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PULSE REFLECTION IN A RANDOM WAVEGUIDE WITH A TURNING POINT*

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Abstract. We present an analysis of wave propagation and reflection in an acoustic waveguide with random sound soft boundary and a turning point. The waveguide has slowly bending axis and variable cross section. The variation consists of a slow and monotone change of the width of the waveguide and small and rapid fluctuations of the boundary, on the scale of the wavelength. These fluctuations are modeled as random. The turning point is many wavelengths away from the source, which emits a pulse that propagates toward the turning point, where it is reflected. To focus attention on this reflection, we assume that the waveguide supports a single propagating mode from the source to the turning point, beyond which all the waves are evanescent. We consider a scaling regime where scattering at the random boundary has a significant effect on the reflected pulse. In this regime scattering from the random boundary away from the turning point is negligible, while scattering from the random boundary around the turning point results in a strong, deterministic pulse deformation. The reflected pulse shape is not the same as the emitted one. It is damped, due to scattering at the boundary, and is deformed by dispersion in the waveguide. The reflected pulse also carries a random phase.

Key words. turning waves, random waveguide, pulse stabilization

AMS subject classifications. 35Q61, 35R60

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1. Introduction.

Guided waves arise in a wide range of applications in electromagnetics [10], optics and communications [25], underwater acoustics [19], and so on. The classical theory of guided waves relies on the separability of the wave equation in ideal waveguides with straight walls and filled with homogeneous media [29]. It decomposes the wave field in independent waveguide modes, which are special solutions of the wave equation. The modes are either propagating waves along the axis of the waveguide or evanescent waves. They do not interact with each other and have constant amplitudes determined by the source excitation.

We study sound waves in two-dimensional waveguides with varying cross section and slowly bending axis, where the waveguide effect is due to reflecting boundaries, modeled for simplicity as sound soft. The three-dimensional case and other boundary conditions can be treated similarly and do not involve conceptual differences. We refer to [18, 26] for examples of numerical studies of waves in slowly varying waveguides and to [25] for local mode decompositions of the wave field, where the modes are coupled, and their amplitudes vary along the waveguide axis. An analysis of such a decomposition is given in [2, 31], and the transition of propagating modes to evanescent ones at turning points in slowly changing waveguides is studied in [5]. Here we analyze this wave transition at a turning point in a random waveguide with small and rapid random fluctuations of the boundary on the scale of the wavelength, in addition to the slow variations.

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The wave field is generated by a source which emits a pulse with central frequency ω_o and bandwidth $B \ll \omega_o$. It is the superposition of a countable set of modes, of which only finitely many propagate. To focus attention on the turning point, we consider a central frequency ω_o such that there is a single propagating mode between the source and the turning point. We also assume that the slow variation of the waveguide width is monotone, so that no propagation occurs beyond the turning point. Due to energy conservation, the propagating mode is reflected at the turning point and returns to the source location. The goal of the paper is to analyze the pulse shape carried by this reflected wave.

Sound wave propagation in random waveguides is analyzed in [19, 11, 14, 13, 16] for the case of waveguides filled with a random medium and in [4, 7, 17] for the case of waveguides with random perturbations of straight boundaries. We also refer to [25, 3] for the analysis of electromagnetic waves in random waveguides. A main difficulty arising in the extension of these results to random waveguides with slowly varying cross section is due to the turning points, studied in this paper.

An analysis of random multiple scattering of turning waves is given in [20, 21], in the context of wave propagation in randomly layered media. These results are relevant to our study, specially the stochastic averaging theorem in [21]. In this paper we derive from first principles a stochastic equation for the reflection coefficient of the propagating mode in the random waveguide and study in detail its statistics, using the limit theorem in [21]. To characterize the reflected pulse, we carry out a multifrequency analysis of the reflection coefficient whose phase has a nontrivial random frequency dependence. We quantify the standard deviation of the random fluctuations of the boundary that triggers strong modifications of the amplitude and shape of the reflected pulse. We show that such random fluctuations have negligible effect on the pulse away from the turning point, but near the turning point the effect is strong and leads to a deterministic pulse deformation and damping. This pulse stabilization result is similar but different from the ones obtained in layered media in [27, 9, 23, 22], in locally layered media in [30], in time-dependent layered media in [8], and in three-dimensional random media in [15]. In these references the medium is random, not the boundary, there is no turning point, and pulse deformation is observed when the standard deviation of the random fluctuations is larger than the one considered here. In this paper we explain why the random fluctuations have a stronger effect close to the turning point than away from it.

The paper is organized as follows. We begin in section 2 with the formulation of the problem and state the pulse stabilization result in section 3. The proof of this result is in section 4. We end with a summary in section 5.

2. Formulation of the problem. We describe in section 2.1 the setup of the problem and define in section 2.2 the scaling regime. Then we give in section 2.3 the mode decomposition of the wave field and derive the stochastic differential equation satisfied by the propagating mode. The remainder of the paper is concerned with the analysis of this equation.

2.1. Setup. Consider a two-dimensional waveguide occupying the semi-infinite domain Ω, with sound soft boundary $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+$ consisting of the union of two curves, as illustrated in Figure 1. We refer to $\partial\Omega^-$ as the bottom boundary and to $\partial\Omega^+$ as the top boundary. The waveguide has a slowly bending axis parametrized by the arc length z. Ideally, $\partial\Omega^-$ and $\partial\Omega^+$ would be symmetric with respect to this axis, but the top boundary is perturbed by small fluctuations. The waveguide is filled

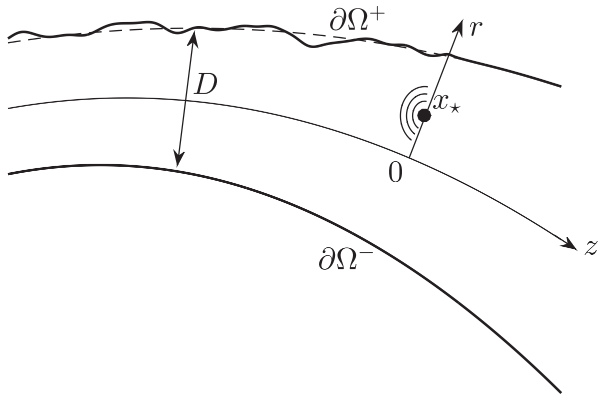

Fig. 1. *Illustration of a waveguide with monotonically increasing width D and bending axis parametrized by the arc length z. The boundary $\partial\Omega$ is the union of the curves $\partial\Omega^-$ (the bottom boundary) and $\partial\Omega^+$ (the top boundary). The top boundary is perturbed by small fluctuations modeled with a random process. The source of waves is at $\boldsymbol{x}_\star$. The waves first propagate toward negative z in the form of a left-going propagating mode, they are reflected at the turning point, and they propagate back toward positive z in the form of a right-going mode.*

with a homogeneous medium with wave speed c, and the excitation is due to a point source at location $\boldsymbol{x}_\star \in \Omega$, that emits the pulse

$$f(t) = \cos(\omega_o t)F(Bt). \tag{2.1}$$

This pulse is modeled by a periodic carrier signal at frequency ω_o and a real-valued, smooth envelope function F of dimensionless argument. Its Fourier transform $\widehat{F}$ is supported in the interval $[-\pi, \pi]$, so the Fourier transform of (2.1),

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} dt\, \cos(\omega_o t)F(Bt)e^{i\omega t} = \frac{1}{2B}\left[\widehat{F}\Big(\frac{\omega-\omega_o}{B}\Big) + \widehat{F}\Big(\frac{\omega+\omega_o}{B}\Big)\right], \tag{2.2}$$

is supported in the frequency interval $[\omega_o - \pi B, \omega_o + \pi B]$ centered at ω_o, with bandwidth B, and its negative image $[-\omega_o - \pi B, -\omega_o + \pi B]$. Since F is real valued,

$$\widehat{F}\Big(\frac{\omega+\omega_o}{B}\Big) = \overline{\widehat{F}\Big(\frac{-\omega-\omega_o}{B}\Big)}, \tag{2.3}$$

where the bar denotes complex conjugate. We take $B \ll \omega_o$, so that $\lambda_o = 2\pi c/\omega_o$ approximates the wavelength at all frequencies ω in the support of $\widehat{f}(\omega)$, and suppose that λ_o is small with respect to the arc length distance of order L from the source to the turning point.

The wave field is modeled by the acoustic pressure $p(t, \boldsymbol{x})$, the solution of the wave equation

$$\Big(\Delta - \frac{1}{c^2}\partial_t^2\Big)p(t,\boldsymbol{x}) = f(t)\delta(\boldsymbol{x}-\boldsymbol{x}_\star), \qquad \boldsymbol{x} \in \Omega, \quad t \in \mathbb{R}, \tag{2.4}$$

with homogeneous Dirichlet boundary conditions

$$p(t,\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega, \quad t \in \mathbb{R}. \tag{2.5}$$

Prior to the excitation the medium is quiescent,

$$p(t,\boldsymbol{x}) \equiv 0, \qquad t \ll 0. \tag{2.6}$$

It is convenient to write (2.4)–(2.5) in the orthogonal curvilinear coordinate system with axes along $\boldsymbol{\tau}(z/L)$ and $\boldsymbol{n}(z/L)$, the unit tangent and normal vectors to the axis of the waveguide, at arc length z. These vectors change slowly in z, on the length scale $L \gg \lambda_o$, according to the Frenet–Serret formulas

$$\partial_z \boldsymbol{\tau}\Big(\frac{z}{L}\Big) = \frac{1}{L}\kappa\Big(\frac{z}{L}\Big)\boldsymbol{n}\Big(\frac{z}{L}\Big), \qquad \partial_z \boldsymbol{n}\Big(\frac{z}{L}\Big) = -\frac{1}{L}\kappa\Big(\frac{z}{L}\Big)\boldsymbol{\tau}\Big(\frac{z}{L}\Big), \tag{2.7}$$

where $\kappa(z/L)$ is the curvature. We parametrize the points $\boldsymbol{x} \in \Omega$ by (r, z), using

$$\boldsymbol{x} = \boldsymbol{x}_{\|}(z) + r\boldsymbol{n}\Big(\frac{z}{L}\Big), \tag{2.8}$$

where $\boldsymbol{x}_{\|}(z)$ is on the waveguide axis, at arc length z, and r is the coordinate in the direction of the normal at z. This coordinate lies in the interval $[r^-(z), r^+(z)]$, with $r^-(z)$ at the bottom boundary $\partial\Omega^-$

$$r^-(z) = -\frac{D(z/L)}{2} \tag{2.9}$$

and $r^+(z)$ at the randomly perturbed top boundary[1] $\partial\Omega^+$

$$r^+(z) = \frac{D(z/L)}{2}\left[1 + 1_{(-\infty,0)}(z)\sigma\nu\Big(\frac{z}{\ell}\Big)\right]. \tag{2.10}$$

Here $D(z/L)$ is the width of the unperturbed waveguide, a smooth (at least three times continuously differentiable) and monotonically increasing function that varies slowly in z, on the scale L. The top boundary has small and rapid random fluctuations on the left of the source, on the scale ℓ called the correlation length, and $1_{(-\infty,0)}(z)$ is the indicator function of the negative axis $z < 0$, smoothed near the origin. The fluctuations are modeled by the zero-mean stationary process ν of dimensionless argument, with autocorrelation function

$$\mathcal{R}(\zeta) = \mathbb{E}\big[\nu(\zeta)\nu(0)\big]. \tag{2.11}$$

This process is mixing, with rapidly decaying mixing rate, as defined, for example, in [28, section 2], and it is bounded, with bounded first two derivatives, almost surely. We normalize ν so that

$$\mathcal{R}(0) = 1, \qquad \int_{-\infty}^{\infty} d\zeta\, \mathcal{R}(\zeta) = 1 \text{ [or } O(1)], \tag{2.12}$$

and control the amplitude of the fluctuations in (2.10) by the standard deviation σ and their spatial scale by the correlation length ℓ. The Fourier transform of $\mathcal{R}$

$$\widehat{\mathcal{R}}(k) = \int_{-\infty}^{\infty} d\zeta\, e^{ik\zeta}\mathcal{R}(\zeta) = \int_{-\infty}^{\infty} d\zeta \cos(k\zeta)\mathcal{R}(\zeta)$$

is the power spectral density of the stationary process ν. It is an even and nonnegative function.

[1]The results of this paper extend readily to perturbations of both boundaries: the top one by the random process $\nu(z/\ell)$ as in (2.10) and the bottom one by another, independent random process $\nu^-(z/\ell)$. Such a setup was considered in [4], where it was shown that the effect of the two random boundaries is additive, due to the independence of the processes ν and ν^-. The same conclusion holds here, so for simplicity we suppose that only the top boundary is perturbed.

In the curvilinear coordinate system the source is located at $(r_\star, z = 0)$, and the wave equation (2.4) becomes

$$\left[\partial_r^2 - \frac{\frac{1}{L}\kappa\left(\frac{z}{L}\right)\partial_r}{1-\frac{r}{L}\kappa\left(\frac{z}{L}\right)} + \frac{\partial_z^2}{\left[1-\frac{r}{L}\kappa\left(\frac{z}{L}\right)\right]^2} + \frac{\frac{r}{L^2}\kappa'\left(\frac{z}{L}\right)\partial_z}{\left[1-\frac{r}{L}\kappa\left(\frac{z}{L}\right)\right]^3} - \frac{1}{c^2}\partial_t^2\right] p(t,r,z)$$

$$= \left|1 - \frac{r_\star}{L}\kappa(0)\right|^{-1} f(t)\delta(z)\delta(r-r_\star) \tag{2.13}$$

for $t \in \mathbb{R}$, $z \in \mathbb{R}$, and $r \in (r^-(z), r^+(z))$, with boundary conditions (2.5) given by

$$p(t, r^-(z), z) = p(t, r^+(z), z) = 0 \quad \forall\, t \in \mathbb{R},\ z \in \mathbb{R}. \tag{2.14}$$

Here κ' denotes the derivative of the curvature and we used the parametrization (2.8) of the points in the waveguide, with

$$\partial_r \boldsymbol{x} = \boldsymbol{n}\left(\frac{z}{L}\right), \qquad \partial_z \boldsymbol{x} = \left[1 - \frac{r}{L}\kappa\left(\frac{z}{L}\right)\right]\boldsymbol{\tau}\left(\frac{z}{L}\right),$$

and Lamé coefficients

$$h_r = |\partial_r \boldsymbol{x}| = 1, \qquad h_z = |\partial_z \boldsymbol{x}| = \left|1 - \frac{r}{L}\kappa\left(\frac{z}{L}\right)\right|,$$

to write the Laplacian

$$\Delta = \frac{1}{h_r h_z}\left[\partial_r\left(\frac{h_r h_z}{h_r^2}\partial_r\right) + \partial_z\left(\frac{h_r h_z}{h_z^2}\partial_z\right)\right]$$

and the Dirac delta at $\boldsymbol{x}_\star$,

$$\delta(\boldsymbol{x} - \boldsymbol{x}_\star) = \frac{1}{h_r h_z}\delta(z)\delta(r - r_\star).$$

The problem is to analyze the wave field $p(t, r, z = 0)$ at time $t > T_f$, where T_f is the duration of the emitted pulse $f(t)$. This models the reflected wave in the random section of the waveguide, which contains the turning point.

2.2. The scaling regime. We define here a scaling regime where the random boundary fluctuations have a significant effect on the reflected wave. The regime is defined by the standard deviation σ of the random fluctuations and the relation between the important length scales in the problem: the central wavelength $\lambda_o = 2\pi c/\omega_o$, the correlation length ℓ of the random fluctuations, the scale L of the slow variations of the waveguide, and the width D of the cross section.

The length scales are ordered as

$$L \gg D \sim \lambda_o \sim \ell, \tag{2.15}$$

where $\sim$ denotes "of the same order as." In this scaling regime the central wavelength is of the same order as the correlation length of the medium and is much smaller than the typical propagation distance, so that the waves interact efficiently with the boundary fluctuations. We model (2.15) using the small, dimensionless parameter

$$\varepsilon = \frac{\ell}{L} \ll 1 \tag{2.16}$$

and use asymptotic analysis in the limit $\varepsilon \to 0$ to characterize the reflected wave.

The relation between the waveguide width $D(z/L)$ and the central wavelength λ_o determines the number

$$N(z) = \lfloor 2D(z/L)/\lambda_o \rfloor$$

of propagating modes in the local mode decomposition of the wave $p(t, r, z)$, at given z, where $\lfloor\ \rfloor$ denotes the integer part. To simplify the analysis we assume that the central frequency ω_o of the pulse is such that $N(z) = 1$ for $z \in (z_T(\omega_o), 0)$, where $z_T(\omega_o) < 0$ is the arc length at the turning point, satisfying

$$\lambda_o = 2D\left(z_T(\omega_o)/L\right). \tag{2.17}$$

The turning point is assumed simple, meaning that $D'(z_T(\omega_o)/L) > 0$, and by the monotonicity of $D(z/L)$ we have $N(z) = 0$ for $z < z_T(\omega_o)$. Consistent with the slow variations of the waveguide on the scale L, we suppose that $|z_T(\omega_o)| \sim L$.

We know from the study [4] of waveguides with randomly perturbed straight boundaries that the interaction of the waves with the boundary fluctuations gives an order one net scattering effect over the distance L scaled as in (2.16), when the standard deviation of the fluctuations is of the order $\sqrt{\varepsilon}$. Thus, we take

$$\sigma = \sqrt{\varepsilon}\sigma_\varepsilon \tag{2.18}$$

with σ_ε at most of order one with respect to ε. It will be adjusted later, so that the effect of the random fluctuations of the waveguide boundary on the reflected pulse is of order one as $\varepsilon \to 0$.

The duration T_f of $f(t)$ is inverse proportional to the bandwidth B and must be much smaller than the travel time from the source to the turning point and back; otherwise $f(t)$ would not be a pulse. This implies the scaling relation

$$\frac{1}{\omega_o} \ll \frac{1}{B} \ll \frac{L}{c} \sim \frac{1}{\varepsilon\omega_o}, \tag{2.19}$$

where we used (2.15) and $B \ll \omega_o$. We show in section 4 that the characterization of the probability distribution of the reflected pulse involves the joint distribution of the reflection coefficients at frequencies spaced by $O(B)$. We choose

$$\frac{B}{\omega_o} \sim \sqrt{\varepsilon} \tag{2.20}$$

so that (2.19) is satisfied and the phases of the frequency-dependent reflection coefficients have statistically dependent and independent components. This gives the pulse stabilization result after Fourier synthesis.

2.2.1. The scaled variables. We scale the arc length z by L and the waveguide width and cross-range coordinate r by ℓ,

$$\tilde{z} = z/L, \quad \tilde{r} = r/\ell, \quad \tilde{D}(\tilde{z}) = D(z/L)/\ell. \tag{2.21}$$

The scaled frequency is

$$\tilde{\omega} = \omega\frac{\ell}{c}. \tag{2.22}$$

The central frequency is $\tilde{\omega}_o = \omega_o \ell/c = 2\pi\ell/\lambda_o$. By (2.20) the bandwidth is such that $B\ell/c \sim \sqrt{\varepsilon}$ so we introduce the scaled bandwidth $\tilde{B}$ defined by

$$\tilde{B} = \frac{B\ell}{c}\frac{1}{\sqrt{\varepsilon}}. \tag{2.23}$$

The scaled wavenumber $k(\omega) = \omega/c$ is

$$\tilde{k}(\tilde{\omega}) = k(\omega)\ell. \tag{2.24}$$

All scaled quantities are of order one in the scaling regime described just above.

2.2.2. The scaled equation. We assume henceforth that the variables are scaled and simplify the notation by dropping the tilde. We take the Fourier transform of (2.13) with respect to time and denote by $\widehat{p}$ the wave field in the scaled variables. After multiplying the resulting equation by $L^2(1-\varepsilon r\kappa(z))^2$ we obtain

$$\begin{aligned}\Bigg[\partial_z^2 + \frac{(1-\varepsilon r\kappa(z))^2}{\varepsilon^2}(k^2(\omega)+\partial_r^2) \\ -\frac{\kappa(z)(1-\varepsilon r\kappa(z))}{\varepsilon}\partial_r + \frac{\varepsilon r\kappa'(z)}{(1-\varepsilon r\kappa(z))}\partial_z\Bigg]\widehat{p}(\omega,r,z) \\ = \frac{\widehat{f}^\varepsilon(\omega)}{\varepsilon}\delta(r-r_\star)\delta(z)\end{aligned} \tag{2.25}$$

with

$$\widehat{f}^\varepsilon(\omega) = \frac{(1-\varepsilon r_\star\kappa(0))}{2\sqrt{\varepsilon}B}\left[\widehat{F}\left(\frac{\omega-\omega_o}{\sqrt{\varepsilon}B}\right) + \overline{\widehat{F}\left(\frac{-\omega-\omega_o}{\sqrt{\varepsilon}B}\right)}\right] \tag{2.26}$$

and homogeneous Dirichlet boundary conditions

$$\widehat{p}(\omega, r^\pm(z), z) = 0, \qquad r^-(z) = -\frac{D(z)}{2}, \quad r^+(z) = \frac{D(z)}{2}\left[1+\sqrt{\varepsilon}\sigma_\varepsilon\nu\Big(\frac{z}{\varepsilon}\Big)\right] \tag{2.27}$$

for all ω in the support of $\widehat{f}^\varepsilon(\omega)$ and $z \in \mathbb{R}$. The wave is outgoing at $z > 0$, because there are no random fluctuations there, and decays exponentially (is evanescent) at $z < z_T(\omega_o)$.

2.3. Mode decomposition. To define the mode decomposition, we change coordinates to map the random boundary fluctuations to the coefficients of the wave equation (2.25). This way we obtain a linear differential operator $\mathcal{L}^\varepsilon$ that has an asymptotic expansion in ε and acts on functions that vanish at the unperturbed boundary $r = \pm D(z)/2$ for all z. The modes are defined using the spectral decomposition of the leading part of $\mathcal{L}^\varepsilon$, and they have random amplitudes satisfying stochastic differential equations driven by the process ν, with excitation given by jump conditions at $z = 0$, where the source lies.

2.3.1. The random change of coordinates. We use the following change of coordinates that maps the random boundary fluctuations to the wave operator:

$$r = \rho + \frac{(2\rho + D(z))}{4}\sqrt{\varepsilon}\sigma_\varepsilon\nu\Big(\frac{z}{\varepsilon}\Big) \qquad \forall\, z < 0, \tag{2.28}$$

where ρ is in the unperturbed domain $[-D(z)/2, D(z)/2]$. There are no random fluctuations for $z > 0$, so $r = \rho$ there. Substituting in (2.25) and using the chain rule, we obtain after straightforward calculations that

$$\widehat{p}^{\varepsilon}(\omega,\rho,z) = \widehat{p}\left(\omega, \rho + \frac{(2\rho + D(z))}{4}\sqrt{\varepsilon}\sigma_{\varepsilon}\nu\Big(\frac{z}{\varepsilon}\Big), z\right) \tag{2.29}$$

satisfies the equation

$$\mathcal{L}^{\varepsilon}\widehat{p}^{\varepsilon}(\omega,\rho,z) = \frac{\widehat{f}^{\varepsilon}(\omega)}{\varepsilon}\delta(\rho - r_{\star})\delta(z) \quad \forall\, \rho \in \left(-\frac{D(z)}{2}, \frac{D(z)}{2}\right), \quad z \in \mathbb{R}, \tag{2.30}$$

and the boundary conditions (2.27) become

$$\widehat{p}^{\varepsilon}(\omega, \pm D(z)/2, z) = 0. \tag{2.31}$$

The operator $\mathcal{L}^{\varepsilon}$ is given by

$$\begin{aligned}\mathcal{L}^{\varepsilon} = {} & \left(\sum_{j=0}^{2} \varepsilon^{j/2-2}\mathcal{L}_j\right) + \partial_z^2 - \frac{(2\rho + D(z))}{2} \\ & \times \left[\frac{\sigma_{\varepsilon}}{\sqrt{\varepsilon}}\nu'\Big(\frac{z}{\varepsilon}\Big) - \frac{\sigma_{\varepsilon}^2}{2}\nu\Big(\frac{z}{\varepsilon}\Big)\nu'\Big(\frac{z}{\varepsilon}\Big) + O(\sqrt{\varepsilon})\right]\partial_{\rho z}^2 \\ & + O(\varepsilon)\partial_z + O(\varepsilon^{-1/2})\partial_{\rho}^2 + O(\varepsilon^{-1/2})\partial_{\rho},\end{aligned} \tag{2.32}$$

where $\mathcal{L}_j$ are differential operators with respect to ρ, with coefficients that depend on z. These operators depend on ε only through σ_{ε} and the argument of ν, ν', and ν''. The leading operator $\mathcal{L}_0$ is

$$\mathcal{L}_0 = k^2(\omega) + \partial_{\rho}^2, \tag{2.33}$$

its first perturbation depends linearly on the random process ν,

$$\mathcal{L}_1 = -\sigma_{\varepsilon}\nu\Big(\frac{z}{\varepsilon}\Big)\partial_{\rho}^2 - \sigma_{\varepsilon}\nu''\Big(\frac{z}{\varepsilon}\Big)\frac{(2\rho + D(z))}{4}\partial_{\rho}, \tag{2.34}$$

and the second perturbation is quadratic in ν,

$$\begin{aligned}\mathcal{L}_2 = {} & \frac{3\sigma_{\varepsilon}^2}{4}\nu^2\Big(\frac{z}{\varepsilon}\Big)\partial_{\rho}^2 + \sigma_{\varepsilon}^2\nu'^2\Big(\frac{z}{\varepsilon}\Big)\left[\frac{(2\rho + D(z))^2}{16}\partial_{\rho}^2 + \frac{(2\rho + D(z))}{4}\partial_{\rho}\right] \\ & + \sigma_{\varepsilon}^2\nu\Big(\frac{z}{\varepsilon}\Big)\nu''\Big(\frac{z}{\varepsilon}\Big)\frac{(2\rho + D(z))}{8}\partial_{\rho} - \kappa(z)\left[2\rho(k^2(\omega) + \partial_{\rho}^2) + \partial_{\rho}\right].\end{aligned} \tag{2.35}$$

2.3.2. The waveguide modes. The self-adjoint operator $\mathcal{L}_0$, acting on functions that vanish at $\rho = \pm D(z)/2$ for any fixed z, has the eigenfunctions

$$y_j(\rho,z) = \left[\frac{2}{D(z)}\right]^{1/2}\sin\left[\frac{(2\rho + D(z))}{2}\mu_j(z)\right], \quad \mu_j(z) = \frac{\pi j}{D(z)}, \quad j = 1, 2, \dots, \tag{2.36}$$

and eigenvalues $k^2(\omega) - \mu_j^2(z)$, for $j = 1, 2, \dots$ The eigenfunctions form an orthonormal L^2 basis in $[-D(z)/2, D(z)/2]$, so we can decompose the wave field at any z as

$$\widehat{p}^{\varepsilon}(\omega,\rho,z) = \sum_{j=1}^{\infty}\widehat{u}_j^{\varepsilon}(\omega,z)y_j(\rho,z), \tag{2.37}$$

where $\widehat{u}_j^\varepsilon(\omega,z)$ are waves in one dimension, called the waveguide modes. Substituting (2.37) in (2.30), using the orthogonality of the eigenfunctions and the identities given in Appendix A, we obtain

$$\begin{aligned}(2.38)\quad &\left[\partial_z^2 + \frac{k^2(\omega)-\mu_j^2(z)}{\varepsilon^2}\right]\widehat{u}_j^\varepsilon(\omega,z)\\ &+\frac{\sigma_\varepsilon}{\varepsilon^{3/2}}\left[\mu_j^2(z)\nu\Big(\frac{z}{\varepsilon}\Big)+\frac{1}{4}\nu''\Big(\frac{z}{\varepsilon}\Big)\right]\widehat{u}_j^\varepsilon(\omega,z)+\frac{\sigma_\varepsilon}{2\varepsilon^{1/2}}\nu'\Big(\frac{z}{\varepsilon}\Big)\partial_z\widehat{u}_j^\varepsilon(\omega,z)\\ &-\frac{\sigma_\varepsilon^2}{\varepsilon}\left\{\frac{3\mu_j^2(z)}{4}\nu^2\Big(\frac{z}{\varepsilon}\Big)+\left[\frac{1}{8}+\frac{(\pi j)^2}{12}\right]\nu'^2\Big(\frac{z}{\varepsilon}\Big)+\frac{1}{8}\nu\Big(\frac{z}{\varepsilon}\Big)\nu''\Big(\frac{z}{\varepsilon}\Big)\right\}\widehat{u}_j^\varepsilon(\omega,z)\\ &-\frac{\sigma_\varepsilon^2}{4}\nu\Big(\frac{z}{\varepsilon}\Big)\nu'\Big(\frac{z}{\varepsilon}\Big)\partial_z\widehat{u}_j^\varepsilon(\omega,z)=\mathcal{C}_j^\varepsilon\left(\omega,z,\{\widehat{u}_q^\varepsilon\}_{q\neq j}\right)\end{aligned}$$

for $z<0$. Here we neglected the remainder of order $\varepsilon^{1/2}$ and denoted by $\mathcal{C}_j^\varepsilon$ the coupling terms that depend on the modes $\widehat{u}_q^\varepsilon$ for $q\neq j$. The curvature $\kappa(z)$ of the axis of the waveguide appears only in these terms. The equations for $z>0$ are simpler, because there are no random fluctuations in the right-hand side. They are obtained from (2.38) by setting to zero all the terms that depend on the process ν.

The first term in the wave equations (2.38) shows that the jth mode is a propagating wave when $\mu_j^2(z)<k^2(\omega)$, and it is evanescent when the opposite inequality holds. By our scaling assumptions we have a single propagating mode for $z<0$, the one indexed by $j=1$. This interacts with the evanescent modes via the coupling term $\mathcal{C}_1^\varepsilon$. We refer to [4, section 3.3] for the analysis of such an interaction. It shows that the evanescent modes can be expressed in terms of u_1^ε, so that we can close the wave equation for this propagating mode. We do not give here this calculation, because it is basically the same as in [4]. The result is that the contribution of the evanescent modes consists of an additional term in the equation for $\widehat{u}_1^\varepsilon$, that is similar to the quadratic one in the fluctuations, written in the curly brackets in (2.38). We will see in section 4 that this term is negligible when σ_ε is scaled so that the reflected pulse retains a deterministic shape. For the sake of brevity, we do not include the contribution of the evanescent modes which play no role in the end.

2.3.3. The equation for the propagating mode. We can simplify the equation for the propagating mode $\widehat{u}_1^\varepsilon$ using integrating factors by redefining the unknown

$$(2.39)\quad \widehat{u}^\varepsilon(\omega,z)=\widehat{u}_1^\varepsilon(\omega,z)\exp\left[\frac{\varepsilon^{1/2}\sigma_\varepsilon}{4}\nu\Big(\frac{z}{\varepsilon}\Big)-\frac{\varepsilon\sigma_\varepsilon^2}{16}\nu^2\Big(\frac{z}{\varepsilon}\Big)\right]=\widehat{u}_1^\varepsilon(\omega,z)\big[1+O\big(\varepsilon^{1/2}\big)\big].$$

Substituting in (2.38) for $j=1$, we obtain

$$(2.40)\quad \partial_z^2\widehat{u}^\varepsilon(\omega,z)+\left[\frac{k^2(\omega)-\mu^2(z)}{\varepsilon^2}+\frac{\sigma_\varepsilon\mu^2(z)}{\varepsilon^{3/2}}\nu\Big(\frac{z}{\varepsilon}\Big)+\frac{\sigma_\varepsilon^2}{\varepsilon}g^\varepsilon(\omega,z)\right]\widehat{u}^\varepsilon(\omega,z)=0$$

for $z<0$, with the simplified notation

$$(2.41)\quad \mu(z)=\mu_1(z)=\frac{\pi}{D(z)},\qquad g^\varepsilon(\omega,z)=-\frac{3}{4}\mu^2(z)\nu^2\left(\frac{z}{\varepsilon}\right)-\left(\frac{\pi^2}{12}+\frac{1}{16}\right)\nu'^2\left(\frac{z}{\varepsilon}\right),$$

where the contribution of the evanescent waves is not written as it vanishes in our scaling regime. The excitation comes from the jump conditions at the source, with $\widehat{f}^\varepsilon(\omega)$ defined in (2.26),

$$\widehat{u}^\varepsilon(\omega,0^+) - \widehat{u}^\varepsilon(\omega,0^-) = 0, \tag{2.42}$$

$$\partial_z\widehat{u}^\varepsilon(\omega,0^+) - \partial_z\widehat{u}^\varepsilon(\omega,0^-) = \varepsilon^{-1}\widehat{f}^\varepsilon(\omega)y_1(r_\star,0). \tag{2.43}$$

The remainder of the paper is concerned with the analysis of the solution of (2.40), with initial condition defined by (2.42)–(2.43), outgoing condition at $z > 0$, and exponential decay beyond the turning point, where the mode is evanescent.

3. The reflection coefficient and statement of results. We begin in section 3.1 with the decomposition of $\widehat{u}^\varepsilon(\omega,z)$ in forward and backward-going waves. This allows us to define the reflection coefficient in section 3.2 and then state the pulse stabilization result in section 3.3. This result is derived in section 4 under the assumption that the turning point $z_T(\omega)$ of the mode $\widehat{u}^\varepsilon(\omega,z)$ is simple for any ω in the support of $\widehat{f}^\varepsilon(\omega)$. The frequency-dependent turning point $z_T(\omega)$ is defined by

$$k(\omega) = \mu(z_T(\omega)) = \frac{\pi}{D(z_T(\omega))}, \tag{3.1}$$

and it is unique due to the monotonicity of $D(z)$.

3.1. The forward- and backward-going waves. Let us write (2.40) as a first-order system of stochastic differential equations

$$\begin{aligned}\partial_z\begin{pmatrix}\widehat{u}^\varepsilon(\omega,z)\\ \widehat{v}^\varepsilon(\omega,z)\end{pmatrix} &= \frac{i}{\varepsilon}\begin{pmatrix}0 & 1\\ k^2(\omega)-\mu^2(z) & 0\end{pmatrix}\begin{pmatrix}\widehat{u}^\varepsilon(\omega,z)\\ \widehat{v}^\varepsilon(\omega,z)\end{pmatrix}\\ &\quad + \left[\frac{i\sigma_\varepsilon}{\sqrt{\varepsilon}}\mu^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + i\sigma_\varepsilon^2 g^\varepsilon(\omega,z)\right]\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\widehat{u}^\varepsilon(\omega,z)\\ \widehat{v}^\varepsilon(\omega,z)\end{pmatrix}\end{aligned} \tag{3.2}$$

for the vector with components $\widehat{u}^\varepsilon(\omega,z)$ and $\widehat{v}^\varepsilon(\omega,z) = -i\varepsilon\partial_z\widehat{u}^\varepsilon(\omega,z)$. Let also $\boldsymbol{M}^\varepsilon(\omega,z)$ be a flow of smooth and invertible matrices, and define the vector

$$\begin{pmatrix}\widehat{a}^\varepsilon(\omega,z)\\ \widehat{b}^\varepsilon(\omega,z)\end{pmatrix} = \boldsymbol{M}^{\varepsilon,-1}(\omega,z)\begin{pmatrix}\widehat{u}^\varepsilon(\omega,z)\\ \widehat{v}^\varepsilon(\omega,z)\end{pmatrix}, \tag{3.3}$$

which satisfies equations

$$\begin{aligned}\partial_z\begin{pmatrix}\widehat{a}^\varepsilon(\omega,z)\\ \widehat{b}^\varepsilon(\omega,z)\end{pmatrix} &= \boldsymbol{M}^{\varepsilon,-1}(\omega,z)\left\{\frac{i}{\varepsilon}\begin{pmatrix}0 & 1\\ k^2(\omega)-\mu^2(z) & 0\end{pmatrix}\boldsymbol{M}^\varepsilon(\omega,z) - \partial_z\boldsymbol{M}^\varepsilon(\omega,z)\right.\\ &\quad \left. + \left[\frac{i\sigma_\varepsilon}{\sqrt{\varepsilon}}\mu^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + i\sigma_\varepsilon^2 g^\varepsilon(\omega,z)\right]\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\boldsymbol{M}^\varepsilon(\omega,z)\right\}\begin{pmatrix}\widehat{a}^\varepsilon(\omega,z)\\ \widehat{b}^\varepsilon(\omega,z)\end{pmatrix},\end{aligned} \tag{3.4}$$

derived from (3.2), where $\boldsymbol{M}^{\varepsilon,-1}$ denotes the inverse of $\boldsymbol{M}^\varepsilon$. The purpose of the decomposition (3.3) is to remove the leading deterministic coupling term in (3.4) by a proper choice of $\boldsymbol{M}^\varepsilon(\omega,z)$, so that we can analyze the effect of the random fluctuations. Then, we can associate the random fields $\widehat{a}^\varepsilon(\omega,z)$ and $\widehat{b}^\varepsilon(\omega,z)$ to the amplitudes of the forward- and backward-going waves for the mode $\widehat{u}^\varepsilon(\omega,z)$, at $z > z_T(\omega)$.

3.1.1. The propagator. The leading coupling term in (3.4) vanishes when $\boldsymbol{M}^\varepsilon(\omega,z) = \boldsymbol{M}^\varepsilon_\star(\omega,z)$, the exact propagator matrix in the unperturbed, slowly changing waveguide. This is the solution of the flow problem

$$\partial_z\boldsymbol{M}^\varepsilon_\star(\omega,z) = \frac{i}{\varepsilon}\begin{pmatrix}0 & 1\\ k^2(\omega)-\mu^2(z) & 0\end{pmatrix}\boldsymbol{M}^\varepsilon_\star(\omega,z), \quad z<0,$$

with $\boldsymbol{M}^\varepsilon_\star(\omega, z=0)$ chosen so that we have the usual wave decomposition at $z=0$, as in a waveguide with straight boundaries. We work with an approximate propagator, which does not make the first line in the right-hand side of (3.4) exactly zero, but it ensures that its contribution to (3.4) converges to zero in the limit $\varepsilon \to 0$, uniformly in z, and its expression is explicit.

As in [24], $\boldsymbol{M}^\varepsilon(\omega, z)$ is the WKB approximation of $\boldsymbol{M}^\varepsilon_\star(\omega, z)$. It is a matrix with structure

$$\boldsymbol{M}^\varepsilon(\omega, z) = \begin{pmatrix} M^\varepsilon_{11}(\omega, z) & -\overline{M^\varepsilon_{11}(\omega, z)} \\ M^\varepsilon_{21}(\omega, z) & \overline{M^\varepsilon_{21}(\omega, z)} \end{pmatrix}, \qquad z < 0, \tag{3.5}$$

where we recall that the bar denotes complex conjugate. The structure in (3.5) is like in waveguides with straight boundaries and ensures energy conservation, as follows later in the section. The entries in (3.5) are defined in terms of the function

$$\phi_\omega(z) = \begin{cases} \displaystyle\int_{z_T(\omega)}^{z} dz' \sqrt{k^2(\omega) - \mu^2(z')}, & z_T(\omega) \le z \le 0, \\ \displaystyle -\int_{z}^{z_T(\omega)} dz' \sqrt{\mu^2(z') - k^2(\omega)}, & z < z_T(\omega), \end{cases} \tag{3.6}$$

which in turn defines

$$\eta^\varepsilon_\omega(z) = \begin{cases} \varepsilon^{-2/3} \left[3\phi_\omega(z)/2\right]^{2/3}, & z_T(\omega) \le z \le 0, \\ -\,\varepsilon^{-2/3} \left[-3\phi_\omega(z)/2\right]^{2/3}, & z < z_T(\omega), \end{cases} \tag{3.7}$$

and

$$Q_\omega(z) = \begin{cases} \dfrac{\left[3\phi_\omega(z)/2\right]^{1/6}}{\left[k^2(\omega) - \mu^2(z)\right]^{1/4}}, & z_T(\omega) \le z \le 0, \\ \dfrac{\left[-3\phi_\omega(z)/2\right]^{1/6}}{\left[\mu^2(z) - k^2(\omega)\right]^{1/4}}, & z < z_T(\omega). \end{cases} \tag{3.8}$$

Note that $Q_\omega(z)$ is positive and at least twice continuously differentiable, and at the turning point it satisfies

$$Q_\omega(z_T(\omega)) = \gamma_\omega^{-1/6}, \quad \partial_z Q_\omega(z_T(\omega)) = \frac{\theta_\omega}{5\gamma_\omega^{7/6}}, \quad \partial_z^2 Q_\omega(z_T(\omega)) = \frac{3\rho_\omega}{7\gamma_\omega^{7/6}} + \frac{9\theta_\omega^2}{35\gamma_\omega^{13/6}}, \tag{3.9}$$

where

$$\gamma_\omega = -\partial_z[\mu^2(z)]\big|_{z=z_T(\omega)} = \frac{2k^3(\omega)}{\pi} D'(z_T(\omega)) > 0, \tag{3.10}$$

$$\theta_\omega = \frac{1}{2}\partial_z^2[\mu^2(z)]\Big|_{z=z_T(\omega)}, \qquad \rho_\omega = \frac{1}{6}\partial_z^3[\mu^2(z)]\Big|_{z=z_T(\omega)}. \tag{3.11}$$

The function $\eta^\varepsilon_\omega(z)$ vanishes at the turning point, and its derivative is given by

$$\partial_z \eta^\varepsilon_\omega(z) = \varepsilon^{-2/3} Q_\omega^{-2}(z) \qquad \forall\, z < 0. \tag{3.12}$$

The entries of the propagator matrix (3.5) are defined by

$$M^\varepsilon_{11}(\omega, z) = \varepsilon^{-1/6}\sqrt{\pi} Q_\omega(z) e^{-i\phi_\omega(0)/\varepsilon + i\pi/4} \left[A_i(-\eta^\varepsilon_\omega(z)) - iB_i(-\eta^\varepsilon_\omega(z))\right] \tag{3.13}$$

and

$$
\begin{aligned}
M_{21}^{\varepsilon}(\omega, z) = & -i\varepsilon\partial_z M_{11}^{\varepsilon}(\omega, z) \\
= & -\frac{\varepsilon^{1/6}\sqrt{\pi}}{Q_\omega(z)} e^{-i\phi_\omega(0)/\varepsilon - i\pi/4} \left[A_i'(-\eta_\omega^\varepsilon(z)) - iB_i'(-\eta_\omega^\varepsilon(z))\right] \\
& + \varepsilon^{5/6}\sqrt{\pi} Q_\omega'(z) e^{-i\phi_\omega(0)/\varepsilon - i\pi/4} \left[A_i(-\eta_\omega^\varepsilon(z)) - iB_i(-\eta_\omega^\varepsilon(z))\right]
\end{aligned}
\tag{3.14}
$$

in terms of the Airy functions [1, chapter 10] denoted by A_i and B_i.

The next lemma, proved in Appendix B, shows that $\boldsymbol{M}^\varepsilon(\omega, z)$ approximates the exact propagator and that it is an invertible matrix with constant determinant.

LEMMA 3.1. *The matrix-valued process* (3.5)*, with entries defined by* (3.13)–(3.14)*, satisfies*

$$
\partial_z \boldsymbol{M}^\varepsilon(\omega, z) = \frac{i}{\varepsilon}\begin{pmatrix} 0 & 1 \\ k^2(\omega) - \mu^2(z) & 0 \end{pmatrix} \boldsymbol{M}^\varepsilon(\omega, z) - \frac{i\varepsilon Q_\omega''(z)}{Q_\omega(z)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \boldsymbol{M}^\varepsilon(\omega, z) \tag{3.15}
$$

and

$$
\det \boldsymbol{M}^\varepsilon(\omega, z) = 2 \tag{3.16}
$$

for all $z < 0$*.*

The next lemma, proved in Appendix B, describes the propagator as z approaches 0, where the source lies.

LEMMA 3.2. *When* $z < 0$ *and* $|z| \ll 1$*, the entries* (3.13)–(3.14) *of* $\boldsymbol{M}^\varepsilon(\omega, z)$ *have the following asymptotic expansions in* ε*:*

$$
M_{11}^\varepsilon(\omega, z) = [k^2(\omega) - \mu^2(0)]^{-1/4}\left\{\exp\left[\frac{i}{\varepsilon}\big(\phi_\omega(z) - \phi_\omega(0)\big)\right] + O(\varepsilon)\right\}
$$

and

$$
M_{21}^\varepsilon(\omega, z) = [k^2(\omega) - \mu^2(0)]^{1/4}\left\{\exp\left[\frac{i}{\varepsilon}\big(\phi_\omega(z) - \phi_\omega(0)\big)\right] + O(\varepsilon)\right\}.
$$

The leading terms in these expansions are the entries of the propagator in waveguides with straight boundaries and width $D(0)$*.*

Using this result in (3.3) and (3.5), we obtain a wave decomposition like in waveguides with straight walls [12, chapter 20]. The wave field is, for $|z| \ll 1$,

$$
\begin{aligned}
\widehat{u}^\varepsilon(\omega, z) \approx [k^2(\omega) - \mu^2(0)]^{-1/4} \Bigg[& \widehat{a}^\varepsilon(\omega, z) \exp\left(\frac{i}{\varepsilon}\int_0^z dz' \sqrt{k^2(\omega) - \mu^2(z')}\right) \\
& -\widehat{b}^\varepsilon(\omega, z) \exp\left(-\frac{i}{\varepsilon}\int_0^z dz' \sqrt{k^2(\omega) - \mu^2(z')}\right)\Bigg],
\end{aligned}
\tag{3.17}
$$

and its derivative is

$$
\begin{aligned}
\partial_z \widehat{u}^\varepsilon(\omega, z) \approx \frac{i}{\varepsilon}[k^2(\omega) - \mu^2(0)]^{1/4} \Bigg[& \widehat{a}^\varepsilon(\omega, z) \exp\left(\frac{i}{\varepsilon}\int_0^z dz' \sqrt{k^2(\omega) - \mu^2(z')}\right) \\
& + \widehat{b}^\varepsilon(\omega, z) \exp\left(-\frac{i}{\varepsilon}\int_0^z dz' \sqrt{k^2(\omega) - \mu^2(z')}\right)\Bigg]
\end{aligned}
\tag{3.18}
$$

with relative error of order ε.

In the vicinity of the turning point, for $|z - z_T(\omega)| = O(\varepsilon^{2/3})$, the Airy functions and their derivatives are bounded, as are $Q_\omega(z)$ and its derivatives, described in (3.9). We obtain that the entries in the first row of $\boldsymbol{M}^\varepsilon(\omega, z)$ are large, of order $\varepsilon^{-1/6}$, and the entries in the second row are small, of order $\varepsilon^{1/6}$.

Beyond the turning point, at $z_T(\omega) - z \gg O(\varepsilon^{2/3})$, the entries of $\boldsymbol{M}^\varepsilon(\omega, z)$ grow exponentially, as stated in the next lemma, proved in Appendix B. The mode $\widehat{u}^\varepsilon(\omega, z)$ is evanescent in this region of the waveguide and must be exponentially decaying away from $z_T(\omega)$. This is ensured by carefully chosen boundary conditions of the mode amplitudes, as explained in the next section.

LEMMA 3.3. *When $z_T(\omega) - z \gg O(\varepsilon^{2/3})$, the entries of the approximate propagator $\boldsymbol{M}^\varepsilon(\omega, z)$ have the following asymptotic expansions in ε:*

$$M_{11}^\varepsilon(\omega, z) \approx [\mu^2(z) - k^2(\omega)]^{-1/4} \exp\left[\frac{1}{\varepsilon}\int_z^{z_T(\omega)} dz'\, \sqrt{\mu^2(z') - k^2(\omega)} - \frac{i\phi_\omega(0)}{\varepsilon} - \frac{i\pi}{4}\right]$$

and

$$M_{21}^\varepsilon(\omega, z) \approx [\mu^2(z) - k^2(\omega)]^{1/4} \exp\left[\frac{1}{\varepsilon}\int_z^{z_T(\omega)} dz'\, \sqrt{\mu^2(z') - k^2(\omega)} - \frac{i\phi_\omega(0)}{\varepsilon} + \frac{i\pi}{4}\right]$$

with relative error of order ε.

The Airy function A_i and its derivative A_i' decay exponentially in this region and are negligible. The asymptotic expansions above are determined by B_i and B_i'.

3.1.2. The mode amplitudes. To derive the system of differential equations satisfied by the mode amplitudes, we use Lemma 3.1 and the inverse of the propagator

$$\boldsymbol{M}^{\varepsilon,-1}(\omega, z) = \frac{1}{2}\begin{pmatrix} \overline{M_{21}^\varepsilon(\omega, z)} & \overline{M_{11}^\varepsilon(\omega, z)} \\ -M_{21}^\varepsilon(\omega, z) & M_{11}^\varepsilon(\omega, z) \end{pmatrix} \tag{3.19}$$

in (3.4). We obtain that

$$\partial_z \begin{pmatrix} \widehat{a}^\varepsilon(\omega, z) \\ \widehat{b}^\varepsilon(\omega, z) \end{pmatrix} = \boldsymbol{H}^\varepsilon(\omega, z) \begin{pmatrix} \widehat{a}^\varepsilon(\omega, z) \\ \widehat{b}^\varepsilon(\omega, z) \end{pmatrix}, \quad z < 0, \tag{3.20}$$

with matrix-valued random process

$$\boldsymbol{H}^\varepsilon(\omega, z) = \begin{pmatrix} H_{11}^\varepsilon(\omega, z) & \overline{H_{21}^\varepsilon(\omega, z)} \\ H_{21}^\varepsilon(\omega, z) & -H_{11}^\varepsilon(\omega, z) \end{pmatrix}, \tag{3.21}$$

and entries equal, up to negligible terms, to

$$H_{11}^\varepsilon(\omega, z) = \frac{1}{2}\left[\frac{i\sigma_\varepsilon}{\sqrt{\varepsilon}}\mu^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + i\sigma_\varepsilon^2 g^\varepsilon(\omega, z)\right] \Big|M_{11}^\varepsilon(\omega, z)\Big|^2, \tag{3.22}$$

$$H_{21}^\varepsilon(\omega, z) = \frac{1}{2}\left[\frac{i\sigma_\varepsilon}{\sqrt{\varepsilon}}\mu^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + i\sigma_\varepsilon^2 g^\varepsilon(\omega, z)\right] \Big(M_{11}^\varepsilon(\omega, z)\Big)^2. \tag{3.23}$$

To specify the solution of (3.20), we need boundary conditions. At $z = 0$ we obtain from the jump conditions (2.42)–(2.43), equations (3.17)–(3.18), and the outgoing condition $\widehat{b}^\varepsilon(\omega, 0^+) = 0$ that

$$\widehat{b}^\varepsilon(\omega, 0^-) = \frac{iC_F}{\sqrt{\varepsilon}B}\left[\widehat{F}\left(\frac{\omega - \omega_o}{\sqrt{\varepsilon}B}\right) + \overline{\widehat{F}\left(\frac{-\omega - \omega_o}{\sqrt{\varepsilon}B}\right)}\right] \tag{3.24}$$

with constant

$$C_F = \frac{y_1(r_\star, 0)}{4[k^2(\omega_o) - \mu^2(0)]^{1/4}}. \tag{3.25}$$

Here we neglected the $O(\varepsilon^{1/2})$ residual in the definition (2.26) of $\widehat{f}^\varepsilon(\omega)$ and in the expansion of $k(\omega)$ for $\omega = \omega_o + O(\varepsilon^{1/2})$. The forward-going wave amplitudes at $z = 0$ satisfy the relation

$$a^\varepsilon(\omega, 0^-) - a^\varepsilon(\omega, 0^+) = b^\varepsilon(\omega, 0^-), \tag{3.26}$$

and we need one more boundary condition. This will ensure that the wave is exponentially decaying away from the turning point.

The asymptotic expansion of the propagator at $z < z_T(\omega)$, given in Lemma 3.3, shows that $\boldsymbol{M}^\varepsilon(\omega, z)$ has exponentially growing terms, due to the Airy function B_i. To compensate this growth, we introduce here a boundary condition at some z_b far enough from the turning point,[2] satisfying $z_T(\omega) - z_\text{b} \gg O(\varepsilon^{2/3})$. Definitions (3.3) and (3.13)–(3.14) give that

$$\begin{aligned} \widehat{u}^\varepsilon(\omega, z_\text{b}) &= \varepsilon^{-1/6} C^\varepsilon B_i(-\eta^\varepsilon_\omega(z_\text{b})) \left[a^\varepsilon(\omega, z_\text{b}) - i e^{2i\phi_\omega(0)/\varepsilon} b^\varepsilon(\omega, z_\text{b})\right] \\ &\quad + i\varepsilon^{-1/6} C^\varepsilon A_i(-\eta^\varepsilon_\omega(z_\text{b})) \left[a^\varepsilon(\omega, z_\text{b}) + i e^{2i\phi_\omega(0)/\varepsilon} b^\varepsilon(\omega, z_\text{b})\right] \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \partial_z \widehat{u}^\varepsilon(\omega, z_\text{b}) &= -\varepsilon^{-5/6} C^\varepsilon \left[Q_\omega^{-2}(z_\text{b}) B_i'(-\eta^\varepsilon_\omega(z_\text{b})) - \varepsilon \frac{Q_\omega'(z_\text{b})}{Q_\omega(z_\text{b})} B_i(-\eta^\varepsilon_\omega(z_\text{b}))\right] \\ &\quad \times \left[a^\varepsilon(\omega, z_\text{b}) - i e^{2i\phi_\omega(0)/\varepsilon} b^\varepsilon(\omega, z_\text{b})\right] - i\varepsilon^{-5/6} C^\varepsilon Q_\omega^{-2}(z_\text{b}) A_i'(-\eta^\varepsilon_\omega(z_\text{b})) \\ &\quad \times \left[a^\varepsilon(\omega, z_\text{b}) + i e^{2i\phi_\omega(0)/\varepsilon} b^\varepsilon(\omega, z_\text{b})\right] [1 + O(\varepsilon)] \end{aligned} \tag{3.28}$$

with constant

$$C^\varepsilon = \sqrt{\pi} Q_\omega(z_\text{b}) e^{-i\phi_\omega(0)/\varepsilon - i\pi/4}.$$

We set to zero the coefficients of B_i and B_i' in these expressions, to get an exponentially small wave field, and obtain the boundary condition

$$a^\varepsilon(\omega, z_\text{b}) = i e^{2i\phi_\omega(0)/\varepsilon} b^\varepsilon(\omega, z_\text{b}). \tag{3.29}$$

3.2. The reflection coefficient. The mode amplitudes define the reflection coefficient

$$\widehat{R}^\varepsilon(\omega, z) = \frac{\widehat{a}^\varepsilon(\omega, z)}{\widehat{b}^\varepsilon(\omega, z)}, \tag{3.30}$$

which is a complex number with modulus one. This is because the structure (3.21) of the matrix $\boldsymbol{H}^\varepsilon(\omega, z)$ in (3.20) ensures $\partial_z[|\widehat{a}^\varepsilon(\omega, z)|^2 - |\widehat{b}^\varepsilon(\omega, z)|^2] = 0$, which gives the flux energy conservation equation

$$|\widehat{a}^\varepsilon(\omega, z)|^2 - \left|\widehat{b}^\varepsilon(\omega, z)\right|^2 = \text{constant} \qquad \forall\, z \in (z_\text{b}, 0), \tag{3.31}$$

[2] We show in section 4 that the result does not depend on the value of the fictitious boundary z_b.

where the constant must equal zero by (3.29). Thus, we can write (3.30) in the form

$$\widehat{R}^\varepsilon(\omega, z) = i \exp\left[2i\frac{\phi_\omega(0)}{\varepsilon} + i\psi_\omega^\varepsilon(z)\right] \tag{3.32}$$

with real-valued, random phase $\psi_\omega^\varepsilon(z)$. It satisfies the differential equation

$$\begin{aligned}\partial_z \psi_\omega^\varepsilon(z) = \frac{2\pi Q_\omega^2(z)}{\varepsilon^{1/3}}\left[\frac{\sigma_\varepsilon}{\sqrt{\varepsilon}}\mu^2(z)\nu\Big(\frac{z}{\varepsilon}\Big) + \sigma_\varepsilon^2 g^\varepsilon(\omega, z)\right]\left[A_i^2(-\eta_\omega^\varepsilon(z)) + B_i^2(-\eta_\omega^\varepsilon(z))\right] \\ \times \cos^2\left\{\frac{\psi_\omega^\varepsilon(z)}{2} - \arg\left[A_i(-\eta_\omega^\varepsilon(z)) + iB_i(-\eta_\omega^\varepsilon(z))\right]\right\}, \quad z > z_{\rm b},\end{aligned} \tag{3.33}$$

derived from (3.20) and (3.30)–(3.32), with homogeneous boundary condition

$$\psi_\omega^\varepsilon(z_{\rm b}) = 0. \tag{3.34}$$

We are particularly interested in the phase $\psi_\omega^\varepsilon(z = 0)$, which defines the frequency-dependent amplitude of the reflected wave at the source.

3.3. The pulse stabilization result. Equations (2.37), (3.17), (3.24), (3.30), and (3.32) give that the reflected pressure wave at $z = 0^-$ is given by

$$\begin{aligned}p_{\rm ref}^\varepsilon(t, \rho, 0^-) &= y_1(\rho, 0)\int \frac{d\omega}{2\pi}\frac{1}{[k^2(\omega) - \mu^2(0)]^{1/4}} e^{-i\omega t/\varepsilon}\widehat{a}^\varepsilon(\omega, 0^-) \\ &\approx -\frac{y_1(\rho, 0)y_1(r_\star, 0)}{2\sqrt{k^2(\omega_o) - \mu^2(0)}} f_{\rm ref}^\varepsilon(t)\big[1 + O(\sqrt{\varepsilon})\big]\end{aligned} \tag{3.35}$$

with reflected pulse

$$f_{\rm ref}^\varepsilon(t) = {\rm Re}\left\{\int \frac{dw}{2\pi B}\widehat{F}\Big(\frac{w}{B}\Big)\exp\left[\frac{i[2\phi_{\omega_o+\sqrt{\varepsilon}w}(0) - (\omega_o + \sqrt{\varepsilon}w)t]}{\varepsilon} + i\psi_{\omega_o+\sqrt{\varepsilon}w}^\varepsilon(0)\right]\right\}.$$

The wave emerging at the right of the source, at $z = 0^+$, is

$$p_{\rm ref}^\varepsilon(t, \rho, 0^+) = y_1(\rho, 0)\int \frac{d\omega}{2\pi}\frac{1}{[k^2(\omega) - \mu^2(0)]^{1/4}} e^{-i\omega t/\varepsilon}\widehat{a}^\varepsilon(\omega, 0^+) \tag{3.36}$$

with $a^\varepsilon(\omega, 0^+)$ obtained from (3.26). It is the superposition of the reflected wave (3.35) and the direct wave that has no interaction with the random section of the waveguide and propagates from the source in the forward direction.

To describe $f_{\rm ref}^\varepsilon(t)$ in the limit $\varepsilon \to 0$, we eliminate first the large deterministic phase of the integrand. For this purpose, we expand

$$2\phi_{\omega_o+\sqrt{\varepsilon}w}(0) = 2\phi_{\omega_o}(0) + \sqrt{\varepsilon}wT_{\omega_o} + \varepsilon w^2\beta_{\omega_o} + O(\varepsilon^{3/2}),$$

where

$$T_{\omega_o} = 2\partial_\omega\phi_\omega(0)\big|_{\omega=\omega_o} = \frac{2k^2(\omega_o)}{\omega_o}\int_{z_T(\omega_o)}^0 \frac{dz}{\sqrt{k^2(\omega_o) - \mu^2(z)}} \tag{3.37}$$

is the travel time of the propagating mode from the source to the turning point and back, and

$$\begin{aligned}\beta_{\omega_o} &= \partial_\omega^2\phi_\omega(0)\big|_{\omega=\omega_o} \\ &= \frac{2k^4(\omega_o)}{\omega_o^2\gamma_{\omega_o}}\left\{\frac{1}{\sqrt{k^2(\omega_o) - \mu^2(0)}} + \int_{z_T(\omega_o)}^0 dz\,\frac{\mu(z)[\mu(z)\mu'(z_T(\omega_o)) - \mu(z_T(\omega_o))\mu'(z)]}{k(\omega_o)[k^2(\omega_o) - \mu^2(z)]^{3/2}}\right\}\end{aligned}$$

is an effective dispersion coefficient as we will see below. When we observe $f^\varepsilon_{\rm ref}$ around time T_{ω_o}, in a time window of order $\sqrt{\varepsilon}$, which corresponds to the scaled support of the emitted pulse, we obtain

$$f^\varepsilon_{\rm ref}\left(T_{\omega_o}+\sqrt{\varepsilon}t\right)={\rm Re}\left\{\exp\left[i\left(2\phi_{\omega_o}(0)-\omega_o T_{\omega_o}-\omega_o\sqrt{\varepsilon}t\right)/\varepsilon\right]\mathcal{F}^\varepsilon_{\rm ref}(t)\right\}. \tag{3.38}$$

This oscillates at carrier frequency ω_o/ε, like the emitted pulse, and its envelope

$$\mathcal{F}^\varepsilon_{\rm ref}(t)=\int\frac{dw}{2\pi B}\widehat{F}\Big(\frac{w}{B}\Big)\exp\Big[iw^2\beta_{\omega_o}+i\psi^\varepsilon_{\omega_o+\sqrt{\varepsilon}w}(0)-iwt\Big] \tag{3.39}$$

is described in the next theorem.

THEOREM 3.4. *Suppose that the standard deviation σ_ε of the random fluctuations is of the order $|\ln\varepsilon|^{-1/2}$, so that*

$$\upsilon^2_{\omega_o}=\frac{k^4(\omega_o)}{\gamma_{\omega_o}}\widehat{\mathcal{R}}(0)\lim_{\varepsilon\to 0}\sigma^2_\varepsilon\ln\left(\frac{\phi_{\omega_o}(0)}{\varepsilon}\right) \tag{3.40}$$

is finite, where γ_{ω_o} is defined in (3.10) *and $\widehat{\mathcal{R}}(0)>0$, because it is the power spectral density of the fluctuations ν, evaluated at zero. Then, as $\varepsilon\to 0$, $\mathcal{F}^\varepsilon_{\rm ref}(t)$ converges in distribution, in the space of continuous functions on compact sets in $\mathbb{R}$, to*

$$\mathcal{F}_{\rm ref}(t)=\exp\left(i\psi_{\omega_o}-\frac{\upsilon^2_{\omega_o}}{6}\right)\int\frac{dw}{2\pi B}\,\widehat{F}\Big(\frac{w}{B}\Big)\exp\left(iw^2\beta_{\omega_o}-iwt\right), \tag{3.41}$$

where ψ_{ω_o} is a Gaussian random variable with mean zero and variance $2\upsilon^2_{\omega_o}/3$.

This is the pulse stabilization result, proved in section 4. It says that aside from the random phase $\psi_{\omega_o}\sim\mathcal{N}(0,2\upsilon^2_{\omega_o}/3)$, the envelope of the reflected pulse is deterministic. It differs from the envelope $F(Bt)$ of the emitted pulse by the damping factor $\exp[-\upsilon^2_{\omega_o}/6]$ and the deformation by the second-order dispersive term $\beta_{\omega_o}w^2$ in the phase.

Remark 3.5. The convergence stated in Theorem 3.4 holds in the space of continuous functions endowed with the topology induced by the supremum norm over the compact sets. It does not hold in L^2. Equation (3.41) describes a reflected pulse with damped amplitude. Its energy is smaller than the energy of the incoming pulse, which may seem surprising because no energy can be transmitted beyond the turning point and there is no dissipation in the medium, so all the incoming energy should be reflected. This is what we have before taking the limit $\varepsilon\to 0$, since the reflection coefficient has modulus one. Theorem 3.4 describes only the coherent reflected pulse, which is observed around the time T_{ω_o}, at the time scale of the incoming pulse width. The theorem does not describe the coda wave, consisting of the incoherent, small-amplitude, longlasting wave fluctuations that arrive after the coherent reflected pulse. These carry the remainder of the energy. When $\upsilon_{\omega_o}\ll 1$, the incoherent wave fluctuations are negligible, but when $\upsilon_{\omega_o}\gg 1$, they carry most of the energy.

Remark 3.6. Theorem 3.4 assumes that the scaled standard deviation σ_ε of the random fluctuations of the boundary is small, of order $|\ln\varepsilon|^{-1/2}$. In the absence of the turning point such fluctuations would have a negligible effect on the wave. The results [4, 17] in waveguides with random perturbations of straight boundaries show that (1) the fluctuations have a net scattering effect when $\sigma_\varepsilon=O(1)$, and (2) the slower the modes propagate along the waveguide axis, the stronger this effect. We have a single propagating mode, which slows down as it approaches the turning point, meaning that its group velocity along z tends to zero. Because the mode hovers around $z_T(\omega_o)$, it scatters repeatedly at the random boundary, which is why the net scattering effect can be observed at the smaller standard deviation $\sigma_\varepsilon = O(|\ln \varepsilon|^{-1/2})$.

Theorem 3.4 is proved in the following section. Roughly speaking, the proof is based on a diffusion-approximation result that describes the joint distribution of the frequency-dependent phases of the reflection coefficients in the limit $\varepsilon \to 0$. These phases are random and become asymptotically Gaussian distributed, and their covariance function (as a function of the frequency) exhibits an interesting feature: the frequency-dependent phases have a common random component and they also have uncorrelated and identically distributed components. Since the time-dependent profile of the reflected wave is the superposition of many frequency-dependent reflection coefficients by Fourier synthesis, the common phase gives the random phase in the time-dependent profile in (3.41), while the uncorrelated phases average out and give the damping term.

4. Derivation of the pulse stabilization result.

We begin in section 4.1 with the single-frequency asymptotic analysis of the random phase $\psi_\omega^\varepsilon(z)$, which defines the reflection coefficient (3.32). The multifrequency analysis of $\psi_\omega^\varepsilon(z)$ is in section 4.3, and the proof of Theorem 3.4 is completed in section 4.4. We assume throughout the section that σ_ε is of order $|\ln \varepsilon|^{-1/2}$, as stated in Theorem 3.4.

4.1. Single-frequency analysis.

To analyze the random phase $\psi_\omega^\varepsilon(z)$ in the limit $\varepsilon \to 0$, we change variables so that (3.33) takes a form that can be analyzed with the diffusion limit theorem in [21]. The change of variables is

$$z \to \zeta := \varepsilon^{1/3} \eta_\omega^\varepsilon(z) \tag{4.1}$$

with η_ω^ε defined in (3.7). The inverse of this mapping is $z = Z_\omega(\varepsilon^{1/3}\zeta)$ in terms of the function $Z_\omega : \mathbb{R} \to \mathbb{R}$, defined pointwise as the unique solution of

$$\phi_\omega\big(Z_\omega(\xi)\big) = \frac{2}{3}\mathrm{sgn}(\xi)\big|\xi\big|^{3/2} \qquad \forall\, \xi \in \mathbb{R} \tag{4.2}$$

with ϕ_ω given in (3.6) and "sgn" denoting the sign function. Equivalently, in differential equation form, $Z_\omega(\xi)$ is the unique solution of

$$\partial_\xi Z_\omega(\xi) = Q_\omega^2\big(Z_\omega(\xi)\big) \quad \text{for} \quad \xi \neq 0, \qquad Z_\omega(0) = z_T(\omega). \tag{4.3}$$

We denote the phase after the change of variables (4.1) with the same symbol ψ_ω^ε and obtain using (3.12) that (3.33) becomes

$$\begin{aligned} \partial_\zeta \psi_\omega^\varepsilon(\zeta) = {} & \frac{2J_\omega^2(\varepsilon^{1/3}\zeta)V(\varepsilon^{-1/3}\zeta)}{\sqrt{|\zeta|}} \\ & \times \left[\frac{\sigma_\varepsilon}{\varepsilon^{1/3}}\mu^2\left(Z_\omega(\varepsilon^{1/3}\zeta)\right)\nu\left(\frac{Z_\omega(\varepsilon^{1/3}\zeta)}{\varepsilon}\right) + \sigma_\varepsilon^2 \varepsilon^{1/6} g^\varepsilon\big(\omega, Z_\omega(\varepsilon^{1/3}\zeta)\big)\right] \\ & \times \cos^2\left\{\frac{\psi_\omega^\varepsilon(\zeta)}{2} - \arg[A_i + iB_i](-\varepsilon^{-1/3}\zeta)\right\} \end{aligned} \tag{4.4}$$

with

$$J_\omega(\xi) = Q_\omega^2(Z_\omega(\xi)), \qquad V(\xi) = \pi\sqrt{|\xi|}\left[A_i^2(-\xi) + B_i^2(-\xi)\right], \tag{4.5}$$

and the shortened notation

$$\arg[A_i + iB_i](-\xi) = \arg[A_i(-\xi) + iB_i(-\xi)].$$

The turning point lies at $\zeta = 0$, and the source is at

$$\zeta_{\rm s}^\varepsilon = \varepsilon^{1/3}\eta_\omega^\varepsilon(0) = \varepsilon^{-1/3}[3\phi_\omega(0)/2]^{2/3} = O(\varepsilon^{-1/3}). \tag{4.6}$$

The boundary point $z_{\rm b}$ is mapped to

$$\zeta_{\rm b}^\varepsilon = -\varepsilon^{-1/3}[-3\phi_\omega(z_{\rm b})/2]^{2/3}, \tag{4.7}$$

and we have the boundary condition

$$\psi_\omega^\varepsilon(\zeta_{\rm b}^\varepsilon) = 0. \tag{4.8}$$

The quadratic term in the fluctuations, modeled by g^ε in (4.4), is negligible in the limit $\varepsilon \to 0$, because it gives a contribution that can be bounded by

$$O(\sigma_\varepsilon^2 \varepsilon^{1/6}\sqrt{\zeta_{\rm s}^\varepsilon - \zeta_{\rm b}^\varepsilon}) = O(\sigma_\varepsilon^2) = O(1/|\ln\varepsilon|).$$

We neglect it henceforth and simplify (4.4) to

$$\begin{aligned}\partial_\zeta \psi_\omega^\varepsilon(\zeta) = {}& \frac{2\sigma_\varepsilon}{\varepsilon^{1/3}\sqrt{|\zeta|}} J_\omega^2(\varepsilon^{1/3}\zeta)V(\varepsilon^{-1/3}\zeta)\mu^2\big(Z_\omega(\varepsilon^{1/3}\zeta)\big)\nu\left(\frac{Z_\omega(\varepsilon^{1/3}\zeta)}{\varepsilon}\right)\\ &\times \cos^2\left\{\frac{\psi_\omega^\varepsilon(\zeta)}{2} - \arg[A_i+iB_i](-\varepsilon^{-1/3}\zeta)\right\}.\end{aligned} \tag{4.9}$$

To understand how $\psi_\omega^\varepsilon(\zeta)$ evolves from the boundary value (4.8), let us start from $\zeta_{\rm b}^\varepsilon$ and consider first points that are far on the left of the turning point, at $\zeta < 0$ satisfying $|\zeta| \gg O(\varepsilon^{1/3})$. The function V defined in (4.5) is large at these points, as given by the asymptotic expansions of the Airy function B_i in Appendix B.3,

$$V(\varepsilon^{-1/3}\zeta) \approx \pi\sqrt{\varepsilon^{-1/3}|\zeta|}\, B_i^2(\varepsilon^{-1/3}|\zeta|) \approx \exp\left[\frac{4|\zeta|^{3/2}}{3\varepsilon^{1/2}}\right].$$

We also have the expansion

$$\arg\Big[A_i(-\varepsilon^{-1/3}\zeta) + iB_i(-\varepsilon^{-1/3}\zeta)\Big] \approx \frac{\pi}{2} - \frac{1}{2}\exp\left[-\frac{4|\zeta|^{3/2}}{3\varepsilon^{1/2}}\right],$$

and since the phase starts from zero by (4.8),

$$\cos^2\left\{\frac{\psi_\omega^\varepsilon(\zeta)}{2} - \arg\Big[A_i(-\varepsilon^{-1/3}\zeta) + iB_i(-\varepsilon^{-1/3}\zeta)\Big]\right\} \approx \frac{1}{4}\exp\left[-\frac{8|\zeta|^{3/2}}{3\varepsilon^{1/2}}\right].$$

This makes the right-hand side in (4.9) exponentially small, so the phase remains essentially zero on the left of the turning point. Moreover, the phase is independent of the precise value of $\zeta_{\rm b}^\varepsilon$ at which we prescribe the boundary condition (4.8).

Now consider the $O(\varepsilon^{1/3})$ vicinity of the turning point, where we can set $\zeta = \varepsilon^{1/3}\tilde\zeta$ with $\tilde\zeta = O(1)$, to rewrite (4.9) for $\widetilde\psi_\omega^\varepsilon(\tilde\zeta) = \psi_\omega^\varepsilon(\varepsilon^{1/3}\tilde\zeta)$ as

$$\begin{aligned}\partial_{\tilde\zeta}\widetilde\psi_\omega^\varepsilon(\tilde\zeta) = {}& \frac{2\pi\sigma_\varepsilon k^2(\omega)}{\varepsilon^{1/6}\gamma_\omega^{2/3}}\Big[A_i^2(-\tilde\zeta) + B_i^2(-\tilde\zeta)\Big]\nu\left(\frac{z_T(\omega)}{\varepsilon} + \frac{\tilde\zeta}{\varepsilon^{1/3}\gamma_\omega^{1/3}}\right)\\ &\times\cos^2\left\{\frac{\widetilde\psi_\omega^\varepsilon(\tilde\zeta)}{2} - \arg[A_i+iB_i](-\tilde\zeta)\right\} + \dots,\end{aligned} \tag{4.10}$$

with the dots denoting negligible terms. Here we used (4.3), and

$$J_\omega(0) = \gamma_\omega^{-1/3}, \qquad \mu^2(Z_\omega(0)) = \mu^2(z_T(\omega)) = k^2(\omega).$$

If σ_ε were order one, the right-hand side in (4.10) would be in the usual diffusion approximation form with

$$\widetilde{\nu}^\varepsilon(\tilde{\zeta}) = \frac{1}{\varepsilon^{1/6}} \nu \left(\frac{z_T(\omega)}{\varepsilon} + \frac{\tilde{\zeta}}{\varepsilon^{1/3}\gamma_\omega^{1/3}} \right)$$

behaving like white noise in the limit $\varepsilon \to 0$, for $\tilde{\zeta}$ of order one [28]. But in our case $\sigma_\varepsilon = O(|\ln \varepsilon|^{-1/2})$ tends to zero as $\varepsilon \to 0$, so the fluctuations are negligible in the $O(\varepsilon^{1/3})$ vicinity of the turning point.

The net scattering effect at the random boundary comes from the long interval

$$\mathcal{I}^\varepsilon = \Big\{ \zeta \in \mathbb{R} \ \text{ s.t. } \ O(\varepsilon^{1/3}) < \zeta \le \zeta_{\rm s}^\varepsilon \Big\}$$

that grows as $\varepsilon^{-1/3}$ in the limit $\varepsilon \to 0$ by (4.6). The asymptotic expansions of the Airy functions at large negative arguments given in Appendix B.2 show that in $\mathcal{I}^\varepsilon$ we have

$$V(\varepsilon^{-1/3}\zeta) \approx 1 + o(1) \tag{4.11}$$

and

$$\arg\Big[A_i(-\varepsilon^{-1/3}\zeta) + iB_i(-\varepsilon^{-1/3}\zeta) \Big] \approx \frac{\pi}{4} - \frac{2}{3}(\varepsilon^{-1/3}\zeta)^{3/2} + o(1). \tag{4.12}$$

One can verify that these are excellent approximations for all $\varepsilon^{-1/3}\zeta > 3$, so we can take $\mathcal{I}^\varepsilon = (\zeta_-^\varepsilon, \zeta_{\rm s}^\varepsilon]$, with $\zeta_-^\varepsilon = 3\varepsilon^{1/3}$, and simplify (4.9) as

$$\begin{aligned} \partial_\zeta \psi_\omega^\varepsilon(\zeta) = {} & \frac{\sigma_\varepsilon J_\omega^2(\varepsilon^{1/3}\zeta)\mu^2\big(Z_\omega(\varepsilon^{1/3}\zeta)\big)}{\varepsilon^{1/3}\sqrt{\zeta}} \nu\left(\frac{Z_\omega(\varepsilon^{1/3}\zeta)}{\varepsilon} \right) \\ & \times \left\{ 1 + \sin\left[\psi_\omega^\varepsilon(\zeta) + \frac{4}{3}(\varepsilon^{-1/3}\zeta)^{3/2} \right] \right\}. \end{aligned} \tag{4.13}$$

4.2. The diffusion limit for the single-frequency case. To see how to apply the limit theorem in [21] to (4.9), imagine that we discretize the interval $\mathcal{I}^\varepsilon$ at points $\zeta^{(j)} = \zeta_-^\varepsilon + j\Delta_\zeta$ separated by $\Delta_\zeta = O(1)$, for $j = 0, \dots, n^\varepsilon$, and

$$n^\varepsilon = \lfloor (\zeta_{\rm s}^\varepsilon - \zeta_-^\varepsilon)/\Delta_\zeta \rfloor = O(\varepsilon^{-1/3}). \tag{4.14}$$

If the argument of $\lfloor \ \rfloor$ in this equation is not integer, the length of the last interval is adjusted so that $\zeta_{\rm s}^\varepsilon = \zeta^{(n^\varepsilon)}$.

In each subinterval $[\zeta^{(j)}, \zeta^{(j+1)}]$, we expand the argument of ν in (4.13) as

$$\frac{Z_\omega(\varepsilon^{1/3}\zeta)}{\varepsilon} \approx \frac{Z_\omega(\varepsilon^{1/3}\zeta^{(j)})}{\varepsilon} + J_\omega\big(\varepsilon^{1/3}\zeta^{(j)}\big)\frac{\zeta - \zeta^{(j)}}{\varepsilon^{2/3}} + \left(\frac{\zeta - \zeta^{(j)}}{\varepsilon^{2/3}} \right)^2 O(\varepsilon),$$

where $\varepsilon^{1/3}\zeta^{(j)}$ is order one by (4.14), and we used (4.3) and (4.5). Similarly, the argument of the sin in (4.13) is

$$(\varepsilon^{-1/3}\zeta)^{3/2} \approx \left(\varepsilon^{-1/3}\zeta^{(j)}\right)^{3/2} + \frac{3\sqrt{\varepsilon^{1/3}\zeta^{(j)}}}{2}\frac{\zeta-\zeta^{(j)}}{\varepsilon^{2/3}} + \left(\frac{\zeta-\zeta^{(j)}}{\varepsilon^{2/3}}\right)^2 \frac{O(\varepsilon)}{\sqrt{\varepsilon^{1/3}\zeta^{(j)}}}.$$

This makes (4.13) of the same form as in [21], with the small parameter ϵ there replaced by our $\varepsilon^{1/3}$ and the process ν satisfying by assumption the strong mixing conditions in [21]. Thus, we can use the limit theorem in [21, section III] for the joint process $(\nu^\varepsilon(\zeta), \zeta^\varepsilon(\zeta))$ on the state space $\mathbb{R} \times [0, 3\pi/2]$, with

$$\nu^\varepsilon(\zeta) = \nu\left(\frac{Z_\omega(\varepsilon^{1/3}\zeta)}{\varepsilon}\right), \qquad \zeta^\varepsilon(\zeta) = \left(\varepsilon^{-1/3}\zeta\right)^{3/2}. \tag{4.15}$$

The torus $[0, 3\pi/2]$ arises because the right-hand side in (4.13) is periodic in $\zeta^\varepsilon(z)$.

The next lemma, proved in Appendix C, describes the distribution of the phase $\psi^\varepsilon_\omega(\zeta^\varepsilon_{\rm s})$ at the location $\zeta^\varepsilon_{\rm s}$ of the source.

LEMMA 4.1. *$\psi^\varepsilon_\omega(\zeta^\varepsilon_{\rm s})$ is asymptotically Gaussian distributed in the limit $\varepsilon \to 0$, with mean zero and variance*

$$\upsilon^2_\omega = \frac{k^4(\omega)}{\gamma_\omega}\widehat{\mathcal{R}}(0)\lim_{\varepsilon\to 0}\sigma^2_\varepsilon \ln\left(\frac{\phi_\omega(0)}{\varepsilon}\right). \tag{4.16}$$

4.3. Multifrequency analysis. The expression (3.39) of the envelope of the reflected pulse involves the random phases $\psi^\varepsilon_\omega(z=0)$ at frequencies $\omega = \omega_o + \sqrt{\varepsilon}w$, with w in the support of $\widehat{F}(w/B)$, the Fourier transform of the envelope of the emitted pulse. Here we describe the asymptotic distribution of these phases at m such distinct frequencies.

Let us introduce the notation

$$\psi^\varepsilon_j(z) = \psi^\varepsilon_{\omega_o+\sqrt{\varepsilon}w_j}(z) \tag{4.17}$$

and consider the random process $(\psi^\varepsilon_1(z), \ldots, \psi^\varepsilon_m(z))$. Each $\psi^\varepsilon_j(z)$ satisfies (3.33) with $\omega = \omega_o + \sqrt{\varepsilon}w_j$ and boundary condition $\psi^\varepsilon_j(z_{\rm b}) = 0$. We proceed as in the previous section and change variables to transform the problem into one that can be analyzed with the diffusion limit theorem in [21]. The change of variables is similar to (4.1), but since we have multiple frequencies that are close to ω_o, we take

$$z \to \zeta := \varepsilon^{1/3}\eta^\varepsilon_{\omega_o}(z) \tag{4.18}$$

with inverse transform given by $z = Z_{\omega_o}(\varepsilon^{1/3}\zeta)$ with Z_{ω_o} defined in (4.2) for $\omega = \omega_o$. By definitions (3.7) and (4.2) we have the expansion

$$\begin{aligned}\eta^\varepsilon_{\omega_o+\sqrt{\varepsilon}w}(Z_{\omega_o}(\xi)) &= \operatorname{sgn}(\xi)\varepsilon^{-2/3}\left[\operatorname{sgn}(\xi)\frac{3}{2}\phi_{\omega_o+\sqrt{\varepsilon}w}(Z_{\omega_o}(\xi))\right]^{2/3} \\ &= \varepsilon^{-2/3}\left[\xi + \varepsilon^{1/2}w\mathcal{K}(\xi)\right] + o(1),\end{aligned} \tag{4.19}$$

uniformly in ξ, up to $|\xi| = O(1)$, where

$$\mathcal{K}(\xi) = \frac{1}{|\xi|^{1/2}}\partial_\omega\phi_\omega(Z_{\omega_o}(\xi))\big|_{\omega=\omega_o}. \tag{4.20}$$

The function

$$\mathcal{K}(\xi) = \begin{cases} \dfrac{k^2(\omega_o)}{\omega_o \xi^{1/2}} \displaystyle\int_{z_T(\omega_o)}^{Z_{\omega_o}(\xi)} \frac{dz}{\sqrt{k^2(\omega_o) - \mu^2(z)}} & \text{if } \xi > 0, \\ \dfrac{k^2(\omega_o)}{\omega_o |\xi|^{1/2}} \displaystyle\int_{Z_{\omega_o}(\xi)}^{z_T(\omega_o)} \frac{dz}{\sqrt{\mu^2(z) - k^2(\omega_o)}} & \text{if } \xi < 0, \end{cases} \tag{4.21}$$

is continuous and it is equal to $2k^2(\omega_o)/(\gamma_{\omega_o}^{2/3}\omega_o)$ at the turning point, where $\xi = 0$.

Using the change of variables (4.18) and the expansion (4.19) for $\xi = \varepsilon^{1/3}\zeta$ in (3.33) we obtain

$$\begin{aligned} \partial_\zeta \psi_j^\varepsilon(\zeta) = &\frac{2\sigma_\varepsilon}{\varepsilon^{1/3}\sqrt{|\zeta|}} J_{\omega_o}^2(\varepsilon^{1/3}\zeta) V(\varepsilon^{-1/3}\zeta) \mu^2\big(Z_{\omega_o}(\varepsilon^{1/3}\zeta)\big)\nu\left(\frac{Z_{\omega_o}(\varepsilon^{1/3}\zeta)}{\varepsilon}\right) \\ &\times \cos^2\left\{\frac{\psi_\omega^\varepsilon(\zeta)}{2} - \arg\left[A_i + iB_i\right]\big(-\varepsilon^{-1/3}\zeta - \varepsilon^{-1/6} w_j \mathcal{K}(\varepsilon^{1/3}\zeta)\big)\right\}, \end{aligned} \tag{4.22}$$

where we neglected the terms that have no contribution as $\varepsilon \to 0$. This equation is basically the same as (4.9) analyzed in the previous section, except that the phases in the argument of the cosine change with j.

The discussion in the previous section applies verbatim here, and we conclude the same way that we need to consider only $\zeta \in (\zeta_-^\varepsilon, \zeta_{\rm s}^\varepsilon]$, with

$$\zeta_-^\varepsilon = 3\varepsilon^{1/3}, \qquad \zeta_{\rm s}^\varepsilon = \varepsilon^{-1/3}\left[3\phi_{\omega_o}(0)/2\right]^{2/3} = O(\varepsilon^{-1/3}),$$

where (4.22) takes the form

$$\begin{aligned} \partial_\zeta \psi_j^\varepsilon(\zeta) = &\frac{\sigma_\varepsilon}{\varepsilon^{1/3}\sqrt{|\zeta|}} J_{\omega_o}^2(\varepsilon^{1/3}\zeta) \mu^2\big(Z_{\omega_o}(\varepsilon^{1/3}\zeta)\big)\nu\left(\frac{Z_{\omega_o}(\varepsilon^{1/3}\zeta)}{\varepsilon}\right) \\ &\times \left\{1 + \sin\left[\psi_j^\varepsilon(\zeta) + \frac{4}{3}\big(\varepsilon^{-1/3}\zeta + \varepsilon^{-1/6} w_j \mathcal{K}(\varepsilon^{1/3}\zeta)\big)^{3/2}\right]\right\}. \end{aligned} \tag{4.23}$$

This leads to the asymptotic distribution of the phases stated in the next lemma, proved in Appendix C.2.

Lemma 4.2. *The vector*

$$\boldsymbol{\Psi}^\varepsilon(\zeta_{\rm s}^\varepsilon) = (\psi_1^\varepsilon(\zeta_{\rm s}^\varepsilon), \ldots, \psi_m^\varepsilon(\zeta_{\rm s}^\varepsilon)) \tag{4.24}$$

converges in distribution in the limit $\varepsilon \to 0$ to a Gaussian vector with mean zero and covariance matrix

$$\mathbf{C} = \frac{\upsilon_{\omega_o}^2}{3}(\mathbf{I}_m + 2\mathbf{J}_m), \tag{4.25}$$

where $\upsilon_{\omega_o}^2$ is defined in (3.40), *$\mathbf{I}_m$ is the $m \times m$ identity matrix, and $\mathbf{J}_m$ is the $m \times m$ matrix with all entries equal to one.*

4.4. Proof of pulse stabilization. By Lemma 4.2 and definition (3.39) of the envelope of the reflected pulse, for any $t_1, \ldots, t_m \in \mathbb{R}$ we can calculate the finite-order moments

$$\begin{aligned} \mathbb{E}\left[\prod_{j=1}^m \mathcal{F}_{\rm ref}^\varepsilon(t_j)\right] = &\int \frac{dw_1}{2\pi B}\widehat{F}\Big(\frac{w_1}{B}\Big) \ldots \int \frac{dw_m}{2\pi B}\widehat{F}\Big(\frac{w_m}{B}\Big) \exp\left[i\sum_{j=1}^m \big(w_j^2 \beta_{\omega_o} - w_j t_j\big)\right] \\ &\times \mathbb{E}\left[\exp\left(i\sum_{j=1}^m \psi_j^\varepsilon(\zeta_{\rm s}^\varepsilon)\right)\right] \end{aligned} \tag{4.26}$$

in the limit $\varepsilon \to 0$. The process (4.24) is Gaussian in this limit, with covariance (4.25), so the expectation in (4.26) is given by

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\exp\left(i\sum_{j=1}^m \psi_j^\varepsilon(\zeta_{\rm s}^\varepsilon)\right)\right] = \exp\left[-\frac{m(2m+1)\upsilon_{\omega_o}^2}{6}\right]. \tag{4.27}$$

The right-hand side can also be written in terms of the random Gaussian phase ψ_{ω_o} with mean zero and variance $2\upsilon_{\omega_o}^2/3$ as

$$\exp\left[-\frac{m(2m+1)\upsilon_{\omega_o}^2}{6}\right] = \mathbb{E}\left[\prod_{j=1}^m \exp\left(-\frac{\upsilon_{\omega_o}^2}{6} + i\psi_{\omega_o}\right)\right]. \tag{4.28}$$

By substituting into (4.26), we obtain the convergence of the finite-order moments

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\prod_{j=1}^m \mathcal{F}_{\rm ref}^\varepsilon(t_j)\right] = \mathbb{E}\left[\prod_{j=1}^m \mathcal{F}_{\rm ref}(t_j)\right] \tag{4.29}$$

for $\mathcal{F}_{\rm ref}(t)$ defined in (3.41).

The convergence result stated in Theorem 3.4 follows from (4.29), once we prove tightness of the process $\mathcal{F}_{\rm ref}^\varepsilon(t)$ in the space of continuous functions on compact sets in $\mathbb{R}$, as shown in [6, Chapter 2].

We obtain from definition (3.39) and the triangle inequality that $\mathcal{F}_{\rm ref}^\varepsilon(t)$ is bounded independent of ε, uniformly in $t \in [0,T]$, where T is finite. Moreover, for any $\Delta_t > 0$,

$$\begin{aligned} |\mathcal{F}_{\rm ref}^\varepsilon(t+\Delta_t) - \mathcal{F}_{\rm ref}^\varepsilon(t)| &= \left|\int \frac{dw}{2\pi B}\,\widehat{F}\Big(\frac{w}{B}\Big) e^{iw^2\beta_{\omega_o} + i\psi^\varepsilon_{\omega_o+\sqrt{\varepsilon}w}(\zeta_{\rm s}^\varepsilon) - iw(t+\Delta_t)}\left(1 - e^{iw\Delta_t}\right)\right| \\ &\le \int \frac{dw}{2\pi B}\left|\widehat{F}\left(\frac{w}{B}\right)\right|\left|1 - \exp(iw\Delta_t)\right|. \end{aligned} \tag{4.30}$$

Note that

$$\left|1 - e^{iw\Delta_t}\right| = 2|\sin(w\Delta_t/2)| \le |w|\Delta_t$$

and that $w\widehat{F}(w/B)$ is absolutely integrable by the assumption of compact support $[-\pi,\pi]$ of $\widehat{F}(w)$. Substituting in (4.30), we conclude that there exists a constant C, independent of ε and t, that bounds the modulus of continuity

$$\sup_{|t'-t|\le\Delta_t} |\mathcal{F}_{\rm ref}^\varepsilon(t') - \mathcal{F}_{\rm ref}^\varepsilon(t)| \le C\Delta_t.$$

This implies that $\left(\mathcal{F}_{\rm ref}^\varepsilon(t)\right)_{t\in\mathbb{R}}$ is tight [6, Chapter 2]. This completes the proof of Theorem 3.4.

5. Summary. In this paper we studied the reflection of a pulse in a random waveguide with a turning point. The waveguide has reflecting boundaries, a slowly bending axis, and variable cross section. The variation consists of small-amplitude, random fluctuations of the boundary and a slow and monotone change of the opening of the waveguide. The pulse is emitted by a point source and is modeled as usual by a carrier oscillatory signal multiplying a smooth envelope. The carrier wavelength is similar to the width of the cross section of the waveguide, so that the emitted wave is a superposition of a single propagating mode and infinitely many evanescent modes. The turning point is many wavelengths away from the source and marks the limit of propagation of the mode in the waveguide, meaning that once the wave reaches it, it is reflected back. The goal of the paper is to characterize in detail this reflection.

We derived from first principles, starting with the wave equation in the waveguide, a stochastic differential equation for the reflection coefficient, driven by the random fluctuations of the boundary. We showed how this equation can be studied asymptotically, for a small carrier wavelength with respect to the distance of propagation, using stochastic diffusion limits. We also quantified the amplitude of the random fluctuations of the boundary under which the reflected pulse is strongly affected and we explain why it maintains a deterministic shape, i.e., it is stabilized. The reflected pulse oscillates at the same central frequency as the emitted one, but it has a different envelope that is damped and deformed due to scattering at the random boundary.

Appendix A. Useful identities. Here we give a few identities satisfied by the eigenfunctions (2.36) for all $z \in \mathbb{R}$. The first identity is just the statement that the eigenfunctions are orthonormal,

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z) y_q(\rho,z) = \delta_{jq}, \tag{A.1}$$

where δ_{jq} is the Kronecker delta symbol. The second identity,

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, \rho y_j^2(\rho,z) = 0, \tag{A.2}$$

is due to the fact that the integrand is odd. The third identity follows from the fundamental theorem of calculus,

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_\rho y_j(\rho,z) = \frac{1}{2}\int_{-D(z)/2}^{D(z)/2} d\rho\, \partial_\rho y_j^2(\rho,z) = 0, \tag{A.3}$$

because the eigenfunctions vanish at $\rho = \pm D(z)/2$. The fourth identity is

$$\begin{aligned}\int_{-D(z)/2}^{D(z)/2} & d\rho\, [2\rho + D(z)] y_j(\rho,z)\partial_\rho y_j(\rho,z) \\ &= \int_{-D(z)/2}^{D(z)/2} d\rho\, \rho \partial_\rho y_j^2(\rho,z) \\ &= \int_{-D(z)/2}^{D(z)/2} d\rho \left\{ \partial_\rho \left[\rho y_j^2(\rho,z) \right] - y_j^2(\rho,z) \right\} = -1,\end{aligned} \tag{A.4}$$

where we used integration by parts. The fifth identity is

$$\int_{-D(z)/2}^{D(z)/2} d\rho\, y_j(\rho,z)\partial_z y_j(\rho,z) = 0, \tag{A.5}$$

and to derive it, we take the derivative with respect to z in (A.1), for $q = j$, and obtain that

$$
\begin{aligned}
0 &= \partial_z \int_{-D(z)/2}^{D(z)/2} d\rho \, y_j^2(\rho, z) \\
&= 2 \int_{-D(z)/2}^{D(z)/2} d\rho \, y_j(\rho, z) \partial_z y_j(\rho, z) \\
&\quad + \frac{D'(z)}{2} \left[y_j^2(D(z)/2, z) - y_j^2(-D(z)/2, z) \right] \\
&= 2 \int_{-D(z)/2}^{D(z)/2} d\rho \, y_j(\rho, z) \partial_z y_j(\rho, z).
\end{aligned}
$$

The last identity follows from (A.1), (A.2), and the substitution of (2.36) in the remaining integral that is evaluated explicitly,

$$
\begin{aligned}
\text{(A.6)} \qquad & \int_{-D(z)/2}^{D(z)/2} d\rho [2\rho + D(z)]^2 y_j^2(\rho, z) \\
&= D^2(z) + \frac{8}{D(z)} \int_{-D(z)/2}^{D(z)/2} d\rho \, \rho^2 \sin^2 \left[\left(\frac{\rho}{D(z)} + \frac{1}{2} \right) \pi j \right] \\
&= D^2(z) \left[\frac{4}{3} - \frac{2}{(\pi j)^2} \right].
\end{aligned}
$$

Appendix B. Properties of the propagator. Here we prove the statements of Lemmas 3.1–3.3, which describe the approximate propagator $\boldsymbol{M}^\varepsilon(\omega, z)$.

B.1. Proof of Lemma 3.1. To derive (3.15), we need to show that

$$
\text{(B.1)} \qquad \partial_z M_{11}^\varepsilon(\omega, z) = \frac{i}{\varepsilon} M_{21}^\varepsilon(\omega, z)
$$

and

$$
\text{(B.2)} \qquad \partial_z M_{21}^\varepsilon(\omega, z) = \frac{i}{\varepsilon} [k^2(\omega) - \mu^2(z)] M_{11}^\varepsilon(\omega, z) - i\varepsilon \frac{Q_\omega''(z)}{Q_\omega(z)} M_{11}^\varepsilon(\omega, z)
$$

for all $z < 0$. Equation (B.1) is just the definition (3.14) of $M_{21}^\varepsilon(\omega, z)$. Equation (B.2) follows by taking the derivative in the right-hand side of (3.14), using (3.12) and the equations satisfied by the Airy functions [1, chapter 10],

$$
\text{(B.3)} \qquad A_i''(-\eta_\omega^\varepsilon(z)) = -\eta_\omega^\varepsilon(z) A_i(-\eta_\omega^\varepsilon(z)), \qquad B_i''(-\eta_\omega^\varepsilon(z)) = -\eta_\omega^\varepsilon(z) B_i(-\eta_\omega^\varepsilon(z)).
$$

The determinant of the matrix $\boldsymbol{M}^\varepsilon(\omega, z)$ defined in (3.5) is given by

$$
\text{(B.4)} \qquad \det \boldsymbol{M}^\varepsilon(\omega, z) = 2 \, \mathrm{Re} \left[M_{11}^\varepsilon(\omega, z) \overline{M_{21}^\varepsilon(\omega, z)} \right],
$$

where Re[ ] denotes the real part. We calculate using definitions (3.13)–(3.14) that

$$
\begin{aligned}
M_{11}^\varepsilon(\omega, z) \overline{M_{21}^\varepsilon(\omega, z)} = &- i\pi \left[A_i(-\eta_\omega^\varepsilon(z)) - iB_i(-\eta_\omega^\varepsilon(z)) \right] \left[A_i'(-\eta_\omega^\varepsilon(z)) + iB_i'(-\eta_\omega^\varepsilon(z)) \right] \\
&+ i\pi \varepsilon^{2/3} Q_\omega(z) Q_\omega'(z) \left[A_i^2(-\eta_\omega^\varepsilon(z)) + B_i^2(-\eta_\omega^\varepsilon(z)) \right],
\end{aligned}
$$

and taking the real part we have

$$
\mathrm{Re} \left[M_{11}^\varepsilon(\omega, z) \overline{M_{21}^\varepsilon(\omega, z)} \right] = \pi \left[A_i(-\eta_\omega^\varepsilon(z)) B_i'(-\eta_\omega^\varepsilon(z)) - A_i'(-\eta_\omega^\varepsilon(z)) B_i(-\eta_\omega^\varepsilon(z)) \right].
$$

The term in the square bracket in the right-hand side is the Wronskian of the Airy functions which is constant and equal to $1/\pi$. Equation (3.16) follows after substituting the result in (B.4).

B.2. Proof of Lemma 3.2. When $z \nearrow 0$, the function (3.6) is order one and $\eta^\varepsilon_\omega(z)$ defined in (3.7) satisfies $\eta^\varepsilon_\omega(z) = O(\varepsilon^{-2/3}) \gg 1$. The asymptotic expansions of the Airy functions at large and negative argument [1, chapter 10] give that

$$A_i(-\eta^\varepsilon_\omega(z)) \approx \frac{1}{\sqrt{\pi}\big(\eta^\varepsilon_\omega(z)\big)^{1/4}} \left\{ \sin\left[\frac{2}{3}\big(\eta^\varepsilon_\omega(z)\big)^{3/2} + \frac{\pi}{4}\right] + O\left(\big(\eta^\varepsilon_\omega(z)\big)^{-3/2}\right)\right\}, \tag{B.5}$$

$$A_i'(-\eta^\varepsilon_\omega(z)) \approx -\frac{\big(\eta^\varepsilon_\omega(z)\big)^{1/4}}{\sqrt{\pi}} \left\{ \cos\left[\frac{2}{3}\big(\eta^\varepsilon_\omega(z)\big)^{3/2} + \frac{\pi}{4}\right] + O\left(\big(\eta^\varepsilon_\omega(z)\big)^{-3/2}\right)\right\}, \tag{B.6}$$

and

$$B_i(-\eta^\varepsilon_\omega(z)) \approx \frac{1}{\sqrt{\pi}\big(\eta^\varepsilon_\omega(z)\big)^{1/4}} \left\{ \cos\left[\frac{2}{3}\big(\eta^\varepsilon_\omega(z)\big)^{3/2} + \frac{\pi}{4}\right] + O\left(\big(\eta^\varepsilon_\omega(z)\big)^{-3/2}\right)\right\}, \tag{B.7}$$

$$B_i'(-\eta^\varepsilon_\omega(z)) \approx \frac{\big(\eta^\varepsilon_\omega(z)\big)^{1/4}}{\sqrt{\pi}} \left\{ \sin\left[\frac{2}{3}\big(\eta^\varepsilon_\omega(z)\big)^{3/2} + \frac{\pi}{4}\right] + O\left(\big(\eta^\varepsilon_\omega(z)\big)^{-3/2}\right)\right\}. \tag{B.8}$$

We also get from definitions (3.7) and (3.8) that

$$\big(\eta^\varepsilon_\omega(z)\big)^{1/4} = \varepsilon^{-1/6}[k^2(\omega) - \mu^2(z)]^{1/4} Q_\omega(z) \tag{B.9}$$

and

$$\frac{2}{3}\big(\eta^\varepsilon_\omega(z)\big)^{3/2} = \varepsilon^{-1}\phi_\omega(z). \tag{B.10}$$

The statement of the lemma follows by straightforward calculations from these results and definitions (3.13)–(3.14).

B.3. Proof of Lemma 3.3. When $z < z_T(\omega)$ and $|z_T(\omega) - z| \gg O(\varepsilon^{2/3})$, the function $\eta^\varepsilon_\omega(z)$ defined in (3.7) is negative valued and $|\eta^\varepsilon_\omega(z)| \gg 1$. Then, we have from the asymptotic expansions of the Airy functions at large and positive argument that [1, chapter 10]

$$A_i(|\eta^\varepsilon_\omega(z)|) \approx \frac{1}{2\sqrt{\pi}|\eta^\varepsilon_\omega(z)|^{1/4}} e^{-\frac{2}{3}|\eta^\varepsilon_\omega(z)|^{3/2}} \left[1 + O\left(\big|\eta^\varepsilon_\omega(z)\big|^{-3/2}\right)\right], \tag{B.11}$$

$$A_i'(|\eta^\varepsilon_\omega(z)|) \approx -\frac{\big|\eta^\varepsilon_\omega(z)\big|^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}|\eta^\varepsilon_\omega(z)|^{3/2}} \left[1 + O\left(\big|\eta^\varepsilon_\omega(z)\big|^{-3/2}\right)\right], \tag{B.12}$$

and

$$B_i(|\eta^\varepsilon_\omega(z)|) \approx \frac{1}{\sqrt{\pi}|\eta^\varepsilon_\omega(z)|^{1/4}} e^{\frac{2}{3}|\eta^\varepsilon_\omega(z)|^{3/2}} \left[1 + O\left(\big|\eta^\varepsilon_\omega(z)\big|^{-3/2}\right)\right], \tag{B.13}$$

$$B_i'(|\eta^\varepsilon_\omega(z)|) \approx \frac{\big|\eta^\varepsilon_\omega(z)\big|^{1/4}}{\sqrt{\pi}} e^{\frac{2}{3}|\eta^\varepsilon_\omega(z)|^{3/2}} \left[1 + O\left(\big|\eta^\varepsilon_\omega(z)\big|^{-3/2}\right)\right]. \tag{B.14}$$

We also get from definitions (3.7) and (3.8) that

$$\big|\eta^\varepsilon_\omega(z)\big|^{1/4} = \varepsilon^{-1/6}[\mu^2(z) - k^2(\omega)]^{1/4} Q_\omega(z) \tag{B.15}$$

and

$$\frac{2}{3}\big|\eta^\varepsilon_\omega(z)\big|^{3/2} = \varepsilon^{-1}|\phi_\omega(z)|. \tag{B.16}$$

The statement of the lemma follows by straightforward calculations from these results and definitions (3.13)–(3.14).

Appendix C. Details on the diffusion limit.

Here we derive the diffusion limit results stated in Lemmas 4.1 and 4.2.

C.1. Single-frequency diffusion limit. Let us write (4.13) in the form

$$\partial_z \psi^\varepsilon_\omega(\zeta) = \frac{1}{\varepsilon^{1/3}} \mathfrak{F}\big(\varepsilon^{1/3}\zeta, \nu^\varepsilon(\zeta), \zeta^\varepsilon(\zeta), \psi^\varepsilon_\omega(\zeta)\big) \tag{C.1}$$

with real-valued function $\mathfrak{F}$ defined pointwise by

$$\begin{aligned}\mathfrak{F}\big(\varepsilon^{1/3}\zeta, \nu^\varepsilon(\zeta), \zeta^\varepsilon(\zeta), \psi^\varepsilon_\omega(\zeta)\big) = \frac{\sigma_\varepsilon}{\sqrt{|\zeta|}} J^2_\omega(\varepsilon^{1/3}\zeta)\mu^2\big(Z_\omega(\varepsilon^{1/3}\zeta)\big)\nu^\varepsilon(\zeta) \\ \times \left\{1 + \sin\left[\psi^\varepsilon_\omega(\zeta) + \frac{4}{3}\zeta^\varepsilon(\zeta)\right]\right\}\end{aligned} \tag{C.2}$$

for the joint process $(\nu^\varepsilon(z), \zeta^\varepsilon(\zeta))$ defined in (4.15).

The theorem in [21, section III] states that the distribution of $\psi^\varepsilon_\omega(\omega)$ can be described in the limit $\varepsilon \to 0$, in the long interval $(\zeta^\varepsilon_-, \zeta^\varepsilon_{\rm s}]$, by the distribution of the diffusion process with infinitesimal generator $\mathscr{L}^\varepsilon_\zeta$. This is the second-order differential operator $\mathscr{L}^\varepsilon_\zeta : \mathscr{C}^2 \to \mathscr{C}^0$ given by

$$\begin{aligned}\mathscr{L}^\varepsilon_\zeta = \int_0^\infty d\xi \, \Big\langle \mathbb{E}\Big[\mathfrak{F}\big(\varepsilon^{1/3}\zeta, \nu^\varepsilon(\zeta), \zeta^\varepsilon(\zeta), \psi\big) \\ \times \partial_\psi\Big(\mathfrak{F}\big(\varepsilon^{1/3}\zeta, \nu^\varepsilon(\zeta+\varepsilon^{2/3}\xi), \zeta^\varepsilon(\zeta+\varepsilon^{2/3}\xi), \psi\big)\partial_\psi\Big)\Big]\Big\rangle_\zeta,\end{aligned} \tag{C.3}$$

where $\langle\ \rangle_\zeta$ denotes the average over the torus, and $\mathscr{C}^q$ is the space of real-valued functions of ψ with bounded and continuous derivatives up to order $q \geq 0$. Note that $\varepsilon^{1/3}\zeta$ is of order one in our domain.

Recalling definition (4.15) of ν^ε and the autocorrelation (2.11) of ν, we have

$$\mathbb{E}\left[\nu^\varepsilon(\zeta)\nu^\varepsilon(\zeta+\varepsilon^{2/3}\xi)\right] = \mathcal{R}\left(\frac{Z_\omega(\varepsilon^{1/3}\zeta+\varepsilon\xi)}{\varepsilon} - \frac{Z_\omega(\varepsilon^{1/3}\zeta)}{\varepsilon}\right) \approx \mathcal{R}\left(\xi J_\omega(\varepsilon^{1/3}\zeta)\right)$$

with error of order ε. The averages over the torus are

$$\left\langle \sin\left(\psi + \frac{4}{3}\zeta^\varepsilon(\zeta)\right)\sin\left(\psi + \frac{4}{3}\zeta^\varepsilon(\zeta+\varepsilon^{2/3}\xi)\right)\right\rangle_\zeta \approx \frac{1}{2}\cos\left[2\left(\varepsilon^{1/3}\zeta\right)^{1/2}\xi\right]$$

with error of order ε, and similarly,

$$\left\langle \sin\left(\psi + \frac{4}{3}\zeta^\varepsilon(\zeta)\right)\cos\left(\psi + \frac{4}{3}\zeta^\varepsilon(\zeta+\varepsilon^{2/3}\xi)\right)\right\rangle_\zeta \approx -\frac{1}{2}\sin\left[2\left(\varepsilon^{1/3}\zeta\right)^{1/2}\xi\right] \tag{C.4}$$

and

$$\left\langle \sin\left(\psi + \frac{4}{3}\zeta^\varepsilon(\zeta)\right)\right\rangle_\zeta = 0. \tag{C.5}$$

Substituting in the expression of $\mathscr{L}^\varepsilon_\zeta$, and changing variables of integration we obtain

$$\mathscr{L}^\varepsilon_\zeta \approx a^\varepsilon(\zeta)\partial^2_\psi + b^\varepsilon(\zeta)\partial_\psi, \tag{C.6}$$

where

$$a^\varepsilon(\zeta) = \frac{\sigma^2_\varepsilon J^3_\omega(\varepsilon^{1/3}\zeta)\mu^4(Z_\omega(\varepsilon^{1/3}\zeta))}{2\zeta}\left[\widehat{\mathcal{R}}(0) + \frac{1}{2}\widehat{\mathcal{R}}\left(\frac{2(\varepsilon^{1/3}\zeta)^{1/2}}{J_\omega(\varepsilon^{1/3}\zeta)}\right)\right], \tag{C.7}$$

$$b^\varepsilon(\zeta) = -\frac{\sigma^2_\varepsilon J^3_\omega(\varepsilon^{1/3}\zeta)\mu^4(Z_\omega(\varepsilon^{1/3}\zeta))}{2\zeta}\left[\int_0^\infty d\xi\,\mathcal{R}(\xi)\sin\left(\frac{2(\varepsilon^{1/3}\zeta)^{1/2}\xi}{J_\omega(\varepsilon^{1/3}\zeta)}\right)\right]. \tag{C.8}$$

Recall that $\widehat{\mathcal{R}}$ is the power spectral density, the Fourier transform of $\mathcal{R}$.

Since the generator $\mathscr{L}_\zeta^\varepsilon$ is a parabolic operator with constant coefficients, the corresponding process is Gaussian. The random variable $\psi_\omega(\zeta_{\rm s}^\varepsilon)$ is asymptotically Gaussian with mean

$$\mathcal{M} = \lim_{\varepsilon\to 0} \int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, b^\varepsilon(\zeta)$$

and variance

$$\mathcal{V} = \lim_{\varepsilon\to 0} 2 \int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, a^\varepsilon(\zeta).$$

Let us denote

$$a(\zeta) = \frac{3\sigma_\varepsilon^2 k^4(\omega)}{4\gamma_\omega \zeta}\widehat{\mathcal{R}}(0), \tag{C.9}$$

where we recall that $J_\omega(0) = \gamma_\omega^{-1/3}$ and that $\mu(Z_\omega(0)) = \mu(z_T(\omega)) = k(\omega)$. Using the estimates

$$J_\omega^3(\varepsilon^{1/3}\zeta)\mu^4(Z_\omega(\varepsilon^{1/3}\zeta)) - \frac{k^4(\omega)}{\gamma_\omega} = O(\varepsilon^{1/3}\zeta),$$
$$\widehat{\mathcal{R}}\left(\frac{2(\varepsilon^{1/3}\zeta)^{1/2}}{J_\omega(\varepsilon^{1/3}\zeta)}\right) - \widehat{\mathcal{R}}(0) = O\left((\varepsilon^{1/3}\zeta)^{1/2}\right),$$

and

$$\sin\left(\frac{2(\varepsilon^{1/3}\zeta)^{1/2}\xi}{J_\omega(\varepsilon^{1/3}\zeta)}\right) = O\left((\varepsilon^{1/3}\zeta)^{1/2}\right),$$

for ξ in the support of $\mathcal{R}$, we obtain using the dominated convergence theorem and $\sigma_\varepsilon^2 = O(|\ln\varepsilon|^{-1})$ that

$$\lim_{\varepsilon\to 0}\left[\int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, b^\varepsilon(\zeta)\right] = 0, \qquad \lim_{\varepsilon\to 0}\left[\int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, a^\varepsilon(\zeta) - \int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, a(\zeta)\right] = 0, \tag{C.10}$$

which shows that $\mathcal{M} = 0$ and $\mathcal{V}$ is given by

$$\mathcal{V} = 2\lim_{\varepsilon\to 0}\int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, a(\zeta) = \frac{3k^4(\omega)}{2\gamma_\omega}\widehat{\mathcal{R}}(0)\left[\lim_{\varepsilon\to 0}\sigma_\varepsilon^2 \ln\left(\frac{\zeta_{\rm s}^\varepsilon}{\zeta_-^\varepsilon}\right)\right],$$

with $\zeta_{\rm s}^\varepsilon$ defined in (4.6) and $\zeta_-^\varepsilon = 3\varepsilon^{1/3}$. We also have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon^2\ln\left(\frac{\zeta_{\rm s}^\varepsilon}{\zeta_-^\varepsilon}\right) = \lim_{\varepsilon\to 0}\sigma_\varepsilon^2\ln\left[\frac{\varepsilon^{-2/3}}{3}\left(\frac{3}{2}\phi_\omega(0)\right)^{2/3}\right] = \frac{2}{3}\lim_{\varepsilon\to 0}\sigma_\varepsilon^2\ln\left(\frac{\phi_\omega(0)}{\varepsilon}\right),$$

and therefore

$$\mathcal{V} = \frac{k^4(\omega)}{\gamma_\omega}\widehat{\mathcal{R}}(0)\left[\lim_{\varepsilon\to 0}\sigma_\varepsilon^2\ln\left(\frac{\phi_\omega(0)}{\varepsilon}\right)\right]. \tag{C.11}$$

This gives the asymptotic variance of the random phase $\psi_\omega(\zeta_{\rm s}^\varepsilon)$ at the source location $\zeta_{\rm s}^\varepsilon$ and completes the proof of Lemma 4.1.

C.2. Multifrequency diffusion limit. With the same argument as in section 4.2, we conclude that we can analyze the process

$$\boldsymbol{\Psi}^{\varepsilon}(\zeta) = (\psi_1^{\varepsilon}(\zeta), \dots, \psi_m^{\varepsilon}(\zeta)) \tag{C.12}$$

using the diffusion limit theorem in [21, section III]. To apply the theorem, let us gather (4.23) in the system

$$\partial_z \boldsymbol{\Psi}^{\varepsilon}(\zeta) = \frac{1}{\varepsilon^{1/3}} \boldsymbol{\mathfrak{F}}\big(\varepsilon^{1/3}\zeta, \nu^{\varepsilon}(\zeta), \zeta_1^{\varepsilon}(\zeta), \dots, \zeta_m^{\varepsilon}(\zeta), \boldsymbol{\Psi}^{\varepsilon}(\zeta)\big), \tag{C.13}$$

with vector-valued function $\boldsymbol{\mathfrak{F}}$ taking values in $\mathbb{R}^m$, with components

$$\begin{aligned} \mathfrak{F}_j\big(\varepsilon^{1/3}\zeta, \nu^{\varepsilon}(\zeta), \zeta_1^{\varepsilon}(\zeta), \dots, \zeta_m^{\varepsilon}(\zeta), \Psi^{\varepsilon}(\zeta)\big) = \frac{\sigma_{\varepsilon}}{\sqrt{|\zeta|}} J_{\omega_o}^2(\varepsilon^{1/3}\zeta)\mu^2\big(Z_{\omega_o}(\varepsilon^{1/3}\zeta)\big)\nu^{\varepsilon}(\zeta) \\ \times \left\{1 + \sin\left[\psi_j^{\varepsilon}(\zeta) + \frac{4}{3}\zeta_j^{\varepsilon}(\zeta)\right]\right\}, \end{aligned} \tag{C.14}$$

for $j = 1, \dots, m$. Recall that $\varepsilon^{1/3}\zeta$ is order one, because the domain is long, of order $\varepsilon^{-1/3}$. The system (C.13) is driven by the joint process $(\nu^{\varepsilon}(z), \zeta_1^{\varepsilon}(\zeta), \dots \zeta_m^{\varepsilon}(\zeta))$ on the state space $\mathbb{R} \times [0, 3\pi/2] \times \dots \times [0, 3\pi/2]$, with

$$\nu^{\varepsilon}(z) = \nu\left(\frac{Z_{\omega_o}(\varepsilon^{1/3}\zeta)}{\varepsilon}\right), \tag{C.15}$$

lying in $\mathbb{R}$ and

$$\zeta_j^{\varepsilon}(z) = \left[\varepsilon^{-1/3}\zeta + \varepsilon^{-1/6} w_j \mathcal{K}(\varepsilon^{1/3}\zeta)\right]^{3/2}, \qquad j = 1, \dots, m, \tag{C.16}$$

on the torus $[0, 3\pi/2]$.

The infinitesimal generator $\mathcal{L}_{\zeta}^{\varepsilon}$ is a second-order differential operator defined on real-valued functions $f(\psi_1, \dots, \psi_m)$, that are twice continuously differentiable, with bounded derivatives up to order two. It is given by

$$\begin{aligned} \mathscr{L}_{\zeta}^{\varepsilon} = \sum_{j,q=1}^{m} \int_0^{\infty} d\xi \, \Big\langle \mathbb{E}\Big[\mathfrak{F}_j\big(\varepsilon^{1/3}\zeta, \nu^{\varepsilon}(\zeta), \zeta^{\varepsilon}(\zeta), \psi_1, \dots, \psi_m\big) \\ \times \partial_{\psi_j}\Big(\mathfrak{F}_q\big(\varepsilon^{1/3}\zeta, \nu^{\varepsilon}(\zeta + \varepsilon^{2/3}\xi), \zeta^{\varepsilon}(\zeta + \varepsilon^{2/3}\xi), \psi_1, \dots, \psi_m\big)\partial_{\psi_q}\Big)\Big]\Big\rangle_{\zeta}, \end{aligned} \tag{C.17}$$

where $\langle\ \rangle_{\zeta}$ denotes again the average over the torus. We obtain as in the previous section that

$$\mathbb{E}\left[\nu^{\varepsilon}(\zeta)\nu^{\varepsilon}(\zeta + \varepsilon^{2/3}\xi)\right] = \mathcal{R}\left(\frac{Z_{\omega_o}(\varepsilon^{1/3}\zeta + \varepsilon\xi)}{\varepsilon} - \frac{Z_{\omega_o}(\varepsilon^{1/3}\zeta)}{\varepsilon}\right) \approx \mathcal{R}\left(\xi J_{\omega_o}(\varepsilon^{1/3}\zeta)\right),$$

with error of order ε, and

$$\left\langle \sin\left(\psi_j + \frac{4}{3}\zeta_j^{\varepsilon}(\zeta)\right) \sin\left(\psi_q + \frac{4}{3}\zeta_q^{\varepsilon}(\zeta + \varepsilon^{2/3}\xi)\right)\right\rangle_{\zeta} \approx \frac{1}{2}\cos\left[2\left(\varepsilon^{1/3}\zeta\right)^{1/2}\xi\right]\delta_{jq},$$

$$\left\langle \sin\left(\psi_j + \frac{4}{3}\zeta_j^{\varepsilon}(\zeta)\right) \cos\left(\psi_q + \frac{4}{3}\zeta_q^{\varepsilon}(\zeta + \varepsilon^{2/3}\xi)\right)\right\rangle_{\zeta} \approx -\frac{1}{2}\sin\left[2\left(\varepsilon^{1/3}\zeta\right)^{1/2}\xi\right]\delta_{jq},$$

where δ_{jq} is the Kronecker delta. Substituting in (C.17), and changing variables of integration, we get the following expression of the infinitesimal generator:

$$\mathscr{L}_\zeta^\varepsilon \approx \sum_{j,q=1}^m a_{jq}^\varepsilon(\zeta)\partial^2_{\psi_j\psi_q} + \sum_{j=1}^m b_j^\varepsilon(\zeta)\partial_{\psi_j}, \tag{C.18}$$

where

$$a_{jq}^\varepsilon(\zeta) = \frac{\sigma_\varepsilon^2 J_{\omega_o}^3(\varepsilon^{1/3}\zeta)\mu^4(Z_{\omega_o}(\varepsilon^{1/3}\zeta))}{2\zeta}\left[\widehat{\mathcal{R}}(0) + \frac{1}{2}\widehat{\mathcal{R}}\left(\frac{2(\varepsilon^{1/3}\zeta)^{1/2}}{J_{\omega_o}(\varepsilon^{1/3}\zeta)}\right)\delta_{jq}\right], \tag{C.19}$$

$$b_j^\varepsilon(\zeta) = -\frac{\sigma_\varepsilon^2 J_{\omega_o}^3(\varepsilon^{1/3}\zeta)\mu^4(Z_{\omega_o}(\varepsilon^{1/3}\zeta))}{2\zeta}\int_0^\infty d\xi\,\mathcal{R}(\xi)\sin\left(\frac{2(\varepsilon^{1/3}\zeta)^{1/2}\xi}{J_\omega(\varepsilon^{1/3}\zeta)}\right). \tag{C.20}$$

Since the generator $\mathscr{L}_\zeta^\varepsilon$ is a parabolic operator with constant coefficients, the corresponding process is Gaussian. The random vector $\boldsymbol{\Psi}^{\boldsymbol{\varepsilon}}(\zeta_{\rm s}^\varepsilon)$ is asymptotically Gaussian with mean $(\mathcal{M}_j)_{j=1}^m$ given by

$$\mathcal{M}_j = \lim_{\varepsilon\to 0}\int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, b_j^\varepsilon(\zeta)$$

and covariance matrix $(\mathcal{V}_{jq})_{j,q=1}^m$ given by

$$\mathcal{V}_{jq} = \lim_{\varepsilon\to 0} 2\int_{\zeta_-^\varepsilon}^{\zeta_{\rm s}^\varepsilon} d\zeta\, a_{jq}^\varepsilon(\zeta).$$

Proceeding as in the previous section we find that $\mathcal{M}_j = 0$ and

$$\mathcal{V}_{jq} = \upsilon_{\omega_o}^2\frac{(\delta_{jq}+2)}{3},$$

which completes the proof of Lemma 4.2.

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