OPTION PRICING UNDER FAST-VARYING AND ROUGH STOCHASTIC VOLATILITY

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Abstract. Recent empirical studies suggest that the volatilities associated with financial time series exhibit short range correlations. This entails that the volatility process is very rough and its autocorrelation exhibits sharp decay at the origin. Another classic stylistic feature often assumed for the volatility is that it is "mean reverting". In this paper it is shown that the price impact of a fast mean reverting rough volatility model coincides with that associated with fast mean reverting Markov stochastic volatility models. This then reconciles the empirical observation of rough volatility paths with the good fit of the implied volatility surface to models of fast mean reverting Markov volatilities. Moreover, the result conforms with recent numerical results regarding rough stochastic volatility models. It extends the scope of models for which the asymptotic results of fast mean reverting Markov volatilities are valid. The paper concludes with a general discussion of fractional volatility asymptotics and their interrelation. The regimes discussed there include fast and slow volatility factors with strong or small volatility fluctuations and with the limits not commuting in general. The notion of a characteristic term structure exponent is introduced, this exponent governs the implied volatility term structure in the various asymptotic regimes.

Key words. Stochastic volatility, Short range correlation, Mean reversion, Fractional Ornstein Uhlenbeck process, Hurst exponent.

AMS subject classifications. 91G80, 60H10, 60G22, 60K37.

1. Introduction.

The assumption that the volatility is constant, as in the standard Black-Scholes model, is not realistic. Indeed, practically, in order to match observed prices, one needs to use an implied volatility that depends on the pricing parameters. Therefore, a consistent parameterization of the implied volatility is needed so that, after calibration of the implied volatility model to liquid contracts, it can be used for pricing of less liquid contracts written on the same underlying. Stochastic volatility models have been introduced because they give such consistent parameterizations of the implied volatility. Here we will consider a specific class of stochastic volatility models and identify the associated parameterization of the price correction and associated implied volatility correction that follow from the volatility fluctuations.

Empirical studies suggest that the volatility may exhibit a "multiscale" character as in Bollerslev et al. (2013); Breidt et al. (1998); Chronopoulou and Viens (2012); Cont (2001, 2005); Engle and Patton (2001); Oh et al. (2008). That is, correlations that decay as a power law in offset rather than as an exponential function as in a Markov process. Recent empirical evidence in particular shows that stochastic volatility often should be modeled as being *rough* with rapidly decaying correlations at the origin Gatheral et al. (2016). In Funahashi and Kijima (2017) it was shown numerically that the implied volatility correction for a rough fractional stochastic volatility model tends to the correction associated with the Markov case in the regime of relatively long time to maturity. This is consistent with the analytic result derived in this paper where we consider a *fast mean reverting rough* volatility. In this paper, using the martingale method, we get an analytical expression for the option price and the corresponding implied volatility in the regime when the volatility process is fast mean reverting and rough. The main conclusion is that the corrections

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to the option prices and the corresponding implied volatilities have exactly the same forms as in the mixing case (when the stochastic volatility is Markov).

A main technical aspect of our derivation is a careful analysis of the form and properties of the covariation in between the Brownian motion and the martingale process being the conditional square volatility shift over the time epoch of interest, see Eq. (4.11) below. It is important in this context that we incorporate *leverage* in our model so that these processes are correlated leading to a non-trivial covariation. Another main aspect of our modeling is that we model the stochastic volatility fluctuations as being *stationary*. In a number of recent works a model for the volatility has been introduced where the initial time plays a special role leading to a non-stationary process which is artificial from the modeling viewpoint. However, we show here that in fact in the regime of rapid mean reversion the asymptotic results of the non-stationary case coincides with those of the stationary case considered here since the volatility process then "forgets" its initial state.

A central aspect of our analysis is the notion of time scales and time scale separation. It is then important to identify a reference time scale. Here we will use the characteristic diffusion time:

$$\bar{\tau} = \frac{2}{\bar{\sigma}^2},$$

as the reference time, where $\bar{\sigma}$ is the effective volatility, see Eqs. (3.2) and (4.6) below. Then we consider a regime when the time to maturity is on the scale of the characteristic diffusion time, while the mean reversion time of the volatility fluctuations is small relative to the characteristic diffusion time scale. We remark that a number of recent work on the other hand consider implied volatility asymptotics in a regime of relatively short time to maturity, see for instance Gulisashvili (2015) and references therein. The case with contracts that are moreover close to the money are discussed in for instance Bayer et al (2017) and Fukasawa (2017). In Fukasawa (2017) a time-homogeneous model is used deriving from a representation of fractional Brownian motion due to Muravlev, Muralev (2011).

In Garnier and Sølna (2015) we considered the situations when the multiscale stochastic volatility has *small amplitude* or when it has *slow variations*, with the latter case corresponding to a relatively large mean reversion time for the volatility fluctuations. The corrections to the option prices and the corresponding implied volatilities are identified there and the situation is then qualitatively different from the one considered here in that the correction to the price has a fractional behavior in time to maturity. The characteristic *term structure exponent* then reflects the roughness of the underlying volatility path, see Section 6 below. In Garnier and Sølna (2016) we considered the case with stochastic volatility fluctuations that are *rapid mean reverting* and that have a standard deviation of the same order as the mean, as we do here. However, in Garnier and Sølna (2016) the stochastic volatility fluctuations are so called long range so that the paths are smoother than those associated with the Markov case. Then the "persistence" of the volatility fluctuations leads to a fractional term structure and it also leads to a random component of the price correction that is adapted to the filtration generated by the price process and whose covariance structures can be identified in detail. As we explain below in our modeling the Hurst exponent H parameterizes the smoothness of the paths with $H < 1/2$ corresponding to short-range or rough paths considered here and $H > 1/2$ producing the long range case. Indeed both regimes have been identified from the empirical perspective. We refer to for instance Gatheral et al. (2016) for observations of rough volatility, while in Chronopoulou and Viens (2012) cases of long-range volatility are reported. Long-range volatility situations have beem also reported for currencies in Walther et al. (2017), for commodities in Charfeddine (2014) and for equity index in Chia et al. (2015), while analysis of electricity markets data typically gives $H < 1/2$ as in Simonsen (2002); Rypdal and Lovsletten (2013); Bennedsen (2015). We believe that both the rough and the long-range cases are important and can be seen depending on the specific market and regime. Taken together with the present paper we then have a generalization of the two-factor model of Fouque et al. (2003, 2011) to the case of processes with multiscale fluctuations, and for general smoothness of the volatility factor, that is either short range or long range. Here we consider fractional volatility in the context of option pricing, see Fouque and Hu (2017) and Fouque and Hu (2017b) for applications to portfolio optimization.

In Section 6 we summarize the form of the fractional term structure exponent as it depends on the smoothness of volatility fluctuations, fluctuation magnitude, and the time scale of mean reversion. Otherwise the outline of the paper is as follows: In Section 2 we introduce the volatility factor in terms of a fractional Ornstein-Uhlenbeck process and in Section 3 the full stochastic volatility model. In Section 4 we present the main result and its proof. A number of technical lemmas that are used in the proof are presented in the appendices.

2. The Rapid Fractional Ornstein-Uhlenbeck Process. We use a rapid fractional Ornstein-Uhlenbeck (fOU) process as the volatility factor and describe here how this process can be represented in terms of a fractional Brownian motion. Since fractional Brownian motion can be expressed in terms of ordinary Brownian motion we also arrive at an expression for the rapid fOU process as a filtered version of Brownian motion.

A fractional Brownian motion (fBM) is a zero-mean Gaussian process $(W_t^H)_{t\in\mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \tag{2.1}$$

where σ_H is a positive constant.

We use the following moving-average stochastic integral representation of the fBM Mandelbrot and Van Ness (1968):

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s, \tag{2.2}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion over $\mathbb{R}$. Then indeed $(W_t^H)_{t\in\mathbb{R}}$ is a zero-mean Gaussian process with the covariance (2.1) and we have

$$\begin{aligned}\sigma_H^2 &= \frac{1}{\Gamma(H+\frac{1}{2})^2}\Big[\int_0^\infty \big((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\big)^2 ds + \frac{1}{2H}\Big]\\ &= \frac{1}{\Gamma(2H+1)\sin(\pi H)}.\end{aligned} \tag{2.3}$$

We introduce the ε-scaled fractional Ornstein-Uhlenbeck process (fOU) as

$$Z_t^\varepsilon = \varepsilon^{-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H = \varepsilon^{-H} W_t^H - \varepsilon^{-1-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} W_s^H ds. \tag{2.4}$$

Thus, the fractional OU process is in fact a fractional Brownian motion with a restoring force towards zero. It is a zero-mean, stationary Gaussian process, with variance

$$\mathbb{E}[(Z_t^\varepsilon)^2] = \sigma_{\rm ou}^2, \text{ with } \sigma_{\rm ou}^2 = \frac{1}{2}\Gamma(2H+1)\sigma_H^2, \tag{2.5}$$

that is independent of ε, and covariance:

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_{\rm ou}^2 \mathcal{C}_Z\Big(\frac{s}{\varepsilon}\Big),$$

that is a function of s/ε only, with

$$\begin{aligned}\mathcal{C}_Z(s) &= \frac{1}{\Gamma(2H+1)}\Big[\frac{1}{2}\int_{\mathbb{R}} e^{-|v|}|s+v|^{2H}dv - |s|^{2H}\Big] \\ &= \frac{2\sin(\pi H)}{\pi}\int_0^\infty \cos(sx)\frac{x^{1-2H}}{1+x^2}dx.\end{aligned} \tag{2.6}$$

This shows that ε is the natural scale of variation of the fOU Z_t^ε. Note that the random process Z_t^ε is nor a martingale, neither a Markov process. For $H \in (0,1/2)$ it possesses short-range correlation properties in the sense that its correlation function is rough at zero:

$$\mathcal{C}_Z(s) = 1 - \frac{1}{\Gamma(2H+1)}s^{2H} + o\big(s^{2H}\big), \qquad s \ll 1, \tag{2.7}$$

while it is integrable and it decays as s^{2H-2} at infinity:

$$\mathcal{C}_Z(s) = \frac{1}{\Gamma(2H-1)}s^{2H-2} + o\big(s^{2H-2}\big), \qquad s \gg 1. \tag{2.8}$$

Using Eqs. (2.2) and (2.4) we arrive at the moving-average integral representation of the scaled fOU as:

$$Z_t^\varepsilon = \sigma_{\rm ou}\int_{-\infty}^t \mathcal{K}^\varepsilon(t-s)dW_s, \tag{2.9}$$

where

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}\Big(\frac{t}{\varepsilon}\Big), \qquad \mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big[t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}}e^{-s}ds\Big]. \tag{2.10}$$

The main properties of the kernel $\mathcal{K}$ in our context are the following ones (valid for any $H \in (0,1/2)$):

(i) $\mathcal{K} \in L^2(0,\infty)$ with $\int_0^\infty \mathcal{K}^2(u)du = 1$ and $\mathcal{K} \in L^1(0,\infty)$.

(ii) For small times $t \ll 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big(t^{H-\frac{1}{2}} + O\big(t^{H+\frac{1}{2}}\big)\Big). \tag{2.11}$$

(iii) For large times $t \gg 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H-\frac{1}{2})}\Big(t^{H-\frac{3}{2}} + O\big(t^{H-\frac{5}{2}}\big)\Big). \tag{2.12}$$

Remark. The results presented in this paper can be generalized to any stochastic volatility model of the form (3.2) and (2.9) provided $\mathcal{K}^\varepsilon(t) = \mathcal{K}(t/\varepsilon)/\sqrt{\varepsilon}$ is such that the kernel $\mathcal{K}$ satisfies the properties (i)-(ii)-(iii) up to multiplicative constants.

3. The Stochastic Volatility Model. The price of the risky asset follows the stochastic differential equation:

$$dX_t = \sigma_t^\varepsilon X_t dW_t^*. \tag{3.1}$$

The stochastic volatility is

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{3.2}$$

where Z_t^ε is the scaled fOU with Hurst parameter $H \in (0, 1/2)$ introduced in the previous section which is adapted to the Brownian motion W_t. Moreover, W_t^* is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1-\rho^2} B_t, \tag{3.3}$$

where the Brownian motion B_t is independent of W_t.

The function F is assumed to be one-to-one, positive-valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration $\mathcal{F}_t$ generated by (B_t, W_t) is also the one generated by X_t. Indeed, it is equivalent to the one generated by (W_t^*, W_t), or (W_t^*, Z_t^ε). Since F is one-to-one, it is equivalent to the one generated by $(W_t^*, \sigma_t^\varepsilon)$. Since F is positive-valued, it is equivalent to the one generated by $(W_t^*, (\sigma_t^\varepsilon)^2)$, or X_t.

As we have discussed above, the volatility driving process Z_t^ε has short-range correlation properties. As we now show the volatility process σ_t^ε inherits this property.

LEMMA 3.1. *We denote, for $j = 1, 2$:*

$$\left\langle F^j \right\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^j p(z) dz, \qquad \left\langle F'^j \right\rangle = \int_{\mathbb{R}} F'(\sigma_{\rm ou} z)^j p(z) dz, \tag{3.4}$$

where $p(z)$ is the pdf of the standard normal distribution.

1. *The process σ_t^ε is a stationary random process with mean $\mathbb{E}[\sigma_t^\varepsilon] = \langle F \rangle$ and variance $\mathrm{Var}(\sigma_t^\varepsilon) = \langle F^2 \rangle - \langle F \rangle^2$, independently of ε.*
2. *The covariance function of the process σ_t^ε is of the form*

$$\mathrm{Cov}\big(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon\big) = \big(\left\langle F^2 \right\rangle - \left\langle F \right\rangle^2\big) \mathcal{C}_\sigma\Big(\frac{s}{\varepsilon}\Big), \tag{3.5}$$

where the correlation function $\mathcal{C}_\sigma$ satisfies $\mathcal{C}_\sigma(0) = 1$ and

$$\mathcal{C}_\sigma(s) = 1 - \frac{1}{\Gamma(2H+1)} \frac{\sigma_{\rm ou}^2 \left\langle F'^2 \right\rangle}{\left\langle F^2 \right\rangle - \left\langle F \right\rangle^2} s^{2H} + o\big(s^{2H}\big), \qquad \text{for } s \ll 1, \tag{3.6}$$

$$\mathcal{C}_\sigma(s) = \frac{1}{\Gamma(2H-1)} \frac{\sigma_{\rm ou}^2 \left\langle F' \right\rangle^2}{\left\langle F^2 \right\rangle - \left\langle F \right\rangle^2} s^{2H-2} + o\big(s^{2H-2}\big), \qquad \text{for } s \gg 1. \tag{3.7}$$

Consequently, the process σ_t^ε has short-range correlation properties and its covariance function is integrable.

Proof. The fact that σ_t^ε is a stationary random process with mean $\langle F \rangle$ is straightforward in view of the definition (3.2) of σ_t^ε.

For any t, s, the vector $\sigma_{\rm ou}^{-1}(Z_t^\varepsilon, Z_{t+s}^\varepsilon)$ is a Gaussian random vector with mean $(0, 0)$ and 2×2 covariance matrix:

$$\mathbf{C}^\varepsilon = \begin{pmatrix} 1 & \mathcal{C}_Z(s/\varepsilon) \\ \mathcal{C}_Z(s/\varepsilon) & 1 \end{pmatrix}.$$

Therefore, denoting $F_c(z) = F(\sigma_{\rm ou} z) - \langle F\rangle$, the covariance function of the process σ_t^ε is

$$
\begin{aligned}
{\rm Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) &= \mathbb{E}\big[F_c(\sigma_{\rm ou}^{-1} Z_t^\varepsilon) F_c(\sigma_{\rm ou}^{-1} Z_{t+s}^\varepsilon)\big] \\
&= \frac{1}{2\pi\sqrt{\det \mathbf{C}^\varepsilon}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big(- \frac{(z_1,z_2){\mathbf{C}^{\varepsilon}}^{-1}(z_1,z_2)^T}{2}\Big) dz_1 dz_2 \\
&= \Psi\Big(\mathcal{C}_Z\Big(\frac{s}{\varepsilon}\Big)\Big),
\end{aligned}
$$

with

$$
\Psi(C) = \frac{1}{2\pi\sqrt{1-C^2}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big(- \frac{z_1^2+z_2^2 - 2C z_1 z_2}{2(1-C^2)}\Big) dz_1 dz_2 \, .
$$

This shows that ${\rm Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon)$ is a function of s/ε only.
The function Ψ can be expanded in powers of $1-C$ for C close to one:

$$
\begin{aligned}
\Psi(C) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} F_c\Big(z \frac{\sqrt{1+C}}{\sqrt{2}} + \zeta \frac{\sqrt{1-C}}{\sqrt{2}}\Big) F_c\Big(z \frac{\sqrt{1+C}}{\sqrt{2}} - \zeta \frac{\sqrt{1-C}}{\sqrt{2}}\Big) \\
&\quad \times \exp\Big(-\frac{z^2}{2} - \frac{\zeta^2}{2}\Big) dz d\zeta \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_c(z)^2 \exp\Big(-\frac{z^2}{2}\Big) dz \\
&\quad + (1-C) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_c'(z)^2 \exp\Big(-\frac{z^2}{2}\Big) dz + O((1-C)^2), \qquad 1 - C \ll 1,
\end{aligned}
$$

which gives with (2.7) the form (3.6) of the correlation function for σ_t^ε.
The function Ψ can be expanded in powers of C for small C:

$$
\begin{aligned}
\Psi(C) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big(-\frac{z_1^2+z_2^2}{2}\Big) dz_1 dz_2 \\
&\quad - \frac{1}{2\pi} \iint_{\mathbb{R}^2} z_1 z_2 F_c(z_1) F_c(z_2) \exp\Big(-\frac{z_1^2+z_2^2}{2}\Big) dz_1 dz_2 + O(C^2), \qquad C \ll 1,
\end{aligned}
$$

which gives with (2.8) the form (3.7) of the correlation function for σ_t^ε. □

4. The Option Price. We aim at computing the option price defined as the martingale

$$
M_t = \mathbb{E}\big[h(X_T) | \mathcal{F}_t\big], \tag{4.1}
$$

where h is a smooth function and $t \leq T$. In fact weaker assumptions are possible for h, as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than h, as is discussed in (Garnier and Sølna , 2015, Section 4).

We introduce the operator

$$
\mathcal{L}_{\rm BS}(\sigma) = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_x^2, \tag{4.2}
$$

that is, the standard Black-Scholes operator at zero interest rate and constant volatility σ.

We next exploit the fact that the price process is a martingale to obtain an approximation, via constructing an explicit function $Q_t^\varepsilon(x)$ so that $Q_T^\varepsilon(x) = h(x)$ and so that $Q_t^\varepsilon(X_t)$ is a martingale to first-order corrected terms. Then, indeed $Q_t^\varepsilon(X_t)$ gives the approximation for M_t to this order.

The following proposition gives the first-order correction to the expression for the martingale M_t in the regime of ε small.

PROPOSITION 4.1. *We have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} \sup_{t\in[0,T]} \mathbb{E}\left[|M_t - Q_t^\varepsilon(X_t)|^2\right]^{1/2} = 0, \tag{4.3}$$

where

$$Q_t^\varepsilon(x) = Q_t^{(0)}(x) + \varepsilon^{1/2}\rho Q_t^{(1)}(x), \tag{4.4}$$

$Q_t^{(0)}(x)$ *is deterministic and given by the Black-Scholes formula with constant volatility* $\overline{\sigma}$*,*

$$\mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x), \tag{4.5}$$

with

$$\overline{\sigma}^2 = \left\langle F^2 \right\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^2 p(z) dz, \tag{4.6}$$

$p(z)$ *the pdf of the standard normal distribution,*
$Q_t^{(1)}(x)$ *is the deterministic correction*

$$Q_t^{(1)}(x) = (T-t)\overline{D}\big(x\partial_x(x^2\partial_x^2)\big)Q_t^{(0)}(x), \tag{4.7}$$

with the coefficient $\overline{D}$ *defined by*

$$\overline{D} = \sigma_{\rm ou}\int_0^\infty \Big[\iint_{\mathbb{R}^2} F(\sigma_{\rm ou} z)(FF')(\sigma_{\rm ou} z') p_{\mathcal{C}_Z(s)}(z,z') dz dz'\Big]\mathcal{K}(s) ds, \tag{4.8}$$

$p_C(z,z')$ *the pdf of the bivariate normal distribution with mean zero and covariance matrix* $\begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix}$*, and* $\mathcal{C}_Z(s)$ *given by (2.6).*

This proposition shows that the result is similar to the mixing (Markov) case addressed in Fouque et al. (2000, 2011). In the fast-varying framework, the short-range correlation property of the stochastic volatility is not visible to leading order nor in the first correction. This is in contrast to the slowly-varying case addressed in Garnier and Sølna (2015) and this is the main result of this paper.

Proof. For any smooth function $q_t(x)$, we have by Itô's formula

$$\begin{aligned} dq_t(X_t) &= \partial_t q_t(X_t)dt + \big(x\partial_x q_t\big)(X_t)\sigma_t^\varepsilon dW_t^* + \frac{1}{2}\big(x^2\partial_x^2 q_t\big)(X_t)(\sigma_t^\varepsilon)^2 dt \\ &= \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)q_t(X_t)dt + \big(x\partial_x q_t\big)(X_t)\sigma_t^\varepsilon dW_t^*, \end{aligned}$$

the last term being a martingale. Therefore, by (4.5), we have

$$dQ_t^{(0)}(X_t) = \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt + dN_t^{(0)}, \tag{4.9}$$

with $N_t^{(0)}$ a martingale:

$$dN_t^{(0)} = \big(x\partial_x Q_t^{(0)}\big)(X_t)\sigma_t^\varepsilon dW_t^*.$$

Let ϕ_t^ε be defined by

$$\phi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_t^T \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds\big|\mathcal{F}_t\Big]. \tag{4.10}$$

We have

$$\phi_t^\varepsilon = \psi_t^\varepsilon - \frac{1}{2}\int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds,$$

where the martingale ψ_t^ε is defined by

$$\psi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_0^T \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds\big|\mathcal{F}_t\Big]. \tag{4.11}$$

We can write

$$\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt = \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon - \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t^\varepsilon.$$

By Itô's formula:

$$\begin{aligned} d\big[\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t^\varepsilon + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* \\ & + \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle \phi^\varepsilon, W^*\right\rangle_t. \end{aligned}$$

Since $\mathcal{L}_{\rm BS}(\sigma_t^\varepsilon) = \mathcal{L}_{\rm BS}(\overline{\sigma}) + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)$ and $\mathcal{L}_{\rm BS}(\overline{\sigma})\big(x^2\partial_x^2\big)Q_t^{(0)}(x) = 0$, this gives

$$\begin{aligned} d\big[\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & -\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\ & + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle \phi^\varepsilon, W^*\right\rangle_t \\ & + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon. \end{aligned}$$

We have $\left\langle \phi^\varepsilon, W^*\right\rangle_t = \left\langle \psi^\varepsilon, W^*\right\rangle_t = \rho\left\langle \psi^\varepsilon, W\right\rangle_t$ and therefore

$$\begin{aligned} d\big[(\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = {} & -\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\ & + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ & + \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle \psi^\varepsilon, W\right\rangle_t \\ & + dN_t^{(1)}, \end{aligned}$$

where $N_t^{(1)}$ is a martingale,

$$dN_t^{(1)} = \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon.$$

Therefore

$$d\big[Q_t^{(0)}(X_t)+\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big]=\frac{1}{2}\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\big((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\big)\phi_t^\varepsilon dt$$
$$+\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\vartheta_t^\varepsilon dt$$
$$+dN_t^{(0)}+dN_t^{(1)}. \tag{4.12}$$

Here, we have introduced the covariation increments

$$d\left\langle\psi^\varepsilon,W\right\rangle_t=\vartheta_t^\varepsilon dt, \tag{4.13}$$

defined in Lemma A.1.

The deterministic function $Q_t^{(1)}$ defined by (4.7) satisfies

$$\mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(1)}(x)=-\overline{D}\big(x\partial_x(x^2\partial_x^2)\big)Q_t^{(0)}(x),\qquad Q_T^{(1)}(x)=0.$$

Applying Itô's formula

$$\begin{aligned}dQ_t^{(1)}(X_t)&=\mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)Q_t^{(1)}(X_t)dt+\big(x\partial_xQ_t^{(1)}\big)(X_t)\sigma_t^\varepsilon dW_t^*\\&=\mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(1)}(X_t)dt+\frac{1}{2}\big((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt\\&\quad+\big(x\partial_xQ_t^{(1)}\big)(X_t)\sigma_t^\varepsilon dW_t^*\\&=\frac{1}{2}\big((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt-\big(x\partial_x(x^2\partial_x^2)\big)Q_t^{(0)}(X_t)\overline{D}dt+dN_t^{(2)},\end{aligned}$$

where $N_t^{(2)}$ is a martingale,

$$dN_t^{(2)}=\big(x\partial_xQ_t^{(1)}\big)(X_t)\sigma_t^\varepsilon dW_t^*.$$

Therefore

$$\begin{aligned}&d\big[Q_t^{(0)}(X_t)+\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)+\varepsilon^{1/2}\rho Q_t^{(1)}(X_t)\big]\\&=\frac{1}{2}\big(x^2\partial_x^2(x^2\partial_x^2)\big)Q_t^{(0)}(X_t)\big((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\big)\phi_t^\varepsilon dt+\frac{\varepsilon^{1/2}}{2}\rho\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)\big((\sigma_t^\varepsilon)^2-\overline{\sigma}^2\big)dt\\&\quad+\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\big(\sigma_t^\varepsilon\vartheta_t^\varepsilon-\varepsilon^{1/2}\overline{D}\big)dt\\&\quad+dN_t^{(0)}+dN_t^{(1)}+\varepsilon^{1/2}\rho dN_t^{(2)}.\end{aligned} \tag{4.14}$$

We next show that the first three terms of the right-hand side of (4.14) are smaller than $\varepsilon^{1/2}$. We introduce for any $t\in[0,T]$:

$$R_{t,T}^{(1)}=\int_t^T\frac{1}{2}\big(x^2\partial_x^2(x^2\partial_x^2)\big)Q_s^{(0)}(X_s)\big((\sigma_s^\varepsilon)^2-\overline{\sigma}^2\big)\phi_s^\varepsilon ds, \tag{4.15}$$

$$R_{t,T}^{(2)}=\int_t^T\frac{\varepsilon^{1/2}}{2}\rho\big(x^2\partial_x^2\big)Q_s^{(1)}(X_s)\big((\sigma_s^\varepsilon)^2-\overline{\sigma}^2\big)ds, \tag{4.16}$$

$$R_{t,T}^{(3)}=\int_t^T\rho\big(x\partial_x(x^2\partial_x^2)\big)Q_s^{(0)}(X_s)\big(\vartheta_s^\varepsilon\sigma_s^\varepsilon-\varepsilon^{1/2}\overline{D}\big)ds. \tag{4.17}$$

We will show that, for $j=1,2,3$,

$$\lim_{\varepsilon\to0}\varepsilon^{-1/2}\sup_{t\in[0,T]}\mathbb{E}\big[(R_{t,T}^{(j)})^2\big]^{1/2}=0. \tag{4.18}$$

Step 1: Proof of (4.18) for $j = 1$.
Since $Q^{(0)}$ is smooth and bounded and F is bounded, there exists C such that

$$\sup_{t\in[0,T]} \mathbb{E}\big[(R^{(1)}_{t,T})^2\big] \le CT \int_0^T \mathbb{E}\big[(\phi^\varepsilon_s)^2\big] ds.$$

By Lemma A.4 we get the desired result.

Step 2: Proof of (4.18) for $j = 2$.
We denote

$$Y^{(2)}_s = \rho\big(x^2\partial_x^2\big)Q^{(1)}_s(X_s)$$

and

$$\kappa^\varepsilon_t = \frac{\varepsilon^{1/2}}{2}\int_0^t \big((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\big)ds, \tag{4.19}$$

so that

$$R^{(2)}_{t,T} = \int_t^T Y^{(2)}_s \frac{d\kappa^\varepsilon_s}{ds} ds.$$

Note that $Y^{(2)}_s$ is a bounded semimartingale with bounded quadratic variations. Let N be a positive integer. We denote $t_k = t + (T-t)k/N$. We then have

$$\begin{aligned}
R^{(2)}_{t,T} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} Y^{(2)}_s \frac{d\kappa^\varepsilon_s}{ds} ds = R^{(2,a)}_{t,T} + R^{(2,b)}_{t,T},\\
R^{(2,a)}_{t,T} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} Y^{(2)}_{t_k} \frac{d\kappa^\varepsilon_s}{ds} ds = \sum_{k=0}^{N-1} Y^{(2)}_{t_k}\big(\kappa^\varepsilon_{t_{k+1}} - \kappa^\varepsilon_{t_k}\big),\\
R^{(2,b)}_{t,T} &= \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \big(Y^{(2)}_s - Y^{(2)}_{t_k}\big)\frac{d\kappa^\varepsilon_s}{ds} ds.
\end{aligned}$$

Then, on the one hand

$$\mathbb{E}\big[(R^{(2,a)}_{t,T})^2\big]^{1/2} \le 2\sum_{k=0}^{N} \|Y^{(2)}\|_\infty \mathbb{E}[(\kappa^\varepsilon_{t_k})^2]^{1/2} \le 2(N+1)\|Y^{(2)}\|_\infty \sup_{s\in[0,T]} \mathbb{E}[(\kappa^\varepsilon_s)^2]^{1/2},$$

so that, by Lemma A.5,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} \sup_{t\in[0,T]} \mathbb{E}\big[(R^{(2,a)}_{t,T})^2\big]^{1/2} = 0.$$

On the other hand

$$\begin{aligned}
\mathbb{E}\big[(R^{(2,b)}_{t,T})^2\big]^{1/2} &\le \varepsilon^{1/2}\|F\|_\infty^2 \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \mathbb{E}\big[\big(Y^{(2)}_s - Y^{(2)}_{t_k}\big)^2\big]^{1/2} ds\\
&\le K\varepsilon^{1/2}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} (s-t_k)^{1/2} ds = \frac{2KT^{3/2}\varepsilon^{1/2}}{3\sqrt{N}}.
\end{aligned}$$

Therefore, we get

$$\limsup_{\varepsilon\to 0}\varepsilon^{-1/2}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(2)}_{t,T})^2\big]^{1/2}\le \limsup_{\varepsilon\to 0}\varepsilon^{-1/2}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(2,b)}_{t,T})^2\big]^{1/2}\le \frac{2KT^{3/2}}{3\sqrt{N}}.$$

Since this is true for any N, we get the desired result.

Step 3: Proof of (4.18) for $j=3$.
We repeat the same arguments as in the previous step. It remains to show that

$$\widetilde{\kappa}^\varepsilon_t=\int_0^t \big(\vartheta^\varepsilon_s\sigma^\varepsilon_s-\varepsilon^{1/2}\overline{D}\big)ds$$

satisfies

$$\lim_{\varepsilon\to 0}\varepsilon^{-1/2}\sup_{t\in[0,T]}\mathbb{E}\big[(\widetilde{\kappa}^\varepsilon_t)^2\big]^{1/2}=0.$$

Since $(a+b)^2\le 2a^2+2b^2$, we have

$$\mathbb{E}\big[(\widetilde{\kappa}^\varepsilon_t)^2\big]\le 2\int_0^t ds\int_0^t ds'\mathrm{Cov}\big(\vartheta^\varepsilon_s\sigma^\varepsilon_s,\vartheta^\varepsilon_{s'}\sigma^\varepsilon_{s'}\big)+2\Big(\int_0^t \big(\mathbb{E}[\vartheta^\varepsilon_s\sigma^\varepsilon_s]-\varepsilon^{1/2}\overline{D}\big)ds\Big)^2.$$

By Lemma A.2, items 1 and 3, and dominated convergence theorem, the first term of the right-hand side is $o(\varepsilon)$ uniformly in $t\in[0,T]$. By Lemma A.2, item 2, the second term of the right-hand side is $o(\varepsilon)$ uniformly in $t\in[0,T]$. This gives the desired result.

We can now complete the proof of Proposition 4.1. We introduce the approximation:

$$\widetilde{Q}^\varepsilon_t(x)=Q^{(0)}_t(x)+\phi^\varepsilon_t\big(x^2\partial_x^2\big)Q^{(0)}_t(x)+\varepsilon^{1/2}\rho Q^{(1)}_t(x).$$

We then have

$$\widetilde{Q}^\varepsilon_T(x)=h(x),$$

because $Q^{(0)}_T(x)=h(x)$, $\phi^\varepsilon_T=0$, and $Q^{(1)}_T(x)=0$. Let us denote

$$R_{t,T}=R^{(1)}_{t,T}+R^{(2)}_{t,T}+R^{(3)}_{t,T}, \tag{4.20}$$

$$N_t=\int_0^t dN^{(0)}_s+dN^{(1)}_s+\varepsilon^{1/2}\rho dN^{(2)}_s. \tag{4.21}$$

By (4.14) we have

$$\widetilde{Q}^\varepsilon_T(X_t)-\widetilde{Q}^\varepsilon_t(X_t)=R_{t,T}+N_T-N_t.$$

Therefore

$$\begin{aligned}M_t=\mathbb{E}\big[h(X_T)|\mathcal{F}_t\big]&=\mathbb{E}\big[\widetilde{Q}^\varepsilon_T(X_T)|\mathcal{F}_t\big]=\widetilde{Q}^\varepsilon_t(X_t)+\mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big]+\mathbb{E}\big[N_T-N_t|\mathcal{F}_t\big]\\&=\widetilde{Q}^\varepsilon_t(X_t)+\mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big],\end{aligned} \tag{4.22}$$

which gives the desired result since $\mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big]$ and ϕ^ε_t are uniformly of order $o(\varepsilon^{1/2})$ in L^2 (see Lemma A.4 for ϕ^ε_t). □

5. The implied volatility. We now compute and discuss the implied volatility associated with the price approximation given in Proposition 4.1. This implied volatility is the volatility that when used in the constant volatility Black-Scholes pricing formula gives the same price as the approximation, to the order of the approximation. The implied volatility in the context of the European option introduced in the previous section is then given by

$$I_t = \bar{\sigma} + \varepsilon^{1/2}\rho\overline{D}\Big[\frac{1}{2\bar{\sigma}} + \frac{\log(K/X_t)}{\bar{\sigma}^3(T-t)}\Big] + o(\varepsilon^{1/2}). \tag{5.1}$$

The expression (5.1) is in agreement with the one obtained in (Fouque et al. , 2000, Eq. (5.55)) with a stochastic volatility that is an ordinary Ornstein-Uhlenbeck process, that is, a Markovian process with correlations decaying exponentially fast. See for instance Fouque et al. (2003, 2004, 2011) and references therein for data calibration examples.

6. Summary Fractional Stochastic Volatility Asymptotics. This paper together with Garnier and Sølna (2015, 2016) discuss different fractional stochastic volatility models with $H < 1/2$ or $H > 1/2$. We summarize here some main aspects.

6.1. Characteristic Term Structure Exponent. We write the implied volatility associated with a European Call Option for strike K, maturity T, current time t, and current value for the underlying X_t (as in Eq. (5.1)) in the general form:

$$I_t = \sigma_{t,T} + \Delta\sigma\left[\left(\frac{\tau}{\bar{\tau}}\right)^{\zeta(H)} + \left(\frac{\tau}{\bar{\tau}}\right)^{\zeta(H)-1}\log\left(\frac{K}{X_t}\right)\right], \tag{6.1}$$

$$\sigma_{t,T} = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T (\sigma_s)^2 ds \big| \mathcal{F}_t\Big]^{1/2} = \bar{\sigma} + \tilde{\sigma}_{t,T}, \tag{6.2}$$

where σ_s is the volatility path, $\bar{\tau}$ is the characteristic diffusion time defined by

$$\bar{\tau} = \frac{2}{\bar{\sigma}^2}, \tag{6.3}$$

and $\tau = T - t$ the time to maturity. Note that in the regimes we consider we assume that the time to maturity is on the characteristic diffusion time scale. We refer to $\zeta(H)$ as the *characteristic term structure exponent*. Note that $\tilde{\sigma}_{t,T}$ is the price path predicted volatility correction relative to the time horizon and is a stochastic process adapted to the filtration generated by the underlying price process. This is a correction term that reflects the multiscale nature of the volatility fluctuations and the "memory" aspect of this process. The second correction term in Eq. (6.1) involving $\Delta\sigma$ is a skewness correction and vanish in the case $\rho = 0$. As mentioned above it is natural to let the characteristic diffusion time be the reference time scale, if we denote the mean reversion time of the volatility fluctuations by $\tau_{\rm mr}$ then we have considered two main multiscale asymptotic regimes in the context of the characteristic term structure exponent:

- *Slow mean reverting* volatility fluctuations, $\tau_{\rm mr} \gg \bar{\tau}$, (see Garnier and Sølna (2015)). In this case:

$$\zeta(H) = H + \frac{1}{2}. \tag{6.4}$$

- *Fast mean reverting* volatility fluctuations, $\tau_{\rm mr} \ll \bar{\tau}$, (see Garnier and Sølna (2016) for $H \in (1/2, 1)$ and this paper for $H \in (0, 1/2)$). In this case:

$$\zeta(H) = \max\Big(H - \frac{1}{2}, 0\Big). \tag{6.5}$$

Thus, we see that in the case of fast mean reversion leading to a *singular perturbation* expansion we have a fractional characteristic term structure exponent only in the case $H > 1/2$ when we have long range correlation. While in the slow mean reversion case leading to a *regular perturbation* expansion we have a fractional characteristic term structure exponent for all values of the Hurst exponent H.

In Garnier and Sølna (2015) we considered also the case of small volatility fluctuations whose time of mean reversion is of the same order as the characteristic diffusion time. This leads to an asymptotic regime where the characteristic term structure exponent is replaced by a more general characteristic term structure factor of the form (assuming a fractional Ornstein Uhlenbeck volatility factor):

$$\Big(\frac{\tau}{\bar{\tau}}\Big)^{\zeta(H)} \to \mathcal{A}\Big(\frac{\tau}{\bar{\tau}}, \frac{\tau}{\tau_{\rm mr}}\Big) = \Big(\frac{\tau}{\bar{\tau}}\Big)^{H+1/2}\Big\{1 - \int_0^{\tau/\tau_{\rm mr}} e^{-v}\Big(1 - \frac{v}{\tau/\tau_{\rm mr}}\Big)^{H+\frac{3}{2}} dv\Big\}.$$

We then have in a subsequent limit of either relatively slow ($\tau_{\rm mr} \gg \tau$) or fast ($\tau_{\rm mr} \ll \tau$) mean reversion:

$$\mathcal{A}(\tau) \propto \begin{cases} \big(\frac{\tau}{\bar{\tau}}\big)^{H+1/2} & \text{for } \tau \ll \tau_{\rm mr}, \\ \big(\frac{\tau}{\bar{\tau}}\big)^{H-1/2} & \text{for } \tau \gg \tau_{\rm mr}, \end{cases} \tag{6.6}$$

where we have a fractional term structure for all values of H. It follows that the characteristic term structure exponent is consistent with the result (6.4) obtained in the slow mean reverting limit. It is also consistent with the result (6.5) obtained in the fast mean reverting limit, but only in the case $H \in (1/2, 1)$. There is no contradiction in the case $H \in (0, 1/2)$ because there is no fundamental reason that would justify that the limits "small amplitude" and "fast mean reversion" are exchangeable. This means that the prediction (6.6) for $H \in (0, 1/2)$ in the limit "small amplitude" and then "fast mean reversion" does not capture the leading-order contribution of the limit "fast mean reversion" that is independent of time to maturity, but is negligible for small-amplitude volatility fluctuations. Note that when the standard deviation of the volatility fluctuations is of the same order as the mean volatility and the time to maturity is of the same order as the mean reversion time, then the implied volatility reflects the particular structure of the model, see for instance the analysis of the Heston (Heston (1993)) model in Alòs and Yang (2014). Note also that with a model for how the implied volatility depends on the Hurst exponent we can actually estimate the Hurst exponent based on recordings of the implied volatility. An example with estimation of the Hurst exponent based on a spot volatility proxy deriving from implied volatility is in Livieri et al. (2017) and yields a rough volatility regime. A calibration example for H using VIX futures is in Jacquier et al (2017) and yields again a rough volatility regime.

6.2. Flapping of the Implied Surface. Regarding the price path predicted volatility correction $\tilde{\sigma}_{t,T}$ which depends on the price history we have the following picture in the regime of rapid mean reversion:

- *Rough volatility fluctuations, $H < 1/2$:*

$$\tilde{\sigma}_{t,T} = o(\Delta\sigma). \tag{6.7}$$

- *Smooth volatility fluctuations, $H > 1/2$:*

$$\tilde{\sigma}_{t,T} = O(\Delta\sigma). \tag{6.8}$$

In the case of smooth volatility fluctuations we discuss in detail in Garnier and Sølna (2016) the statistical structure of the "t-T" process $\tilde{\sigma}_{t,T}$. We remark that indeed in the scaling addressed in this paper we have $\Delta\sigma/\bar{\sigma} \ll 1$. In the regime of *slow mean reversion* as discussed in Garnier and Sølna (2015) we have that $\tilde{\sigma}_{t,T} = O(\bar{\sigma})$ since then the current level of volatility plays a central role.

7. Conclusion. We have considered rough fractional stochastic volatility models. Such modeling is motivated by a number of recent empirical findings that the volatility is not well modeled by a Markov process with exponentially decaying correlations and certainly not by a constant. Rather it should be modeled as a stochastic process with correlations that are rapidly decaying at the origin, qualitatively faster than the decay that can be associated with a Markov process. In general such models are challenging to use since the volatility factor is not a Markov process nor a martingale so we do not have a pricing partial differential equation. However, here we consider the situation when the volatility is rapidly mean reverting in the sense that its mean reversion time is short relative to the characteristic diffusion time of the price process. In this limit the pricing problem and associated implied volatility surface can be reduced to a parametric form corresponding to that of the Markovian case. An important aspect of our modeling is that we model the rough stochastic volatility as being a stationary process. Many if not most papers on this subject have hitherto used a non-stationary framework where the "time zero" plays a special role. In the case of processes with "memory" of the past, rather than being Markov, we consider this aspect to be crucial from the modeling viewpoint. Indeed, in the general case the history (in principle observable from the underlying price path) impacts the implied volatility. However, in the regime of rapid mean reversion the impact of the price history becomes lower order relative to the leading correction associated with the stochastic volatility which is explicit and which is identified in this paper.

It is important to note that this picture in fact breaks down in the case of long-range stochastic volatility when the volatility factor paths are smoother than in the Markovian case and when their correlations decay slower than in the Markovian case. It also breaks down in the asymptotic context when the mean reversion time is of the same order as the characteristic diffusion time of the price process, but volatility fluctuations have small standard deviation compared to the mean. In these cases both with short (rough volatility) and long range correlations the model for the price correction and the implied volatility changes structure and leads to a picture with a fractional term structure. These results are derived in Garnier and Sølna (2015, 2016) and summarized in Section 6 herein.

These observations then serve to partly explain why parameterizations for the implied surface deriving from a Markovian modeling have been successful in capturing the implied volatility surface despite empirical observations that refute the Markovian framework. Finally, this analytic result confirms the results of the recent paper Funahashi and Kijima (2017) when the price associated with a rough stochastic volatility model was computed numerically.

Appendix A. Technical Lemmas. We denote

$$G(z) = \frac{1}{2}\big(F(z)^2 - \overline{\sigma}^2\big). \tag{A.1}$$

The martingale ψ_t^ε defined by (4.11) has the form

$$\psi_t^\varepsilon = \mathbb{E}\Big[\int_0^T G(Z_s^\varepsilon)ds\big|\mathcal{F}_t\Big]. \tag{A.2}$$

LEMMA A.1. *$(\psi_t^\varepsilon)_{t\in[0,T]}$ is a square-integrable martingale and*

$$d\left\langle \psi^\varepsilon, W\right\rangle_t = \vartheta_t^\varepsilon dt, \qquad \vartheta_t^\varepsilon = \sigma_{\rm ou}\int_t^T \mathbb{E}\big[G'(Z_s^\varepsilon)|\mathcal{F}_t\big]\mathcal{K}^\varepsilon(s-t)ds. \tag{A.3}$$

Proof. See Lemma B.1 Garnier and Sølna (2016). ◻

The important properties of the random process ϑ_t^ε are stated in the following lemma.

LEMMA A.2.

1. *The exists a constant K_T such that, for any $t\in[0,T]$, we have almost surely*

$$\big|\sigma_t^\varepsilon\vartheta_t^\varepsilon\big| \le K_T\varepsilon^{1/2}. \tag{A.4}$$

2. *For any $t\in[0,T]$, we have*

$$\mathbb{E}[\sigma_t^\varepsilon\vartheta_t^\varepsilon] = \varepsilon^{1/2}\overline{D} + \widetilde{D}_t^\varepsilon, \tag{A.5}$$

where $\overline{D}$ is the deterministic constant (4.8) and $\widetilde{D}_t^\varepsilon$ is smaller than $\varepsilon^{1/2}$:

$$\sup_{\varepsilon\in[0,1]}\sup_{t\in[0,T]}\varepsilon^{-1/2}\big|\widetilde{D}_t^\varepsilon\big| < \infty, \tag{A.6}$$

and

$$\forall t\in[0,T), \quad \lim_{\varepsilon\to 0}\varepsilon^{-1/2}\big|\widetilde{D}_t^\varepsilon\big| = 0. \tag{A.7}$$

3. *For any $0\le t<t'<T$, we have*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\Big|{\rm Cov}\big(\sigma_t^\varepsilon\vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon\vartheta_{t'}^\varepsilon\big)\Big| = 0. \tag{A.8}$$

Proof. Using the expression (A.3) of ϑ_t^ε:

$$\big|\vartheta_t^\varepsilon\sigma_t^\varepsilon\big| \le \sigma_{\rm ou}\|F\|_\infty\|G'\|_\infty\int_0^\infty|\mathcal{K}^\varepsilon(s)|ds$$

The proof of the first item follows from the fact that $\mathcal{K}^\varepsilon(t)=\mathcal{K}(t/\varepsilon)/\sqrt{\varepsilon}$, $\mathcal{K}\in L^1(0,\infty)$

The expectation of $\sigma_t^\varepsilon\vartheta_t^\varepsilon$ is equal to

$$\begin{aligned}
\mathbb{E}\big[\sigma_t^\varepsilon\vartheta_t^\varepsilon\big] &= \sigma_{\rm ou}\int_t^T\mathbb{E}\big[F(Z_t^\varepsilon)G'(Z_s^\varepsilon)\big]\mathcal{K}^\varepsilon(s-t)ds\\
&= \sigma_{\rm ou}\varepsilon^{1/2}\int_0^{(T-t)/\varepsilon}\mathbb{E}\big[F(Z_0^\varepsilon)G'(Z_{\varepsilon s}^\varepsilon)\big]\mathcal{K}(s)ds\\
&= \sigma_{\rm ou}\varepsilon^{1/2}\int_0^{(T-t)/\varepsilon}\Big[\iint_{\mathbb{R}^2}F(\sigma_{\rm ou}z)G'(\sigma_{\rm ou}z')p_{\mathcal{C}_Z(s)}(z,z')dzdz'\Big]\mathcal{K}(s)ds,
\end{aligned}$$

with p_C defined in Proposition 4.1.

Therefore the difference

$$\mathbb{E}\big[\sigma_t^\varepsilon \vartheta_t^\varepsilon\big] - \varepsilon^{1/2}\overline{D} = \sigma_{\rm ou}\varepsilon^{1/2}\int_{(T-t)/\varepsilon}^{\infty} \mathbb{E}\big[F(Z_0^\varepsilon)G'(Z_{\varepsilon s}^\varepsilon)\big]\mathcal{K}(s)ds$$

can be bounded by

$$\big|\mathbb{E}\big[\sigma_t^\varepsilon \vartheta_t^\varepsilon\big] - \varepsilon^{1/2}\overline{D}\big| \leq \|F\|_\infty \|G'\|_\infty \sigma_{\rm ou}\varepsilon^{1/2}\int_{(T-t)/\varepsilon}^{\infty} |\mathcal{K}(s)|ds, \tag{A.9}$$

which gives the second item since $\mathcal{K} \in L^1(0,\infty)$.

Let us consider $0 \leq t \leq t' \leq T$. We have

$$\begin{aligned}\mathbb{E}\big[\sigma_t^\varepsilon \vartheta_t^\varepsilon \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon\big] = \sigma_{\rm ou}^2 \int_t^T ds \mathcal{K}^\varepsilon(s-t)\int_{t'}^T ds' \mathcal{K}^\varepsilon(s'-t') \\ \times \mathbb{E}\Big[\mathbb{E}\big[F(Z_t^\varepsilon)G'(Z_s^\varepsilon)|\mathcal{F}_t\big]\mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_{t'}\big]\Big],\end{aligned}$$

so we can write

$$\begin{aligned}{\rm Cov}\big(\sigma_t^\varepsilon \vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon\big) = \sigma_{\rm ou}^2 \int_t^T ds \mathcal{K}^\varepsilon(s-t)\int_{t'}^T ds' \mathcal{K}^\varepsilon(s'-t') \\ \times \mathbb{E}\Big[\mathbb{E}\big[F(Z_t^\varepsilon)G'(Z_s^\varepsilon)|\mathcal{F}_t\big]\mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_{t}\big]\Big] \\ - \mathbb{E}\Big[\mathbb{E}\big[F(Z_t^\varepsilon)G'(Z_s^\varepsilon)|\mathcal{F}_t\big]\mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)\big]\Big],\end{aligned}$$

and therefore

$$\begin{aligned}\big|{\rm Cov}\big(\sigma_t^\varepsilon \vartheta_t^\varepsilon, \sigma_{t'}^\varepsilon \vartheta_{t'}^\varepsilon\big)\big| \leq \sigma_{\rm ou}^2 \|F\|_\infty \|G'\|_\infty \int_t^T ds |\mathcal{K}^\varepsilon(s-t)|\int_{t'}^T ds' |\mathcal{K}^\varepsilon(s'-t')| \\ \times \mathbb{E}\Big[\big(\mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t\big] - \mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)\big]\big)^2\Big]^{1/2}.\end{aligned}$$

We can write for any $\tau > t$:

$$Z_\tau^\varepsilon = A_{t\tau}^\varepsilon + B_{t\tau}^\varepsilon, \quad A_{t\tau}^\varepsilon = \sigma_{\rm ou}\int_{-\infty}^t \mathcal{K}^\varepsilon(\tau-u)dW_u, \quad B_{t\tau}^\varepsilon = \sigma_{\rm ou}\int_t^\tau \mathcal{K}^\varepsilon(\tau-u)dW_u,$$

where $A_{t\tau}^\varepsilon$ is $\mathcal{F}_t$ adapted while $B_{t\tau}^\varepsilon$ is independent from $\mathcal{F}_t$. Therefore ($s' \geq t' \geq t$)

$$\begin{aligned}&\mathbb{E}\Big[\big(\mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t\big] - \mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)\big]\big)^2\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)|\mathcal{F}_t\big]^2\Big] - \mathbb{E}\big[F(Z_{t'}^\varepsilon)G'(Z_{s'}^\varepsilon)\big]^2 \\ &= \mathbb{E}\Big[F(A_{tt'}^\varepsilon + B_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + B_{ts'}^\varepsilon)F(A_{tt'}^\varepsilon + \tilde{B}_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + \tilde{B}_{ts'}^\varepsilon) \\ &\quad - F(A_{tt'}^\varepsilon + B_{tt'}^\varepsilon)G'(A_{ts'}^\varepsilon + B_{ts'}^\varepsilon)F(\tilde{A}_{tt'}^\varepsilon + \tilde{B}_{tt'}^\varepsilon)G'(\tilde{A}_{ts'}^\varepsilon + \tilde{B}_{ts'}^\varepsilon)\Big],\end{aligned}$$

where $(\tilde{A}_{tt'}^\varepsilon, \tilde{B}_{tt'}^\varepsilon, \tilde{A}_{ts'}^\varepsilon, \tilde{B}_{ts'}^\varepsilon)$ is an independent copy of $(A_{tt'}^\varepsilon, B_{tt'}^\varepsilon, A_{ts'}^\varepsilon, B_{ts'}^\varepsilon)$. We can then write

$$
\begin{aligned}
&\mathbb{E}\Big[\big(\mathbb{E}\big[F(Z^\varepsilon_{t'})G'(Z^\varepsilon_{s'})|\mathcal{F}_t\big]-\mathbb{E}\big[F(Z^\varepsilon_{t'})G'(Z^\varepsilon_{s'})\big]\big)^2\Big]\\
&\le \|F\|_\infty\|G'\|_\infty\mathbb{E}\Big[\big(F(A^\varepsilon_{tt'}+\tilde{B}^\varepsilon_{tt'})G'(A^\varepsilon_{ts'}+\tilde{B}^\varepsilon_{ts'})-F(\tilde{A}^\varepsilon_{tt'}+\tilde{B}^\varepsilon_{tt'})G'(\tilde{A}^\varepsilon_{ts'}+\tilde{B}^\varepsilon_{ts'})\big)^2\Big]^{1/2}\\
&\le C\Big(\mathbb{E}\big[(A^\varepsilon_{tt'}-\tilde{A}^\varepsilon_{tt'})^2\big]^{1/2}+\mathbb{E}\big[(A^\varepsilon_{ts'}-\tilde{A}^\varepsilon_{ts'})^2\big]^{1/2}\Big)\\
&\le 2C\Big(\mathbb{E}\big[(A^\varepsilon_{tt'})^2\big]^{1/2}+\mathbb{E}\big[(A^\varepsilon_{ts'})^2\big]^{1/2}\Big)\\
&\le 2C\Big[\Big(\sigma_{\rm ou}^2\int_{-\infty}^t\mathcal{K}^\varepsilon(t'-u)^2du\Big)^{1/2}+\Big(\sigma_{\rm ou}^2\int_{-\infty}^t\mathcal{K}^\varepsilon(s'-u)^2du\Big)^{1/2}\Big]\\
&\le 4C\sigma^\varepsilon_{t'-t,\infty}\le C_1\big(1\wedge(\varepsilon/(t'-t))^{1-H}\big),
\end{aligned}
$$

where we used Lemma A.6 in the last inequality. Then, using the fact that $\mathcal{K}\in L^1$, this gives

$$
\begin{aligned}
\big|\mathrm{Cov}\big(\sigma^\varepsilon_t\vartheta^\varepsilon_t,\sigma^\varepsilon_{t'}\vartheta^\varepsilon_{t'}\big)\big| &\le C_2\int_t^T ds|\mathcal{K}^\varepsilon(s-t)|\int_{t'}^T ds'|\mathcal{K}^\varepsilon(s'-t')|\big(1\wedge(\varepsilon/(t'-t))\big)^{(1-H)/2}\big)\\
&\le C_3\varepsilon\big(1\wedge(\varepsilon/(t'-t))\big)^{(1-H)/2}\big),
\end{aligned}
$$

which proves the third item. ∎

Lemma A.3. *For any smooth function f with bounded derivative, we have*

$$
\mathrm{Var}\big(\mathbb{E}\big[f(Z^\varepsilon_t)|\mathcal{F}_0\big]\big)\le\|f'\|^2_\infty(\sigma^\varepsilon_{t,\infty})^2. \tag{A.10}
$$

Proof. The conditional distribution of Z^ε_t given $\mathcal{F}_0$ is Gaussian with mean

$$
\mathbb{E}\big[Z^\varepsilon_t|\mathcal{F}_0\big]=\sigma_{\rm ou}\int_{-\infty}^0\mathcal{K}^\varepsilon(t-u)dW_u
$$

and variance

$$
\mathrm{Var}\big(Z^\varepsilon_t|\mathcal{F}_0\big)=(\sigma^\varepsilon_{0,t})^2=\sigma_{\rm ou}^2\int_0^t\mathcal{K}^\varepsilon(u)^2du.
$$

Therefore

$$
\mathrm{Var}\big(\mathbb{E}\big[f(Z^\varepsilon_t)|\mathcal{F}_0\big]\big)=\mathrm{Var}\Big(\int_{\mathbb{R}}f\big(\mathbb{E}\big[Z^\varepsilon_t|\mathcal{F}_0\big]+\sigma^\varepsilon_{0,t}z\big)p(z)dz\Big),
$$

where $p(z)$ is the pdf of the standard normal distribution. The random variable $\mathbb{E}\big[Z^\varepsilon_t|\mathcal{F}_0\big]$ is Gaussian with mean zero and variance $(\sigma^\varepsilon_{t,\infty})^2$ so that

$$
\begin{aligned}
\mathrm{Var}\big(\mathbb{E}\big[f(Z^\varepsilon_t)|\mathcal{F}_0\big]\big) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}dzdz'p(z)p(z')\int_{\mathbb{R}}\int_{\mathbb{R}}dudu'p(u)p(u')\\
&\quad\times\Big[f\big(\sigma^\varepsilon_{t,\infty}u+\sigma^\varepsilon_{0,t}z\big)-f\big(\sigma^\varepsilon_{t,\infty}u'+\sigma^\varepsilon_{0,t}z\big)\Big]\\
&\quad\times\Big[f\big(\sigma^\varepsilon_{t,\infty}u+\sigma^\varepsilon_{0,t}z'\big)-f\big(\sigma^\varepsilon_{t,\infty}u'+\sigma^\varepsilon_{0,t}z'\big)\Big]\\
&\le\|f'\|^2_\infty(\sigma^\varepsilon_{t,\infty})^2\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}dudu'p(u)p(u')(u-u')^2\\
&=\|f'\|^2_\infty(\sigma^\varepsilon_{t,\infty})^2,
\end{aligned}
$$

which is the desired result. □

The random term ϕ_t^ε defined by (4.10) has the form

$$\phi_t^\varepsilon = \mathbb{E}\Big[\int_t^T G(Z_s^\varepsilon)ds\big|\mathcal{F}_t\Big], \tag{A.11}$$

with G defined in (A.1).

LEMMA A.4. *For any $t \leq T$, ϕ_t^ε is a zero-mean random variable with standard deviation of order ε^{1-H}:*

$$\sup_{\varepsilon\in[0,1]}\sup_{t\in[0,T]} \varepsilon^{2H-2}\mathbb{E}[(\phi_t^\varepsilon)^2] < \infty. \tag{A.12}$$

Proof. For $t \in [0,T]$ the second moment of ϕ_t^ε is:

$$\begin{aligned}\mathbb{E}\big[(\phi_t^\varepsilon)^2\big] &= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T G(Z_s^\varepsilon)ds\big|\mathcal{F}_t\Big]^2\Big] \\ &= \int_0^{T-t} ds \int_0^{T-t} ds' \mathrm{Cov}\big(\mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_0\big], \mathbb{E}\big[G(Z_{s'}^\varepsilon)|\mathcal{F}_0\big]\big).\end{aligned}$$

We have by Lemma A.3

$$\mathbb{E}\big[(\phi_t^\varepsilon)^2\big] \leq \Big(\int_0^{T-t} ds \mathrm{Var}\big(\mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_0\big]\big)^{1/2}\Big)^2 \leq \|G'\|_\infty^2\Big(\int_0^{T-t} ds\sigma_{s,\infty}^\varepsilon\Big)^2.$$

In view of Lemma A.6 we then have

$$\mathbb{E}\big[(\phi_t^\varepsilon)^2\big] \leq C_T\big(\varepsilon + \varepsilon^{1-H}\big)^2 \leq 4C_T\varepsilon^{2-2H},$$

uniformly in $t \leq T$ and $\varepsilon \in [0,1]$ for some constant C_T. □

LEMMA A.5. *Let us define for any $t \in [0,T]$:*

$$\kappa_t^\varepsilon = \frac{\varepsilon^{1/2}}{2}\int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds = \varepsilon^{1/2}\int_0^t G(Z_s^\varepsilon)ds, \tag{A.13}$$

as in (4.19). We have

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \varepsilon^{-1/2}\mathbb{E}\big[(\kappa_t^\varepsilon)^2\big]^{1/2} = 0. \tag{A.14}$$

Proof. Since the expectation $\mathbb{E}[G(Z_0^\varepsilon)] = 0$, we have

$$\mathbb{E}\big[(\kappa_t^\varepsilon)^2\big] = \varepsilon\mathbb{E}\Big[\Big(\int_0^t G(Z_s^\varepsilon)ds\Big)^2\Big] = 2\varepsilon\int_0^t ds(t-s)\mathrm{Cov}\big(G(Z_s^\varepsilon), G(Z_0^\varepsilon)\big)ds.$$

We have moreover

$$\begin{aligned}\big|\mathrm{Cov}\big(G(Z_s^\varepsilon), G(Z_0^\varepsilon)\big)\big| &= \big|\mathbb{E}\big[\big(\mathbb{E}[G(Z_s^\varepsilon)|\mathcal{F}_0] - \mathbb{E}[G(Z_s^\varepsilon)]\big)G(Z_0^\varepsilon)\big]\big| \\ &\leq \|G\|_\infty\mathrm{Var}\big(\mathbb{E}[G(Z_s^\varepsilon)|\mathcal{F}_0]\big)^{1/2}.\end{aligned}$$

By Lemma A.3 we obtain

$$\big|\mathrm{Cov}\big(G(Z_s^\varepsilon), G(Z_0^\varepsilon)\big)\big| \leq \|G\|_\infty\|G'\|_\infty\sigma_{s,\infty}^\varepsilon.$$

In view of Lemma A.6 we then have

$$\mathbb{E}\big[(\kappa_t^\varepsilon)^2\big] \le C_T\varepsilon\big(\varepsilon+\varepsilon^{1-H}\big) \le 2C_T\varepsilon^{2-H},$$

uniformly in $t\in[0,T]$ and $\varepsilon\in[0,1]$, which gives the desired result. □

LEMMA A.6. *Define*

$$\sigma_{t,\infty}^\varepsilon = \sigma_{\rm ou}\left(\int_t^\infty \mathcal{K}^\varepsilon(s)^2 ds\right)^{1/2}, \tag{A.15}$$

Then there exists $C>0$ such that

$$\sigma_{t,\infty}^\varepsilon \le C\big(1\wedge(\varepsilon/t)^{1-H}\big). \tag{A.16}$$

Proof. This follows from $|\mathcal{K}(s)|\le Ks^{H-\frac{3}{2}}$ for $s\ge 1$ and $\mathcal{K}\in L^2$. □

Appendix B. An Alternative Model. In Comte and Renault (1998); Funahashi and Kijima (2017) the authors consider a stochastic volatility model that is a kind of fractional Orstein-Uhlenbeck process, but they consider the following representation of the fractional Brownian motion:

$$W_t^{H,0} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}dW_s^0, \tag{B.1}$$

where $(W_t^0)_{t\in\mathbb{R}^+}$ is a standard Brownian motion over $\mathbb{R}^+$. $(W_t^{H,0})_{t\in\mathbb{R}^+}$ is a zero-mean self-similar Gaussian process in the sense that $(\alpha^H W_{t/\alpha}^{H,0})_{t\in\mathbb{R}^+} \overset{d}{=} (W_t^{H,0})_{t\in\mathbb{R}^+}$, but it is not stationary. Its variance is

$$\mathbb{E}\big[(W_t^{H,0})^2\big] = \frac{1}{2H\Gamma(H+\frac{1}{2})^2}t^{2H},$$

while the variance of its increment is (for $s>0$) :

$$\mathbb{E}\big[(W_{t+s}^{H,0}-W_t^{H,0})^2\big] = \frac{1}{\Gamma(H+\frac{1}{2})^2}\Big[\int_0^{t/s}\big((1+u)^{H-\frac{1}{2}}-u^{H-\frac{1}{2}}\big)^2du+\frac{1}{2H}\Big]s^{2H},$$

which has the following behavior

$$\mathbb{E}\big[(W_{t+s}^{H,0}-W_t^{H,0})^2\big] \overset{t\to+\infty}{\longrightarrow} \frac{1}{\Gamma(2H+1)\sin(\pi H)}s^{2H}.$$

This model is special because time 0 plays a special role, and we think it is desirable to deal with the stationary situation addressed in this paper. However, it turns out that the two models give the same result in the fast-varying case. Indeed, the modified fractional Ornstein-Uhlenbeck process corresponding to (B.1) is (to be compared with (2.4)):

$$\begin{aligned} Z_t^{\varepsilon,0} &= Z_0e^{-t/\varepsilon}+\varepsilon^{-H}\int_0^t e^{-\frac{t-s}{\varepsilon}}dW_s^{H,0} \\ &= Z_0e^{-t/\varepsilon}+\varepsilon^{-H}W_t^{H,0}-\varepsilon^{-H-1}\int_0^t e^{-\frac{t-s}{\varepsilon}}W_s^{H,0}ds, \end{aligned} \tag{B.2}$$

where Z_0 is considered as a constant as in Comte and Renault (1998); Funahashi and Kijima (2017). In terms of the Brownian motion W_t^0 this reads:

$$Z_t^{\varepsilon,0} = Z_0 e^{-t/\varepsilon} + \sigma_{\rm ou} \int_0^t \mathcal{K}^\varepsilon(t-s) dW_s^0, \tag{B.3}$$

where $\mathcal{K}^\varepsilon$ is defined in (2.10). It is a Gaussian process with the following covariance $(t, s \geq 0)$:

$$\mathrm{Cov}\big(Z_t^{\varepsilon,0}, Z_{t+s}^{\varepsilon,0}\big) = \sigma_{\rm ou}^2 \mathcal{C}^0_{t/\varepsilon}\Big(\frac{s}{\varepsilon}\Big),$$

that is a function of t/ε and s/ε with

$$\mathcal{C}_t^0(s) = \frac{\int_0^t \mathcal{K}(u)\mathcal{K}(u+s)du}{\int_0^\infty \mathcal{K}(u)^2 du}.$$

Note that

$$\mathcal{C}_t^0(s) \stackrel{t+\infty}{\longrightarrow} \frac{\int_0^\infty \mathcal{K}(u)\mathcal{K}(u+s)du}{\int_0^\infty \mathcal{K}(u)^2 du} = \mathcal{C}_Z(s),$$

with $\mathcal{C}_Z$ defined by (2.6). In other words, except for a small period of time just after time 0 which is of duration of the order of ε, the modified process has the same behavior as the one introduced in this paper. One can then check the detailed calculations carried out in this paper and find that Proposition 4.1 still holds true with the modified model $Z_t^{\varepsilon,0}$.

Appendix C.

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