac{T}{\pi\sqrt{\alpha_o}}\sqrt{\frac{L}{z}} = O(T), \tag{4.18}$$

where we used definitions (2.14) and (4.7). We call $\mathcal{T}_z$ the decoherence time and note that it is of the order of the life span T of the random fluctuations of the medium.

The decay of the coherence function in $|\boldsymbol{x}|$ means that the waves propagate in a beam with radius R_z, which evolves in z as described in (4.15) and satisfies

$$R_z \approx \begin{cases} \dfrac{z}{\sqrt{2}\ell_s k_o} & \text{for} \quad \ell_s \ll \mathcal{D}_z, \\ \sqrt{\dfrac{\vartheta}{3}}\dfrac{z^{3/2}}{k_o\ell} & \text{for} \quad \ell_s \gg \mathcal{D}_z. \end{cases} \tag{4.19}$$

This shows that the transition from diffraction based beam spreading to scattering based beam spreading happens around the critical propagation distance

$$z^* = \frac{1}{\vartheta}\Big(\frac{\ell}{\ell_s}\Big)^2 = L\Big(\frac{\gamma_s}{\gamma}\Big)^2\frac{1}{\pi^2\vartheta_o}. \tag{4.20}$$

This expression is derived from equation $\ell_s = \mathcal{D}_{z^\star}$ and definitions (2.14) and (4.7), and it shows that $z^\star/L$ is finite in our regime.[2]

Note that when $z \gg z^*$, i.e., $\mathcal{D}_z \ll \ell_s$, the coefficients (4.16) become

$$\mathcal{D}_{1z} \approx \sqrt{2}\ell_s, \quad \mathcal{D}_{2z} \approx 2\mathcal{D}_z, \quad H_z \approx 1, \tag{4.21}$$

and the coherence function satisfies

$$\frac{|C(\Delta t, \Delta\boldsymbol{x}, \boldsymbol{x}, z)|}{|C(0,\mathbf{0},\mathbf{0},z)|} \approx \exp\Big(- \frac{\Delta t^2}{2\mathcal{T}_z^2} - \frac{|\boldsymbol{x}|^2}{2R_z^2} - \frac{|\Delta\boldsymbol{x}|^2}{4\ell_s^2} - \frac{|\Delta\boldsymbol{x} - \boldsymbol{v}_o\Delta t|^2}{8\mathcal{D}_z^2}\Big). \tag{4.22}$$

Thus, the spatial spreading and decorrelation of the wave field for $z \gg z^*$ are governed by the parameters R_z, ℓ_s, and $\mathcal{D}_z$, with R_z given by the second case in (4.19) and $\mathcal{D}_z$ given in (4.17). These parameters scale with the propagation distance $z < L$ as $R_z \sim z^{3/2}$ and $\mathcal{D}_z \sim z^{-1/2}$. The temporal decorrelation is on the scale $\mathcal{T}_z \sim z^{-1/2}$.

4.3. Estimation of the Wigner transform. Suppose that we have a receiver array centered at $(\boldsymbol{x}_o, z)$, with aperture in the cross-range plane modeled by the appodization function

$$\mathscr{A}(\boldsymbol{x}) = \exp\Big(-\frac{|\boldsymbol{x}-\boldsymbol{x}_o|^2}{2(\varkappa/k_o)^2}\Big). \tag{4.23}$$

[2]Recall from section 3.2.2 that the paraxial regime is obtained in the limit $\gamma \to 0$ so that $\gamma/\gamma_s = \ell_s/\ell$ remains finite. Here we allow the ratio ℓ_s/ℓ to be large or small but independent of γ, which tends to zero.

The linear size of the array is modeled by the standard deviation $\varkappa/k_o$, with dimensionless $\varkappa > 0$ defining the diameter of the array expressed in units of λ_o.

Recalling the wave decomposition (3.1) and that $\beta(\boldsymbol{k}) \sim k_o$ in the paraxial regime, we define the estimated mode amplitudes by

$$\begin{aligned} a_{\rm est}(\omega,\boldsymbol{k},z) &= \frac{k_o e^{-i\beta(\boldsymbol{k})z}}{i\omega_o\rho_o}\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, \mathscr{A}(\boldsymbol{x}) p(t,\boldsymbol{x},z) e^{i(\omega+\omega_o)t} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= \Big(\frac{\varkappa^2}{2\pi k_o^2}\Big)^{d/2} \int_{\mathbb{R}^d} \mathrm{d}\widetilde{\boldsymbol{k}}\, a(\omega,\boldsymbol{k}+\widetilde{\boldsymbol{k}},z) e^{i[\beta(\boldsymbol{k}+\widetilde{\boldsymbol{k}})-\beta(\boldsymbol{k})]z + i\widetilde{\boldsymbol{k}}\cdot\boldsymbol{x}_o - \frac{\varkappa^2|\widetilde{\boldsymbol{k}}|^2}{2k_o^2}}. \end{aligned} \tag{4.24}$$

With these amplitudes we calculate the estimated Wigner transform

$$W_{\rm est}(\omega,\boldsymbol{k},\boldsymbol{x},z) = \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} e^{i\boldsymbol{q}\cdot(\nabla\beta(\boldsymbol{k})z+\boldsymbol{x})} a_{\rm est}\Big(\omega,\boldsymbol{k}+\frac{\boldsymbol{q}}{2},z\Big)\overline{a_{\rm est}\Big(\omega,\boldsymbol{k}-\frac{\boldsymbol{q}}{2},z\Big)} \tag{4.25}$$

and obtain after carrying out the integrals and using the approximation

$$\Big[\beta\Big(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\Big) - \beta\Big(\boldsymbol{k}-\frac{\boldsymbol{q}}{2}\Big)\Big] z \approx \boldsymbol{q}\cdot\nabla\beta(\boldsymbol{k})z$$

that

$$W_{\rm est}(\omega,\boldsymbol{k},\boldsymbol{x},z) \approx \Big(\frac{\varkappa^2}{\pi k_o^2}\Big)^{d/2} e^{-\frac{k_o^2|\boldsymbol{x}-\boldsymbol{x}_o|^2}{\varkappa^2}} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{K}\, e^{-\frac{\varkappa^2|\boldsymbol{K}|^2}{k_o^2}} W(\omega,\boldsymbol{k}+\boldsymbol{K},\boldsymbol{x},z). \tag{4.26}$$

We can now use the expression (4.8) in this equation to obtain an explicit approximation for $W_{\rm est}$. Equivalently, we can substitute (4.12) in (4.26) and obtain after integrating in $\boldsymbol{K}$ that

$$W_{\rm est}(\omega,\boldsymbol{k},\boldsymbol{x},z) \approx e^{-\frac{k_o^2|\boldsymbol{x}-\boldsymbol{x}_o|^2}{\varkappa^2}} \int_{\mathbb{R}} \mathrm{d}\Delta t \int_{\mathbb{R}^d} \mathrm{d}\Delta\boldsymbol{x}\, C(\Delta t,\Delta\boldsymbol{x},\boldsymbol{x},z) e^{i\omega\Delta t - i\Delta\boldsymbol{x}\cdot\boldsymbol{k} - \frac{k_o^2|\Delta\boldsymbol{x}|^2}{4\varkappa^2}} \tag{4.27}$$

with C given in (4.13).

Remark 4.1. Note from (4.9) that the time integration that defines the coherence function is over a time interval determined by the pulse duration T_s, which is larger than the coherence time T of the medium by assumption (4.2). If we interpret the wave as a train of T_s/T pulses of total duration T, each individual pulse travels through uncorrelated layers of medium because the correlation radius of the medium ℓ is much smaller than c_oT. This follows from the fact that $\ell/(c_oT) = \varepsilon/(\eta\gamma)$ and $\varepsilon \ll \gamma\eta$. Thus, $C(\Delta t, \Delta\boldsymbol{x}, \boldsymbol{x}, z)$ is the superposition of approximately T_s/T uncorrelated components and its statistical fluctuations are small by the law of large numbers. Moreover, we conclude from (4.27) that the estimated Wigner transform is approximately equal to its expectation, up to fluctuations of relative standard deviation that is smaller than $\sqrt{T/T_s}$.

4.4. Source localization. We now show how we can use the estimated Wigner transform to localize the source. Recall that we use the system of coordinates with origin at the center of the source. Thus, the location $(\boldsymbol{x}_o, z)$ of the center of the array relative to the source is unknown and the goal of imaging is to estimate it. We begin in section 4.4.1 with the estimation of the direction of arrival of the waves at the array, and then describe the localization in range in section 4.4.2. These two estimates determine the source location in the cross-range plane, as well.

4.4.1. Direction of arrival estimation. We can estimate the direction of arrival of the waves from the peak (maximum) in $\boldsymbol{k}$ of the imaging function

$$\mathcal{O}_{\rm DoA}(\boldsymbol{k},z) = \int_{\mathbb{R}} \frac{{\rm d}\omega}{2\pi} W_{\rm est}(\omega,\boldsymbol{k},\boldsymbol{x}_o,z), \tag{4.28}$$

determined by the estimated Wigner transform at the center of the array of receivers. If the medium were homogeneous, the maximum of $\boldsymbol{k}\mapsto \mathcal{O}_{\rm DoA}(\boldsymbol{k},z)$ would be at the cross-range wave vector $\boldsymbol{k}^* = k_o\frac{\boldsymbol{x}_o}{z}$, and the width of the peak (the resolution) would be $1/(\sqrt{2}\varkappa)$. However, cumulative scattering in the random medium gives a different result, as we now explain.

Substituting (4.27) in (4.28) and using the expression (4.13), we obtain after evaluating the integrals that

$$\frac{\mathcal{O}_{\rm DoA}(\boldsymbol{k},z)}{\max_{\boldsymbol{k}'}\mathcal{O}_{\rm DoA}(\boldsymbol{k}',z)} = \exp\left\{-\frac{1}{2\vartheta^2_{\rm DoA}(z)}\left|\frac{\boldsymbol{k}-\boldsymbol{k}(z)}{k_o}\right|^2\right\} \tag{4.29}$$

with
(4.30)

$$\vartheta_{\rm DoA}(z) = \left\{\frac{1}{3\mathcal{D}_z^2k_o^2}\left(\frac{1+\frac{\ell_s^2}{2\mathcal{D}_z^2}}{1+\frac{2\ell_s^2}{3\mathcal{D}_z^2}}\right)+\frac{1}{2\varkappa^2}\right\}^{1/2}, \qquad \boldsymbol{k}(z) = k_o\frac{\boldsymbol{x}_o}{z}\left(\frac{1+\frac{\ell_s^2}{\mathcal{D}_z^2}}{1+\frac{2\ell_s^2}{3\mathcal{D}_z^2}}\right).$$

Therefore, the maximum of $\boldsymbol{k}\mapsto \mathcal{O}_{\rm DoA}(\boldsymbol{k},z)$ is at the cross-range wave vector $\boldsymbol{k}(z)$ and the width of the peak (the resolution) is determined by $\vartheta_{\rm DoA}(z)$. This resolution improves for larger array aperture (i.e., $\varkappa$) and deteriorates as z increases. Depending on the magnitude of z relative to the critical range z^* defined in (4.20), we distinguish three cases:

1. In the case $z \ll z^*$, i.e., $\ell_s \ll \mathcal{D}_z$, the intensity travels along the deterministic characteristic, meaning that $\mathcal{O}_{\rm DoA}(\boldsymbol{k},z)$ peaks at

$$\boldsymbol{k}(z) \approx k_o\frac{\boldsymbol{x}_o}{z}. \tag{4.31}$$

However, the resolution is worse than in the homogeneous medium,

$$\vartheta_{\rm DoA}(z) \approx \left\{\frac{1}{3\mathcal{D}_z^2k_o^2}+\frac{1}{2\varkappa^2}\right\}^{1/2}, \tag{4.32}$$

with $\mathcal{D}_z$ defined in (4.17).

2. In the case $z \gg z^*$, i.e., $\ell_s \gg \mathcal{D}_z$, the peak of $\mathcal{O}_{\rm DoA}(\boldsymbol{k},z)$ is at the cross-range wave vector

$$\boldsymbol{k}(z) \approx \frac{3}{2}k_o\frac{\boldsymbol{x}_o}{z}, \tag{4.33}$$

and the resolution is

$$\vartheta_{\rm DoA}(z) \approx \left\{\frac{1}{4\mathcal{D}_z^2k_o^2}+\frac{1}{2\varkappa^2}\right\}^{1/2}. \tag{4.34}$$

Here the peak corresponds to a straight line characteristic, but with a different slope than in the homogeneous medium. The resolution is also worse than in the homogeneous medium.

3. In the case $z = O(z^*)$, the characteristic can no longer be approximated by a straight line, as seen from (4.30). Nevertheless, we can still estimate the source position from the observed peak $\boldsymbol{k}(z)$, provided that we have an estimate of the range z. The resolution of the estimate of $\boldsymbol{k}(z)$ is $\vartheta_{\rm DoA}(z)$ given by (4.30) that is bounded from below by (4.34) and from above by (4.32).

Remark 4.2. Note that (4.30) is a decreasing function of the array diameter $\varkappa/k_o$, as long as this satisfies $\varkappa/k_o \leq \sqrt{2}\mathcal{D}_z$. Thus, increasing the aperture size beyond the critical value $\sqrt{2}\mathcal{D}_z$ does not bring any resolution improvement.

4.4.2. Range estimation. The results of the previous section show that the direction of arrival estimation is coupled with the estimation of the range z in general, with the exception of the two extreme cases 1 and 2 outlined above.

To estimate the range z, we use the imaging function

$$\mathcal{O}_{\rm range}(t,z) = \int_{\mathbb{R}} \frac{{\rm d}\omega}{2\pi} e^{-i\omega t} \int_{\mathbb{R}^d} \frac{{\rm d}\boldsymbol{k}}{(2\pi)^d} W_{\rm est}(\omega, \boldsymbol{k}, \boldsymbol{x}_o, z) \approx C(t, \boldsymbol{0}, \boldsymbol{x}_o, z), \tag{4.35}$$

derived from (4.27). Substituting the expression (4.13) of the coherence function in this equation we obtain

$$\frac{|\mathcal{O}_{\rm range}(t,z)|}{\max_{t'} |\mathcal{O}_{\rm range}(t',z)|} = \exp\left\{ -\frac{t^2}{2\vartheta^2_{\rm range}(z)} \right\} \tag{4.36}$$

with

$$\vartheta_{\rm range}(z) = \mathcal{T}_z \left\{ 1 + \frac{|\boldsymbol{v}_o|^2 \mathcal{T}_z^2}{\mathcal{D}_z^2} \left(\frac{1 + \frac{\ell_s^2}{6\mathcal{D}_z^2}}{1 + \frac{2\ell_s^2}{3\mathcal{D}_z^2}} \right) \right\}^{-1/2}. \tag{4.37}$$

As a function of t, this peaks at $t = 0$ and its absolute value decays as a Gaussian, with standard deviation $\vartheta_{\rm range}(z)$. If we know the statistics of the medium (the decoherence time $\mathcal{T}_z$ and length $\mathcal{D}_z$) and the magnitude of the cross-range velocity $|\boldsymbol{v}_o|$, then we can determine the range z by estimating the rate of decay of $\mathcal{O}_{\rm range}(t,z)$. Note that the array dimameter $\varkappa/k_o$ plays no role for the range estimation.

Remark 4.3. We can also estimate the mean velocity $\vec{\boldsymbol{v}}_o = (\boldsymbol{v}_o, v_{oz})$ from (4.36) by considering different beam orientations in the case that the source locations and also the medium statistics (the decoherence time $\mathcal{T}_z$ and length $\mathcal{D}_z$) are known. That is to say, with three known beams we can get the vector $\vec{\boldsymbol{v}}_o$, and then we can use it to localize the unknown source using the direction of arrival and range estimation described above. See also section 4.5 for a more detailed analysis of the velocity estimation.

Remark 4.4. If the decoherence time $\mathcal{T}_z$ and length $\mathcal{D}_z$ are not known, they can also be estimated using additional known sources. Definitions (4.17)–(4.18) show that $\mathcal{D}_z z^{1/2}$ and $\mathcal{T}_z z^{1/2}$ are constant with respect to z. Once estimated, these constants can be used in the imaging of the unknown source.

4.5. Single beam lateral velocity estimation. We observe from (4.29) and (4.36)–(4.37) that the source localization depends only on the Euclidian norm $|\boldsymbol{v}_o|$ of the cross-range component of the mean velocity of the medium. We show here that $\boldsymbol{v}_o$ can be obtained with only one beam and, when the receiver array is large and $z \gg z^*$ i.e., $\ell_s \gg D_z$, the velocity estimate is independent of the medium statistics and the source location.

The estimation of $\boldsymbol{v}_o$ is based on the imaging function

$$\begin{aligned}\mathcal{O}_{\rm v}(\boldsymbol{y},t,z) &= \int_{\mathbb{R}} \frac{{\rm d}\omega}{2\pi} e^{-i\omega t} \int_{\mathbb{R}^d} \frac{{\rm d}\boldsymbol{k}}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \int_{\mathbb{R}^d} {\rm d}\boldsymbol{x}\, W_{\rm est}(\omega,\boldsymbol{k},\boldsymbol{x},z) \\ &\approx \exp\Big(-\frac{k_o^2|\boldsymbol{y}|^2}{4\varkappa^2}\Big)\int_{\mathbb{R}^d} {\rm d}\boldsymbol{x}\, \exp\Big(-\frac{k_o^2|\boldsymbol{x}-\boldsymbol{x}_o|^2}{\varkappa^2}\Big) C(t,\boldsymbol{y},\boldsymbol{x},z). \end{aligned} \tag{4.38}$$

Substituting the expression (4.13) of the coherence function and carrying out the integrals we obtain that

$$|\mathcal{O}_{\rm v}(\boldsymbol{y},t,z)| \approx \frac{\sigma^2\pi^{d/2}(\varkappa\ell_s)^d}{2^{2+d}k_o^{d+1}\mathcal{A}_z^d}\exp\Big\{-\frac{t^2}{2\mathcal{T}_z^2} - \frac{|\boldsymbol{y}-s_z t\boldsymbol{v}_o|^2}{2\mathfrak{m}_z^2\mathcal{A}_z^2} - \frac{|t\boldsymbol{v}_o|^2}{\mathfrak{n}_z^2\mathcal{A}_z^2} - \frac{|\boldsymbol{x}_o|^2}{4\mathcal{A}_z^2}\Big\} \tag{4.39}$$

with the effective apperture

$$\mathcal{A}_z^2 = \frac{1}{4}\Big[\Big(\frac{\varkappa}{k_o}\Big)^2 + \Big(\frac{z}{k_o\ell_s}\Big)^2\Big(1+\frac{2\ell_s^2}{3D_z^2}\Big)\Big] \tag{4.40}$$

and dimensionless parameters

$$\begin{aligned}\mathfrak{m}_z^2 &= \frac{8}{1+\frac{2}{3\mathcal{D}_z^2}\big(\frac{z}{k_o\ell_s}\big)^2\Big(1+\frac{\ell_s^2}{2\mathcal{D}_z^2}\Big) + \big(\frac{\varkappa}{k_o\ell_s}\big)^2\Big(1+\frac{2\ell_s^2}{\mathcal{D}_z^2}\Big) + \big(\frac{k_o}{\varkappa}\big)^2\big(\frac{z}{k_o\ell_s}\big)^2\Big(1+\frac{2\ell_s^2}{3\mathcal{D}_z^2}\Big)},\\ \mathfrak{n}_z^2 &= \frac{\mathfrak{m}_z^2}{s_z(q_z-s_z/2)},\\ s_z &= \frac{\mathfrak{m}_z^2}{4\mathcal{D}_z^2}\Big[\Big(\frac{\varkappa}{k_o}\Big)^2 + \frac{1}{2}\Big(\frac{z}{k_o\ell_s}\Big)^2\Big(1+\frac{\ell_s^2}{3\mathcal{D}_z^2}\Big)\Big], \qquad q_z = \frac{\Big(\frac{\varkappa}{k_o}\Big)^2 + \Big(\frac{z}{k_o\ell_s}\Big)^2\Big(1+\frac{\ell_s^2}{6\mathcal{D}_z^2}\Big)}{2\Big(\frac{\varkappa}{k_o}\Big)^2 + \Big(\frac{z}{k_o\ell_s}\Big)^2\Big(1+\frac{\ell_s^2}{3\mathcal{D}_z^2}\Big)}.\end{aligned}$$

These depend on the radii $\varkappa/k_o$ of the array and ℓ_s of the source, the decoherence length $\mathcal{D}_z$, and the ratio $z/(k_o\ell_s)$ that quantifies the cross-range resolution of focusing of a wave using time delay beamforming at a source of radius ℓ_s.

To estimate $\boldsymbol{v}_o$ we can proceed as follows. First, we estimate for each time t the position $\boldsymbol{y}_{\max}(t)$ that maximizes $\boldsymbol{y}\mapsto\mathcal{O}_{\rm v}(\boldsymbol{y};t,z)$. Second, we note from (4.39) that $\boldsymbol{y}_{\max}(t)$ should be a linear function in t of the form $\boldsymbol{y}_{\max}(t) = s_z\boldsymbol{v}_o t$. Therefore, we can estimate $s_z\boldsymbol{v}_o$ with a weighted linear least squares regression of $\boldsymbol{y}_{\max}(t)$ with respect to t. In practice s_z is likely unknown. However, in the case of a large receiver array with radius satisfying

$$\frac{\varkappa}{k_o} \gg \max\Big\{\frac{z}{k_o\ell_s},\frac{z}{k_o\mathcal{D}_z}\Big\}, \tag{4.41}$$

and for $z\gg z^\star$, so that $\ell_s\gg\mathcal{D}_z$, we obtain that $s_z\approx 1$. Thus, the least squares regression gives an unbiased estimate of $\boldsymbol{v}_o$.

In view of (4.39), the least squares regression can be carried out over a time interval with length of the order of $\min(\mathcal{T}_z,\mathfrak{n}_z\mathcal{A}_z/|\boldsymbol{v}_o|)$. Beyond this critical time the function $\mathcal{O}_{\rm v}$ vanishes. Therefore, as long as $|\boldsymbol{v}_o|<\mathfrak{n}_z\mathcal{A}_z/\mathcal{T}_z$, the velocity resolution is

$${\rm res}_{\rm v} = \frac{\mathfrak{m}_z\mathcal{A}_z}{s_z\mathcal{T}_z} \approx \frac{\mathcal{D}_z}{\mathcal{T}_z}, \tag{4.42}$$

where the approximation is for a large array and $\ell_s\gg\mathcal{D}_z$. If $|\boldsymbol{v}_o|$ is larger than $\mathfrak{n}_z\mathcal{A}_z/\mathcal{T}_z$, then the resolution is reduced to

$${\rm res}_{\rm v} = \frac{\mathfrak{m}_z\mathcal{A}_z}{s_z\mathfrak{n}_z\mathcal{A}_z/|\boldsymbol{v}_o|} \approx \frac{\mathcal{D}_z}{\mathcal{T}_z}\frac{|\boldsymbol{v}_o|\mathcal{T}_z}{\mathfrak{n}_z\mathcal{A}_z}. \tag{4.43}$$

5. Analysis of the wave field. To derive the results stated in section 3, we begin in section 5.1 with a slight reformulation, which transforms (1.2) into a form that is more convenient for the analysis. We scale the resulting equation in section 5.2, in the regime defined in section 2.2, and then we change coordinates to a moving frame in section 5.3. In this frame we write the wave as a superposition of time-harmonic plane waves with random amplitudes that model the net scattering in the random medium, as described in section 5.4. We explain in section 5.5 that the backward going waves are negligible and use the diffusion approximation theory in section 5.6 to analyze the amplitudes of the forward going waves, in the limit $\varepsilon \to 0$. We end in section 5.8 with the paraxial limit.

5.1. Transformation of the wave equation. Let us define the new potential

$$\psi(t, \vec{\boldsymbol{x}}) = \frac{\sqrt{\rho(t, \vec{\boldsymbol{x}})}}{\sqrt{\rho_o}} \phi(t, \vec{\boldsymbol{x}}), \tag{5.1}$$

and substitute it in (1.2) to obtain the wave equation

$$\begin{aligned} D_t \left[\frac{1}{c^2(t, \vec{\boldsymbol{x}})} D_t \psi(t, \vec{\boldsymbol{x}}) \right] - \frac{D_t \psi(t, \vec{\boldsymbol{x}}) D_t \ln \rho(t, \vec{\boldsymbol{x}})}{c^2(t, \vec{\boldsymbol{x}})} - \Delta_{\vec{\boldsymbol{x}}} \psi(t, \vec{\boldsymbol{x}}) + q(t, \vec{\boldsymbol{x}}) \psi(t, \vec{\boldsymbol{x}}) \\ = \sigma_s \frac{\sqrt{\rho(t, \vec{\boldsymbol{x}})}}{\sqrt{\rho_o}} e^{-i\omega_o t} S\Big(\frac{t}{T_s}, \frac{\boldsymbol{x}}{\ell_s}\Big) \delta(z), \end{aligned} \tag{5.2}$$

for $t \in \mathbb{R}$ and $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^{d+1}$, where $\Delta_{\vec{\boldsymbol{x}}}$ is the Laplacian operator and

$$\begin{aligned} q(t, \vec{\boldsymbol{x}}) = & \frac{\Delta_{\vec{\boldsymbol{x}}} \sqrt{\rho(t, \vec{\boldsymbol{x}})}}{\sqrt{\rho(t, \vec{\boldsymbol{x}})}} - \frac{1}{c^2(t, \vec{\boldsymbol{x}})} \left\{ \frac{D_t^2 \sqrt{\rho(t, \vec{\boldsymbol{x}})}}{\sqrt{\rho(t, \vec{\boldsymbol{x}})}} - \frac{1}{2} \left[D_t \ln \rho(t, \vec{\boldsymbol{x}}) \right]^2 \right\} \\ & - \frac{1}{2} D_t c^{-2}(t, \vec{\boldsymbol{x}}) D_t \ln \rho(t, \vec{\boldsymbol{x}}). \end{aligned} \tag{5.3}$$

The initial condition (1.3) becomes

$$\psi(t, \vec{\boldsymbol{x}}) \equiv 0, \quad t \ll -T_s. \tag{5.4}$$

5.2. Scaled wave equation. We use the scaling regime defined in section 2.2 and denote with primes the dimensionless, order one variables

$$\vec{\boldsymbol{x}} = L \vec{\boldsymbol{x}}', \quad t = T_L t'. \tag{5.5}$$

We also let

$$\vec{\boldsymbol{v}}_o = V \vec{\boldsymbol{v}}_o', \quad c_o = c_o c_o', \quad \omega_o = \omega_o \frac{\omega_o'}{2\pi}, \tag{5.6}$$

where the constants $c_o' = 1$ and $\omega_o' = 2\pi$ are introduced so that the scaled equation is easier to interpret.

In the scaled variables, and using the source amplitude (2.17), the right-hand side in (5.2) becomes

$$\sigma_s \frac{\sqrt{\rho(t, \vec{\boldsymbol{x}})}}{\sqrt{\rho_o}} e^{-i\omega_o t} S\Big(\frac{t}{T_s}, \frac{\boldsymbol{x}}{\ell_s}\Big) \delta(z) = \frac{\big[1 + O(\sqrt{\varepsilon})\big]}{\varepsilon \eta_s L^2} \Big(\frac{\gamma_s}{\varepsilon}\Big)^d e^{-i \frac{\omega_o'}{\varepsilon} t'} S\Big(\frac{t'}{\eta_s}, \frac{\gamma_s \boldsymbol{x}'}{\varepsilon}\Big) \delta(z'). \tag{5.7}$$

We also have from definitions (2.1)–(2.3) that the random coefficients take the form

$$\frac{\vec{\boldsymbol{v}}(t,\vec{\boldsymbol{x}})}{V} = \vec{\boldsymbol{v}}'(t',\vec{\boldsymbol{x}}') = \vec{\boldsymbol{v}}'_o + \sqrt{\varepsilon\gamma}\,\bar{\sigma}_v\,\vec{\boldsymbol{\nu}}\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big), \tag{5.8}$$

$$\frac{\rho(t,\vec{\boldsymbol{x}})}{\rho_o} = \exp\left[\sqrt{\varepsilon\gamma}\,\bar{\sigma}_\rho\,\nu_\rho\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big)\right], \tag{5.9}$$

$$\frac{c_o^2}{c^2(t,\vec{\boldsymbol{x}})} = \frac{1}{(c'_o)^2}\left[1 + \sqrt{\varepsilon\gamma}\,\bar{\sigma}_c\,\nu_c\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big)\right] \tag{5.10}$$

with scaled standard deviations $\bar{\sigma}_c, \bar{\sigma}_v, \bar{\sigma}_\rho$ defined in (2.14).

The solution ψ of (5.2) must have variations in t' and $\vec{\boldsymbol{x}}'$ on the same scale as the source term and the coefficients (5.8)–(5.10), meaning that $\partial_{t'}\psi \sim 1/\varepsilon$, $|\nabla_{\vec{\boldsymbol{x}}'}\psi| \sim 1/\varepsilon$. From (2.8)–(2.15) we obtain that in the scaled variables we have

$$D_t = \frac{1}{T_L c'_o} D^\varepsilon_{t'} \quad \text{with } D^\varepsilon_{t'} = \partial_{t'} + \varepsilon\vec{\boldsymbol{v}}'(t',\vec{\boldsymbol{x}}')\cdot\nabla_{\vec{\boldsymbol{x}}'}. \tag{5.11}$$

Equation (5.9) gives

$$D_t \ln\rho(t,\vec{\boldsymbol{x}}) = \frac{\sqrt{\varepsilon\gamma}\bar{\sigma}_\rho}{T_L} D^\varepsilon_{t'}\nu_\rho\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big) = \frac{O(\sqrt{\varepsilon})}{T_L}$$

and

$$\begin{aligned}\frac{D_t^2\sqrt{\rho(t,\vec{\boldsymbol{x}})}}{\sqrt{\rho(t,\vec{\boldsymbol{x}})}} &= \frac{\sqrt{\varepsilon\gamma}\bar{\sigma}_\rho}{2T_L^2}(D^\varepsilon_{t'})^2\nu_\rho\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big) + \frac{\varepsilon\gamma\bar{\sigma}_\rho^2}{4T_L^2}\left[D^\varepsilon_{t'}\nu_\rho\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big)\right]^2 \\ &= \frac{O(\sqrt{\varepsilon})}{T_L^2}.\end{aligned}$$

From (5.10) we get

$$D_t\left[\frac{1}{c^2(t,\vec{\boldsymbol{x}})}\right] = \frac{\sqrt{\varepsilon\gamma}\bar{\sigma}_c}{(c_o c'_o)^2 T_L} D^\varepsilon_{t'}\nu_c\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big) = \frac{O(\sqrt{\varepsilon})}{c_o L},$$

and q defined in (5.3) takes the form

$$q(t,\vec{\boldsymbol{x}}) = \frac{\gamma^{5/2}\bar{\sigma}_\rho}{2\varepsilon^{3/2}L^2}\left[Q^\varepsilon\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big) + O(\varepsilon^2)\right] \tag{5.12}$$

with

$$Q^\varepsilon(\tau,\vec{\boldsymbol{r}}) = \Delta_{\vec{\boldsymbol{r}}}\nu_\rho(\tau,\vec{\boldsymbol{r}}) + \frac{\sqrt{\varepsilon\gamma}\bar{\sigma}_\rho}{2}\left|\nabla_{\vec{\boldsymbol{r}}}\nu_\rho(\tau,\vec{\boldsymbol{r}})\right|^2. \tag{5.13}$$

Substituting in (5.2) and multiplying both sides by εL^2, we obtain that the potential denoted by $\psi'(t',\vec{\boldsymbol{x}}')$ in the scaled variables satisfies

$$\begin{aligned}&\varepsilon\left\{\frac{\left[1 + \sqrt{\varepsilon\gamma}\,\bar{\sigma}_c\nu_c\big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\big)\right]}{(c'_o)^2}\partial^2_{t'} + \frac{2\varepsilon}{(c'_o)^2}\vec{\boldsymbol{v}}'_o\cdot\nabla_{\vec{\boldsymbol{x}}'}\partial_{t'} - \Delta_{\vec{\boldsymbol{x}}'}\right\}\psi'(t',\vec{\boldsymbol{x}}') \\ &+ \frac{\bar{\sigma}_\rho\gamma^{5/2}}{2\sqrt{\varepsilon}}Q^\varepsilon\Big(\frac{t'}{\eta}, \frac{\vec{\boldsymbol{x}}' - \varepsilon\vec{\boldsymbol{v}}'_o t'}{\varepsilon/\gamma}\Big)\psi'(t',\vec{\boldsymbol{x}}') \approx \frac{1}{\eta_s}\Big(\frac{\gamma_s}{\varepsilon}\Big)^d e^{-i\frac{\omega'_o}{\varepsilon}t'}S\Big(\frac{t'}{\eta_s}, \frac{\boldsymbol{x}'}{\varepsilon/\gamma_s}\Big)\delta(z')\end{aligned} \tag{5.14}$$

with initial condition obtained from (1.3) and (5.1),

$$\psi'(t', \vec{\boldsymbol{x}}') \equiv 0, \quad t' \ll -\eta_s. \tag{5.15}$$

The approximation in (5.14) is because we neglect $O(\sqrt{\varepsilon})$ terms that tend to zero in the limit $\varepsilon \to 0$. Note in particular that the random perturbations $\vec{\boldsymbol{\nu}}$ of the velocity of the flow appear in these terms and are negligible in our regime.

All variables are assumed scaled in the remainder of the section and we simplify notation by dropping the primes.

5.3. Moving frame. Let us introduce the notation $\vec{\boldsymbol{v}}_o = (\boldsymbol{v}_o, v_{oz})$ for the scaled mean velocity of the ambient flow and change the range coordinate z to

$$\zeta = z - \varepsilon v_{oz} t. \tag{5.16}$$

We denote the potential in this moving frame by

$$u^\varepsilon(t, \boldsymbol{x}, \zeta) = \psi(t, \boldsymbol{x}, \zeta + \varepsilon v_{oz} t) \tag{5.17}$$

and obtain from (5.14) that it satisfies the wave equation

$$\begin{aligned} &\varepsilon \left\{ \frac{\left[1 + \sqrt{\varepsilon\gamma}\, \bar{\sigma}_c \nu_c\left(\frac{t}{\eta}, \frac{\boldsymbol{x} - \varepsilon \boldsymbol{v}_o t}{\varepsilon/\gamma}, \frac{\gamma\zeta}{\varepsilon}\right)\right]}{c_o^2} \partial_t^2 + \frac{2\varepsilon}{c_o^2} \boldsymbol{v}_o \cdot \nabla_{\boldsymbol{x}} \partial_t - \Delta_{\boldsymbol{x}} - \partial_\zeta^2 \right\} u^\varepsilon(t, \boldsymbol{x}, \zeta) \\ &+ \frac{\bar{\sigma}_\rho \gamma^{5/2}}{2\sqrt{\varepsilon}} Q^\varepsilon\Big(\frac{t}{\eta}, \frac{\boldsymbol{x} - \varepsilon \boldsymbol{v}_o t}{\varepsilon/\gamma}, \frac{\gamma\zeta}{\varepsilon}\Big) u^\varepsilon(t, \boldsymbol{x}, \zeta) \approx \Big(\frac{\gamma_s}{\varepsilon}\Big)^d \frac{e^{-i\frac{\omega_o}{\varepsilon}t}}{\eta_s} S\Big(\frac{t}{\eta_s}, \frac{\gamma_s \boldsymbol{x}}{\varepsilon}\Big) \delta(\zeta + \varepsilon v_{oz} t), \end{aligned} \tag{5.18}$$

where again we neglect the terms that become negligible in the limit $\varepsilon \to 0$. The gradient $\nabla_{\boldsymbol{x}}$ and Laplacian $\Delta_{\boldsymbol{x}}$ are in the cross-range variable $\boldsymbol{x} \in \mathbb{R}^d$.

5.4. Wave decomposition. The interaction of the waves with the random medium depends on the frequency and direction of propagation, so we decompose $u^\varepsilon(t, \boldsymbol{x}, \zeta)$ using the Fourier transform

$$\widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, u^\varepsilon(t, \boldsymbol{x}, \zeta) e^{i\left(\frac{\omega_o}{\varepsilon} + \omega\right)t - i\frac{\boldsymbol{k}}{\varepsilon} \cdot \boldsymbol{x}} \tag{5.19}$$

with inverse

$$u^\varepsilon(t, \boldsymbol{x}, \zeta) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi\varepsilon)^d}\, \widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{-i\left(\frac{\omega_o}{\varepsilon} + \omega\right)t + i\frac{\boldsymbol{k}}{\varepsilon} \cdot \boldsymbol{x}}. \tag{5.20}$$

The Fourier transform of (5.18) is

$$\begin{aligned} &\left[-\frac{\beta^2(\boldsymbol{k})}{\varepsilon} - 2k_o\Big(\frac{\omega}{c_o} - \frac{\boldsymbol{v}_o \cdot \boldsymbol{k}}{c_o}\Big) - \varepsilon\partial_\zeta^2\right] \widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) - \frac{\eta\gamma^{1/2-d}}{\sqrt{\varepsilon}} \int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{2\pi} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^d} \\ &\quad \widehat{u}^\varepsilon(\omega', \boldsymbol{k}', \zeta) \left[k_o^2 \bar{\sigma}_c \widehat{\nu}_c\Big(\eta\big(\omega - \omega' - (\boldsymbol{k} - \boldsymbol{k}') \cdot \boldsymbol{v}_o\big), \frac{\boldsymbol{k} - \boldsymbol{k}'}{\gamma}, \frac{\gamma\zeta}{\varepsilon}\Big) \right. \\ &\quad \left. - \frac{\gamma^2 \bar{\sigma}_\rho}{2} \widehat{Q}^\varepsilon\Big(\eta\big(\omega - \omega' - (\boldsymbol{k} - \boldsymbol{k}') \cdot \boldsymbol{v}_o\big), \frac{\boldsymbol{k} - \boldsymbol{k}'}{\gamma}, \frac{\gamma\zeta}{\varepsilon}\Big)\right] \\ &\approx \frac{e^{-i\frac{\omega\zeta}{\varepsilon v_{oz}}}}{\varepsilon \eta_s v_{oz}} \breve{S}\Big(-\frac{\zeta}{\varepsilon \eta_s v_{oz}}, \frac{\boldsymbol{k}}{\gamma_s}\Big), \end{aligned} \tag{5.21}$$

where

$$\beta(\boldsymbol{k}) = \sqrt{k_o^2 - |\boldsymbol{k}|^2}, \quad k_o = \frac{\omega_o}{c_o}, \tag{5.22}$$

and

$$\breve{S}(\tau, \boldsymbol{\kappa}) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\, S(\tau, \boldsymbol{r}) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}}.$$

Note that the right-hand side in (5.21) is supported at $|\boldsymbol{k}| = O(\gamma_s)$, so by keeping γ_s small, we ensure that $\beta(\boldsymbol{k})$ remains real valued in our regime. Physically, this means that $\widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta)$ is a propagating wave, not evanescent.

Note also that if the mean velocity $\vec{\boldsymbol{v}}_o$ is orthogonal to the range direction, the source term satisfies

$$\lim_{v_{oz}\to 0} \frac{e^{-i\frac{\omega\zeta}{\varepsilon v_{oz}}}}{\varepsilon\eta_s v_{oz}} \breve{S}\Big(-\frac{\zeta}{\varepsilon\eta_s v_{oz}}, \frac{\boldsymbol{k}}{\gamma_s}\Big) = \widehat{S}\Big(\eta_s\omega, \frac{\boldsymbol{k}}{\gamma_s}\Big)\delta(\zeta)$$

in the sense of distributions, where

$$\widehat{S}(\omega, \boldsymbol{\kappa}) = \int_{\mathbb{R}} \mathrm{d}\tau \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r}\, S(\tau, \boldsymbol{r}) e^{i\omega\tau - i\boldsymbol{\kappa}\cdot\boldsymbol{r}}.$$

We introduce

$$a^\varepsilon(\omega, \boldsymbol{k}, \zeta) = \Big[\frac{\sqrt{\beta(\boldsymbol{k})}}{2}\widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) + \frac{\varepsilon}{2i\sqrt{\beta(\boldsymbol{k})}}\partial_\zeta \widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta)\Big] e^{-i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}}, \tag{5.23}$$

$$a_-^\varepsilon(\omega, \boldsymbol{k}, \zeta) = \Big[\frac{\sqrt{\beta(\boldsymbol{k})}}{2}\widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) - \frac{\varepsilon}{2i\sqrt{\beta(\boldsymbol{k})}}\partial_\zeta \widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta)\Big] e^{i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}}, \tag{5.24}$$

so that we have the decomposition

$$\widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) = \frac{1}{\sqrt{\beta(\boldsymbol{k})}}\left[a^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}} + a_-^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{-i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}}\right], \tag{5.25}$$

and the complex amplitudes a^ε and a_-^ε satisfy the relation

$$\partial_\zeta a^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}} + \partial_\zeta a_-^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{-i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}} = 0. \tag{5.26}$$

This gives that

$$\partial_\zeta \widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) = \frac{i\sqrt{\beta(\boldsymbol{k})}}{\varepsilon}\left[a^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}} - a_-^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{-i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}}\right] \tag{5.27}$$

and, moreover, that

$$\partial_\zeta^2 \widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) = -\frac{\beta^2(\boldsymbol{k})}{\varepsilon^2}\widehat{u}^\varepsilon(\omega, \boldsymbol{k}, \zeta) + \frac{2i\sqrt{\beta(\boldsymbol{k})}}{\varepsilon}\partial_\zeta a^\varepsilon(\omega, \boldsymbol{k}, \zeta) e^{i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}}. \tag{5.28}$$

The decomposition in (5.25) and (5.20) is a decomposition of $u^\varepsilon(t, \boldsymbol{x}, \zeta)$ into a superposition of plane waves with wave vectors

$$\vec{\boldsymbol{k}}_\pm = (\boldsymbol{k}, \pm\beta(\boldsymbol{k})), \tag{5.29}$$

where the plus sign denotes the waves propagating in the positive range direction and the negative sign denotes the waves propagating in the negative range direction.

The amplitudes a^ε and a_-^ε of these waves are random fields, which evolve in range according to (5.26) and the equation

$$\begin{aligned}
\partial_\zeta a^\varepsilon(\omega,\boldsymbol{k},\zeta) &\approx \frac{ik_o(\omega-\boldsymbol{v}_o\cdot\boldsymbol{k})}{c_o\beta(\boldsymbol{k})}\Big[a^\varepsilon(\omega,\boldsymbol{k},\zeta)+a_-^\varepsilon(\omega,\boldsymbol{k},\zeta)e^{-2i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}}\Big]\\
&+\frac{i\eta\gamma^{1/2-d}}{2\sqrt{\varepsilon}}\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^d}\Big[k_o^2\bar\sigma_c\widehat{\nu}_c\Big(\eta\big(\omega-\omega'-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}_o\big),\frac{\boldsymbol{k}-\boldsymbol{k}'}{\gamma},\frac{\gamma\zeta}{\varepsilon}\Big)\\
&-\frac{\gamma^2\bar\sigma_\rho}{2}\widehat{Q}^\varepsilon\Big(\eta\big(\omega-\omega'-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}_o\big),\frac{\boldsymbol{k}-\boldsymbol{k}'}{\gamma},\frac{\gamma\zeta}{\varepsilon}\Big)\Big]\\
&\times\frac{1}{\sqrt{\beta(\boldsymbol{k})\beta(\boldsymbol{k}')}}\Big[a^\varepsilon(\omega',\boldsymbol{k}',\zeta)e^{i\big[\beta(\boldsymbol{k}')-\beta(\boldsymbol{k})\big]\frac{\zeta}{\varepsilon}}+a_-^\varepsilon(\omega',\boldsymbol{k}',\zeta)e^{-i\big[\beta(\boldsymbol{k}')+\beta(\boldsymbol{k})\big]\frac{\zeta}{\varepsilon}}\Big]\\
&+\frac{i}{2\sqrt{\beta(\boldsymbol{k})}\varepsilon\eta_s v_{oz}}\breve{S}\Big(-\frac{\zeta}{\varepsilon\eta_s v_{oz}},\frac{\boldsymbol{k}}{\gamma_s}\Big)e^{-i\frac{\omega\zeta}{\varepsilon v_{oz}}-i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}},
\end{aligned}\tag{5.30}$$

derived by substituting (5.25)–(5.28) into (5.21).

5.5. Forward scattering approximation. Equation (5.30) shows that the amplitudes a^ε are coupled to each other and to a_-^ε. In our scaling regime, where the cone of directions of propagation has small opening angle controlled by the parameter γ_s, and where the covariance (2.5) of the fluctuations is smooth, the coupling between a^ε and a_-^ε becomes negligible in the limit $\varepsilon\to 0$. We refer to [2, section C.2] and [3, section 5.2] for a more detailed explanation of this fact.

Using the assumption that the random fluctuations are supported at finite range (see section 2), we require that the wave be outgoing at $|\zeta|\to\infty$. This radiation condition and the negligible coupling between a^ε and a_-^ε in the limit $\varepsilon\to 0$ imply that we can neglect the backward going waves, and we can write

$$\widehat{u}^\varepsilon(\omega,\boldsymbol{k},\zeta)\approx\frac{a^\varepsilon(\omega,\boldsymbol{k},\zeta)}{\sqrt{\beta(\boldsymbol{k})}}e^{i\beta(\boldsymbol{k})\frac{\zeta}{\varepsilon}},\quad \zeta>O(\varepsilon).\tag{5.31}$$

The starting value of $a^\varepsilon(\omega,\boldsymbol{k},\zeta)$ is determined by the source term in (5.30), which contributes only for $\zeta=v_{oz}O(\varepsilon)$. For such small ζ, we can change variables $\zeta=\varepsilon\xi$ in (5.30) and obtain that

$$\partial_\xi a^\varepsilon(\omega,\boldsymbol{k},\varepsilon\xi)=\frac{i}{2\sqrt{\beta(\boldsymbol{k})}\eta_s v_{oz}}\breve{S}\Big(-\frac{\xi}{\eta_s v_{oz}},\frac{\boldsymbol{k}}{\gamma_s}\Big)e^{-i\frac{\omega\xi}{v_{oz}}-i\beta(\boldsymbol{k})\xi}+O(\sqrt{\varepsilon}).$$

Integrating in ξ and using that $a^\varepsilon(\omega,\boldsymbol{k},\zeta)$ vanishes for $\zeta\ll -O(\varepsilon)$, we obtain that

$$\begin{aligned}
a^\varepsilon(\omega,\boldsymbol{k},\varepsilon\xi)&\approx\frac{i}{2\sqrt{\beta(\boldsymbol{k})}\eta_s v_{oz}}\int_{\mathbb{R}}\mathrm{d}\xi\,\breve{S}\Big(-\frac{\xi}{\eta_s v_{oz}},\frac{\boldsymbol{k}}{\gamma_s}\Big)e^{-i\frac{\omega\xi}{v_{oz}}-i\beta(\boldsymbol{k})\xi}\\
&=\frac{i}{2\sqrt{\beta(\boldsymbol{k})}}\widehat{S}\Big(\eta_s(\omega+\beta(\boldsymbol{k})v_{oz}),\frac{\boldsymbol{k}}{\gamma_s}\Big).
\end{aligned}$$

We use this expression as the initial condition for the forward going amplitudes

$$a^\varepsilon(\omega,\boldsymbol{k},0+)\approx\frac{i}{2\sqrt{\beta(\boldsymbol{k})}}\widehat{S}\Big(\eta_s(\omega+\beta(\boldsymbol{k})v_{oz}),\frac{\boldsymbol{k}}{\gamma_s}\Big)\tag{5.32}$$

and drop the source term and the backward going amplitudes a_-^ε in (5.30) for range $\zeta>0$.

5.6. The acoustic pressure field in the Markovian limit. By definitions (1.1), (5.1), (5.17), and the scaling relations (5.5), the acoustic pressure is

$$p(T_L t, L\boldsymbol{x}, Lz) \approx -\frac{\rho_o}{T_L}\partial_t u^\varepsilon(t, \boldsymbol{x}, z - \varepsilon v_{oz} t).$$

Furthermore, (5.31) and the Fourier decomposition (5.20) give that

$$\frac{p(T_L t, L\boldsymbol{x}, Lz)}{2\pi c_o \rho_o/\lambda_o} \approx \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi}\int_{\mathbb{R}^d}\frac{\mathrm{d}\boldsymbol{k}}{(2\pi\varepsilon)^d}\frac{i a^\varepsilon(\omega, \boldsymbol{k}, z)}{\sqrt{\beta(\boldsymbol{k})}} e^{-i\left(\frac{\omega_o}{\varepsilon}+\omega+\beta(\boldsymbol{k})v_{oz}\right)t + i\frac{(\boldsymbol{k},\beta(\boldsymbol{k}))}{\varepsilon}\cdot(\boldsymbol{x},z)},$$

where we have used (5.30) to write $a^\varepsilon(\omega, \boldsymbol{k}, z - \varepsilon v_{oz} t) = a^\varepsilon(\omega, \boldsymbol{k}, z) + O(\sqrt{\varepsilon})$.

The shifted scaled frequency $\omega + \beta(\boldsymbol{k})v_{oz}$ appears in the initial condition (5.32), and the random processes $\widehat{\nu}_c$ and $\widehat{Q}^\varepsilon$ in (5.30) depend on $\boldsymbol{k}/\gamma$. Thus, it is convenient to introduce the variables

$$\Omega = \eta\big[\omega + \beta(\boldsymbol{k})v_{oz}\big], \quad \boldsymbol{K} = \boldsymbol{k}/\gamma, \tag{5.33}$$

and rewrite the expression of the pressure as

$$\frac{p(T_L t, L\boldsymbol{x}, Lz)}{\omega_o \rho_o} \approx \int_{\mathbb{R}} \frac{\mathrm{d}\Omega}{2\pi\eta}\int_{\mathbb{R}^d}\frac{\mathrm{d}\boldsymbol{K}}{(2\pi\varepsilon/\gamma)^d}\frac{A^\varepsilon(\Omega, \boldsymbol{K}, z)}{\sqrt{\beta(\gamma\boldsymbol{K})}} e^{-i\left(\frac{\omega_o}{\varepsilon}+\frac{\Omega}{\eta}\right)t + i\frac{(\gamma\boldsymbol{K},\beta(\gamma\boldsymbol{K}))}{\varepsilon}\cdot(\boldsymbol{x},z)} \tag{5.34}$$

with redefined amplitude

$$A^\varepsilon(\Omega, \boldsymbol{K}, z) = i a^\varepsilon\Big(\frac{\Omega}{\eta} - \beta(\gamma\boldsymbol{K})v_{oz}, \gamma\boldsymbol{K}, z\Big) + o(1). \tag{5.35}$$

The $o(1)$ term, which tends to zero as $\varepsilon \to 0$, is used in this definition so that we have an equal sign in the evolution equation for A^ε, derived from (5.30), after neglecting the backward going amplitudes,

$$\begin{aligned}
\partial_z A^\varepsilon(\Omega, \boldsymbol{K}, z) &= \frac{i k_o}{\beta(\gamma\boldsymbol{K})}\Big[\frac{\Omega}{\eta c_o} - \frac{\vec{\boldsymbol{v}}_o}{c_o}\cdot\big(\gamma\boldsymbol{K}, \beta(\gamma\boldsymbol{K})\big)\Big]A^\varepsilon(\Omega, \boldsymbol{K}, z)\\
&+ \frac{i}{2}\sqrt{\frac{\gamma}{\varepsilon}}\int_{\mathbb{R}}\frac{\mathrm{d}\Omega'}{2\pi}\int_{\mathbb{R}^d}\frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d}\frac{A^\varepsilon(\Omega', \boldsymbol{K}', z)}{\sqrt{\beta(\gamma\boldsymbol{K})\beta(\gamma\boldsymbol{K}')}} e^{i\left[\beta(\gamma\boldsymbol{K}')-\beta(\gamma\boldsymbol{K})\right]\frac{z}{\varepsilon}}\\
&\times\Big[k_o^2\bar{\sigma}_c\widehat{\nu}_c\Big(\Omega-\Omega'-\eta\big(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}', \beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')\big)\cdot\vec{\boldsymbol{v}}_o, \boldsymbol{K}-\boldsymbol{K}', \frac{\gamma z}{\varepsilon}\Big)\\
&-\frac{\gamma^2\bar{\sigma}_\rho}{2}\widehat{Q}^\varepsilon\Big(\Omega-\Omega'-\eta\big(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}', \beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')\big)\cdot\vec{\boldsymbol{v}}_o, \boldsymbol{K}-\boldsymbol{K}', \frac{\gamma z}{\varepsilon}\Big)\Big]
\end{aligned} \tag{5.36}$$

for $z > 0$. The initial condition (5.32) becomes

$$A^\varepsilon(\Omega, \boldsymbol{K}, 0+) = A_o(\Omega, \boldsymbol{K}) = -\frac{1}{2\sqrt{\beta(\gamma\boldsymbol{K})}}\widehat{S}\Big(\frac{\eta_s}{\eta}\Omega, \frac{\gamma}{\gamma_s}\boldsymbol{K}\Big). \tag{5.37}$$

5.7. The Markovian limit. Let $L^2(\mathscr{O}, \mathbb{C})$ be the space of complex-valued, square-integrable functions defined on the set

$$\mathscr{O} = \{\Omega \in \mathbb{R}\} \times \{\boldsymbol{K} \in \mathbb{R}^d, \gamma|\boldsymbol{K}| < k_o\} \tag{5.38}$$

and denote $\boldsymbol{A}^\varepsilon(z) = (A^\varepsilon(\Omega, \boldsymbol{K}, z))_{(\Omega,\boldsymbol{K})\in\mathscr{O}}$ for $z \geq 0$. From (5.36) we obtain the conservation of energy relation

$$\partial_z \int_{\mathscr{O}} \mathrm{d}\Omega\mathrm{d}\boldsymbol{K}\, |A^\varepsilon(\Omega, \boldsymbol{K}, z)|^2 = 0, \tag{5.39}$$

so the Markov process $\boldsymbol{A}^\varepsilon(z) \in L^2(\mathscr{O}, \mathbb{C})$ lives on the surface of the ball with center at the origin and ε independent radius $R_{\boldsymbol{A}}$ defined by

$$R_{\boldsymbol{A}}^2 = \int_{\mathscr{O}} \mathrm{d}\Omega \mathrm{d}\boldsymbol{K}\, |A^\varepsilon(\Omega, \boldsymbol{K}, z)|^2 = \int_{\mathscr{O}} \mathrm{d}\Omega \mathrm{d}\boldsymbol{K}\, |A_o(\Omega, \boldsymbol{K})|^2. \tag{5.40}$$

We describe the Markovian limit $\varepsilon \to 0$ in Appendix A. The result is that the process of $\boldsymbol{A}^\varepsilon(z)$ converges weakly in $\mathcal{C}([0,\infty), L^2)$ to a Markov process whose infinitesimal generator can be identified. The first and second moments of the limit process are described below.

5.7.1. The mean amplitude. The expectation of $A^\varepsilon(\Omega, \boldsymbol{K}, z)$ in the limit $\varepsilon \to 0$ is given by

$$\lim_{\varepsilon \to 0} \mathbb{E}\big[A^\varepsilon(\Omega, \boldsymbol{K}, z)\big] = A_o(\Omega, \boldsymbol{K}) e^{i\theta(\Omega, \boldsymbol{K})z + D(\boldsymbol{K})z}, \tag{5.41}$$

where

$$\theta(\Omega, \boldsymbol{K}) = \frac{k_o}{\beta(\gamma \boldsymbol{K})} \Big[\frac{\Omega}{\eta c_o} - \frac{\vec{\boldsymbol{v}}_o}{c_o} \cdot (\gamma \boldsymbol{K}, \beta(\gamma \boldsymbol{K}))\Big] + \frac{\gamma^3 \bar{\sigma}_\rho^2}{8\beta(\gamma \boldsymbol{K})} \Delta_{\vec{\boldsymbol{r}}} \mathscr{R}_{\rho\rho}(0, \vec{\boldsymbol{r}})\big|_{\vec{\boldsymbol{r}}=\boldsymbol{0}} \tag{5.42}$$

is a real phase and

$$\begin{aligned} D(\boldsymbol{K}) = -\int_{|\boldsymbol{K}'|<k_o/\gamma} \frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d} \frac{1}{4\beta(\gamma \boldsymbol{K})\beta(\gamma \boldsymbol{K}')} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{r} \int_0^\infty \mathrm{d}r_z\, e^{-i\left(\boldsymbol{K}-\boldsymbol{K}', \frac{\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')}{\gamma}\right)\cdot \vec{\boldsymbol{r}}} \\ \times \left\{ k_o^4 \bar{\sigma}_c^2 \mathscr{R}_{cc}(0, \vec{\boldsymbol{r}}) + \frac{\gamma^4 \bar{\sigma}_\rho^2}{4} \Delta_{\vec{\boldsymbol{r}}}^2 \mathscr{R}_{\rho\rho}(0, \vec{\boldsymbol{r}}) - k_o^2 \gamma^2 \bar{\sigma}_c \bar{\sigma}_\rho \Delta_{\vec{\boldsymbol{r}}} \mathscr{R}_{c\rho}(0, \vec{\boldsymbol{r}}) \right\} \end{aligned} \tag{5.43}$$

with $\vec{\boldsymbol{r}} = (\boldsymbol{r}, r_z)$. Moreover, $\mathrm{Re}\big[D(\boldsymbol{K})\big] < 0$, because

$$\int_{\mathbb{R}^{d+1}} \mathrm{d}\boldsymbol{r}\, e^{-i\vec{\boldsymbol{K}}\cdot\vec{\boldsymbol{r}}} \left\{ k_o^4 \bar{\sigma}_c^2 \mathscr{R}_{cc}(0, \vec{\boldsymbol{r}}) + \frac{\gamma^4 \bar{\sigma}_\rho^2}{4} \Delta_{\vec{\boldsymbol{r}}}^2 \mathscr{R}_{\rho\rho}(0, \vec{\boldsymbol{r}}) - k_o^2 \gamma^2 \bar{\sigma}_c \bar{\sigma}_\rho \Delta_{\vec{\boldsymbol{r}}} \mathscr{R}_{c\rho}(0, \vec{\boldsymbol{r}}) \right\} \geq 0$$

is the power spectral density of the process

$$X(t, \vec{\boldsymbol{r}}) = k_0^2 \bar{\sigma}_c \nu_c(t, \vec{\boldsymbol{r}}) - \frac{\bar{\sigma}_\rho \gamma^2}{2} \Delta_{\vec{\boldsymbol{r}}} \nu_\rho(t, \vec{\boldsymbol{r}}) \tag{5.44}$$

in the variable $\vec{\boldsymbol{r}}$ (for fixed t), which is nonnegative by Bochner's theorem. Thus, the mean amplitude decays on the range scale

$$\mathscr{S}(\boldsymbol{K}) = -\frac{1}{\mathrm{Re}\big[D(\boldsymbol{K})\big]}, \tag{5.45}$$

called the scattering mean free path. In the relatively high frequency regime the damping is mainly due to the fluctuations of the wave speed, while in the relatively low frequency regime the damping is mainly due to the fluctuations of the density. This damping is the mathematical manifestation of the randomization of the wave due to cumulative scattering.

Recall that we have assumed $\bar{\sigma}_\rho = O(1)$. Therefore, in the regime $\gamma \ll 1$, the $\mathscr{R}_{cc}$ term dominates in (5.43).

5.7.2. The mean intensity. The expectation of the intensity

$$I(\Omega, \boldsymbol{K}, z) = \lim_{\varepsilon \to 0} \mathbb{E}\big[|A^\varepsilon(\Omega, \boldsymbol{K}, z)|^2\big] \tag{5.46}$$

satisfies

$$\partial_z I(\Omega, \boldsymbol{K}, z) = \int_{\mathscr{O}} \frac{\mathrm{d}\Omega'}{2\pi} \frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d} Q(\Omega, \Omega', \boldsymbol{K}, \boldsymbol{K}') \left[I(\Omega', \boldsymbol{K}', z) - I(\Omega, \boldsymbol{K}, z)\right], \tag{5.47}$$

for $z > 0$, with initial condition obtained from (5.37):

$$I(\Omega, \boldsymbol{K}, 0) = |A_o(\Omega, \boldsymbol{K})|^2. \tag{5.48}$$

Denoting the power spectrum of $X(t, \vec{\boldsymbol{r}})$ in (5.44) by $P(\Omega, \vec{\boldsymbol{K}})$, and letting

$$\left(\widetilde{\Omega}, \vec{\boldsymbol{K}}\right) = \left(\Omega - \eta\big(\gamma \boldsymbol{K}, \beta(\gamma \boldsymbol{K})\big) \cdot \vec{\boldsymbol{v}}_o, \boldsymbol{K}, \frac{\beta(\gamma \boldsymbol{K})}{\gamma}\right),$$

the kernel in (5.47) is given by

$$Q(\Omega, \Omega', \boldsymbol{K}, \boldsymbol{K}') = \frac{P(\widetilde{\Omega} - \widetilde{\Omega}', \vec{\boldsymbol{K}} - \vec{\boldsymbol{K}}')}{4\beta(\gamma \boldsymbol{K})\beta(\gamma \boldsymbol{K}')},$$

that is explicitly

$$Q(\Omega, \Omega', \boldsymbol{K}, \boldsymbol{K}') = \Bigg\{ k_o^4 \bar{\sigma}_c^2 \widetilde{\mathscr{R}}_{cc} + \frac{\gamma^4 \bar{\sigma}_\rho^2}{4} \left[|\boldsymbol{K} - \boldsymbol{K}'|^2 + \Big(\frac{\beta(\gamma \boldsymbol{K}) - \beta(\gamma \boldsymbol{K}')}{\gamma} \Big)^2 \right]^2 \widetilde{\mathscr{R}}_{\rho\rho}$$
$$+ k_o^2 \gamma^2 \bar{\sigma}_c \bar{\sigma}_\rho \left[|\boldsymbol{K} - \boldsymbol{K}'|^2 + \Big(\frac{\beta(\gamma \boldsymbol{K}) - \beta(\gamma \boldsymbol{K}')}{\gamma} \Big)^2 \right] \widetilde{\mathscr{R}}_{\rho c} \Bigg\} \frac{1}{4\beta(\gamma \boldsymbol{K})\beta(\gamma \boldsymbol{K}')}, \tag{5.49}$$

where $\widetilde{\mathscr{R}}_{cc}$ is the power spectral density (3.9), evaluated as

$$\widetilde{\mathscr{R}}_{cc} = \widetilde{\mathscr{R}}_{cc}\Big(\Omega - \Omega' - \eta\big(\gamma \boldsymbol{K} - \gamma \boldsymbol{K}', \beta(\gamma \boldsymbol{K}) - \beta(\gamma \boldsymbol{K}')\big) \cdot \vec{\boldsymbol{v}}_o, \boldsymbol{K} - \boldsymbol{K}', \frac{\beta(\gamma \boldsymbol{K}) - \beta(\gamma \boldsymbol{K}')}{\gamma}\Big),$$

and similarly for $\widetilde{\mathscr{R}}_{\rho c}$ and $\widetilde{\mathscr{R}}_{\rho\rho}$.

Note that the kernel satisfies

$$\int_{\mathscr{O}} \frac{\mathrm{d}\Omega'}{2\pi} \frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d} Q(\Omega, \Omega', \boldsymbol{K}, \boldsymbol{K}') = -2\mathrm{Re}[\mathscr{Q}(\boldsymbol{K})] = \frac{2}{\mathscr{S}(\boldsymbol{K})}, \tag{5.50}$$

where $\mathscr{S}(\boldsymbol{K})$ is the scattering mean free path defined in (5.45).

5.7.3. The Wigner transform. The wave amplitudes decorrelate at distinct frequencies $\Omega \neq \Omega'$ and wave vectors $\boldsymbol{K} \neq \boldsymbol{K}'$, meaning that

$$\lim_{\varepsilon \to 0} \mathbb{E}\big[A^\varepsilon(\Omega, \boldsymbol{K}, z)\overline{A^\varepsilon}(\Omega', \boldsymbol{K}', z)\big] = \lim_{\varepsilon \to 0} \mathbb{E}\big[A^\varepsilon(\Omega, \boldsymbol{K}, z)\big] \lim_{\varepsilon \to 0} \mathbb{E}\big[\overline{A^\varepsilon}(\Omega', \boldsymbol{K}', z)\big]. \tag{5.51}$$

The right-hand side is the product of the means of the mode amplitudes, which decay on the range scale defined by the scattering mean free path (5.45).

However, the amplitudes are correlated for $|\Omega - \Omega'| = O(\varepsilon)$ and $|\boldsymbol{K} - \boldsymbol{K}'| = O(\varepsilon)$. We are interested in the second moment

$$\mathbb{E}\Big[A^\varepsilon\Big(\Omega, \boldsymbol{K} + \frac{\varepsilon \boldsymbol{q}}{2}, z\Big)\overline{A^\varepsilon\Big(\Omega, \boldsymbol{K} - \frac{\varepsilon \boldsymbol{q}}{2}, z\Big)}\Big],$$

whose Fourier transform in $\boldsymbol{q}$ gives the energy density resolved over frequencies and directions of propagation. This is the Wigner transform defined by

$$W^\varepsilon(\Omega,\boldsymbol{K},\boldsymbol{x},z) = \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} e^{i\boldsymbol{q}\cdot(\nabla\beta(\gamma\boldsymbol{K})z+\boldsymbol{x})}\mathbb{E}\Big[A^\varepsilon\Big(\Omega,\boldsymbol{K}+\frac{\varepsilon\boldsymbol{q}}{2},z\Big)\overline{A^\varepsilon\Big(\Omega,\boldsymbol{K}-\frac{\varepsilon\boldsymbol{q}}{2},z\Big)}\Big]. \tag{5.52}$$

We show in Appendix A.3 that the Wigner transform converges in the limit $\varepsilon \to 0$ to $W(\Omega,\boldsymbol{K},\boldsymbol{x},z)$, the solution of the transport equation

$$\begin{aligned}[\partial_z - \nabla\beta(\gamma\boldsymbol{K})\cdot\nabla_{\boldsymbol{x}}]\, W(\Omega,\boldsymbol{K},\boldsymbol{x},z) = \int_{\mathscr{O}} \frac{\mathrm{d}\Omega'}{2\pi}\frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d}\, Q(\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}') \\ \times [W(\Omega',\boldsymbol{K}',\boldsymbol{x},z) - W(\Omega,\boldsymbol{K},\boldsymbol{x},z)]\,,\end{aligned} \tag{5.53}$$

for $z > 0$, with initial condition

$$W(\Omega,\boldsymbol{K},\boldsymbol{x},0) = |A_o(\Omega,\boldsymbol{K})|^2\delta(\boldsymbol{x}). \tag{5.54}$$

The transport equation (3.12) in the physical scales is obtained from (5.54) as explained in Appendix B. In the next section we will show how this equation simplifies in the paraxial regime, when $\gamma \ll 1$. This is the result used for the imaging applications discussed in section 4.

5.8. The paraxial limit. Equation (5.53) shows that the energy is transported on the characteristic

$$\boldsymbol{x} = -\gamma\frac{\boldsymbol{K}}{\beta(\gamma\boldsymbol{K})}z, \tag{5.55}$$

parametrized by z, and depending on the wave vector $\boldsymbol{K}$. Here $|\boldsymbol{x}|/z = O(\gamma)$ quantifies the opening angle of the cone (beam) of propagation with axis z. We write this explicitly as

$$\boldsymbol{X} = \boldsymbol{x}/\gamma, \quad \text{where } |\boldsymbol{X}| = O(1). \tag{5.56}$$

The paraxial regime corresponds to a narrow beam, modeled by $\gamma \to 0$ and

$$\Gamma = \gamma/\gamma_s = O(1). \tag{5.57}$$

At the range $z = 0$ of the source we have from (5.54) and (5.37) that

$$W(\Omega,\boldsymbol{K},\boldsymbol{x},0) = \frac{\big|\widehat{S}\big(\frac{\eta_s}{\eta}\Omega,\Gamma\boldsymbol{K}\big)\big|^2}{4\gamma^d k_o}\delta(\boldsymbol{X}), \tag{5.58}$$

and to obtain a finite limit as $\gamma \to 0$ we rescale the Wigner transform as

$$\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},z) = \gamma^d W(\Omega,\boldsymbol{K},\gamma\boldsymbol{X},z). \tag{5.59}$$

We also change variables in (5.53),

$$\Omega - \Omega' - \eta\big(\gamma(\boldsymbol{K}-\boldsymbol{K}'),\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')\big)\cdot\vec{\boldsymbol{v}}_o \rightsquigarrow \Omega', \qquad \boldsymbol{K}-\boldsymbol{K}' \rightsquigarrow \boldsymbol{K}',$$

and obtain the transport equation

$$\begin{aligned}\Big[\partial_z + \frac{\boldsymbol{K}}{\beta(\gamma\boldsymbol{K})}\cdot\nabla_{\boldsymbol{X}}\Big]\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},z) &= \frac{k_o^4}{4}\int_{\mathscr{O}}\frac{\mathrm{d}\Omega'}{2\pi}\frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d}\frac{1}{\beta(\gamma\boldsymbol{K})\beta(\gamma\boldsymbol{K}')}\\ &\times\Big[\bar\sigma_c^2\widetilde{\mathscr{R}}_{cc}\Big(\Omega',\boldsymbol{K}',\frac{\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}')}{\gamma}\Big)+O(\gamma^2)\Big]\\ &\times\Big[\mathcal{W}\left(\Omega-\Omega'-\eta\gamma\boldsymbol{v}_o\cdot\boldsymbol{K}'-\eta v_{oz}(\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}')),\boldsymbol{K}-\boldsymbol{K}',\boldsymbol{X},z\right)\\ &\quad-\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},z)\Big]\end{aligned} \tag{5.60}$$

for $z>0$ and a finite $\gamma\ll 1$, where $O(\gamma^2)$ denotes the $\widetilde{\mathscr{R}}_{\rho c}$ and $\widetilde{\mathscr{R}}_{\rho\rho}$ terms in the kernel (5.49).

Recall that $\gamma \ll 1$ so, in order to observe a significant effect of the ambient motion, we rescale the transversal speed as

$$\boldsymbol{v}_o = \frac{\boldsymbol{V}_o}{\eta\gamma} \quad \text{with } |\boldsymbol{V}_o| = O(1). \tag{5.61}$$

With a similar scaling of the range velocity

$$v_{oz} = \frac{V_{oz}}{\eta\gamma} \quad \text{with } |V_{oz}| = O(1), \tag{5.62}$$

we obtain that the range motion plays no role in (5.60) as $\gamma \to 0$, because

$$\beta(\gamma\boldsymbol{K}) = k_o + O(\gamma^2), \qquad \beta(\gamma\boldsymbol{K}) - \beta(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}') = O(\gamma^2).$$

The transport equation satisfied by the Wigner transform $\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},z)$ in the paraxial limit $\gamma\to 0$ is

$$\begin{aligned}\Big[\partial_z + \frac{\boldsymbol{K}}{k_o}\cdot\nabla_{\boldsymbol{X}}\Big]\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},z) &= \frac{k_o^2}{4}\int_{\mathbb{R}^d}\frac{\mathrm{d}\boldsymbol{K}'}{(2\pi)^d}\int_{\mathbb{R}}\frac{\mathrm{d}\Omega'}{2\pi}\,\bar\sigma_c^2\widetilde{\mathscr{R}}_{cc}(\Omega',\boldsymbol{K}',0)\\ &\times\big[\mathcal{W}\left(\Omega-\Omega'-\boldsymbol{K}'\cdot\boldsymbol{V}_o,\boldsymbol{K}-\boldsymbol{K}',\boldsymbol{X},z\right)-\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},z)\big],\end{aligned} \tag{5.63}$$

for $z>0$, with initial condition

$$\mathcal{W}(\Omega,\boldsymbol{K},\boldsymbol{X},0) = \frac{\big|\widehat{S}\big(\frac{\eta_s}{\eta}\Omega,\Gamma\boldsymbol{K}\big)\big|^2}{4k_o}\delta(\boldsymbol{X}). \tag{5.64}$$

The transport equation (3.22) in the physical scales is obtained from (5.63) using the scaling relations explained in Appendix B.

6. Summary. We introduced an analysis of sound wave propagation in a time dependent random medium which moves due to an ambient flow at speed $\vec{\boldsymbol{v}}(t,\vec{\boldsymbol{x}})$ and is modeled by the wave speed $c(t,\vec{\boldsymbol{x}})$ and mass density $\rho(t,\vec{\boldsymbol{x}})$. The random fields $\vec{\boldsymbol{v}}(t,\vec{\boldsymbol{x}})$, $c(t,\vec{\boldsymbol{x}})$, and $\rho(t,\vec{\boldsymbol{x}})$ have small, statistically correlated fluctuations about the constant values $\vec{\boldsymbol{v}}_o$, c_o, and ρ_o, on the length scale ℓ and time scale T and with the fluctuations modeled in terms of stationary random fields. The analysis starts from Pierce's equation, which is obtained from the linearization of the fluid dynamics equations about an ambient flow, and applies to waves with central wavelength $\lambda_o \ll \ell$. The excitation is from a stationary source with radius ℓ_s, which emits a narrowband signal of duration T_s.

The analysis is in a forward wave propagation regime to a large distance (range) $L \gg \ell$, within a cone with small opening angle. Using the diffusion approximation theory, we showed that the coherent part (the expectation) of the wave decays exponentially in $L/\mathscr{S}$ and quantified the frequency and wavevector dependent scattering mean free path $\mathscr{S}$. We also derived transport equations for the energy density (Wigner transform) of the wave, which show explicitly the effect of the ambient flow and net scattering in the time dependent random medium.

We used the wave propagation theory to study the inverse problem of localizing (imaging) the source from measurements at a stationary array of receivers located at range L. This study is in the regime of paraxial wave propagation, where the Wigner transform can be computed explicitly, and assumes a large range $L \gg \mathscr{S}$, so that the wave is incoherent due to strong scattering in the random medium. The temporal variation of the medium is at time scale $T \ll T_s$, and it has two beneficial effects for imaging. First, it causes broadening of the bandwidth of the recorded waves, which leads to improved travel time estimation and consequently, better range resolution. Second, it allows a robust (statistically stable) estimation of the Wigner transform from the array measurements. We presented an explicit analysis of imaging based on this Wigner transform and showed how one can estimate the source location, the mean velocity $\vec{\boldsymbol{v}}_o$, and the statistics of the random medium.

Appendix A. The Markovian limit theorem. In this appendix we obtain the $\varepsilon \to 0$ limit of the Markov process $\boldsymbol{A}^\varepsilon(z) = (A^\varepsilon(\Omega, \boldsymbol{K}, z))_{(\Omega,\boldsymbol{K})\in\mathscr{O}}$, which lies on the surface of the sphere with radius $R_{\boldsymbol{A}}$ given in (5.40). The set $\mathscr{O}$ is defined by (5.38). The process $\boldsymbol{A}^\varepsilon(z)$ starts from

$$\boldsymbol{A}^\varepsilon(0) = \left(-\frac{1}{2\sqrt{\beta(\gamma\boldsymbol{K})}}\widehat{S}\Big(\frac{\eta_s}{\eta}\Omega, \frac{\gamma}{\gamma_s}\boldsymbol{K}\Big)\right)_{(\Omega,\boldsymbol{K})\in\mathscr{O}}, \tag{A.1}$$

which is independent of ε, and evolves at $z > 0$ according to the stochastic equation

$$\frac{\mathrm{d}\boldsymbol{A}^\varepsilon}{\mathrm{d}z} = \mathcal{G}\Big(\frac{z}{\varepsilon}, \frac{z}{\varepsilon}\Big)\boldsymbol{A}^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{F}\Big(\frac{z}{\varepsilon}, \frac{z}{\varepsilon}\Big)\boldsymbol{A}^\varepsilon. \tag{A.2}$$

Here $\mathcal{G}$ and $\mathcal{F}$ are integral operators

$$[\mathcal{G}(z,\zeta)\boldsymbol{A}](\Omega,\boldsymbol{K}) = \int_{\mathscr{O}} \mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\, G(z,\zeta,\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}')A(\Omega',\boldsymbol{K}'),$$
$$[\mathcal{F}(z,\zeta)\boldsymbol{A}](\Omega,\boldsymbol{K}) = \int_{\mathscr{O}} \mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\, F(z,\zeta,\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}')A(\Omega',\boldsymbol{K}'),$$

with kernels depending on the random processes $\nu_c(\tau,\vec{\boldsymbol{r}})$ and $Q^\varepsilon(\tau,\vec{\boldsymbol{r}})$. Recall the definition (5.13) of $Q^\varepsilon(\tau,\vec{\boldsymbol{r}})$. We rewrite it here as

$$Q^\varepsilon(\tau,\vec{\boldsymbol{r}}) = Q^{(0)}(\tau,\vec{\boldsymbol{r}}) + \frac{\sqrt{\varepsilon\gamma}\bar{\sigma}_\rho}{2}Q^{(1)}(\tau,\vec{\boldsymbol{r}}) \tag{A.3}$$

with

$$Q^{(0)}(\tau,\vec{\boldsymbol{r}}) = \Delta_{\vec{\boldsymbol{r}}}\nu_\rho(\tau,\vec{\boldsymbol{r}}) \quad \text{and} \quad Q^{(1)}(\tau,\vec{\boldsymbol{r}}) = \left|\nabla_{\vec{\boldsymbol{r}}}\nu_\rho(\tau,\vec{\boldsymbol{r}})\right|^2. \tag{A.4}$$

The kernel G has a deterministic part supported at $\Omega' = \Omega$ and $\boldsymbol{K}' = \boldsymbol{K}$ and a random part determined by $Q^{(1)}$,

$$\begin{aligned} G(z,\zeta,\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}') &= \frac{ik_o}{\beta(\gamma\boldsymbol{K})}\Big[\frac{\Omega}{\eta c_o} - \frac{\vec{\boldsymbol{v}}_o}{c_o}\cdot\big(\gamma\boldsymbol{K},\beta(\gamma\boldsymbol{K})\big)\Big]\delta(\Omega'-\Omega)\delta(\boldsymbol{K}'-\boldsymbol{K}) \\ &\quad - \frac{i\gamma^3\bar{\sigma}_\rho^2}{8(2\pi)^{d+1}\sqrt{\beta(\gamma\boldsymbol{K})\beta(\gamma\boldsymbol{K}')}}\exp\big\{i\big[\beta(\gamma\boldsymbol{K}')-\beta(\gamma\boldsymbol{K})\big]\zeta\big\} \\ &\quad \times \widehat{Q}^{(1)}\Big(\Omega-\Omega'-\eta\big(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}',\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')\big)\cdot\vec{\boldsymbol{v}}_o,\boldsymbol{K}-\boldsymbol{K}',\gamma z\Big). \end{aligned} \tag{A.5}$$

The kernel F is determined by ν_c and $Q^{(0)}$,

$$\begin{aligned} F(z,\zeta,\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}') &= \frac{i\sqrt{\gamma}}{2(2\pi)^{d+1}\sqrt{\beta(\gamma\boldsymbol{K})\beta(\gamma\boldsymbol{K}')}}\exp\big\{i[\beta(\gamma\boldsymbol{K}')-\beta(\gamma\boldsymbol{K})]\zeta\big\} \\ &\quad \times\Bigg[k_o^2\bar{\sigma}_c\widehat{\nu}_c\Big(\Omega-\Omega'-\eta\big(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}',\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')\big)\cdot\vec{\boldsymbol{v}}_o,\boldsymbol{K}-\boldsymbol{K}',\gamma z\Big) \\ &\quad - \frac{\gamma^2\bar{\sigma}_\rho}{2}\widehat{Q}^{(0)}\Big(\Omega-\Omega'-\eta\big(\gamma\boldsymbol{K}-\gamma\boldsymbol{K}',\beta(\gamma\boldsymbol{K})-\beta(\gamma\boldsymbol{K}')\big)\cdot\vec{\boldsymbol{v}}_o,\boldsymbol{K}-\boldsymbol{K}',\gamma z\Big)\Bigg]. \end{aligned} \tag{A.6}$$

The random process $\boldsymbol{A}^\varepsilon(z)$ is Markov with generator

$$\begin{aligned} \mathcal{L}^\varepsilon f(\boldsymbol{A},\overline{\boldsymbol{A}}) &= \int_{\mathscr{O}^2}\frac{1}{\sqrt{\varepsilon}}F\Big(\frac{z}{\varepsilon},\frac{z}{\varepsilon},\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}'\Big)\frac{\delta f}{\delta A(\Omega,\boldsymbol{K})}A(\Omega',\boldsymbol{K}')\mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\mathrm{d}\Omega\mathrm{d}\boldsymbol{K} \\ &\quad + \int_{\mathscr{O}^2}\frac{1}{\sqrt{\varepsilon}}\overline{F}\Big(\frac{z}{\varepsilon},\frac{z}{\varepsilon},\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}'\Big)\frac{\delta f}{\delta\overline{A}(\Omega,\boldsymbol{K})}\overline{A}(\Omega',\boldsymbol{K}')\mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\mathrm{d}\Omega\mathrm{d}\boldsymbol{K} \\ &\quad + \int_{\mathscr{O}^2}G\Big(\frac{z}{\varepsilon},\frac{z}{\varepsilon},\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}'\Big)\frac{\delta f}{\delta A(\Omega,\boldsymbol{K})}A(\Omega',\boldsymbol{K}')\mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\mathrm{d}\Omega\mathrm{d}\boldsymbol{K} \\ &\quad + \int_{\mathscr{O}^2}\overline{G}\Big(\frac{z}{\varepsilon},\frac{z}{\varepsilon},\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}'\Big)\frac{\delta f}{\delta\overline{A}(\Omega,\boldsymbol{K})}\overline{A}(\Omega',\boldsymbol{K}')\mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\mathrm{d}\Omega\mathrm{d}\boldsymbol{K}, \end{aligned}$$

where $\delta f/\delta A(\Omega,\boldsymbol{K})$ denotes the variational derivative, defined as follows. If φ is a smooth function and

$$\begin{aligned} f(\boldsymbol{A},\overline{\boldsymbol{A}}) &= \int\cdots\int\varphi(\Omega_1,\ldots,\Omega_{n+m},\boldsymbol{K}_1,\ldots,\boldsymbol{K}_{n+m})\prod_{j=1}^n A(\Omega_j,\boldsymbol{K}_j) \\ &\quad\times\prod_{j=n+1}^{n+m}\overline{A}(\Omega_j,\boldsymbol{K}_j)\prod_{j=1}^{n+m}\mathrm{d}\Omega_j\mathrm{d}\boldsymbol{K}_j, \end{aligned}$$

then we have

$$\begin{aligned} \frac{\delta f}{\delta A(\Omega,\boldsymbol{K})} &= \sum_{l=1}^n\int\cdots\int\varphi(\Omega_1,\ldots,\Omega_{n+m},\boldsymbol{K}_1,\ldots,\boldsymbol{K}_{n+m})\mid_{\Omega_l=\Omega,\boldsymbol{K}_l=\boldsymbol{K}}\prod_{j=1,j\neq l}^n A(\Omega_j,\boldsymbol{K}_j) \\ &\quad\times\prod_{j=n+1}^{n+m}\overline{A}(\Omega_j,\boldsymbol{K}_j)\prod_{j=1,j\neq l}^{n+m}\mathrm{d}\Omega_j\mathrm{d}\boldsymbol{K}_j \end{aligned}$$

and

$$\frac{\delta f}{\delta \overline{A}(\Omega, \boldsymbol{K})} = \sum_{l=n+1}^{n+m} \int \cdots \int \varphi(\Omega_1, \dots, \Omega_{n+m}, \boldsymbol{K}_1, \dots, \boldsymbol{K}_{n+m}) \,|_{\Omega_l=\Omega, \boldsymbol{K}_l=\boldsymbol{K}} \prod_{j=1}^{n} A(\Omega_j, \boldsymbol{K}_j) \times \prod_{j=n+1, j\neq l}^{n+m} \overline{A}(\Omega_j, \boldsymbol{K}_j) \prod_{j=1, j\neq l}^{n+m} \mathrm{d}\Omega_j \mathrm{d}\boldsymbol{K}_j.$$

The linear combinations of such functions f form an algebra that is dense in $\mathcal{C}(L^2)$ and is convergence determining. We can also extend the class of functions to include generalized functions φ of the form

$$\varphi(\Omega_1, \dots, \Omega_{2n}, \boldsymbol{K}_1, \dots, \boldsymbol{K}_{2n}) = \Phi(\Omega_1, \dots, \Omega_n, \boldsymbol{K}_1, \dots, \boldsymbol{K}_n) \times \prod_{j=1}^{n} \delta(\Omega_{n+j} - \Omega_j) \delta(\boldsymbol{K}_{n+j} - \boldsymbol{K}_j),$$

where Φ is a smooth function.

Applying the diffusion approximation theory described in [9, Chapter 6] and [18, 17], we obtain the limit generator

$$\begin{aligned}
\mathcal{L} f(\boldsymbol{A}, \overline{\boldsymbol{A}}) = & \int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty} \frac{1}{Z} \int_0^Z \mathrm{d}h \int_{\mathscr{O}^4} \mathrm{d}\Omega_1' \mathrm{d}\boldsymbol{K}_1' \mathrm{d}\Omega_2' \mathrm{d}\boldsymbol{K}_2' \mathrm{d}\Omega_1 \mathrm{d}\boldsymbol{K}_1 \mathrm{d}\Omega_2 \mathrm{d}\boldsymbol{K}_2 \\
& \times \Big\{ \mathbb{E}\big[F(0, h, \Omega_1, \Omega_1', \boldsymbol{K}_1, \boldsymbol{K}_1') F(\zeta, \zeta + h, \Omega_2, \Omega_2', \boldsymbol{K}_2, \boldsymbol{K}_2')\big] \\
& \quad \times \frac{\delta^2 f}{\delta A(\Omega_1, \boldsymbol{K}_1) \delta A(\Omega_2, \boldsymbol{K}_2)} A(\Omega_1', \boldsymbol{K}_1') A(\Omega_2', \boldsymbol{K}_2') \\
& \quad + \mathbb{E}\big[F(0, h, \Omega_1, \Omega_1', \boldsymbol{K}_1, \boldsymbol{K}_1') \overline{F}(\zeta, \zeta + h, \Omega_2, \Omega_2', \boldsymbol{K}_2, \boldsymbol{K}_2')\big] \\
& \quad \times \frac{\delta^2 f}{\delta A(\Omega_1, \boldsymbol{K}_1) \delta \overline{A}(\Omega_2, \boldsymbol{K}_2)} A(\Omega_1', \boldsymbol{K}_1') \overline{A}(\Omega_2', \boldsymbol{K}_2') \\
& \quad + \mathbb{E}\big[\overline{F}(0, h, \Omega_1, \Omega_1', \boldsymbol{K}_1, \boldsymbol{K}_1') F(\zeta, \zeta + h, \Omega_2, \Omega_2', \boldsymbol{K}_2, \boldsymbol{K}_2')\big] \\
& \quad \times \frac{\delta^2 f}{\delta \overline{A}(\Omega_1, \boldsymbol{K}_1) \delta A(\Omega_2, \boldsymbol{K}_2)} \overline{A}(\Omega_1', \boldsymbol{K}_1') A(\Omega_2', \boldsymbol{K}_2') \\
& \quad + \mathbb{E}\big[\overline{F}(0, h, \Omega_1, \Omega_1', \boldsymbol{K}_1, \boldsymbol{K}_1') \overline{F}(\zeta, \zeta + h, \Omega_2, \Omega_2', \boldsymbol{K}_2, \boldsymbol{K}_2')\big] \\
& \quad \times \frac{\delta^2 f}{\delta \overline{A}(\Omega_1, \boldsymbol{K}_1) \delta \overline{A}(\Omega_2, \boldsymbol{K}_2)} \overline{A}(\Omega_1', \boldsymbol{K}_1') \overline{A}(\Omega_2', \boldsymbol{K}_2') \Big\} \\
& + \int_0^\infty \mathrm{d}\zeta \lim_{Z\to\infty} \frac{1}{Z} \int_0^Z \mathrm{d}h \int_{\mathscr{O}^3} \mathrm{d}\Omega_1' \mathrm{d}\boldsymbol{K}_1' \mathrm{d}\Omega_1 \mathrm{d}\boldsymbol{K}_1 \mathrm{d}\Omega' \mathrm{d}\boldsymbol{K}' \\
& \times \Big\{ \mathbb{E}\big[F(0, h, \Omega', \Omega_1', \boldsymbol{K}', \boldsymbol{K}_1') F(\zeta, \zeta + h, \Omega_1, \Omega', \boldsymbol{K}_1, \boldsymbol{K}')\big] \frac{\delta f}{\delta A(\Omega_1, \boldsymbol{K}_1)} A(\Omega_1', \boldsymbol{K}_1') \\
& \quad + \mathbb{E}\big[\overline{F}(0, h, \Omega', \Omega_1', \boldsymbol{K}', \boldsymbol{K}_1') \overline{F}(\zeta, \zeta + h, \Omega_1, \Omega', \boldsymbol{K}_1, \boldsymbol{K}')\big] \frac{\delta f}{\delta \overline{A}(\Omega_1, \boldsymbol{K}_1)} \overline{A}(\Omega_1', \boldsymbol{K}_1') \Big\} \\
& + \lim_{Z\to\infty} \frac{1}{Z} \int_0^Z \mathrm{d}h \int_{\mathscr{O}^2} \mathrm{d}\Omega \mathrm{d}\boldsymbol{K} \mathrm{d}\Omega' \mathrm{d}\boldsymbol{K}' \Big\{ \mathbb{E}\big[G(0, h, \Omega, \Omega', \boldsymbol{K}, \boldsymbol{K}')\big] \\
& \quad \times \frac{\delta f}{\delta A(\Omega, \boldsymbol{K})} A(\Omega', \boldsymbol{K}') + \mathbb{E}\big[\overline{G}(0, h, \Omega, \Omega', \boldsymbol{K}, \boldsymbol{K}')\big] \frac{\delta f}{\delta \overline{A}(\Omega, \boldsymbol{K})} \overline{A}(\Omega', \boldsymbol{K}') \Big\}.
\end{aligned} \tag{A.7}$$

The expectations in the expression of the generator can be computed with

$$\mathbb{E}\big[\widehat{\nu}_c(\Omega,\boldsymbol{K},\zeta)\widehat{\nu}_c(\Omega',\boldsymbol{K}',0)\big] = (2\pi)^{d+1}\delta(\boldsymbol{K}+\boldsymbol{K}')\delta(\Omega+\Omega')\widehat{\mathscr{R}}_{cc}(\Omega,\boldsymbol{K},\zeta), \tag{A.8}$$

$$\mathbb{E}\big[\overline{\widehat{\nu}_c}(\Omega,\boldsymbol{K},\zeta)\widehat{\nu}_c(\Omega',\boldsymbol{K}',0)\big] = (2\pi)^{d+1}\delta(\boldsymbol{K}-\boldsymbol{K}')\delta(\Omega-\Omega')\overline{\widehat{\mathscr{R}}_{cc}(\Omega,\boldsymbol{K},\zeta)}, \tag{A.9}$$

$$\mathbb{E}\big[\widehat{\nu}_c(\Omega,\boldsymbol{K},\zeta)\overline{\widehat{\nu}_c}(\Omega',\boldsymbol{K}',0)\big] = (2\pi)^{d+1}\delta(\boldsymbol{K}-\boldsymbol{K}')\delta(\Omega-\Omega')\widehat{\mathscr{R}}_{cc}(\Omega,\boldsymbol{K},\zeta), \tag{A.10}$$

$$\mathbb{E}\big[\overline{\widehat{\nu}_c}(\Omega,\boldsymbol{K},\zeta)\overline{\widehat{\nu}_c}(\Omega',\boldsymbol{K}',0)\big] = (2\pi)^{d+1}\delta(\boldsymbol{K}+\boldsymbol{K}')\delta(\Omega+\Omega')\overline{\widehat{\mathscr{R}}_{cc}(\Omega,\boldsymbol{K},\zeta)}, \tag{A.11}$$

and similarly for $\widehat{\nu}_\rho$. Note here that both ν_c and $\widehat{\mathscr{R}}_{cc}$ are real. We also have

$$\widehat{Q}^{(0)}(\Omega,\boldsymbol{K},z) = \big(-|\boldsymbol{K}|^2+\partial_z^2\big)\widehat{\nu}_\rho(\Omega,\boldsymbol{K},z) \tag{A.12}$$

and

$$\mathbb{E}\big[\widehat{Q}^{(1)}(\Omega,\boldsymbol{K},z)\big] = -(2\pi)^{d+1}\delta(\Omega)\delta(\boldsymbol{K})\Delta_{\vec{\boldsymbol{r}}}\mathscr{R}_{\rho\rho}(0,\vec{\boldsymbol{r}})\big|_{\vec{\boldsymbol{r}}=\boldsymbol{0}}. \tag{A.13}$$

A.1. The mean amplitude. To calculate the mean of the limit process, we let

$$f(\boldsymbol{A},\overline{\boldsymbol{A}}) = \int_{\mathscr{O}} \mathrm{d}\Omega\mathrm{d}\boldsymbol{K}\,\varphi(\Omega,\boldsymbol{K})A(\Omega,\boldsymbol{K})$$

so that

$$\frac{\delta f}{\delta A(\Omega_1,\boldsymbol{K}_1)} = \varphi(\Omega_1,\boldsymbol{K}_1), \qquad \frac{\delta f}{\delta \overline{A}(\Omega_1,\boldsymbol{K}_1)} = 0,$$

and all second variational derivatives are zero.

From the expression (A.7), definitions (A.5)–(A.6), and the expectations (A.8)–(A.13) we obtain

$$\mathcal{L}f(\boldsymbol{A},\overline{\boldsymbol{A}}) = \int_{\mathscr{O}} \mathrm{d}\Omega\mathrm{d}\boldsymbol{K}\,\big[i\theta(\Omega,\boldsymbol{K})+D(\boldsymbol{K})\big]\varphi(\Omega,\boldsymbol{K})A(\Omega,\boldsymbol{K}) \tag{A.14}$$

with θ and D given in (5.42) and (5.43). This gives the result (5.41).

A.2. The mean intensity. To characterize the mean intensity of the limit process, we let

$$\begin{aligned} f(\boldsymbol{A},\overline{\boldsymbol{A}}) &= \int_{\mathscr{O}} \mathrm{d}\Omega\mathrm{d}\boldsymbol{K}\,\varphi(\Omega,\boldsymbol{K})|A(\Omega,\boldsymbol{K})|^2 \\ &= \int_{\mathscr{O}^2} \mathrm{d}\Omega\mathrm{d}\boldsymbol{K}\mathrm{d}\Omega'\mathrm{d}\boldsymbol{K}'\,\varphi(\Omega,\boldsymbol{K})\delta(\Omega-\Omega')\delta(\boldsymbol{K}-\boldsymbol{K}')A(\Omega,\boldsymbol{K})\overline{A}(\Omega',\boldsymbol{K}') \end{aligned}$$

so that

$$\frac{\delta f}{\delta A(\Omega_1,\boldsymbol{K}_1)} = \overline{A}(\Omega_1,\boldsymbol{K}_1)\varphi(\Omega_1,\boldsymbol{K}_1), \qquad \frac{\delta f}{\delta \overline{A}(\Omega_1,\boldsymbol{K}_1)} = A(\Omega_1,\boldsymbol{K}_1)\varphi(\Omega_1,\boldsymbol{K}_1),$$

$$\frac{\delta^2 f}{\delta \overline{A}(\Omega_1,\boldsymbol{K}_1)\delta A(\Omega_2,\boldsymbol{K}_2)} = \varphi(\Omega_2,\boldsymbol{K}_2)\delta(\Omega_2-\Omega_1)\delta(\boldsymbol{K}_2-\boldsymbol{K}_1),$$

and all other second variational derivatives are zero.

Using the expectation (A.13) and definition (A.5) in (A.7), we obtain that the G dependent terms make no contribution. Furthermore, using the expectations

(A.8)–(A.11) and (A.12) we get

$$\mathcal{L}f(\boldsymbol{A},\overline{\boldsymbol{A}}) = -\int_{\mathscr{O}} \frac{\mathrm{d}\Omega_1 \mathrm{d}\boldsymbol{K}_1}{(2\pi)^{d+1}}\,\varphi(\Omega_1,\boldsymbol{K}_1)|A(\Omega_1,\boldsymbol{K}_1)|^2 \int_{\mathscr{O}} \mathrm{d}\Omega_1' \mathrm{d}\boldsymbol{K}_1' Q(\Omega_1,\Omega_1',\boldsymbol{K}_1,\boldsymbol{K}_1')$$
$$+ \int_{\mathscr{O}} \frac{\mathrm{d}\Omega_1 \mathrm{d}\boldsymbol{K}_1}{(2\pi)^{d+1}}\,\varphi(\Omega_1,\boldsymbol{K}_1) \int_{\mathscr{O}} \mathrm{d}\Omega_1' \mathrm{d}\boldsymbol{K}_1'\,|A(\Omega_1',\boldsymbol{K}_1')|^2 Q(\Omega_1,\Omega_1',\boldsymbol{K}_1,\boldsymbol{K}_1')$$

with kernel defined in (5.49). This gives the equation satisfied by the mean intensity.

A.3. Wave decorrelation and the Wigner transform. To study the second moments at distinct frequencies Ω, Ω' and wave vectors $\boldsymbol{K}$ and $\boldsymbol{K}'$, we let

$$f(\boldsymbol{A},\overline{\boldsymbol{A}}) = \int_{\mathscr{O}^2} \mathrm{d}\Omega \mathrm{d}\boldsymbol{K} \mathrm{d}\Omega' \mathrm{d}\boldsymbol{K}'\,\varphi(\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}')A(\Omega,\boldsymbol{K})\overline{A}(\Omega',\boldsymbol{K}').$$

Then, we have

$$\frac{\delta f}{\delta A(\Omega_1,\boldsymbol{K}_1)} = \int_{\mathscr{O}} \mathrm{d}\Omega' \mathrm{d}\boldsymbol{K}'\,\overline{A}(\Omega',\boldsymbol{K}')\varphi(\Omega_1,\Omega',\boldsymbol{K}_1,\boldsymbol{K}'),$$
$$\frac{\delta f}{\delta \overline{A}(\Omega_1,\boldsymbol{K}_1)} = \int_{\mathscr{O}} \mathrm{d}\Omega' \mathrm{d}\boldsymbol{K}'\,A(\Omega',\boldsymbol{K}')\varphi(\Omega',\Omega_1,\boldsymbol{K}',\boldsymbol{K}_1),$$
$$\frac{\delta^2 f}{\delta A(\Omega_1,\boldsymbol{K}_1)\delta \overline{A}(\Omega_2,\boldsymbol{K}_2)} = \varphi(\Omega_1,\Omega_2,\boldsymbol{K}_1,\boldsymbol{K}_2),$$

and all other second variational derivatives are zero.

Substituting in (A.7) and using the expectations (A.8)–(A.13) we obtain that

$$\mathcal{L}f(\boldsymbol{A},\overline{\boldsymbol{A}}) = \int_{\mathscr{O}} \mathrm{d}\Omega \mathrm{d}\boldsymbol{K} \int_{|\boldsymbol{K}'|<k_o} \mathrm{d}\boldsymbol{K}'\,\varphi(\Omega,\Omega',\boldsymbol{K},\boldsymbol{K}')$$
$$\times \big[i\theta(\Omega,\boldsymbol{K}) - i\theta(\Omega',\boldsymbol{K}') + D(\boldsymbol{K}) + \overline{D}(\boldsymbol{K}')\big] A(\Omega,\boldsymbol{K})\overline{A}(\Omega',\boldsymbol{K}') \tag{A.15}$$

with $\theta(\Omega,\boldsymbol{K})$ and $D(\boldsymbol{K})$ defined in (5.42) and (5.43). This gives the decorrelation result (5.51).

Finally, to study the Wigner transform, we use (5.36) to obtain an evolution equation for

$$\mathscr{W}^\varepsilon(\Omega,\boldsymbol{K},t,\boldsymbol{x},z) = \int_{\mathbb{R}} \frac{\mathrm{d}w}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} e^{-iwt+i\boldsymbol{q}\cdot[\boldsymbol{x}+\nabla\beta(\boldsymbol{K})z]}$$
$$\times A^\varepsilon\Big(\Omega + \frac{\varepsilon w}{2}, \boldsymbol{K} + \frac{\varepsilon \boldsymbol{q}}{2}, z\Big)\overline{A}^\varepsilon\Big(\Omega - \frac{\varepsilon w}{2}, \boldsymbol{K} - \frac{\varepsilon \boldsymbol{q}}{2}, z\Big),$$

and then analyze the limit $\varepsilon \to 0$ of $\mathscr{W}^\varepsilon$ with the same approach as described in this appendix. The Wigner transform (5.52) is

$$W^\varepsilon(\Omega,\boldsymbol{K},\boldsymbol{x},z) = \int_{\mathbb{R}} \mathrm{d}t\,\mathbb{E}\big[\mathscr{W}^\varepsilon(\Omega,\boldsymbol{K},t,\boldsymbol{x},z)\big],$$

and this converges in the limit to the solution $W(\Omega,\boldsymbol{K},\boldsymbol{x},z)$ of (5.53)–(5.54).

Appendix B. The transport equation in the physical scales. To distinguish between the scaled and unscaled variables, we resurrect the notation of section 5.2 with the unscaled variables denoted by primes.

We begin with the pressure field (5.34),

$$p(T_L t', L\boldsymbol{x}', Lz') \approx \frac{i\omega_o \rho_o}{\varepsilon^d} \int \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^d} \frac{a^{\varepsilon\,\prime}(\omega' - \beta'(\boldsymbol{k}')v'_z, \boldsymbol{k}', z')}{\sqrt{\beta'(\boldsymbol{k}')}} \times e^{-i\left(\frac{\omega'_o}{\varepsilon} + \omega'\right)t' + i\frac{\boldsymbol{k}'}{\varepsilon}\cdot\boldsymbol{x}' + i\frac{\beta'(\boldsymbol{k}')}{\varepsilon}z'}. \tag{B.1}$$

The scaling relations (2.8)–(2.15) and (5.5)–(5.6) give

$$\begin{aligned}
\frac{\omega'_o t'}{\varepsilon} &= \frac{2\pi}{\lambda_o/L}\frac{t}{T_L} = \omega_o t,\\
\omega' t' &= \omega' \frac{t}{T_L} = \omega t, \quad \text{i.e., } \omega' = \omega T_L,\\
\frac{\boldsymbol{k}'}{\varepsilon}\cdot\boldsymbol{x}' &= \frac{\boldsymbol{k}'}{\lambda_o/L}\cdot\frac{\boldsymbol{x}}{L} = \frac{\boldsymbol{k}'}{\lambda_o}\cdot\boldsymbol{x} = \boldsymbol{k}\cdot\boldsymbol{x}, \quad \text{i.e., } \boldsymbol{k}' = \lambda_o \boldsymbol{k},\\
\beta'(\boldsymbol{k}') &= \sqrt{(k'_o)^2 - |\boldsymbol{k}'|^2} = \lambda_o\sqrt{k_o^2 - |\boldsymbol{k}|^2} = \lambda_o \beta(\boldsymbol{k}),\\
\omega' - \beta'(\boldsymbol{k}')v'_{oz} &= T_L\omega - \lambda_o\beta(\boldsymbol{k})\frac{v_{oz}}{(\lambda_o/L)c_o} = T_L[\omega - \beta(\boldsymbol{k})v_{oz}],\\
\frac{\beta'(\boldsymbol{k}')}{\varepsilon}z' &= \frac{\lambda_o\beta(\boldsymbol{k})}{\lambda_o/L}\frac{z}{L} = \beta(\boldsymbol{k})z.
\end{aligned}$$

Equation (B.1) becomes (3.1), with amplitudes

$$a(\omega, \boldsymbol{k}, z) = \frac{T_L L^d}{\sqrt{\lambda_o}} a^{\varepsilon\,\prime}\left(T_L(\omega - \beta(\boldsymbol{k})v_{oz}), \lambda_o\boldsymbol{k}, \frac{z}{L}\right), \tag{B.2}$$

satisfying the initial conditions

$$a(\omega, \boldsymbol{k}, 0) = \frac{T_L L^d}{\lambda_o}\frac{i}{2\sqrt{\beta(\boldsymbol{k})}}\widehat{S}(T_s\omega, \ell_s\boldsymbol{k}) = \frac{i\sigma_s T_s \ell_s^d}{2\sqrt{\beta(\boldsymbol{k})}}\widehat{S}(T_s\omega, \ell_s\boldsymbol{k}), \tag{B.3}$$

derived from (2.17) and (5.32).

It remains to write the transport equation (3.12) for the Wigner transform. To do so, we obtain from definitions (5.33) and the scaling relations above that

$$\begin{aligned}
\Omega' &= \eta[\omega' + \beta'(\boldsymbol{k}')v'_{oz}] = T[\omega + \beta(\boldsymbol{k})v_{oz}],\\
\boldsymbol{K}' &= \frac{\boldsymbol{k}'}{\gamma} = \frac{\lambda_o\boldsymbol{k}}{\lambda_o/\ell} = \ell\boldsymbol{k}.
\end{aligned}$$

We also recall the definition (5.35) of A^ε in terms of a^ε and obtain that

$$\begin{aligned}
W(\omega,\boldsymbol{k},\boldsymbol{x},z) &= \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}\exp\left[i\boldsymbol{q}\cdot\left(\nabla\beta(\boldsymbol{k})z + \boldsymbol{x}\right)\right]\mathbb{E}\left[a\left(\omega, \boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right)\overline{a\left(\omega, \boldsymbol{k} - \frac{\boldsymbol{q}}{2}\right)}\right]\\
&= \left(\frac{T_L L^d}{\sqrt{\lambda_o}}\right)^2 \frac{1}{\lambda_o^d}\int \frac{\mathrm{d}\boldsymbol{q}'}{(2\pi)^d}\exp\left[i\frac{\boldsymbol{q}'}{\lambda_o/L}\cdot\left(\nabla\beta'(\boldsymbol{k}')z' + \boldsymbol{x}'\right)\right]\\
&\quad\times\mathbb{E}\left[A^\varepsilon\left(\Omega' - \eta\beta'(\gamma\boldsymbol{K}')v'_{oz}, \boldsymbol{K}' + \frac{\boldsymbol{q}'}{2}, z'\right)\overline{A^\varepsilon\left(\Omega' - \eta\beta(\gamma\boldsymbol{K}')v'_{oz}, \boldsymbol{K}' - \frac{\boldsymbol{q}'}{2}, z'\right)}\right]
\end{aligned}$$

with Ω' and $\boldsymbol{K}'$ defined as above in terms of ω and $\boldsymbol{k}$, and $z' = z/L$. Since $\varepsilon = \lambda_o/L$, we can change the variable of integration as $\boldsymbol{q}' \rightsquigarrow \boldsymbol{q}'\varepsilon$ and obtain

$$
\begin{aligned}
W(\omega,\boldsymbol{k},\boldsymbol{x},z) &= \frac{T_L^2 L^d}{\lambda_o} W^{\varepsilon\,'}\big(\Omega' - \eta\beta'(\gamma\boldsymbol{K}')v'_{oz}, \boldsymbol{K}', \boldsymbol{x}', z'\big) \\
&\approx \sigma_s^2 T_s^2 \ell_s^{2d} \frac{\lambda_o}{L^d} W'\big(\Omega' - \eta\beta'(\gamma\boldsymbol{K}')v'_{oz}, \boldsymbol{K}', \boldsymbol{x}', z'\big). \qquad \text{(B.4)}
\end{aligned}
$$

Here the approximation is for $\varepsilon \ll 1$, where we have replaced $W^{\varepsilon\,'}$ by its $\varepsilon \to 0$ limit W'.

Using the initial conditions (5.37) and (5.54) and the scaling relations between Ω', $\boldsymbol{K}'$ and ω and $\boldsymbol{k}$, we have

$$
\begin{aligned}
W(\omega,\boldsymbol{k},\boldsymbol{x},0) &= \sigma_s^2 T_s^2 \ell_s^{2d} \frac{\lambda_o}{L^d} \frac{\delta(\boldsymbol{x}/L)}{4\beta'(\boldsymbol{k}')} \Big|\widehat{S}\Big(\eta_s\omega', \frac{\ell_s}{\ell}\boldsymbol{K}'\Big)\Big|^2 \\
&= \sigma_s^2 T_s^2 \ell_s^{2d} \frac{\delta(\boldsymbol{x})}{4\beta(\boldsymbol{k})} \big|\widehat{S}(T_s\omega, \ell_s\boldsymbol{k})\big|^2,
\end{aligned}
$$

as stated in (3.13) and (3.3). The transport equation (3.12) follows from (5.53), using

$$
\partial_{z'} - \nabla\beta'(\gamma\boldsymbol{K}')\cdot\nabla_{\boldsymbol{x}'} = L\left[\partial_z - \nabla\beta(\boldsymbol{k})\cdot\nabla_{\boldsymbol{x}}\right].
$$

Appendix C. Solution of the transport equation in the paraxial regime. To deal with the convolution in (3.22), we Fourier transform in $\omega, \boldsymbol{k}$ and $\boldsymbol{x}$,

$$
\breve{W}(t,\boldsymbol{y},\boldsymbol{q},z) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega t} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} W(\omega,\boldsymbol{k},\boldsymbol{x},z). \qquad \text{(C.1)}
$$

Using definition (3.23) of the scattering kernel and the expression (3.9) of the power spectral density $\widehat{\mathscr{R}}_{cc}$, we write

$$
\begin{aligned}
Q_{\mathrm{par}}(\omega,\boldsymbol{k}) &= \frac{k_o^2\sigma_c^2\ell^{d+1}T}{4} \int_{\mathbb{R}} \mathrm{d}\widetilde{t'}\, e^{i\omega T\widetilde{t'}} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}'\, e^{-i\ell\boldsymbol{k}\cdot\boldsymbol{y}'} \int_{\mathbb{R}} \mathrm{d}z'\, \mathscr{R}_{cc}(\widetilde{t'},\boldsymbol{y}',z') \\
&= \frac{k_o^2\sigma_c^2\ell}{4} \int_{\mathbb{R}} \mathrm{d}t\, e^{i\omega t} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{y}}{\ell}\Big)
\end{aligned}
$$

with $\mathscr{R}$ defined in (3.25). Substituting into (3.22), we obtain

$$
\Big[\partial_z + \frac{\boldsymbol{q}}{k_o}\cdot\nabla_{\boldsymbol{y}}\Big]\breve{W}(t,\boldsymbol{y},\boldsymbol{q},z) = \frac{\sigma_c^2\ell k_o^2}{4}\Big[\mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{y}-\boldsymbol{v}t}{\ell}\Big) - \mathscr{R}(0,\boldsymbol{0})\Big]\breve{W}(t,\boldsymbol{y},\boldsymbol{q},z), \qquad \text{(C.2)}
$$

for $z > 0$, with initial condition obtained from (3.26) and (C.1)

$$
\begin{aligned}
\breve{W}(t,\boldsymbol{y},\boldsymbol{q},0) = \breve{W}_0(t,\boldsymbol{y}) &:= \frac{\sigma_s^2 T_s^2 \ell_s^{2d}}{4k_o} \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega t} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \big|\widehat{S}(T_s\omega, \ell_s\boldsymbol{k})\big|^2 \\
&= \frac{\sigma_s^2 T_s \ell_s^d}{4k_o} \int_{\mathbb{R}} \frac{\mathrm{d}\Omega}{2\pi}\, e^{-i\Omega\frac{t}{T_s}} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{K}}{(2\pi)^d}\, e^{i\boldsymbol{K}\cdot\frac{\boldsymbol{y}}{\ell_s}} \big|\widehat{S}(\Omega,\boldsymbol{K})\big|^2. \qquad \text{(C.3)}
\end{aligned}
$$

Note that this condition is independent of $\boldsymbol{q}$.

Equation (C.2) can be solved by integrating along the characteristic $\boldsymbol{y} = \boldsymbol{y}_0 + \frac{\boldsymbol{q}}{k_o}z$, stemming from $\boldsymbol{y}_0$ at $z = 0$,

$$
\begin{aligned}
\breve{W}\Big(t, \boldsymbol{y}_0 + \frac{\boldsymbol{q}}{k_o}z, \boldsymbol{q}, z\Big) =& \breve{W}_0(t,\boldsymbol{y}_0) \\
&\times \exp\Big\{\frac{\sigma_c^2\ell k_o^2}{4}\int_0^z \mathrm{d}z'\,\Big[\mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{y}_0 + \frac{\boldsymbol{q}}{k_o}z' - \boldsymbol{v}_o t}{\ell}\Big) - \mathscr{R}(0,\boldsymbol{0})\Big]\Big\}.
\end{aligned}
$$

Substituting $\boldsymbol{y}_0 = \boldsymbol{y} - \boldsymbol{q}/k_o z$ in this equation, and inverting the Fourier transform,

$$W(\omega, \boldsymbol{k}, \boldsymbol{x}, z) = \int_{\mathbb{R}} \mathrm{d}t\, e^{i\omega t} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \breve{W}_0\Big(t, \boldsymbol{y} - \frac{\boldsymbol{q}}{k_o} z\Big)$$
$$\times \exp\Big\{\frac{\sigma_c^2 \ell k_o^2}{4} \int_0^z \mathrm{d}z' \Big[\mathscr{R}\Big(\frac{t}{T}, \frac{\boldsymbol{y} - \frac{\boldsymbol{q}}{k_o}(z - z') - \boldsymbol{v}_o t}{\ell}\Big) - \mathscr{R}(0, \boldsymbol{0})\Big]\Big\}. \tag{C.4}$$

The result (3.27) follows after substituting the expression (C.3) into this equation.

Appendix D. Proof of radiative transfer connection. We prove here the result involving (3.17) and (3.18). We start by computing the different terms in (3.18): The first term is

$$\nabla_{\vec{\boldsymbol{k}}} \Omega(\vec{\boldsymbol{k}}) \cdot \nabla_{\vec{\boldsymbol{x}}} V(\omega, \vec{\boldsymbol{k}}, \vec{\boldsymbol{x}}) = \frac{c_o}{k_o} \delta\big(k_z - \beta(\boldsymbol{k})\big) \Big[\big(\partial_z - \nabla_{\boldsymbol{k}} \beta(\boldsymbol{k}) \cdot \nabla_{\boldsymbol{x}}\big) W(\omega, \boldsymbol{k}, \boldsymbol{x}, z)\Big].$$

For the second term we use that if $\vec{\boldsymbol{k}}' = (\boldsymbol{k}', \beta(\boldsymbol{k}'))$ and $\vec{\boldsymbol{k}} = (\boldsymbol{k}, k_z)$, then

$$\delta\big(\Omega(\vec{\boldsymbol{k}}) - \Omega(\vec{\boldsymbol{k}}')\big) = \frac{1}{c_o} \delta\big(|\vec{\boldsymbol{k}}| - k_o\big) = \frac{k_o}{c_o \beta(\boldsymbol{k})} \delta\big(k_z - \beta(\boldsymbol{k})\big)$$

and

$$\int_{\mathbb{R}^{d+1}} \frac{\mathrm{d}\vec{\boldsymbol{k}}'}{(2\pi)^{d+1}} \int \frac{\mathrm{d}\omega'}{2\pi} \mathfrak{S}\big(\omega, \omega', \vec{\boldsymbol{k}}, \vec{\boldsymbol{k}}'\big) V(\omega', \vec{\boldsymbol{k}}', \vec{\boldsymbol{x}}) = \frac{c_o}{k_o} \delta\big(k_z - \beta(\boldsymbol{k})\big)$$
$$\times \int_{\mathcal{O}} \frac{\mathrm{d}\omega' \mathrm{d}\boldsymbol{k}'}{(2\pi)^{d+1}} Q(\omega, \omega', \boldsymbol{k}, \boldsymbol{k}') W(\omega', \boldsymbol{k}', \boldsymbol{x}, z).$$

Similarly, for the third term we have that if $\vec{\boldsymbol{k}} = (\boldsymbol{k}, \beta(\boldsymbol{k}))$ and $\vec{\boldsymbol{k}}' = (\boldsymbol{k}', k_z')$, then

$$\delta\big(\Omega(\vec{\boldsymbol{k}}) - \Omega(\vec{\boldsymbol{k}}')\big) = \frac{1}{c_o} \delta\big(k_o - |\vec{\boldsymbol{k}}|\big) = \frac{k_o}{c_o \beta(\boldsymbol{k}')} \delta\big(k_z' - \beta(\boldsymbol{k}')\big)$$

and

$$\int_{\mathbb{R}^{d+1}} \frac{\mathrm{d}\vec{\boldsymbol{k}}'}{(2\pi)^{d+1}} \int \frac{\mathrm{d}\omega'}{2\pi} \mathfrak{S}\big(\omega, \omega', \vec{\boldsymbol{k}}, \vec{\boldsymbol{k}}'\big) V(\omega, \vec{\boldsymbol{k}}, \vec{\boldsymbol{x}}) = \frac{c_o}{k_o} \delta\big(k_z - \beta(\boldsymbol{k})\big)$$
$$\times \int_{\mathcal{O}} \frac{\mathrm{d}\omega' \mathrm{d}\boldsymbol{k}'}{(2\pi)^{d+1}} Q(\omega, \omega', \boldsymbol{k}, \boldsymbol{k}') W(\omega, \boldsymbol{k}, \boldsymbol{x}, z).$$

Gathering the results and using (3.12), we obtain (3.18).

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