# OPTION PRICING UNDER FAST-VARYING LONG-MEMORY STOCHASTIC VOLATILITY

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**Abstract.** Recent empirical studies suggest that the volatility of an underlying price process may have correlations that decay relatively slowly under certain market conditions. In this paper, the volatility is modeled as a stationary process with long-range correlation properties to capture such a situation and we consider European option pricing. This means that the volatility process is neither a Markov process nor a martingale. However, by exploiting the fact that the price process still is a semimartingale and accordingly using the martingale method, one can get an analytical expression for the option price in the regime when the volatility process is fast mean reverting. The volatility process is here modeled as a smooth and bounded function of a fractional Ornstein Uhlenbeck process and we give the expression for the implied volatility which has a fractional term structure.

**Key words.** Stochastic volatility, Long range correlation, Mean reversion, Fractional Ornstein Uhlenbeck process.

**AMS subject classifications.** 91G80, 60H10, 60G22, 60K37.

## 1. Introduction.

**Stochastic Volatility and the Implied Surface.** Under many market scenarios the assumption that the volatility is constant, as in the standard Black-Scholes model, is not realistic. Practically, this reflects itself in an implied volatility that depends on the pricing parameters. In the context of European options for instance, this means that in order to match observed prices the volatility one needs to use in the Black-Scholes pricing formula depends on time to maturity and log moneyness, with the moneyness here being strike price over current price of underlying. Indeed the implied volatility is a convenient way to parameterize the price of a financial contract relative to a particular underlying. It gives insight about how the market deviates from the ideal Black-Scholes situation and it is a convenient tool for using prices of liquid contracts for calibration and subsequent pricing of less liquid contracts. One may then look for some consistent parameterizations of the implied volatility that correspond to an underlying model for stochastic volatility fluctuations. This is useful for choosing models with respect which one calibrate, moreover, for linking different financial products via implied volatility. For background on stochastic volatility models we refer to the books and surveys: Fouque et al. (2011); Gatheral (2006); Ghysels et al. (1995); Gulisashvili (2012); Henry-Labordére (2009); Rebonato (2004) and the references therein. We also refer to our recent paper on fractional stochastic volatility Garnier and Solna (2015) in the regularly perturbed case for further references on the recent literature on the class of volatility models we consider here.

Empirical studies suggest that the volatility may exhibit a "multi scale" character with long-range correlations: Bollerslev et al. (2013); Breidt et al. (1998); Chronopoulou and Viens (2012); Cont (2001, 2005); Engle and Patton (2001); Oh et al. (2008). That is, correlations that decay as a power law in offset rather than as an exponential function as in a Markov process. We seek to answer the question what parametric forms for the implied volatility such long-range correlations correspond to. In our recent paper Garnier and Solna (2015) we considered this question in the

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context when the magnitude of the volatility fluctuations is small. Here, we consider the situation when the magnitude of the volatility fluctuations is of the same order as the mean volatility. Indeed empirical studies show that the volatility fluctuations may be quite large: Breidt et al. (1998); Cont (2001); Engle and Patton (2001). We stress that the model considered here in fact is quite different from the one in Garnier and Solna (2015). While in Garnier and Solna (2015) the volatility fluctuations were small leading to a (regular) perturbative situation, here the situation is different in that it is the fast mean reversion relative to the maturity time that allows us to push through an asymptotic analysis. However, the presence of long range correlations in this context gives a novel singular perturbation situation and the analysis becomes significantly more involved. The interesting observation is however that the form for the implied volatility surface has a similar structure as in the Markovian case which confirms the robustness of the implied volatility parametric model with respect to underlying price dynamics. There are however some central differences. In particular the long range correlations produce a volatility covariance that is not integrable which in turn gives an implied volatility surface that is a *random field*, and whose statistics we will describe in detail.

As in Garnier and Solna (2015) a main objective of our modeling is to construct a time consistent scheme so that indeed the volatility model is chosen as a stationary process and we consider general times to maturity. In the empirical study in Fouque et al. (2003) it was shown that to fit well the implied volatility it was appropriate to consider a two-time scale model with one slow and one fast volatility factor. In Garnier and Solna (2015) we considered a slow factor, which closely associates with a small fluctuations factor. Here, we now consider a fast factor with large fluctuations. Taken together we then have a generalization of the two-factor model of Fouque et al. (2011, 2003) to the case of processes with long-range correlations. This leads to a fractional term structure of the implied volatility and it was shown in Fouque et al. (2004) that such a term structure may be useful to fit the implied volatility under certain market conditions.

**Long Memory and Fast Mean Reversion.** As mentioned above the asymptotic regime we shall consider here is the situation when the volatility is relatively fast mean reverting, we will denote its time scale by $\varepsilon$ and this is the small parameter in our model. The volatility then decorrelates on the time scale $\varepsilon$.

Stochastic volatility models are most often posed with a volatility driving process that has mixing properties. This means that the volatility driving process at times $t$ and $t + \Delta t$, that is $Z_t^\varepsilon$ and $Z_{t+\Delta t}^\varepsilon$, become rapidly uncorrelated when $\Delta t \to \infty$. In other words the autocovariance function $\mathcal{C}^\varepsilon(\Delta t) = \mathbb{E}[Z_t^\varepsilon Z_{t+\Delta t}^\varepsilon]$ decays rapidly to zero as $\Delta t \to \infty$. More precisely we say that the volatility driving process is mixing if its autocovariance function decays fast enough at infinity so that it is absolutely integrable:

$$\int_0^\infty |\mathcal{C}^\varepsilon(t)| dt < \infty \,. \tag{1.1}$$

In this case we may associate the process with the finite correlation time $t_c = 2\int_0^\infty \mathcal{C}^\varepsilon(t)dt/\mathcal{C}^\varepsilon(0)$.

Stochastic volatility models with long-range correlation properties have recently attracted a lot of attention, as more and more data collected under various situations confirm that this situation can be encountered in many different markets. Qualitatively, the long-range correlation property that we consider here means that the random process has long memory (in contrast with a mixing process). This means that the correlation degree between the random values $Z_t^\varepsilon$ and $Z_{t+\Delta t}^\varepsilon$ taken at two times separated by $\Delta t$ is not completely negligible when $\Delta t \to \infty$. It corresponds to the fact that the autocovariance function has a slow decay at infinity. More precisely we say that the random process $Z_t^\varepsilon$ has the $H$-long-range correlation property if its autocovariance function satisfies:

$$\mathcal{C}^\varepsilon(t) \overset{|t|\to\infty}{\simeq} r_H \Big|\frac{t}{\varepsilon}\Big|^{2H-2}, \tag{1.2}$$

where $r_H > 0$ and $H \in (1/2, 1)$. We will refer to $H$ as the Hurst exponent. Here the correlation time $\varepsilon$ is the critical length scale beyond which the power law behavior (1.2) is valid. Note that the autocovariance function is not integrable since $2H - 2 \in (-1, 0)$, which means that a random process with the $H$-long-range correlation property is not mixing. As we describe in more detail below a common approach for modeling long-range dependence is via using fractional Brownian motion (fBm) processes Mandelbrot and Van Ness (1968). In Chronopoulou and Viens (2012) the authors consider the situation when the volatility is modeled as a discrete and continuous time function of a fractional Ornstein-Uhlenbeck (fOU) process whose shocks are independent from those of the underlying. Their focus is on a tree-based method for computing prices, estimation schemes for model parameters, and a particle filtering technique for the unobserved volatility given discrete observations. They consider some real data examples and find estimated values for the Hurst exponent which is larger than 1/2, in particular in a period after a market crash. In Chronopoulou and Viens (2012); Gulisashvili et al. (2015) small maturity asymptotic results are in particular presented for this model.

As we described above a central objective is to pose a model for stochastic volatility that is useful in the context of calibration and linkage of financial products with respect to implied volatility. Here, we consider just one piece of this challenge which involves characterization of the implied surface for long-range stochastic volatilities. An underlying premiss is that constructing parametric form for the implied volatility that derives from a stochastic volatility that is a stationary process helps in designing a scheme that is time consistent and robust, moreover, provides the appropriate degrees of freedom and a framework for linkage.

One criterion for judging different dynamic models for stochastic volatility is certainly to what extent they can fit observed implied surfaces. One classic challenge regarding fitting of implied surfaces is to capture a relatively strong moneyness dependence for short time to maturity without creating artificial behavior for long maturity. Moreover, to retain a relatively strong parametric dependence for long maturities despite averaging effects that set in regarding this regime: Bollerslev and Mikkelsen (1999); Bollerslev et al. (2013); Comte et al. (2012); Sundarsen et al. (2000). We remark that models involving jumps have been promoted as one approach to meet this challenge Carr and Wu (2003); Mijatovic and Tankov (2016). Recent works show that stochastic volatility models with long-range dependence also provide a promising framework for meeting this challenge. Approaches based on using fractional noises in the description of the stochastic volatility process were used in Comte and Renault (1998); Comte et al. (2012). This provides an approach for endowing the volatility process with high persistence in the long run (long memory with $H > 1/2$) in order to capture the steepness of long term volatility smiles without overincreasing the short run persistence. The model presented in Comte et al. (2012) was recently revisited in Guennoun et al. (2014) where short and long maturity asymptotics are analyzed using large deviations principles. As a further generalization relative to a fractional Brownian motion based model the case of multi fractional Brownian motion based models is considered in Corley et al. (2014). This allows for a non-stationary local regularity or a time dependent Hurst exponent and then the implied volatility depends on weighted averages of the local Hurst exponent.

Long-memory stochastic volatility models are indeed easy to pose, however, their analysis is quite challenging. This is largely due to the fact that the volatility process is then neither a Markov process nor a semimartingale. It is however important to notice that the price process is still a semimartingale and the problem formulation does not entail arbitrage Mendes et al. (2015) as has been argued for some models whose price process itself is driven by fractional processes Bjork and Hult (2005); Rogers (1997); Shiryaev (1998). To get explicit results for the implied volatility a number of asymptotic regimes have been considered. Chief among them has been the regime of short time to maturity. In Alòs and Vives (2007) the authors use Malliavin calculus to decompose option prices as the sum of the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility plus a term due to correlation and a term due to the volatility of the volatility. Their model is a fractional version of the Bates Bates (1996) model. They find that the implied volatility flattens in the long-range dependent case in the limit of small maturity. In Forde and Zhang (2015) the authors use large deviation principles to compute the short time to maturity asymptotic form of the implied volatility. They consider the correlated case with leverage and obtain results that are consistent with those in Alòs and Vives (2007). They consider a stochastic volatility model based on fBm and also more general ones where the volatility process is driven by fBms and which are analyzed using rough path theory. They also consider large time asymptotics for some fractional processes. In Fukasawa (2011) the author discusses the case with small volatility fluctuations and long-range dependence impact on the implied volatility as an application of the general theory he sets forth. He uses a non-stationary "planar" fBm as the volatility factor so that the leading implied volatility surface is identified conditioned on the present value of the implied volatility factor only, while below with a stationary model the surface depends on the path of the volatility factor until the present, reflecting the non-Markovian nature of fBm.

In terms of a computation of prices for general maturities, so far mainly numerical approximations have been available. However, recently in Alòs and Yang (2014); Garnier and Solna (2015) approaches have been presented that give price approximations for general maturities in the case when the volatility fluctuations are relatively small. Here, we present an extension of the analysis in Garnier and Solna (2015) that gives in fact novel explicit approximation formulas for general maturities and with a strongly fluctuating (fast mean reverting) volatility. Taken together the results of Garnier and Solna (2015) and this paper allow to construct a fractional two-time scale stochastic volatility model, generalizing the modeling in Fouque et al. (2003) and they provide a convenient tool to fit both the short- and long-maturity part of the implied surface, moreover, bridging these in the context of a consistent dynamical model for the volatility that allows of linkage of financial products.

In this paper we will only consider the analytic aspects of our model. The fitting with respect to specific data is beyond the scope of this paper and will be presented elsewhere. We remark regarding the implied surface derived here that it is linear in log moneyness. This may seem somewhat restrictive from the point of view of fitting since in many cases a relatively strong skew in log moneyness may be observed in certain markets. This has particularly been case for the stock market, but relatively less so in other markets like fixed income markets. However, if one considers higher order approximations, then this generates also skew effects. A number of other modeling issues like transaction costs, bid-ask spreads and liquidity for instance may also affect the skew shape. Let us also remark that for simplicity we do not incorporate a non-zero interest rate here, nor do we incorporate market price of risk aspects. Let us also remark that the fractional model we set forth here indeed incorporate additional empirical "stylized facts", like heavy tails of returns, volatility clustering and mean reversion, moreover, long memory or volatility persistence. Additionally, we here incorporate the leverage effect. A term coined by Black Black et al. (1976) referring to stock price movements which are correlated (typically negatively) with volatility, as falling stock prices may imply more uncertainty and hence volatility.

**Rapid-Clustering, Long-Memory and the Implied Surface.** We summarize next the main result of the paper from the point of view of calibration. That is, the form of the implied volatility in the context of a stochastic volatily modeled by a fast process with long-range correlation properties. We summarize first some aspects of the modeling.

We consider a continuous time stochastic volatility model that is a smooth function of a Gaussian long-range process. Explicitly, we model the fractional stochastic volatility (fSV) as a smooth function of a fractional Ornstein-Uhlenbeck (fOU) process. The fOU process is a classic model for a stationary process with a fractional long-range correlation structure. This process can be expressed in terms of an integral of a fractional Brownian motion (fBm) process. The distribution of a fBm process is characterized in terms of the Hurst exponent $H \in (0,1)$. The fBm process is locally Hölder continuous of exponent $H'$ for all $H' < H$ and this property is inherited by the fOU process. The fBm process, $W_t^H$, is also self-similar in that

$$\left\{W_{\alpha t}^H, t \in \mathbb{R}\right\} \overset{dist.}{=} \left\{\alpha^H W_t^H, t \in \mathbb{R}\right\} \text{ for all } \alpha > 0. \tag{1.3}$$

The self-similarity property is inherited approximately by the fOU process on scales smaller than the mean reversion time of the fOU process that we will denote by $\varepsilon$ below. In this sense we may refer to the fOU process as a multiscale process on relatively short scales. The case $H \in (1/2, 1)$ that we address in this paper gives a fOU process that is a long-range process. This regime corresponds to a persistent process where consecutive increments of the fBm are positively correlated. The relatively stronger positive correlation for the consecutive increments of the associated fBm process with increasing $H$ values gives a relatively smoother process whose correlations decay relatively slowly. For more details regarding the fBm and fOU processes we refer repectively to Biagini et al. (2008); Coutin (2007); Doukhan et al. (2003); Mandelbrot and Van Ness (1968) and Cheridito et al. (2003); Kaarakka and Salminen (2011).

First note that the volatility driving process is the $\varepsilon$-scaled fractional Ornstein-Uhlenbeck process (fOU) defined by:

$$Z_t^\varepsilon = \varepsilon^{-H} \int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H. \tag{1.4}$$

It is a zero-mean, stationary Gaussian process, that exhibits long-range correlations for the Hurst exponent $H \in (1/2, 1)$. It is important to note that this is a process whose "natural time scale" is $\varepsilon$, this in the sense that the mean reversion time or time before the process reaches it equilibrium distribution scales like $\varepsilon$. It is also important next to note that the decay of the correlations (on the $\varepsilon$ time scale) is polynomial rather than exponential as in the standard Ornstein-Uhlenbeck process. Explicitly, the correlation of the process between times $t$ and $t + \Delta t$ decays as $(\Delta t/\varepsilon)^{2H-2}$. The variance of the process is independent of $\varepsilon$.

In this paper we consider a stochastic volatility model that is a smooth function of the rapidly varying fractional Ornstein-Uhlenbeck process with Hurst coefficient $H \in (1/2, 1)$, it is given by

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{1.5}$$

where $F$ is a smooth, positive, one-to-one, bounded function with bounded derivatives and with an additional technical condition that is given in Eq. (3.5). The process $\sigma_t^\varepsilon$ inherits the long-range correlation properties of the fOU $Z_t^\varepsilon$.

The main result we then set forth in Section 5 is the implied volatility of the European Call Option, it is given by

$$I_t = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T (\sigma_s^\varepsilon)^2 ds \big| \mathcal{F}_t\Big]^{1/2} + \overline{\sigma} a_F \left[ \left(\frac{\tau}{\bar{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\bar{\tau}}\right)^{H-3/2} \log\left(\frac{K}{X_t}\right)\right]. \tag{1.6}$$

Here

$$a_F = \varepsilon^{1-H} \frac{\widetilde{\sigma}\sigma_{\rm ou}\rho \left\langle FF'\right\rangle \bar{\tau}^H}{2^{3/2}\overline{\sigma}\Gamma(H+\frac{3}{2})}, \tag{1.7}$$

with the characteristic diffusion time being

$$\bar{\tau} = \frac{2}{\overline{\sigma}^2}, \tag{1.8}$$

moreover, $\tau = T - t$ is time to maturity and $\rho$ the correlation in between the Brownian motion driving the fBM which in turn drives the fOU and the Brownian motion driving the underlying. Furthermore, we have with $\sigma_{\rm ou}^2 = 1/(2\sin(\pi H))$:

$$\begin{aligned}
\overline{\sigma}^2 = \left\langle F^2\right\rangle &= \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^2 p(z) dz, \\
\widetilde{\sigma} = \left\langle F\right\rangle &= \int_{\mathbb{R}} F(\sigma_{\rm ou} z) p(z) dz, \\
\left\langle FF'\right\rangle &= \int_{\mathbb{R}} F(\sigma_{\rm ou} z) F'(\sigma_{\rm ou} z) p(z) dz,
\end{aligned}$$

with $p(z)$ the pdf of the standard normal distribution. That is, we form moments of the volatility function averaged with respect to the invariant distribution of the fOU process $Z_t^\varepsilon$.

The first term in Eq. (1.6) is indeed the expected effective volatility until maturity conditioned on the present. The second term is a skewness term that is non-zero only when the volatility process and the underlying are correlated so that $\rho$ is non-zero. Note that the exponent of the fractional term structure depends on the Hurst exponent which determines the smoothness and the decorrelation rate of the volatility driving process $Z_t^\varepsilon$. The smoother the process the relatively larger the implied volatility for large times to maturity.

In the fast case presented here with large and fast volatility fluctuations the implied volatility explodes in the regime of short time to maturity since the primary small parameter is $\varepsilon$, the mean reversion time of the volatility driving process. Actually, short time to maturity means time to maturity smaller than the diffusion time (1.8) but larger than the mean reversion time. Therefore short time to maturity involves large volatility fluctuations resulting in a moneyness correction that explodes and dominates the pure maturity term. In the context of short or long times to maturity the conditional expected effective volatility gives a relatively small contribution and we have for short times to maturity

$$I_t \sim a_F \Big[ \Big(\frac{\tau}{\overline{\tau}}\Big)^{H-3/2} \log(K/X_t) \Big], \tag{1.9}$$

respectively in the regime of long times to maturity

$$I_t \sim a_F \Big(\frac{\tau}{\overline{\tau}}\Big)^{H-1/2}. \tag{1.10}$$

We remark here that the fractional scaling in the skewness term in Eq. (1.6) is exactly the fractional scaling that corresponds to the case of relatively large time to maturity and small volatility fluctuations given in Garnier and Solna (2015). That is, with large times to maturity there we have a situation reminiscent of the one we have here with rapid volatility fluctuations, however, here the volatility fluctuations are large as compared to the small volatility fluctuations in Garnier and Solna (2015).

We remark also that the case with a mixing volatility, and hence integrable correlation function for the volatility fluctuations, would correspond to $H \searrow 1/2$. Note, however, that our derivation is valid only for $H \in (1/2, 1)$. If we consider the formula (4.10) for $\sigma_\phi$ that determines the variance of the first term in Eq. (1.6), we can observe that it vanishes when $H \searrow 1/2$, which shows that the first term in Eq. (1.6) becomes to leading order deterministic. Indeed in the limit case $H \searrow 1/2$ we get a result as in (Fouque et al. , 2000, Section 5.2.5) that deals with the mixing case. Explicitly, consider the mixing case when the volatility driving process is an ordinary Ornstein-Uhlenbeck process, moreover, the interest rate and market price of volatility risk are zero as we consider here. Then (Fouque et al. , 2000, Eq. (5.55)) gives the implied volatility in terms of a coefficient $V_3$ defined in (Fouque et al. , 2000, Section 5.2.5):

$$I_t = \overline{\sigma} - V_3 \left[ \frac{1}{2\overline{\sigma}} + \frac{1}{\overline{\sigma}^3 \tau} \log\left( \frac{K}{X_t} \right) \right], \tag{1.11}$$

that has the same form as the formal limit of (1.6) as $H \searrow 1/2$. We remark that the averaging expression giving the coefficient $V_3$ does not correspond to the interpretation we arrive at here by the formal limit $H \in (1/2, 1)$. This is because the singular perturbation situation we consider in fact is "singular" at $H = 1/2$ and ordering of important terms becomes different. Note that it is important from the calibration point of view that we have continuity of the implied volatility parameterization and its form at $H = 1/2$, providing robustness to the asymptotic framework. Note moreover that in the mixing case the implied volatility is deterministic to leading correction order, while the non-integrability of the volatility covariance function makes it a stochastic process in the general long range case.

**Outline.** The outline of the paper is as follows. In Section 2 we describe the fractional Ornstein-Uhlenbeck process and derive some fundamental a priori bounds. In Section 3 we describe the stochastic volatility model. In Section 4 we derive the expression for the price in the fast mean reverting fractional case. The derivation is based on the martingale method. That is, we make an ansatz for the price as a process that has the correct payoff and to leading order is a martingale. Then indeed this process will be the leading order expression for the price with an error that is of the order of the non-martingale part. This approach involves introducing correctors so that the non-martingale part is pushed to a relatively small term and we give the resulting decomposition in Section 4. Based on the expression for the price we derive the associated implied volatility in Section 5 and present finally some concluding remarks in Section 7. We give a convenient Hermite decomposition of the volatility in Appendix A. A number of the technical lemmas are proved in Appendix B.

**2. The Rapid Fractional Ornstein-Uhlenbeck Process.** We use a rapid fractional Ornstein-Uhlenbeck (fOU) process as the volatility factor and describe here how this process can be represented in terms of a fractional Brownian motion. Since fractional Brownian motion can be expressed in terms of ordinary Brownian motion we also arrive at an expression for the rapid fOU process as a filtered version of Brownian motion.

A fractional Brownian motion (fBM) is a zero-mean Gaussian process $(W_t^H)_{t\in\mathbb{R}}$ with the covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{\sigma_H^2}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \tag{2.1}$$

where $\sigma_H$ is a positive constant.

We use the following moving-average stochastic integral representation of the fBM Mandelbrot and Van Ness (1968):

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dW_s, \tag{2.2}$$

where $(W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion over $\mathbb{R}$. Then indeed $(W_t^H)_{t\in\mathbb{R}}$ is a zero-mean Gaussian process with the covariance (2.1) and where we now have

$$\begin{aligned}\sigma_H^2 &= \frac{1}{\Gamma(H+\frac{1}{2})^2}\Big[\int_0^\infty \big((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\big)^2 ds + \frac{1}{2H}\Big] \\ &= \frac{1}{\Gamma(2H+1)\sin(\pi H)}.\end{aligned} \tag{2.3}$$

We introduce the $\varepsilon$-scaled fractional Ornstein-Uhlenbeck process (fOU) as

$$Z_t^\varepsilon = \varepsilon^{-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} dW_s^H = \varepsilon^{-H} W_t^H - \varepsilon^{-1-H}\int_{-\infty}^t e^{-\frac{t-s}{\varepsilon}} W_s^H ds. \tag{2.4}$$

Thus, the fractional OU process is in fact a fractional Brownian motion with a restoring force towards zero. It is a zero-mean, stationary Gaussian process, with variance

$$\mathbb{E}[(Z_t^\varepsilon)^2] = \sigma_{\rm ou}^2, \text{ with } \sigma_{\rm ou}^2 = \frac{1}{2}\Gamma(2H+1)\sigma_H^2, \tag{2.5}$$

that is independent of $\varepsilon$, and covariance:

$$\mathbb{E}[Z_t^\varepsilon Z_{t+s}^\varepsilon] = \sigma_{\rm ou}^2 \mathcal{C}_Z\Big(\frac{s}{\varepsilon}\Big),$$

that is a function of $s/\varepsilon$ only, with

$$\mathcal{C}_Z(s) = \frac{1}{\Gamma(2H+1)} \Big[ \frac{1}{2} \int_{\mathbb{R}} e^{-|v|} |s+v|^{2H} dv - |s|^{2H} \Big]$$
$$= \frac{2\sin(\pi H)}{\pi} \int_0^\infty \cos(sx) \frac{x^{1-2H}}{1+x^2} dx.$$

This shows that $\varepsilon$ is the natural scale of variation of the fOU $Z_t^\varepsilon$. Note that the random process $Z_t^\varepsilon$ is not a martingale, neither a Markov process. For $H \in (1/2, 1)$ it possesses long-range correlation properties:

$$\mathcal{C}_Z(s) = \frac{1}{\Gamma(2H-1)} s^{2H-2} + o\big(s^{2H-2}\big), \qquad s \gg 1. \tag{2.6}$$

This shows that the correlation function is non-integrable at infinity. In this paper we shall focus on the case $H \in (1/2, 1)$.

We remark that if $H = 1/2$, then the standard OU process (synthesized with a standard Brownian motion) is a stationary Gaussian Markov process with an exponential correlation and hence a mixing process. It is possible to simulate paths of the fractional OU process using the Cholesky method (see Figure 2.1) or other well-known methods Bardet et al. (2003); Omre et al. (1993).

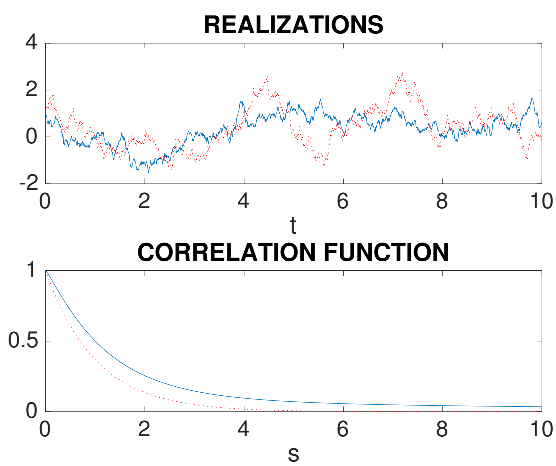


FIG. 2.1. *A realization, $Z_t^\varepsilon$, $t \in (0, 10)$, of the fractional OU process with Hurst index $H = 0.6$ and correlation time $\varepsilon = 1$ shown by (blue) solid line top plot. The dotted (red) line shows a realization of the standard OU process with $H = 1/2$ and $\varepsilon = 1$. The trajectories are more regular when $H$ is larger. The bottom plot shows the corresponding correlation functions, $\mathcal{C}_Z(s)$, and the "heavy" tail of the (blue) solid line of the case $H = 0.6$ gives the long range property.*

Using Eqs. (2.2) and (2.4) we arrive at the moving-average integral representation of the scaled fOU as:

$$Z_t^\varepsilon = \sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^\varepsilon(t-s) dW_s, \tag{2.7}$$

where

$$\mathcal{K}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \mathcal{K}\Big(\frac{t}{\varepsilon}\Big), \qquad \mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \Big[ t^{H-\frac{1}{2}} - \int_0^t (t-s)^{H-\frac{1}{2}} e^{-s} ds \Big]. \tag{2.8}$$

The main properties of the kernel $\mathcal{K}$ in our context are the following ones (valid for any $H \in (1/2, 1)$):
- $\mathcal{K}$ is nonnegative-valued, $\mathcal{K} \in L^2(0,\infty)$ with $\int_0^\infty \mathcal{K}^2(u)du = 1$, but $\mathcal{K} \notin L^1(0,\infty)$.
- for small times $t \ll 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\Big(t^{H-\frac{1}{2}} + O\big(t^{H+\frac{1}{2}}\big)\Big), \tag{2.9}$$

- for large times $t \gg 1$:

$$\mathcal{K}(t) = \frac{1}{\Gamma(H-\frac{1}{2})}\Big(t^{H-\frac{3}{2}} + O\big(t^{H-\frac{5}{2}}\big)\Big), \tag{2.10}$$

and in particular $\mathcal{K}(t) - \frac{1}{\Gamma(H-\frac{1}{2})} t^{H-\frac{3}{2}}$ belongs to $L^1(0,\infty)$.

**3. The Stochastic Volatility Model.** The price of the risky asset follows the stochastic differential equation:

$$dX_t = \sigma_t^\varepsilon X_t dW_t^*, \tag{3.1}$$

where the stochastic volatility is

$$\sigma_t^\varepsilon = F(Z_t^\varepsilon), \tag{3.2}$$

and with $Z_t^\varepsilon$ being the scaled fOU introduced in the previous section which is adapted to the Brownian motion $W_t$. Moreover, $W_t^*$ is a Brownian motion that is correlated to the stochastic volatility through

$$W_t^* = \rho W_t + \sqrt{1-\rho^2} B_t, \tag{3.3}$$

where the Brownian motion $B_t$ is independent of $W_t$.

The function $F$ is assumed to be one-to-one, positive-valued, smooth, bounded and with bounded derivatives. Accordingly, the filtration $\mathcal{F}_t$ generated by $(B_t, W_t)$ is also the one generated by $X_t$. Indeed, it is equivalent to the one generated by $(W_t^*, W_t)$, or $(W_t^*, Z_t^\varepsilon)$. Since $F$ is one-to-one, it is equivalent to the one generated by $(W_t^*, \sigma_t)$. Since $F$ is positive-valued, it is equivalent to the one generated by $(W_t^*, (\sigma_t^\varepsilon)^2)$, or $X_t$.

We denote the Hermite coefficients of the volatility function $F$ with respect to the invariant distribution of the fOU process by $C_k$ :

$$C_k = \int_{\mathbb{R}} H_k(z) F^2(\sigma_{\rm ou} z) p(z) dz, \qquad H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2}, \tag{3.4}$$

with $p(z) = \exp(-z^2/2)/\sqrt{2\pi}$. We use these in Appendix A to derive some technical lemmas and discuss them in more detail there. Here we note that for a technical reason we also require that $F$ satisfies the following condition: there exists some $\alpha > 2$ such that

$$\sum_{k=0}^\infty \frac{\alpha^k C_k^2}{k!} < \infty. \tag{3.5}$$

As we have discussed above, the volatility driving process $Z_t^\varepsilon$ possess long range correlation properties. As we now show the volatility process $\sigma_t^\varepsilon$ itself indeed inherits this property.

LEMMA 3.1. *We denote, for $j = 1, 2$:*

$$\left\langle F^j \right\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^j p(z) dz, \qquad \left\langle F'^j \right\rangle = \int_{\mathbb{R}} F'(\sigma_{\rm ou} z)^j p(z) dz, \tag{3.6}$$

*where $p(z)$ is the pdf of the standard normal distribution.*

1. *The process $\sigma_t^\varepsilon$ is a stationary random process with mean $\mathbb{E}[\sigma_t^\varepsilon] = \langle F \rangle$ and variance ${\rm Var}(\sigma_t^\varepsilon) = \langle F^2 \rangle - \langle F \rangle^2$, independently of $\varepsilon$.*
2. *The covariance function of the process $\sigma_t^\varepsilon$ is of the form*

$$\mathrm{Cov}\big(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon\big) = \big( \langle F^2 \rangle - \langle F \rangle^2 \big) \mathcal{C}_\sigma\Big(\frac{s}{\varepsilon}\Big), \tag{3.7}$$

*where the correlation function $\mathcal{C}_\sigma$ satisfies $\mathcal{C}_\sigma(0) = 1$ and*

$$\mathcal{C}_\sigma(s) = \frac{1}{\Gamma(2H-1)} \frac{\sigma_{\rm ou}^2 \langle F' \rangle^2}{\langle F^2 \rangle - \langle F \rangle^2} s^{2H-2} + o\big(s^{2H-2}\big), \qquad \text{for } s \gg 1. \tag{3.8}$$

Consequently, the process $\sigma_t^\varepsilon$ possesses long-range correlation properties (i.e. its correlation function is not integrable at infinity).

*Proof.* The fact that $\sigma_t^\varepsilon$ is a stationary random process with mean $\langle F \rangle$ is straightforward in view of the definition (3.2) of $\sigma_t^\varepsilon$.

For any $t, s$, the vector $\sigma_{\rm ou}^{-1}(Z_t^\varepsilon, Z_{t+s}^\varepsilon)$ is a Gaussian random vector with mean $(0, 0)$ and $2 \times 2$ covariance matrix:

$$\mathbf{C}^\varepsilon = \begin{pmatrix} 1 & \mathcal{C}_Z(s/\varepsilon) \\ \mathcal{C}_Z(s/\varepsilon) & 1 \end{pmatrix}.$$

Therefore, denoting $F_c(z) = F(\sigma_{\rm ou} z) - \langle F \rangle$, the covariance function of the process $\sigma_t^\varepsilon$ is

$$\begin{aligned}
\mathrm{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon) &= \mathbb{E}\big[F_c(\sigma_{\rm ou}^{-1} Z_t^\varepsilon) F_c(\sigma_{\rm ou}^{-1} Z_{t+s}^\varepsilon)\big] \\
&= \frac{1}{2\pi\sqrt{\det \mathbf{C}^\varepsilon}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big( - \frac{(z_1, z_2) {\mathbf{C}^\varepsilon}^{-1} (z_1, z_2)^T}{2} \Big) dz_1 dz_2 \\
&= \Psi\Big( \mathcal{C}_Z\Big(\frac{s}{\varepsilon}\Big)\Big),
\end{aligned}$$

with

$$\Psi(C) = \frac{1}{2\pi\sqrt{1 - C^2}} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big( - \frac{z_1^2 + z_2^2 - 2C z_1 z_2}{2(1 - C^2)} \Big) dz_1 dz_2 \,.$$

This shows that $\mathrm{Cov}(\sigma_t^\varepsilon, \sigma_{t+s}^\varepsilon)$ is a function of $s/\varepsilon$ only. Moreover, the function $\Psi$ can be expanded in powers of $C$ for small $C$:

$$\begin{aligned}
\Psi(C) = &\frac{1}{2\pi} \iint_{\mathbb{R}^2} F_c(z_1) F_c(z_2) \exp\Big( - \frac{z_1^2 + z_2^2}{2} \Big) dz_1 dz_2 \\
&+ C \frac{1}{2\pi} \iint_{\mathbb{R}^2} z_1 z_2 F_c(z_1) F_c(z_2) \exp\Big( - \frac{z_1^2 + z_2^2}{2} \Big) dz_1 dz_2 + O(C^2), \qquad C \ll 1,
\end{aligned}$$

which gives with (2.6) the form (3.8) of the correlation function for $\sigma_t^\varepsilon$. ∎

**4. The Option Price.** We aim at computing the option price defined as the martingale

$$M_t = \mathbb{E}\big[h(X_T)|\mathcal{F}_t\big], \tag{4.1}$$

where $h$ is a smooth function. In fact weaker assumptions are possible for $h$, as we only need to control the function $Q_t^{(0)}(x)$ defined below rather than $h$.

We introduce the operator

$$\mathcal{L}_{\rm BS}(\sigma) = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_x^2, \tag{4.2}$$

that is, the standard Black-Scholes operator at zero interest rate and (constant) volatility $\sigma$.

We next exploit the fact that the price process is a martingale to obtain an approximation, via constructing an explicit function $Q_t^\varepsilon(x)$ so that $Q_t^\varepsilon(x) = h(x)$ and so that $Q_t^\varepsilon(X_t)$ is a martingale to first-order corrected terms. Then, indeed $Q_t^\varepsilon(X_t)$ gives the approximation for $M_t$ to this order.

The following proposition gives the first-order correction to the expression for the martingale $M_t$ in the regime of $\varepsilon$ small.

PROPOSITION 4.1. *When $\varepsilon$ is small, we have*

$$M_t = Q_t^\varepsilon(X_t) + o(\varepsilon^{1-H}), \tag{4.3}$$

*where*

$$Q_t^\varepsilon(x) = Q_t^{(0)}(x) + \big(x^2\partial_x^2 Q_t^{(0)}(x)\big)\phi_t^\varepsilon + \varepsilon^{1-H}\widetilde{\sigma}\rho Q_t^{(1)}(x), \tag{4.4}$$

*$Q_t^{(0)}(x)$ is deterministic and given by the Black-Scholes formula with constant volatility $\overline{\sigma}$,*

$$\mathcal{L}_{\rm BS}(\overline{\sigma})Q_t^{(0)}(x) = 0, \qquad Q_T^{(0)}(x) = h(x), \tag{4.5}$$

*with*

$$\overline{\sigma}^2 = \big\langle F^2\big\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)^2 p(z)dz, \qquad \widetilde{\sigma} = \langle F\rangle = \int_{\mathbb{R}} F(\sigma_{\rm ou} z)p(z)dz, \tag{4.6}$$

*$p(z)$ the pdf of the standard normal distribution, $\phi_t^\varepsilon$ is the random component*

$$\phi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_t^T \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds\big|\mathcal{F}_t\Big], \tag{4.7}$$

*and $Q_t^{(1)}(x)$ is the deterministic correction*

$$Q_t^{(1)}(x) = x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)D_t, \tag{4.8}$$

*with $D_t$ defined by*

$$D_t = \overline{D}(T-t)^{H+\frac{1}{2}}, \qquad \overline{D} = \frac{\sigma_{\rm ou}\langle FF'\rangle}{\Gamma(H+\frac{3}{2})} = \frac{\sigma_{\rm ou}}{\Gamma(H+\frac{3}{2})}\int_{\mathbb{R}} FF'(\sigma_{\rm ou} z)p(z)dz. \tag{4.9}$$

As shown in Lemma B.3 (first item), as $\varepsilon \to 0$, the zero-mean random variable $\varepsilon^{H-1}\phi_t^\varepsilon$ has a variance that converges to $\sigma_\phi^2(T-t)^{2H}$, with

$$\sigma_\phi^2 = \sigma_{\rm ou}^2 \left\langle FF'\right\rangle^2 \Big( \frac{1}{\Gamma(2H+1)\sin(\pi H)} - \frac{1}{2H\Gamma(H+\frac{1}{2})^2}\Big), \tag{4.10}$$

moreover, it converges in distribution to a Gaussian random variable with mean zero and variance $\sigma_\phi^2(T-t)^{2H}$. This shows that the two corrective terms in (4.4) are of the same order $\varepsilon^{1-H}$, but the first one is random, zero-mean and approximately Gaussian distributed, while the second one is deterministic.

*Proof.* For any smooth function $q_t(x)$, we have by Itô's formula

$$\begin{aligned} dq_t(X_t) &= \partial_t q_t(X_t)dt + \big(x\partial_x q_t\big)(X_t)\sigma_t^\varepsilon dW_t^* + \frac{1}{2}\big(x^2\partial_x^2 q_t\big)(X_t)(\sigma_t^\varepsilon)^2 dt \\ &= \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon) q_t(X_t)dt + \big(x\partial_x q_t\big)(X_t)\sigma_t^\varepsilon dW_t^*, \end{aligned}$$

the last term being a martingale. Therefore, by (4.5), we have

$$dQ_t^{(0)}(X_t) = \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt + dN_t^{(0)}, \tag{4.11}$$

with $N_t^{(0)}$ a martingale:

$$dN_t^{(0)} = \big(x\partial_x Q_t^{(0)}\big)(X_t)\sigma_t^\varepsilon dW_t^*.$$

Let $\phi_t^\varepsilon$ be defined by (4.7). We have

$$\phi_t^\varepsilon = \psi_t^\varepsilon - \frac{1}{2}\int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds,$$

where the martingale $\psi_t^\varepsilon$ is defined by

$$\psi_t^\varepsilon = \mathbb{E}\Big[\frac{1}{2}\int_0^T \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big)ds \big| \mathcal{F}_t\Big]. \tag{4.12}$$

We can write

$$\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt = \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon - \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t^\varepsilon.$$

By Itô's formula:

$$\begin{aligned} d\big[\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] &= \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\phi_t^\varepsilon + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* \\ &\quad + \mathcal{L}_{\rm BS}(\sigma_t^\varepsilon)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &\quad + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle \phi^\varepsilon, W^*\right\rangle_t. \end{aligned}$$

Since $\mathcal{L}_{\rm BS}(\sigma_t^\varepsilon) = \mathcal{L}_{\rm BS}(\overline{\sigma}) + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)$ and $\mathcal{L}_{\rm BS}(\overline{\sigma})\big(x^2\partial_x^2\big)Q_t^{(0)}(x) = 0$, this gives

$$\begin{aligned} d\big[\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] &= -\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\ &\quad + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\ &\quad + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\left\langle \phi^\varepsilon, W^*\right\rangle_t \\ &\quad + \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon. \end{aligned}$$

We have $\langle \phi^\varepsilon, W^* \rangle_t = \langle \psi^\varepsilon, W^* \rangle_t = \rho \langle \psi^\varepsilon, W \rangle_t$ and therefore

$$
\begin{aligned}
d\big[(\phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = & -\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)dt \\
& +\frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\
& +\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\langle \psi^\varepsilon, W\rangle_t \\
& +dN_t^{(1)},
\end{aligned}
$$

where $N_t^{(1)}$ is a martingale,

$$
dN_t^{(1)} = \big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\phi_t^\varepsilon dW_t^* + \big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)d\psi_t^\varepsilon.
$$

Therefore:

$$
\begin{aligned}
d\big[Q_t^{(0)}(X_t) + \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t)\big] = & \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt \\
& +\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon d\langle \psi^\varepsilon, W\rangle_t \\
& +dN_t^{(0)} + dN_t^{(1)}.
\end{aligned}
\tag{4.13}
$$

The deterministic function $Q_t^{(1)}$ defined by (4.8) satisfies

$$
\mathcal{L}_{\mathrm{BS}}(\overline{\sigma})Q_t^{(1)}(x) = -\big(x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)\big)\theta_t, \qquad Q_T^{(1)}(x) = 0,
$$

where $\theta_t = -dD_t/dt$ is such that

$$
d\langle \psi^\varepsilon, W\rangle_t = \big(\varepsilon^{1-H}\theta_t + \widetilde{\theta}_t^\varepsilon\big)dt,
$$

as shown in Lemmas B.1-B.2 with $\widetilde{\theta}_t^\varepsilon$ characterized in Eq. (B.9). Applying Itô's formula

$$
\begin{aligned}
dQ_t^{(1)}(X_t) &= \mathcal{L}_{\mathrm{BS}}(\sigma_t^\varepsilon)Q_t^{(1)}(X_t)dt + \big(x\partial_x Q_t^{(1)}\big)(X_t)\sigma_t^\varepsilon dW_t^* \\
&= \mathcal{L}_{\mathrm{BS}}(\overline{\sigma})Q_t^{(1)}(X_t)dt + \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt \\
&\quad +\big(x\partial_x Q_t^{(1)}\big)(X_t)\sigma_t^\varepsilon dW_t^* \\
&= \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt - \big(x\partial_x\big(x^2\partial_x^2 Q_t^{(0)}(x)\big)\big)\theta_t dt + dN_t^{(2)},
\end{aligned}
$$

where $N_t^{(2)}$ is a martingale,

$$
dN_t^{(2)} = \big(x\partial_x Q_t^{(1)}\big)(X_t)\sigma_t^\varepsilon dW_t^*.
$$

Therefore

$$
\begin{aligned}
& d\big[Q_t^{(0)}(X_t) + \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(X_t) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(X_t)\big] \\
& = \frac{1}{2}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\phi_t^\varepsilon dt + \frac{\varepsilon^{1-H}}{2}\rho\widetilde{\sigma}\big((\sigma_t^\varepsilon)^2 - \overline{\sigma}^2\big)\big(x^2\partial_x^2\big)Q_t^{(1)}(X_t)dt \\
& \quad +\varepsilon^{1-H}\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)(\sigma_t^\varepsilon - \widetilde{\sigma})\theta_t dt + \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q_t^{(0)}(X_t)\sigma_t^\varepsilon\widetilde{\theta}_t^\varepsilon dt \\
& \quad +dN_t^{(0)} + dN_t^{(1)} + \varepsilon^{1-H}\rho\widetilde{\sigma}dN_t^{(2)}.
\end{aligned}
\tag{4.14}
$$

We next show that the first four terms of the right-hand side are of small order $\varepsilon^{1-H}$. We introduce for any $t \in [0,T]$:

$$R^{(1)}_{t,T} = \int_t^T \frac{1}{2}\big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q^{(0)}_s(X_s)\big((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\big)\phi^\varepsilon_s ds, \tag{4.15}$$

$$R^{(2)}_{t,T} = \int_t^T \frac{\varepsilon^{1-H}}{2}\rho\widetilde{\sigma}\big(x^2\partial_x^2\big)Q^{(1)}_s(X_s)\big((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\big)ds, \tag{4.16}$$

$$R^{(3)}_{t,T} = \int_t^T \varepsilon^{1-H}\rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q^{(0)}_s(X_s)\theta_s(\sigma^\varepsilon_s - \widetilde{\sigma})ds, \tag{4.17}$$

$$R^{(4)}_{t,T} = \int_t^T \rho\big(x\partial_x\big(x^2\partial_x^2\big)\big)Q^{(0)}_s(X_s)\sigma^\varepsilon_s\widetilde{\theta}^\varepsilon_s ds. \tag{4.18}$$

We will show that, for $j = 1,2,3,4$,

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(j)}_{t,T})^2\big]^{1/2} = 0. \tag{4.19}$$

*Step 1: Proof of (4.19) for $j = 1$.*
We denote

$$Y^{(1)}_s = \big(x^2\partial_x^2\big(x^2\partial_x^2\big)\big)Q^{(0)}_s(X_s)$$

and

$$\gamma^\varepsilon_t = \frac{1}{2}\int_0^t \big((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\big)\phi^\varepsilon_s ds, \tag{4.20}$$

so that

$$R^{(1)}_{t,T} = \int_t^T Y^{(1)}_s \frac{d\gamma^\varepsilon_s}{ds}ds.$$

Note that $Y^{(1)}_s$ is a bounded semimartingale with bounded quadratic variations, so that its mean square increments $\mathbb{E}[(Y^{(1)}_s - Y^{(1)}_{s'})^2]$ are uniformly bounded by $K|s-s'|$. Let $N$ be a positive integer. We denote $t_k = t + (T-t)k/N$. We have

$$R^{(1)}_{t,T} = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} Y^{(1)}_s\frac{d\gamma^\varepsilon_s}{ds}ds = R^{(1,a)}_{t,T} + R^{(1,b)}_{t,T},$$

$$R^{(1,a)}_{t,T} = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} Y^{(1)}_{t_k}\frac{d\gamma^\varepsilon_s}{ds}ds = \sum_{k=0}^{N-1} Y^{(1)}_{t_k}\big(\gamma^\varepsilon_{t_{k+1}} - \gamma^\varepsilon_{t_k}\big),$$

$$R^{(1,b)}_{t,T} = \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \big(Y^{(1)}_s - Y^{(1)}_{t_k}\big)\frac{d\gamma^\varepsilon_s}{ds}ds.$$

Note that we have by Minkowski's inequality:

$$\mathbb{E}\big[(R^{(1,a)}_{t,T})^2\big]^{1/2} \le 2\sum_{k=0}^{N}\|Y^{(1)}\|_\infty\mathbb{E}[(\gamma^\varepsilon_{t_k})^2]^{1/2} \le 2(N+1)\|Y^{(1)}\|_\infty\sup_{s\in[0,T]}\mathbb{E}[(\gamma^\varepsilon_s)^2]^{1/2},$$

so that, by Lemma B.4, for any fixed $N$:

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(1,a)}_{t,T})^2\big]^{1/2}=0.$$

On the other hand

$$\begin{aligned}\mathbb{E}\big[(R^{(1,b)}_{t,T})^2\big]^{1/2}&\le \|F\|_\infty^2\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\mathbb{E}[\big(Y^{(1)}_s-Y^{(1)}_{t_k}\big)^4]^{1/4}\mathbb{E}[(\phi^\varepsilon_s)^4]^{1/4}ds\\&\le K\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(s-t_k)^{1/2}ds\sup_{s\in[0,T]}\mathbb{E}[(\phi^\varepsilon_s)^4]^{1/4}\\&\le \frac{K'}{\sqrt{N}}\sup_{s\in[0,T]}\mathbb{E}[(\phi^\varepsilon_s)^4]^{1/4}.\end{aligned}$$

Therefore, by Lemma B.3 (fourth item), we get

$$\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(1)}_{t,T})^2\big]^{1/2}\le\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(1,b)}_{t,T})^2\big]^{1/2}\le\frac{K'}{\sqrt{N}}.$$

Since this is true for any $N$, we get the desired result.

*Step 2: Proof of (4.19) for $j=2$.*
We denote

$$Y^{(2)}_s=\rho\widetilde{\sigma}\big(x^2\partial_x^2\big)Q^{(1)}_s(X_s)$$

and

$$\kappa^\varepsilon_t=\frac{\varepsilon^{1-H}}{2}\int_0^t\big((\sigma^\varepsilon_s)^2-\overline{\sigma}^2\big)ds, \tag{4.21}$$

so that

$$R^{(2)}_{t,T}=\int_t^T Y^{(2)}_s\frac{d\kappa^\varepsilon_s}{ds}ds.$$

Note that $Y^{(2)}_s$ is a bounded semimartingale with bounded quadratic variations. Let $N$ be a positive integer. We denote as above $t_k=t+(T-t)k/N$. We then have

$$\begin{aligned}R^{(2)}_{t,T}&=\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}Y^{(2)}_s\frac{d\kappa^\varepsilon_s}{ds}ds=R^{(2,a)}_{t,T}+R^{(2,b)}_{t,T},\\R^{(2,a)}_{t,T}&=\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}Y^{(2)}_{t_k}\frac{d\kappa^\varepsilon_s}{ds}ds=\sum_{k=0}^{N-1}Y^{(2)}_{t_k}\big(\kappa^\varepsilon_{t_{k+1}}-\kappa^\varepsilon_{t_k}\big),\\R^{(2,b)}_{t,T}&=\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\big(Y^{(2)}_s-Y^{(2)}_{t_k}\big)\frac{d\kappa^\varepsilon_s}{ds}ds.\end{aligned}$$

Then, on the one hand

$$\mathbb{E}\big[(R^{(2,a)}_{t,T})^2\big]^{1/2}\le 2\sum_{k=0}^{N}\|Y^{(2)}\|_\infty\mathbb{E}[(\kappa^\varepsilon_{t_k})^2]^{1/2}\le 2(N+1)\|Y^{(2)}\|_\infty\sup_{s\in[0,T]}\mathbb{E}[(\kappa^\varepsilon_s)^2]^{1/2},$$

so that, by Lemma B.6,

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(2,a)}_{t,T})^2\big]^{1/2}=0.$$

On the other hand

$$\begin{aligned}\mathbb{E}\big[(R^{(2,b)}_{t,T})^2\big]^{1/2}&\le \varepsilon^{1-H}\|F\|^2_\infty\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\big[\big(Y^{(2)}_s-Y^{(2)}_{t_k}\big)^2\big]^{1/2}ds\\&\le K\varepsilon^{1-H}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(s-t_k)^{1/2}ds\\&\le\frac{K'\varepsilon^{1-H}}{\sqrt{N}}.\end{aligned}$$

Therefore, we get

$$\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(2)}_{t,T})^2\big]^{1/2}\le\limsup_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(2,b)}_{t,T})^2\big]^{1/2}\le\frac{K'}{\sqrt{N}}.$$

Since this is true for any $N$, we get the desired result.

*Step 3: Proof of (4.19) for* $j=3$.
This proof follows the same lines as the proof of Step 2 with

$$\eta^\varepsilon_t=\varepsilon^{1-H}\int_0^t\big(\sigma^\varepsilon_s-\widetilde{\sigma}\big)ds, \tag{4.22}$$

instead of $\kappa^\varepsilon_t$, and using that $\theta_t$ is bounded. We then get the desired result by Lemma B.5.

*Step 4: Proof of (4.19) for* $j=4$.
We have

$$\mathbb{E}\big[(R^{(4)}_{t,T})^2\big]^{1/2}\le K\int_t^T\mathbb{E}\big[(\widetilde{\theta}^\varepsilon_s)^2\big]^{1/2}ds\le K'\sup_{s\in[0,T]}\mathbb{E}\big[(\widetilde{\theta}^\varepsilon_s)^2\big]^{1/2}.$$

By Lemma B.2,

$$\lim_{\varepsilon\to 0}\varepsilon^{H-1}\sup_{t\in[0,T]}\mathbb{E}\big[(R^{(4)}_{t,T})^2\big]^{1/2}=0.$$

We can now complete the proof of Proposition 4.1. In (4.4) we introduced the approximation:

$$Q^\varepsilon_t(x)=Q^{(0)}_t(x)+\phi^\varepsilon_t\big(x^2\partial_x^2\big)Q^{(0)}_t(x)+\varepsilon^{1-H}\rho\widetilde{\sigma}Q^{(1)}_t(x).$$

We then have

$$Q^\varepsilon_T(x)=h(x),$$

because $Q^{(0)}_T(x)=h(x)$, $\phi^\varepsilon_T=0$, and $Q^{(1)}_T(x)=0$. Let us denote

$$R_{t,T}=R^{(1)}_{t,T}+R^{(2)}_{t,T}+R^{(3)}_{t,T}+R^{(4)}_{t,T}, \tag{4.23}$$

$$N_t=\int_0^t dN^{(0)}_s+dN^{(1)}_s+\varepsilon^{1-H}\rho\widetilde{\sigma}dN^{(2)}_s. \tag{4.24}$$

By (4.14) we have

$$Q^\varepsilon_T(X_t) - Q^\varepsilon_t(X_t) = R_{t,T} + N_T - N_t.$$

Therefore

$$\begin{aligned} M_t &= \mathbb{E}\big[h(X_T)|\mathcal{F}_t\big] = \mathbb{E}\big[Q^\varepsilon_T(X_T)|\mathcal{F}_t\big] = Q^\varepsilon_t(X_t) + \mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big] + \mathbb{E}\big[N_T - N_t|\mathcal{F}_t\big] \\ &= Q^\varepsilon_t(X_t) + \mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big], \end{aligned} \tag{4.25}$$

which gives the desired result since $\mathbb{E}\big[R_{t,T}|\mathcal{F}_t\big]$ is of order $o(\varepsilon^{1-H})$ in $L^2$. □

**5. Call Price Correction and Implied Volatility.** We denote the Black-Scholes call price, with current time $t$, maturity $T$, strike $K$, underlying value $x$, and volatility $\sigma$, by $C_{\rm BS}(t,x;K,T;\sigma)$, so that $Q^{(0)}_t$ in Eq. (4.5) is

$$Q^{(0)}_t(x) = C_{\rm BS}(t,x;K,T;\overline{\sigma}).$$

Indeed, $C_{\rm BS}$ gives an explicit formula for the price in the case with constant volatility. In the situation with a stochastic volatility as considered here no explicit pricing formula exists. However, as shown in Eq. (4.4) we can get an asymptotic expression for the price in the case with the stochastic volatility in Eq. (1.5) as a correction to $Q^{(0)}_t(x)$, the Black-Scholes price evaluated at the effective or "homogenized" volatility $\bar{\sigma}$. Here, we show that this corrected price takes on a rather simple generic form in the two parameters, relative time to maturity and moneyness. This representation then leads to a simple representation for the implied volatility as we show below.

We denote the time to maturity by $\tau = T - t$ and we introduce the characteristic diffusion time $\bar{\tau} = 2/\overline{\sigma}^2$ and the dimensionless effective skewness factor:

$$a_F = \varepsilon^{1-H}\frac{\rho\widetilde{\sigma}\overline{D}\bar{\tau}^H}{2^{3/2}\overline{\sigma}} = \varepsilon^{1-H}\frac{\widetilde{\sigma}\sigma_{\rm ou}\rho\left\langle FF'\right\rangle\bar{\tau}^H}{2^{3/2}\overline{\sigma}\Gamma(H+\frac{3}{2})}, \tag{5.1}$$

with $\overline{\sigma}$, $\widetilde{\sigma}$ and $\overline{D}$ given in Proposition 4.1 and the correlation $\rho$ specified in Eq. (3.3).

LEMMA 5.1. *The price correction in Eq. (4.4), normalized by the strike $K$, can be written in the form*

$$\begin{aligned} &\frac{1}{K}\left(\phi^\varepsilon_t\,(x^2\partial_x^2)Q^{(0)}_t(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q^{(1)}_t(x)\right) \\ &= \left(\frac{e^{-d_1^2/2}\,\frac{x}{K}}{\sqrt{\pi}}\right)\left\{\frac{\phi^\varepsilon_t}{2}\left(\frac{\tau}{\bar{\tau}}\right)^{-1/2} + a_F\left[\left(\frac{\tau}{\bar{\tau}}\right)^H + \left(\frac{\tau}{\bar{\tau}}\right)^{H-1}\log\frac{K}{x}\right]\right\}, \end{aligned} \tag{5.2}$$

*with*

$$d_1 = \frac{\frac{\tau}{\bar{\tau}} - \log\frac{K}{x}}{\sqrt{2\frac{\tau}{\bar{\tau}}}}. \tag{5.3}$$

Here, the dimensionless random and deterministic correction coefficients are small of order

$$\phi^\varepsilon_t = O\left(\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^H\right), \quad a_F = O\left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}, \tag{5.4}$$

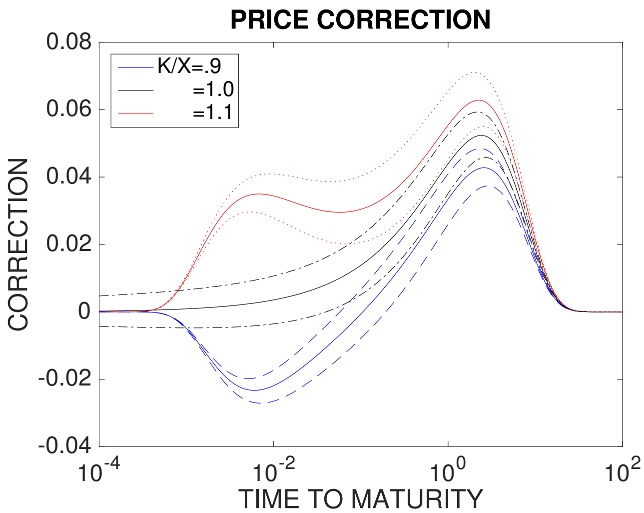


FIG. 5.1. *Price correction as function of relative maturity $\tau/\bar{\tau}$. The three solid lines correspond (from bottom to top) to the mean price correction for $K/X = 0.9$, $1.0$, and $1.1$ respectively. The dashed/dotted lines correspond to the mean $\pm$ one standard deviation. We used here $H = 0.6, a_F = 0.1$ and $\big((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi\big) = 0.04$.*

where we used that $\phi_t^\varepsilon$ as defined in Proposition 4.1 is centered and with standard deviation

$$\mathrm{Var}\big(\phi_t^\varepsilon\big)^{1/2} = \left(\frac{\varepsilon}{\bar{\tau}}\right)^{1-H}\left(\frac{\tau}{\bar{\tau}}\right)^{H}(\bar{\tau}\sigma_\phi) + o(\varepsilon^{1-H}), \tag{5.5}$$

with $\sigma_\phi$ defined by Eq. (4.10) (see also Eq. (B.14) in Lemma B.3). Note that the magnitude of the fluctuations of the implied volatility increases as a fractional power of time to maturity, that is, as $(\tau/\bar{\tau})^{H-1/2}$. We comment in more detail about the statistical structure of the volatility fluctuations in the next section.

It follows from the above that the normalized price correction depends on the two parameters, the moneyness $K/x$ and the relative time to maturity $\tau/\bar{\tau}$, and exhibits a term structure in fractional powers of relative time to maturity.

In Figure 5.1 we show the relative price correction in Eq. (5.2) as function of relative time to maturity $\tau/\bar{\tau}$ for three values of the moneyness $K/x$. The solid lines plot the mean relative price correction and the dashed lines give the mean plus/minus one standard deviation. We used here $H = 0.6, a_F = 0.1$ and $\big((\varepsilon/\bar{\tau})^{(1-H)}\bar{\tau}\sigma_\phi\big) = 0.04$. The mean relative price correction is largest for a mid-range of maturities. For very short times to maturity relative to the effective diffusion time the effect of the volatility fluctuations are small, while for large times the rapid mean reversion "averages" out the effect of the fluctuations. Note that the parameters chosen are not calibrated to market data, this will be considered in another publication.

In Figure 5.2 we show the price correction surface as function of relative maturity $\tau/\bar{\tau}$ and moneyness $K/X$.

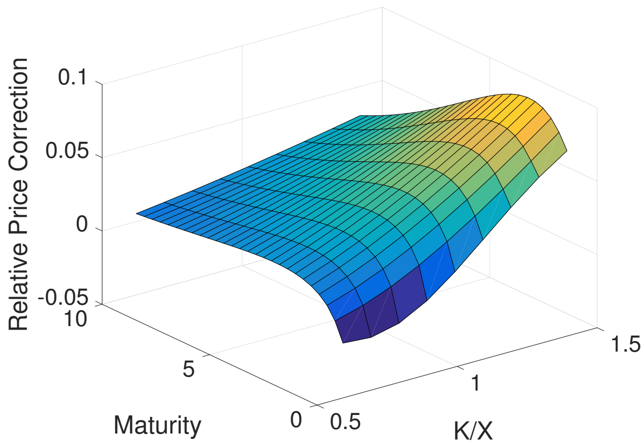


FIG. 5.2. *The price correction surface as function of relative time to maturity and moneyness. The parameters are chosen as in the previous plot.*

*Proof.* For the European call option with payoff $h(x) = (x-K)_+$ we have explicitly

$$C_{\rm BS}(t,x;K,T;\sigma) = x\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) + \frac{\sigma\sqrt{T-t}}{2}\right) \\ -K\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{x}{K}\right) - \frac{\sigma\sqrt{T-t}}{2}\right),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. We then have in particular the "Greek" relationships for the call price:

$$\partial_\sigma C_{\rm BS} = (T-t)\overline{\sigma}x^2\partial_x^2 C_{\rm BS}, \qquad x\partial_x\partial_\sigma C_{\rm BS} = \left(\frac{1}{2} + \frac{\log\frac{K}{x}}{\overline{\sigma}^2(T-t)}\right)\partial_\sigma C_{\rm BS}.$$

We then get

$$x^2\partial_x^2 Q_t^{(0)}(x) = \frac{1}{\overline{\sigma}(T-t)}\partial_{\overline{\sigma}} C_{\rm BS}(t,x;K,T;\overline{\sigma}), \tag{5.6}$$

$$x\partial_x x^2\partial_x^2 Q_t^{(0)}(x) = \left[\frac{1}{2\overline{\sigma}(T-t)} + \frac{\log\frac{K}{x}}{\overline{\sigma}^3(T-t)^2}\right]\partial_{\overline{\sigma}} C_{\rm BS}(t,x;K,T;\overline{\sigma}), \tag{5.7}$$

where the "Vega" is given by

$$\partial_\sigma C_{\rm BS}(t,x;K,T;\overline{\sigma}) = \frac{xe^{-d_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}}, \qquad d_1 = \frac{\frac{1}{2}\sigma^2(T-t) - \log\frac{K}{x}}{\sigma\sqrt{T-t}}. \tag{5.8}$$

Then, with $Q_t^{(1)}(x)$ given in Eq. (4.8) we can identify the form of the price correction as:

$$\phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x) \\ = \phi_t^\varepsilon\left(x^2\partial_x^2\right)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}D(t)x\partial_x x^2\partial_x^2 Q_t^{(0)}(x) \\ = \phi_t^\varepsilon\left(\frac{xe^{-d_1^2/2}}{\overline{\sigma}\sqrt{2\pi(T-t)}}\right) + \varepsilon^{1-H}\left(\frac{x\rho\widetilde{\sigma}\overline{D}e^{-d_1^2/2}}{\sqrt{2\pi}}\right)\left[\frac{(T-t)^H}{2\overline{\sigma}} + \frac{\log\frac{K}{x}}{\overline{\sigma}^3(T-t)^{1-H}}\right], \tag{5.9}$$

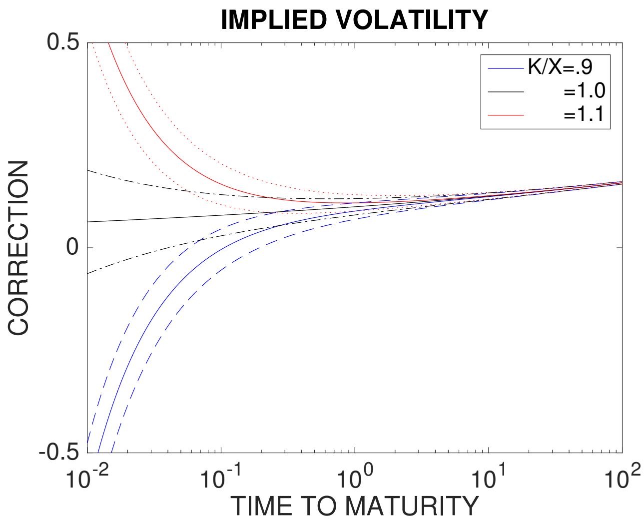


Fig. 5.3. *The implied volatility correction as function of relative time to maturity. The three solid lines correspond (from bottom to top) to the mean implied volatility correction for* $K/X = 0.9$*,* $1.0$*, and* $1.1$ *respectively. The dashed/dotted lines correspond to the mean* $\pm$ *one standard deviation.*

which in turn gives (5.2). □

We next consider the implied volatility associated with the price correction. For the stochastic volatility model in Eq. (1.5) we want to identify the implied volatility $I_t$ so that in terms of the corrected price in Lemma 4.1 we have:

$$C_{\rm BS}(t,x;K,T;I_t) \sim Q_t^{(0)}(x) + \phi_t^\varepsilon\big(x^2\partial_x^2\big)Q_t^{(0)}(x) + \varepsilon^{1-H}\rho\widetilde{\sigma}Q_t^{(1)}(x). \tag{5.10}$$

We define the relative implied volatility correction, $\delta I_t$, by

$$I_t = \overline{\sigma}(1+\delta I_t). \tag{5.11}$$

Lemma 5.2. *The relative implied volatility correction has the form:*

$$\delta I_t = \frac{\phi_t^\varepsilon}{2}\left(\frac{\tau}{\bar\tau}\right)^{-1} + a_F\Big[\left(\frac{\tau}{\bar\tau}\right)^{H-1/2} + \left(\frac{\tau}{\bar\tau}\right)^{H-3/2}\log(K/X_t)\Big] + o(\varepsilon^{1-H}). \tag{5.12}$$

Here $\phi_t^\varepsilon$ is defined in Proposition 4.1 and $a_F$ in Eq. (5.1).

In Figure 5.3 we show the implied volatility correction in Eq. (5.12) as function of relative time to maturity $\tau/\bar\tau$ for three values of the moneyness $K/x$. We used here again $H = 0.6, a_F = 0.1$ and $\big((\varepsilon/\bar\tau)^{(1-H)}\bar\tau\sigma_\phi\big) = 0.04$. Note that due to the form of the "vega" , the sensitivity of the price to the volatility, the form of the implied volatility surface is very different from that of the price correction. In Figure 5.4 we show the implied volatility correction surface as function of relative maturity $\tau/\bar\tau$ and moneyness $K/X$.

*Proof.* We find by using Eqs. (5.9) and (5.8) that the implied volatility is given by

$$I_t = \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} + \varepsilon^{1-H}\widetilde{\sigma}\rho D_t\Big[\frac{1}{2\overline{\sigma}(T-t)} + \frac{\log\frac{K}{X_t}}{\overline{\sigma}^3(T-t)^2}\Big] + o(\varepsilon^{1-H}). \tag{5.13}$$

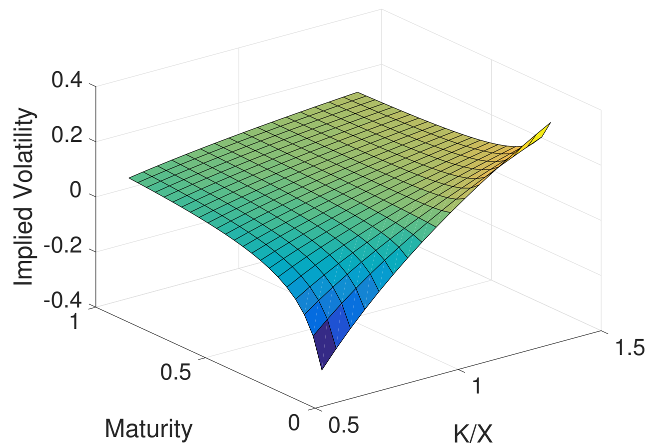


FIG. 5.4. *The mean implied volatility correction surface as function of relative time to maturity and moneyness. The parameters are as in Figure 5.3.*

Since $D_t$ is deterministic and given by (4.9), we can then write

$$I_t = \overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} + \varepsilon^{1-H} \frac{\widetilde{\sigma}\sigma_{\rm ou}\rho \left\langle FF' \right\rangle}{\overline{\sigma}\Gamma(H+\frac{3}{2})} \Big[\frac{1}{2}(T-t)^{H-\frac{1}{2}} + \frac{\log \frac{K}{X_t}}{\overline{\sigma}^2 (T-t)^{\frac{3}{2}-H}}\Big] + o(\varepsilon^{1-H}), \tag{5.14}$$

and the Lemma follows. □

The first two terms in Eq. (5.14) can be combined and rewritten as (up to terms of order $o(\varepsilon^{1-H})$):

$$\overline{\sigma} + \frac{\phi_t^\varepsilon}{\overline{\sigma}(T-t)} = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T (\sigma_s^\varepsilon)^2 ds \big| \mathcal{F}_t\Big]^{1/2} + o(\varepsilon^{1-H}). \tag{5.15}$$

Since $D_t$ is deterministic and given by (4.9), we can then write

$$I_t = \mathbb{E}\Big[\frac{1}{T-t}\int_t^T (\sigma_s^\varepsilon)^2 ds \big| \mathcal{F}_t\Big]^{1/2} + \overline{\sigma} a_F \left[ \left(\frac{\tau}{\overline{\tau}}\right)^{H-1/2} + \left(\frac{\tau}{\overline{\tau}}\right)^{H-3/2} \log\left(\frac{K}{X_t}\right)\right] + o(\varepsilon^{1-H}), \tag{5.16}$$

so that the implied volatility is the expected effective volatility over the remaining time horizon conditioned on the present and with an added skewness correction.

In view of Eq. (5.5), for small time to maturity the fourth term (in $\tau^{H-\frac{3}{2}}$) dominates in (5.12). We remark here that this is related to the fact that the small parameter in our problem is the mean reversion time so that for any order one time to maturity in this regime the volatility will have time to fluctuate and mean revert giving a price correction as in Lemma 5.1. Then with the "Vega", $\partial_\sigma C_{\rm BS}$, being relatively small away from the money, see Eq. (5.8), we get a strong moneyness dependence and the implied volatility blows up for small time-to-maturity.

Moreover, for large time to maturity the third term (in $\tau^{H-\frac{1}{2}}$) dominates in (5.12). The long range dependence gives relatively smooth volatility fluctuations which gives an implied volatility that blows up for large time-to-maturity and with the current value for the underlying being relatively less important in this long maturity regime.

**6. The t-T Process and the Stochastic Implied Surface.** We introduced above the stochastic correction coefficient $\phi^{\varepsilon}_{t,T}$ which gives the random component of the price correction and the implied volatility. Note that if the volatility process had been a Markovian process then the correction we consider here would have been deterministic Fouque et al. (2011). The presence of long range memory in the volatility process means that information from the past (volatility path) must be carried forward and this makes the price correction relative to the price at the homogenized volatility a stochastic process, and correspondingly for the implied volatility.

Note that it follows from Lemma B.3 that as $\varepsilon \to 0$, the random process $\varepsilon^{H-1}\phi^{\varepsilon}_{t,T}/[\sigma_\phi(T-t)^H]$, $t < T$, converges in distribution (in the sense of finite-dimensional distributions) to a Gaussian stochastic process $\psi_{t,T}$, $t < T$, the normalized t-T correction process, with mean zero, variance one, and covariance $\mathbb{E}[\psi_{t,T}\psi_{t',T'}] = \mathcal{C}_\phi(t,t';T,T')$ for any $t \in [0,T)$, $t' \in [0,T')$. The four-parameter function $\mathcal{C}_\phi$ is detailed in Eq. (B.16). We discuss next in more detail the t-T process $\psi_{t,T}$, a two-parameter process of current time $t$ and maturity $T$. This process is scaled to have constant unit variance, however, is a non-stationary Gaussian process supported for $0 < t < T$. As we will see, close to maturity $t \approx T$, the process is relatively strongly affected by the presence of the maturity boundary.

Let us first consider the case of a fixed maturity $T$ and introduce the process

$$\psi_0(t;T) = \psi_{t,T}, \qquad t \in [0,T]. \tag{6.1}$$

On short scales relative to the time to maturity, i.e. for $|t-t'| \ll T-t$, it follows from Eq. (B.16) that the process $(\psi_0(t;T))_{t\in[0,T]}$ decorrelates as

$$\mathbb{E}\big[\psi_0(t;T)\psi_0(t';T)\big] \sim 1 - \frac{|t-t'|}{2(T-t)},$$

that is as a Markov process on short scales. More generally, the autocovariance function of $(\psi_0(t;T))_{t\in[0,T]}$ is

$$\mathbb{E}\big[\psi_0(t;T)\psi_0(t';T)\big] = \mathcal{C}(\Delta_0(t,t';T)),$$
$$\mathcal{C}(\Delta) = \frac{\int_0^\infty du\big[\big(u+\frac{|\Delta|+1}{\sqrt{1-\Delta^2}}\big)^{H-\frac12} - u^{H-\frac12}\big]\big[\big(u+\frac{|\Delta|+1}{\sqrt{1-\Delta^2}}\big)^{H-\frac12} - \big(u+\frac{2|\Delta|}{\sqrt{1-\Delta^2}}\big)^{H-\frac12}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac12} - u^{H-\frac12}\big]^2},$$

with

$$\Delta_0(t,t';T) = \frac{t'-t}{|2T-(t+t')|}. \tag{6.2}$$

Thus the correlation of the process $(\psi_0(t;T))_{t\in[0,T]}$ depends only on this relative separation so that we have a situation with a canonical relative decorrelation that depends only on the times to maturity $\tau = T-t, \tau' = T-t'$. Therefore, we introduce the process $(\psi_1(\tau;T))_{\tau\in[0,T]}$ defined by

$$\psi_1(\tau;T) = \psi_{T-\tau,T}, \qquad \tau \in [0,T]. \tag{6.3}$$

The process $(\psi_1(\tau;T))_{\tau\in[0,T]}$ is Gaussian with mean zero and autocovariance function

$$\mathbb{E}\big[\psi_1(\tau;T)\psi_1(\tau';T)\big] = \mathcal{C}(\Delta_1(\tau,\tau')),$$

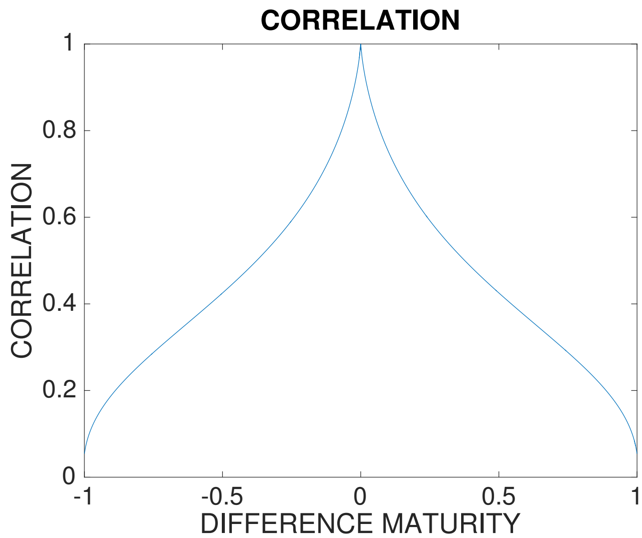


Fig. 6.1. *Autocovariance function of the t-T process $\psi_1(\tau;1)$ as function of relative maturity separation $\Delta_1 = (\tau-\tau')/|\tau+\tau'|$ with $H = 0.6$. The correlation decays approximately linearly at the origin and rapidly as one of the times to maturity goes to zero.*

with $\mathcal{C}$ as above and

$$\Delta_1(\tau,\tau') = \frac{\tau-\tau'}{|\tau+\tau'|}. \tag{6.4}$$

Note that $\Delta_1(\tau,\tau') = \Delta_0(t,t')$ for $\tau = T-t, \tau' = T-t'$. For $|\tau-\tau'| \ll \tau$ the process decorrelates on the scale $\tau$ so that the process fluctuations becomes more rapid close to maturity. In Figure 6.1 we show the correlation function $\Delta_1 \mapsto \mathcal{C}(\Delta_1)$ as function of the relative separation time $\Delta_1 \in [-1,1]$ and $H = 0.6$. The process decorrelates as a Markov process on short scales and indeed as one of the times to maturity goes to zero (relative to the other time) the correlation goes rapidly to zero.

Note that it follows from the expression (6.4) for $\Delta_1$ that it is scale invariant in that $\Delta_1(a\tau,a\tau') = \Delta_1(\tau,\tau')$ for $a > 0$, giving rapid fluctuations for small maturities. The process has indeed a self-similar property. We have in distribution:

$$\big(\psi_1(\tau;1)\big)_{\tau\in[0,1]} \sim \big(\psi_1(\tau T;T)\big)_{\tau\in[0,1]}.$$

In Figure 6.2 we show one realization of the process $\psi_1(\tau;1)$ as a function of time to maturity $\tau$.

One can also investigate the structure of the t-T process for a fixed time to maturity $\tau$, as a function of time $t$. Thus, if we calibrate the price for a given time to maturity, we would like to know how the price correction, respectively the implied volatility, would fluctuate with respect to the current time, or time translation. Accordingly we consider the process

$$\psi_2(t;\tau) = \psi_{t,\tau+t}, \qquad t \geq 0, \tag{6.5}$$

for fixed $\tau > 0$. The process $(\psi_2(t;\tau))_{t\in[0,\infty)}$ is Gaussian with mean zero and auto-

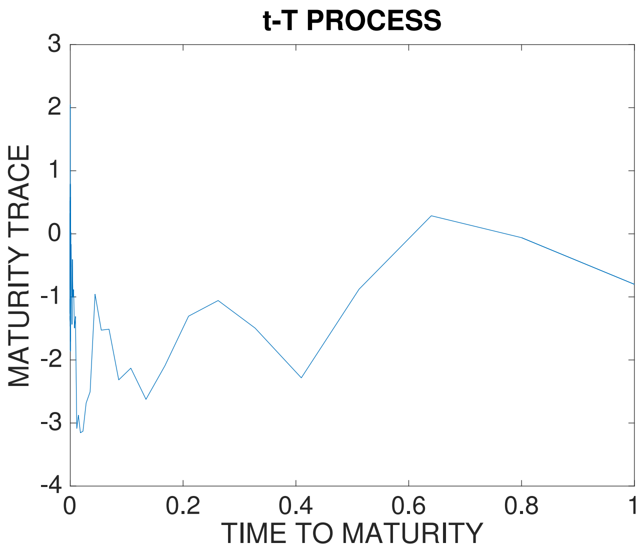


FIG. 6.2. *Realization of the process $\psi_1(\tau;1)$ as function of time to maturity $\tau$ for fixed maturity $T=1$ with $H=0.6$.*

covariance function

$$\mathbb{E}\big[\psi_2(t;\tau)\psi_2(t';\tau)\big] = \mathcal{C}_2(\Delta_2(t,t';\tau)), \tag{6.6}$$

$$\mathcal{C}_2(\Delta) = \frac{\int_0^\infty du\big[(u+1)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]\big[(u+1+|\Delta|)^{H-\frac{1}{2}} - (u+|\Delta|)^{H-\frac{1}{2}}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]^2},$$

with

$$\Delta_2(t,t';\tau) = \frac{t'-t}{\tau}. \tag{6.7}$$

This expression shows that the coherence time of this process scales with time to maturity $\tau$. The process has indeed a self-similar property. We have in distribution:

$$\big(\psi_2(t;1)\big)_{t\in[0,\infty)} \sim \big(\psi_2(\tau t;\tau)\big)_{t\in[0,\infty)}.$$

The autocovariance function of $(\psi_2(t;1))_{t\in[0,\infty)}$ is plotted in Figure 6.3. In the figure note the rapid decay at the origin followed by a long range behavior. This shows how the implied surface decorrelates as we move in time. In Figure 6.4 we show the autocorrelation function in a *log-log* plot with the dashed line corresponding to the asymptotic correlation decay $|t'-t|^{2H-2}$. In Figure 6.5 we show one realization for the process $\psi_2(t;1)$.

Finally, it is of interest to consider the case when we evaluate the stochastic correction factor as function of maturity for fixed current time $t$:

$$\psi_3(\tau;t) = \psi_{t,t+\tau}, \qquad \tau \geq 0. \tag{6.8}$$

The process $(\psi_3(\tau;t))_{\tau\in[0,\infty)}$ is Gaussian with mean zero and autocovariance function

$$\mathbb{E}\big[\psi_3(\tau;t)\psi_3(\tau';t)\big] = \mathcal{C}_3(\Delta_3(\tau,\tau')),$$

$$\mathcal{C}_3(\Delta) = \frac{\int_0^\infty du\big[(u+1/\sqrt{1+|\Delta|})^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]\big[(u+\sqrt{1+|\Delta|})^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big]^2},$$

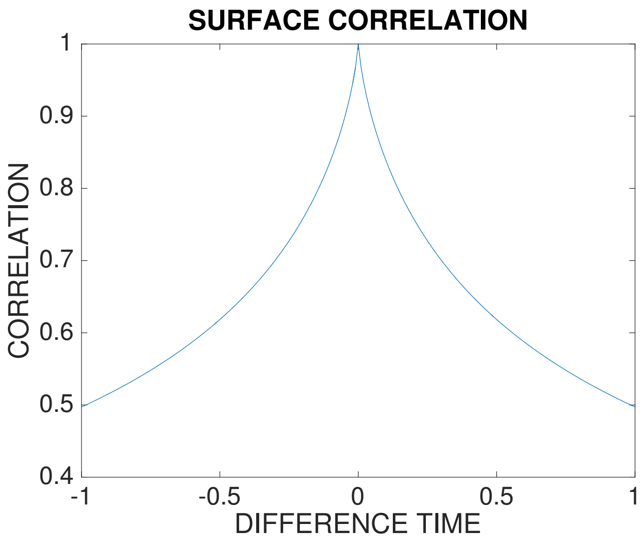


FIG. 6.3. *Autocovariance function of the t-T process $\psi_2(t;1)$ as function of time $t'-t$ for fixed time to maturity $\tau=1$ with $H=0.6$. On the short scales the process decorrelates as a Markov process and on the long scales it exhibits long range correlations.*

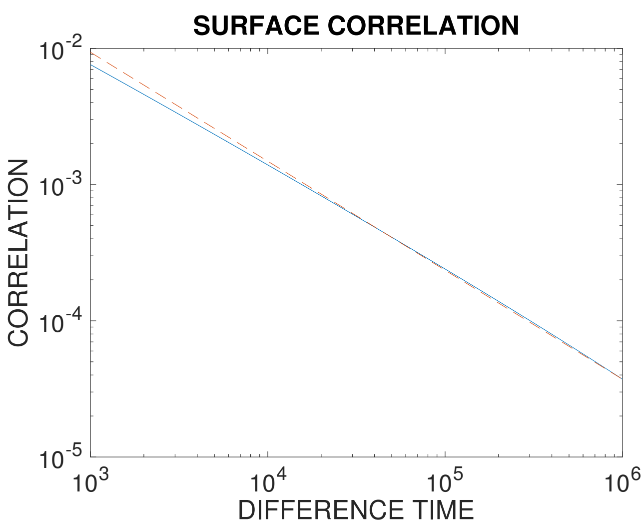


FIG. 6.4. *Autocovariance function of the t-T process $\psi_2(t;1)$ as in Figure 6.3, but on a log-log scale with the dashed line showing the asymptotic decay $|t'-t|^{2H-2}$.*

with

$$\Delta_3(\tau,\tau') = \frac{\tau-\tau'}{\tau\wedge\tau'}. \tag{6.9}$$

This covariance function is plotted in Figure 6.6. Note that it follows from the expression (6.9) for $\Delta_3$ that it is scale invariant in that $\Delta_3(a\tau,a\tau')=\Delta_3(\tau,\tau')$ for $a>0$, so that again the process fluctuates more rapidly for small maturities. The distribution of the process $(\psi_3(\tau;t))_{\tau\in[0,\infty)}$ does not depend on $t$ and it has a self-similar property.

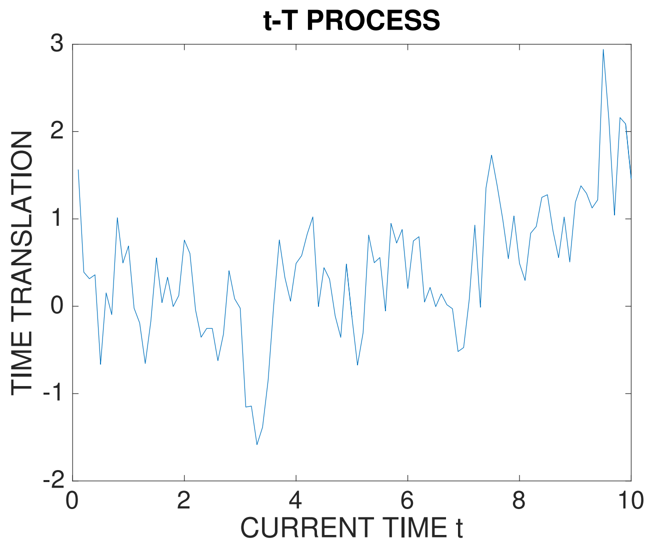


FIG. 6.5. *Realization of the process $\psi_2(t;1)$ with $H = 0.6$.*

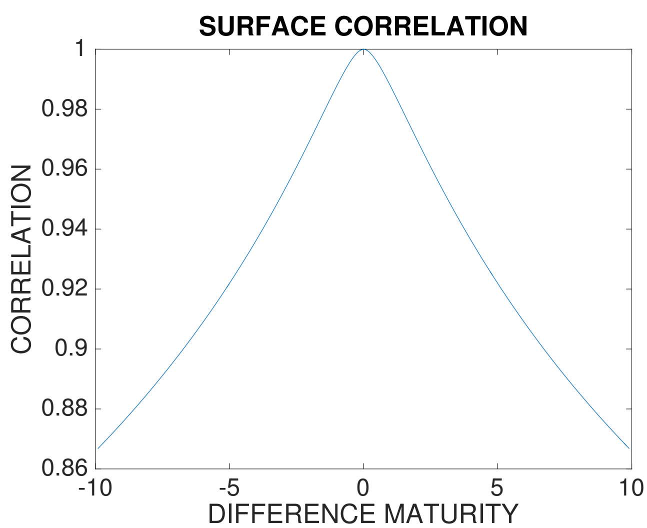


FIG. 6.6. *Autocovariance function of the t-T process $\psi_3(\tau;1)$ as function of the relative maturity separation $\Delta_3 = (\tau - \tau')/(\tau \wedge \tau')$ with $H = 0.6$. Note that the correlation function is smooth at the origin, moreover, exhibits slow decay.*

For any $a > 0$, we have in distribution:

$$\big(\psi_3(\tau;t)\big)_{\tau \in [0,\infty)} \sim \big(\psi_3(a\tau;t)\big)_{\tau \in [0,\infty)}.$$

In Figure 6.7 we show a realization of the process $(\psi_3(\tau;1))_{\tau \in [0,\infty)}$.

**7. Conclusion.** We have considered a continuous time stochastic volatility model with long-range correlation properties. We consider the regime of fast mean reversion. In fact this allows us to derive an explicit expression for the European option price and the implied volatility. Specifically the volatility is a smooth function

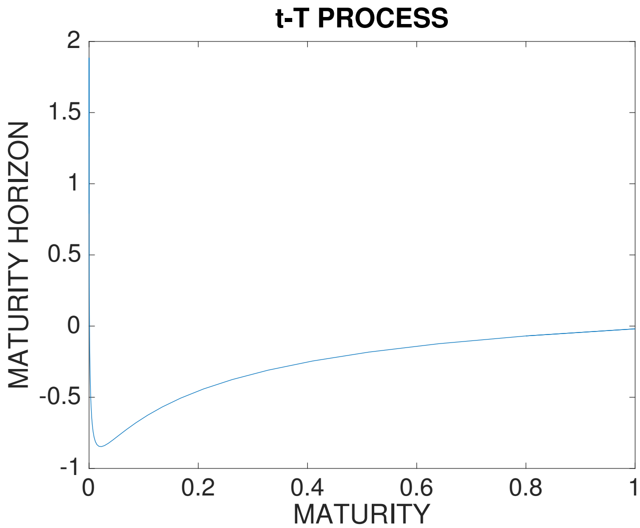


FIG. 6.7. *Realization of the process $\psi_3(\tau;1)$ for fixed current time $t = 1$ and $H = 0.6$, with the smooth and slow decay of the correlations giving a smooth time to maturity dependence.*

of a fractional Ornstein-Uhlenbeck process. The analysis of such a non-Markovian situation is challenging. To the best of our knowledge we present the first analytical expression for the price for general maturities when the volatility fluctuations are order one. So far the price computations for such situations have been based on numerical approximations. The main result from the applied view point is then the form of the fractional term structure we get for the implied volatility surface. Indeed we get an implied volatility that grows large with maturity while generating a strong skew for short maturities consistently with common observations. We stress that in our formulation the mean reversion time will be small compared to any fixed maturity time as we consider a fast mean reverting process. Note finally that we here have considered the case of processes with long-range correlation properties with the Hurst exponent $H > 1/2$ explaining the large growth of implied volatility for large maturity. The situation with short-range correlation properties with the Hurst exponent $H < 1/2$ is different and results for this will be presented elsewhere.

**Appendix A. Hermite Decomposition of the Stochastic Volatility Model.** We denote

$$\widetilde{F}(z) = F(\sigma_{\rm ou} z)^2. \tag{A.1}$$

Because $\mathbb{E}[\widetilde{F}(Z)^2] < \infty$ is finite when $Z$ is a standard normal variable, the function $\widetilde{F}$ can be expanded in terms of the Hermite polynomials

$$H_k(z) = (-1)^k e^{z^2/2} \frac{d^k}{dz^k} e^{-z^2/2} \tag{A.2}$$

and the series

$$\sum_{k=0}^{\infty} \frac{C_k}{k!} H_k(z), \tag{A.3}$$

with

$$C_k = \mathbb{E}\big[H_k(Z)\widetilde{F}(Z)\big] = \int_{\mathbb{R}} H_k(z)\widetilde{F}(z)p(z)dz, \tag{A.4}$$

converges in $L^2(\mathbb{R}, p(z)dz)$ to $\widetilde{F}(z)$. The Hermite polynomials satisfy

$$\mathbb{E}[H_k(Z)H_j(Z)] = \int_{\mathbb{R}} H_k(z)H_j(z)p(z)dz = \delta_{kj}k!,$$

and we have $\sum_{k=0}^{\infty}\frac{C_k^2}{k!} = \mathbb{E}[\widetilde{F}(Z)^2] < \infty$. Note that $C_0 = \langle F^2\rangle$.

LEMMA A.1. *If there exists $\alpha > 2$ such that the function $\widetilde{F}$ defined by (A.1) satisfies*

$$\sum_{k=0}^{\infty}\frac{\alpha^k C_k^2}{k!} < \infty, \tag{A.5}$$

*then the random process*

$$I_t^\varepsilon = \int_0^t F^2(Z_s^\varepsilon) - \langle F^2\rangle\, ds \tag{A.6}$$

*satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}[(I_t^\varepsilon)^4] \le K\varepsilon^{4-4H}, \tag{A.7}$$

*for some constant $K$.*

*Proof.* Denoting $\widetilde{Z}_t^\varepsilon = \sigma_{\rm ou}^{-1}Z_t^\varepsilon$, which is a zero-mean Gaussian process with covariance function $\mathbb{E}[\widetilde{Z}_t^\varepsilon\widetilde{Z}_{t+s}^\varepsilon] = \mathcal{C}_Z(s/\varepsilon)$, we have

$$I_t^\varepsilon = \int_0^t \widetilde{F}(\widetilde{Z}_s^\varepsilon) - \langle F^2\rangle\, ds = \sum_{m=1}^{\infty} C_m I_{t,m}^\varepsilon,$$

where

$$I_{t,m}^\varepsilon = \frac{1}{m!}\int_0^t H_m(\widetilde{Z}_s^\varepsilon)ds, \qquad m \ge 1.$$

From (Taqqu , 1978, Lemma 2.2) the fourth-order moment of $I_{t,m}^\varepsilon$ can be expanded as

$$\mathbb{E}[(I_{t,m}^\varepsilon)^4] = \frac{1}{2^m(2m)!}\sum\int_0^t\cdots\int_0^t dt_1dt_2dt_3dt_4\prod_{\ell=1}^m \mathcal{C}_Z\Big(\frac{t_{i_\ell}-t_{j_\ell}}{\varepsilon}\Big),$$

where the sum is over all indices $i_1, j_1, \dots, i_{2m}, j_{2m}$ such that:
i) $i_1, j_1, \dots, i_{2m}, j_{2m} \in \{1,2,3,4\}$,
ii) $i_1 \neq j_1$, ..., $i_{2m} \neq j_{2m}$,
iii) each number $1,2,3,4$ appears exactly $m$ times in $(i_1, j_1, \dots, i_{2m}, j_{2m})$.
The number $N_{2m}$ of terms in this sum is therefore smaller than $(4m)!/m!^4$ (it would be exactly this cardinal without the second condition, therefore it is smaller than this

number).
Since $\mathcal{C}_Z(s) \leq 1 \wedge K|s|^{2H-2}$ for some constant $K$, we have, for any $t \in [0,T]$,

$$\mathbb{E}[(I^\varepsilon_{t,m})^4] \leq \frac{1}{2^{2m}(2m)!} \sum \int_0^T \cdots \int_0^T dt_1 dt_2 dt_3 dt_4 \prod_{\ell=1}^{2m} 1 \wedge K\big(\frac{|t_{i_\ell} - t_{j_\ell}|}{\varepsilon}\big)^{2H-2}.$$

For each term of the sum, we apply the change of variables $s_1 = t_{i_1}$, $s_2 = t_{j_1}$, $s_3 = t_{\min(\{1,2,3,4\}\setminus\{i_1,j_1\})}$, $s_4 = t_{\max(\{1,2,3,4\}\setminus\{i_1,j_1\})}$. In the product we keep the first term: $K(|s_1 - s_2|/\varepsilon)^{2H-2}$, and the first term that has $s_3$ in it: $K(|s_3 - s_j|/\varepsilon)^{2H-2}$, so that we can write, for any $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}[(I^\varepsilon_{t,m})^4] &\leq \frac{N_{2m}K^2}{2^{2m}(2m)!} \int_0^T \cdots \int_0^T ds_1 ds_2 ds_3 ds_4 \big(\frac{|s_1 - s_2|}{\varepsilon}\big)^{2H-2} \Big[\big(\frac{|s_3 - s_1|}{\varepsilon}\big)^{2H-2} \\ &\qquad + \big(\frac{|s_3 - s_2|}{\varepsilon}\big)^{2H-2} + \big(\frac{|s_3 - s_4|}{\varepsilon}\big)^{2H-2}\Big] \\ &\leq K' \frac{(4m)!}{2^{2m}(2m)!m!^4} \varepsilon^{4-4H},\end{aligned}$$

for some constant $K'$ (that depends on $H$ and $T$), because $s^{2H-2}$ is integrable over $[0,T]$. By Stirling's formula,

$$\frac{(4m)!}{2^{2m}(2m)!m!^4} \simeq \frac{2^{2m}}{m!^2} \frac{1}{\sqrt{2\pi m}}.$$

Therefore, by Minkowski's inequality, for any $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}[(I^\varepsilon_t)^4)]^{1/4} &\leq \sum_{m=1}^\infty |C_m| \mathbb{E}[(I^\varepsilon_m)^4)]^{1/4} \leq K'' \varepsilon^{1-H} \sum_{m=1}^\infty |C_m| \Big(\frac{2^m}{m!}\Big)^{1/2} \\ &\leq K'' \varepsilon^{1-H} \Big(\sum_{m=1}^\infty \frac{\alpha^m C_m^2}{m!}\Big)^{1/2} \Big(\sum_{m=1}^\infty \frac{2^m}{\alpha^m}\Big)^{1/2},\end{aligned}$$

for some constant $K''$, which gives the desired result. □

The hypothesis (A.5) in Lemma A.1 requires some smoothness for the function $\widetilde{F}$. The following lemma gives a sufficient condition.

Lemma A.2. *If the function $\widetilde{F}$ defined by (A.1) is of the form*

$$\widetilde{F}(x) = \int_{-\infty}^x f(y) dy, \tag{A.8}$$

*where the Fourier transform of the function $f$ satisfies $|\hat{f}(\nu)| \leq C \exp(-\nu^2)$ for some $C > 0$, then there exists $K > 0$ such that, for any $k \geq 0$,*

$$\frac{C_k^2}{k!} \leq K 3^{-k}. \tag{A.9}$$

The inequality (A.9) is sufficient to ensure that the hypothesis (A.5) is fulfilled. We may for instance consider :

$$\widetilde{F}(x) = \int_{-\infty}^x e^{-y^2/4} dy \text{ or } \widetilde{F}(x) = \int_{-\infty}^x \mathrm{sinc}^2(y) dy. \tag{A.10}$$

*Proof.* The function $\widetilde{F}$ is of class $\mathcal{C}^\infty$ and we have, for any $k \geq 1$, using integration by parts

$$C_k = \int_{\mathbb{R}} \widetilde{F}(z) H_k(z) p(z) dz = \int_{\mathbb{R}} \widetilde{F}^{(k)}(z) p(z) dz = \int_{\mathbb{R}} f^{(k-1)}(z) p(z) dz$$

By Parseval formula,

$$C_k = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\nu^2/2} (i\nu)^{k-1} \hat{f}(\nu) d\nu.$$

Since $|\hat{f}(\nu)| \leq C \exp(-\nu^2)$,

$$|C_k| \leq C \int_{\mathbb{R}} e^{-3\nu^2/2} |\nu|^{k-1} d\nu = C\Big(\frac{2}{3}\Big)^{\frac{k}{2}} \int_0^\infty e^{-s} s^{\frac{k}{2}-1} ds = C\Big(\frac{2}{3}\Big)^{\frac{k}{2}} \Gamma\Big(\frac{k}{2}\Big),$$

which gives the desired result using Stirling's formula $\Gamma(z) \sim z^{z-1/2} e^{-z} \sqrt{2\pi}$. □

**Appendix B. Technical Lemmas.** We denote

$$G(z) = \frac{1}{2}\big(F(z)^2 - \overline{\sigma}^2\big). \tag{B.1}$$

The martingale $\psi_t^\varepsilon$ defined by (4.12) has the form

$$\psi_t^\varepsilon = \mathbb{E}\Big[\int_0^T G(Z_s^\varepsilon) ds \big| \mathcal{F}_t\Big]. \tag{B.2}$$

Lemma B.1. *$(\psi_t^\varepsilon)_{t\in[0,T]}$ is a square-integrable martingale and*

$$d\left\langle \psi^\varepsilon, W \right\rangle_t = \vartheta_t^\varepsilon dt, \qquad \vartheta_t^\varepsilon = \sigma_{\rm ou} \int_t^T \mathbb{E}\big[G'(Z_s^\varepsilon) | \mathcal{F}_t\big] \mathcal{K}^\varepsilon(s-t) ds. \tag{B.3}$$

An alternative expression of the bracket $\left\langle \psi^\varepsilon, W \right\rangle_t$ is given in (B.5-B.6).

*Proof.* For $t \leq s$, the conditional distribution of $Z_s^\varepsilon$ given $\mathcal{F}_t$ is Gaussian with mean

$$\mathbb{E}\big[Z_s^\varepsilon | \mathcal{F}_t\big] = \sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u$$

and deterministic variance given by

$$\mathrm{Var}\big(Z_s^\varepsilon | \mathcal{F}_t\big) = (\sigma_{0,s-t}^\varepsilon)^2,$$

where we have defined for any $0 \leq s \leq t \leq \infty$:

$$(\sigma_{s,t}^\varepsilon)^2 = \sigma_{\rm ou}^2 \int_s^t \mathcal{K}^\varepsilon(u)^2 du. \tag{B.4}$$

Therefore we have

$$\mathbb{E}\big[G(Z_s^\varepsilon) | \mathcal{F}_t\big] = \int_{\mathbb{R}} F\Big(\sigma_{\rm ou} \int_{-\infty}^t \mathcal{K}^\varepsilon(s-u) dW_u + \sigma_{0,s-t}^\varepsilon z\Big) p(z) dz,$$

where $p(z)$ is the pdf of the standard normal distribution. As a random process in $t$ it is a continuous martingale. By Itô's formula, for any $t \leq s$:

$$\begin{aligned}
\mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_t\big] = &\int_{\mathbb{R}} G\Big(\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\Big)p(z)dz \\
&+\int_0^t\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon du \\
&+\sigma_{\rm ou}\int_0^t\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)dW_u \\
&+\frac{\sigma_{\rm ou}^2}{2}\int_0^t\int_{\mathbb{R}} G''\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)^2du
\end{aligned}$$

and

$$\begin{aligned}
G(Z_s^\varepsilon) &= G\Big(\sigma_{\rm ou}\int_{-\infty}^s \mathcal{K}^\varepsilon(s-v)dW_v\Big) \\
&= \int_{\mathbb{R}} G\Big(\sigma_{\rm ou}\int_{-\infty}^s \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,0}^\varepsilon z\Big)p(z)dz \\
&= \int_{\mathbb{R}} G\Big(\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\Big)p(z)dz \\
&\quad+\int_0^s\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon du \\
&\quad+\sigma_{\rm ou}\int_0^s\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)dW_u \\
&\quad+\frac{\sigma_{\rm ou}^2}{2}\int_0^s\int_{\mathbb{R}} G''\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)^2du.
\end{aligned}$$

Therefore

$$\begin{aligned}
\psi_t^\varepsilon &= \int_0^t G(Z_s^\varepsilon)ds + \int_t^T \mathbb{E}\big[G(Z_s^\varepsilon)|\mathcal{F}_t\big]ds \\
&= \Big[\int_{\mathbb{R}}\int_0^T G\Big(\sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s}^\varepsilon z\Big)dsp(z)dz\Big] \\
&\quad+\int_0^t\Big[\int_u^T\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)zp(z)dz\partial_u\sigma_{0,s-u}^\varepsilon ds\Big]du \\
&\quad+\sigma_{\rm ou}\int_0^t\Big[\int_u^T\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)ds\Big]dW_u \\
&\quad+\frac{\sigma_{\rm ou}^2}{2}\int_0^t\Big[\int_u^T\int_{\mathbb{R}} G''\Big(\sigma_{\rm ou}\int_{-\infty}^u \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-u}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-u)^2ds\Big]du.
\end{aligned}$$

This gives

$$d\left\langle \psi^\varepsilon, W\right\rangle_t = \vartheta_t^\varepsilon dt, \tag{B.5}$$

with

$$\vartheta_t^\varepsilon = \sigma_{\rm ou}\int_t^T\int_{\mathbb{R}} G'\Big(\sigma_{\rm ou}\int_{-\infty}^t \mathcal{K}^\varepsilon(s-v)dW_v + \sigma_{0,s-t}^\varepsilon z\Big)p(z)dz\mathcal{K}^\varepsilon(s-t)ds, \tag{B.6}$$

which can also be written as stated in the Lemma. ◻

The important properties of the random process $\vartheta_t^\varepsilon$ are stated in the following lemma.

Lemma B.2. *For any $t \in [0,T]$, we have*

$$\vartheta_t^\varepsilon = \varepsilon^{1-H}\theta_t + \widetilde{\theta}_t^\varepsilon, \tag{B.7}$$

*where $\theta_t$ is deterministic*

$$\theta_t = \overline{\theta}(T-t)^{H-\frac{1}{2}}, \qquad \overline{\theta} = \frac{\sigma_{\rm ou}\left\langle G'\right\rangle}{\Gamma(H+\frac{1}{2})}, \tag{B.8}$$

*and $\widetilde{\theta}_t^\varepsilon$ is random but smaller than $\varepsilon^{1-H}$:*

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\widetilde{\theta}_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.9}$$

*Proof.* The expectation of $\vartheta_t^\varepsilon$ is equal to

$$\mathbb{E}\big[\vartheta_t^\varepsilon\big] = \sigma_{\rm ou}\left\langle G'\right\rangle \int_0^{T-t} \mathcal{K}^\varepsilon(s)ds = \sigma_{\rm ou}\left\langle G'\right\rangle \sqrt{\varepsilon} \int_0^{(T-t)/\varepsilon} \mathcal{K}(s)ds.$$

Therefore the difference

$$\mathbb{E}\big[\vartheta_t^\varepsilon\big] - \varepsilon^{1-H}\theta_t = \sigma_{\rm ou}\left\langle G'\right\rangle \varepsilon^{1/2} \int_0^{(T-t)/\varepsilon} \mathcal{K}(s) - \frac{s^{H-\frac{3}{2}}}{\Gamma(H-\frac{1}{2})} ds$$

can be bounded by

$$\big|\mathbb{E}\big[\vartheta_t^\varepsilon\big] - \varepsilon^{1-H}\theta_t\big| \le C\varepsilon^{1/2}, \tag{B.10}$$

uniformly in $t \in [0,T]$, for some constant $C$, since $\mathcal{K}(s) - \frac{s^{H-\frac{3}{2}}}{\Gamma(H-\frac{1}{2})}$ is in $L^1$.

We have

$$\begin{aligned}
{\rm Var}(\vartheta_t^\varepsilon) &= \sigma_{\rm ou}^2 \int_t^T ds \int_t^T ds' \mathcal{K}^\varepsilon(s-t)\mathcal{K}^\varepsilon(s'-t){\rm Cov}\big(\mathbb{E}\big[G'(Z_s^\varepsilon)|\mathcal{F}_t\big], \mathbb{E}\big[G'(Z_{s'}^\varepsilon)|\mathcal{F}_t\big]\big) \\
&\le \sigma_{\rm ou}^2 \Big( \int_t^T ds \mathcal{K}^\varepsilon(s-t){\rm Var}\big(\mathbb{E}\big[G'(Z_s^\varepsilon)|\mathcal{F}_t\big]\big)^{1/2}\Big)^2 \\
&= \sigma_{\rm ou}^2 \Big( \int_0^{T-t} ds \mathcal{K}^\varepsilon(s){\rm Var}\big(\mathbb{E}\big[G'(Z_s^\varepsilon)|\mathcal{F}_0\big]\big)^{1/2}\Big)^2.
\end{aligned}$$

The conditional distribution of $Z_t^\varepsilon$ given $\mathcal{F}_0$ is Gaussian with mean

$$\mathbb{E}\big[Z_t^\varepsilon|\mathcal{F}_0\big] = \sigma_{\rm ou}\int_{-\infty}^0 \mathcal{K}^\varepsilon(t-u)dW_u$$

and variance

$${\rm Var}\big(Z_t^\varepsilon|\mathcal{F}_0\big) = (\sigma_{0,t}^\varepsilon)^2 = \sigma_{\rm ou}^2 \int_0^t \mathcal{K}^\varepsilon(u)^2 du.$$

Therefore

$$\mathrm{Var}\big(\mathbb{E}\big[G'(Z_t^\varepsilon)|\mathcal{F}_0\big]\big) = \mathrm{Var}\Big(\int_{\mathbb{R}} G'\big(\mathbb{E}\big[Z_t^\varepsilon|\mathcal{F}_0\big] + \sigma_{0,t}^\varepsilon z\big)p(z)dz\Big).$$

The random variable $\mathbb{E}\big[Z_t^\varepsilon|\mathcal{F}_0\big]$ is Gaussian with mean zero and variance $(\sigma_{t,\infty}^\varepsilon)^2$ so that

$$\begin{aligned}
\mathrm{Var}\big(\mathbb{E}\big[G'(Z_t^\varepsilon)|\mathcal{F}_0\big]\big) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dzdz'p(z)p(z')\int_{\mathbb{R}}\int_{\mathbb{R}} dudu'p(u)p(u') \\
&\quad\times\Big[G'\big(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z\big) - G'\big(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z\big)\Big] \\
&\quad\times\Big[G'\big(\sigma_{t,\infty}^\varepsilon u + \sigma_{0,t}^\varepsilon z'\big) - G'\big(\sigma_{t,\infty}^\varepsilon u' + \sigma_{0,t}^\varepsilon z'\big)\Big] \\
&\leq \|G''\|_\infty^2(\sigma_{t,\infty}^\varepsilon)^2\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dudu'p(u)p(u')(u-u')^2 \\
&= \|G''\|_\infty^2(\sigma_{t,\infty}^\varepsilon)^2.
\end{aligned} \tag{B.11}$$

Therefore

$$\begin{aligned}
\mathrm{Var}(\vartheta_t^\varepsilon)^{1/2} &\leq \|G''\|_\infty\sigma_{\mathrm{ou}}^2\int_0^{T-t} ds\mathcal{K}^\varepsilon(s)\Big(\int_s^\infty du\mathcal{K}^\varepsilon(u)^2\Big)^{1/2} \\
&\leq \|G''\|_\infty\sigma_{\mathrm{ou}}^2\varepsilon^{1/2}\int_0^{(T-t)/\varepsilon} ds\mathcal{K}(s)\Big(\int_s^\infty du\mathcal{K}(u)^2\Big)^{1/2}.
\end{aligned}$$

Since $\mathcal{K}(s) \leq 1 \wedge Ks^{H-\frac{3}{2}}$, this gives

$$\mathrm{Var}(\vartheta_t^\varepsilon)^{1/2} \leq C\left\{\begin{array}{ll} \varepsilon^{1/2} & \text{if } H < 3/4, \\ \varepsilon^{1/2}\ln(\varepsilon) & \text{if } H = 3/4, \\ \varepsilon^{2-2H} & \text{if } H > 3/4, \end{array}\right. \tag{B.12}$$

uniformly in $t \in [0,T]$, for some constant $C$. This completes the proof of the lemma. ∎

The random term $\phi_t^\varepsilon$ defined by (4.7) has the form

$$\phi_{t,T}^\varepsilon = \mathbb{E}\Big[\int_t^T G(Z_s^\varepsilon)ds\big|\mathcal{F}_t\Big]. \tag{B.13}$$

Here we write explicitly the argument $T$ (maturity) as we will compute the correlations of these random terms for different maturities.

Lemma B.3.

1. *For any $t \leq T$, $\phi_{t,T}^\varepsilon$ is a zero-mean random variable with standard deviation of order $\varepsilon^{1-H}$:*

$$\varepsilon^{2H-2}\mathbb{E}[(\phi_{t,T}^\varepsilon)^2] \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_\phi^2(T-t)^{2H}, \tag{B.14}$$

*where $\sigma_\phi$ is defined by (4.10).*

2. *The covariance function of $\phi_{t,T}^\varepsilon$ has the following limit for any $t \leq T$, $t' \leq T'$, with $t \leq t'$:*

$$\varepsilon^{2H-2}\mathbb{E}[\phi_{t,T}^\varepsilon\phi_{t',T'}^\varepsilon] \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_\phi^2(T-t)^H(T'-t')^H\mathcal{C}_\phi(t,t';T,T'), \tag{B.15}$$

*where the limit correlation is*

$$\mathcal{C}_\phi(t,t';T,T') = \frac{\int_0^\infty du\big[(u+r)^{H-\frac12} - u^{H-\frac12}\big]\big[(u+s)^{H-\frac12} - (u+q)^{H-\frac12}\big]}{\int_0^\infty du\big[(1+u)^{H-\frac12} - u^{H-\frac12}\big]^2}, \tag{B.16}$$

*with*

$$q = \frac{t'-t}{\sqrt{(T-t)(T'-t')}}, \qquad r = \frac{\sqrt{T-t}}{\sqrt{T'-t'}}, \qquad s = \frac{T'-t}{\sqrt{(T-t)(T'-t')}}.$$

3. *As $\varepsilon \to 0$, the random process $\varepsilon^{H-1}\phi^\varepsilon_{t,T}$, $t \le T$, converges in distribution (in the sense of finite-dimensional distributions) to a Gaussian random process $\phi_{t,T}$, $t \le T$, with mean zero and covariance $\varepsilon^{2(H-1)}\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2(T-t)^H(T'-t')^H\mathcal{C}_\phi(t,t';T,T')$ for any $t \in [0,T]$, $t' \in [0,T']$, with $t \le t'$.*
4. *The fourth-order moments of $\varepsilon^{H-1}\phi^\varepsilon_{t,T}$ are uniformly bounded: there exists a constant $K_T$ independent of $\varepsilon$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}[(\phi^\varepsilon_{t,T})^4]^{1/4} \le K_T \varepsilon^{1-H}. \tag{B.17}$$

Note that the mean square increment of the limit process $\phi_{t,T}$ satisfies for $t, t+h \in [0,T]$:

$$\begin{aligned}\mathbb{E}\big[(\phi_{t,T} - \phi_{t+h,T})^2\big] &= \frac{\sigma_{\rm ou}^2}{\Gamma(H+\frac12)^2}\int_0^\infty du\big[(T-t-h+u)^{H-\frac12} - u^{H-\frac12}\big]^2 \\ &\quad -\big[(T-t+u)^{H-\frac12} - (u+h)^{H-\frac12}\big]^2 + \big[(u+h)^{H-\frac12} - u^{H-\frac12}\big]^2 \\ &= \frac{\sigma_{\rm ou}^2(T-t)^{2H-1}}{\Gamma(H+\frac12)^2}h + o(h), \quad h \to 0.\end{aligned} \tag{B.18}$$

This shows that the limit Gaussian process $\phi_{t,T}$ has the same local regularity (as a function of $t$) as a standard Brownian motion. We also have for any $t < T \le T+h$:

$$\mathbb{E}\big[(\phi_{t,T+h} - \phi_{t,T})^2\big] = \frac{\sigma_{\rm ou}^2(T-t)^{2H-2}}{(2-2H)\Gamma(H-\frac12)^2}h^2 + o(h^2), \quad h \to 0. \tag{B.19}$$

This shows that the limit Gaussian process $\phi_{t,T}$ is smooth (mean square differentiable) as a function of the maturity $T$.

*Proof.* Let us fix $T_0 > 0$. For $t \in [0,T]$, $t' \in [0,T']$, with $T, T' \le T_0$, and $t \le t'$, the covariance of $\phi^\varepsilon_{t,T}$ is

$$\begin{aligned}\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) &= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T G(Z^\varepsilon_s)ds\big|\mathcal{F}_t\Big]\mathbb{E}\Big[\int_{t'}^{T'} G(Z^\varepsilon_s)ds\big|\mathcal{F}_{t'}\Big]\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[\int_t^T G(Z^\varepsilon_s)ds\big|\mathcal{F}_t\Big]\mathbb{E}\Big[\int_{t'}^{T'} G(Z^\varepsilon_s)ds\big|\mathcal{F}_t\Big]\Big] \\ &= \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \mathrm{Cov}\big(\mathbb{E}\big[G(Z^\varepsilon_s)|\mathcal{F}_0\big], \mathbb{E}\big[G(Z^\varepsilon_{s'})|\mathcal{F}_0\big]\big).\end{aligned}$$

Then, proceeding as in the proof of the previous lemma,

$$\mathrm{Var}(\phi^\varepsilon_{t,T}) \le \Big(\int_0^{T-t} ds \mathrm{Var}\big(\mathbb{E}\big[G(Z^\varepsilon_s)|\mathcal{F}_0\big]\big)^{1/2}\Big)^2 \le \|G'\|_\infty^2\Big(\int_0^{T-t} ds \sigma^\varepsilon_{s,\infty}\Big)^2.$$

Since $\mathcal{K}(s) \leq 1 \wedge K s^{H-\frac{3}{2}}$, this gives

$$
\mathrm{Var}(\phi^\varepsilon_{t,T}) \leq C_{T_0} \varepsilon^{2-2H},
$$

uniformly in $t \leq T \leq T_0$, for some constant $C_{T_0}$. More precisely, for $t \in [0,T]$, $t' \in [0,T']$, with $T, T' \leq T_0$, and $t \leq t'$, we have

$$
\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) = \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}} \int_{\mathbb{R}} dz dz' p(z) p(z') \\
\times \mathbb{E}\Big[ G\Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + \sigma^\varepsilon_{0,s} z \Big) G\Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + \sigma^\varepsilon_{0,s'} z' \Big) \Big].
\end{aligned}
$$

Using the fact that $\langle G \rangle = 0$, we can write

$$
\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) = & \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}} \int_{\mathbb{R}} dz dz' p(z) p(z') \\
& \times \mathbb{E}\Big[ \Big( G\Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + \sigma^\varepsilon_{0,s} z \Big) - G(\sigma_{\rm ou} z) \Big) \\
& \quad \times \Big( G\Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + \sigma^\varepsilon_{0,s'} z' \Big) - G(\sigma_{\rm ou} z') \Big) \Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) = & \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}} \int_{\mathbb{R}} dz dz' p(z) p(z') G'(\sigma_{\rm ou} z) G'(\sigma_{\rm ou} z') \\
& \times \mathbb{E}\Big[ \Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou}) z \Big) \\
& \quad \times \Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + (\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}) z' \Big) \Big] + V^\varepsilon_3,
\end{aligned}
$$

up to a term $V_3^\varepsilon$ which is of order $\varepsilon^{3-3H}$:

$$
\begin{aligned}
V_3^\varepsilon \le\ & 2\|G'\|_\infty \|G''\|_\infty \int_0^{T-t} ds \int_0^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z) p(z') \\
& \times \mathbb{E}\Big[ \Big( \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u) dW_u + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou}) z\Big)^2 \\
& \quad \times \Big| \sigma_{\rm ou} \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u') dW_{u'} + (\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}) z' \Big| \Big] \\
\le\ & C \|G'\|_\infty \|G''\|_\infty \int_0^{T_0-t} ds \int_0^{T_0-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z) p(z') \\
& \times \Big( \sigma_{\rm ou}^2 \int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)^2 du + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})^2 z^2 \Big) \\
& \quad \times \Big( \sigma_{\rm ou}^2 \int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u')^2 du' + (\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou})^2 z'^2 \Big)^{1/2} \\
\le\ & C' \|G'\|_\infty \|G''\|_\infty \Big[ \int_0^{T_0-t} ds \int_{\mathbb{R}} dz p(z) \Big( (\sigma^\varepsilon_{s,\infty})^2 + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})^2 z^2 \Big) \Big]^{3/2} \\
\le\ & C' \|G'\|_\infty \|G''\|_\infty \Big[ \int_0^{T_0-t} ds (\sigma^\varepsilon_{s,\infty})^2 + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})^2 \Big]^{3/2} \\
\le\ & C' \|G'\|_\infty \|G''\|_\infty \Big[ \int_0^{T_0-t} ds 2 \sigma_{\rm ou} (\sigma_{\rm ou} - \sigma^\varepsilon_{0,s}) \Big]^{3/2} \\
\le\ & C'' \|G'\|_\infty \|G''\|_\infty \varepsilon^{3-3H},
\end{aligned}
$$

where we have used $(\sigma^\varepsilon_{s,\infty})^2 + (\sigma^\varepsilon_{0,s})^2 = \sigma_{\rm ou}^2$ and

$$
\begin{aligned}
|\sigma_{\rm ou} - \sigma^\varepsilon_{0,s}| &= \sigma_{\rm ou} \Big( 1 - \Big( \int_0^{s/\varepsilon} \mathcal{K}(u)^2 du \Big)^{1/2} \Big) = \sigma_{\rm ou} \Big( 1 - \Big( 1 - \int_{s/\varepsilon}^\infty \mathcal{K}(u)^2 du \Big)^{1/2} \Big) \\
&\le \sigma_{\rm ou} \int_{s/\varepsilon}^\infty \mathcal{K}(u)^2 du \le \sigma_{\rm ou} \Big( 1 \wedge K \Big( \frac{s}{\varepsilon} \Big)^{2H-2} \Big). \qquad \text{(B.20)}
\end{aligned}
$$

This gives

$$
\begin{aligned}
\mathrm{Cov}(\phi^\varepsilon_{t,T}, \phi^\varepsilon_{t',T'}) =& \int_0^{T-t} ds \int_{t'-t}^{T'-t} ds' \int_{\mathbb{R}}\int_{\mathbb{R}} dz dz' p(z) p(z') G'(\sigma_{\rm ou} z) G'(\sigma_{\rm ou} z') \\
& \times \Big( \sigma_{\rm ou}^2 \int_0^\infty \mathcal{K}^\varepsilon(s+u) \mathcal{K}^\varepsilon(s'+u) du + (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou})(\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}) z z' \Big) + V_3^\varepsilon \\
=& V_1^\varepsilon \left\langle G' \right\rangle^2 + V_2^\varepsilon \sigma_{\rm ou}^2 \left\langle G'' \right\rangle^2 + V_3^\varepsilon,
\end{aligned}
$$

with

$$
\begin{aligned}
V_1^\varepsilon &= \sigma_{\rm ou}^2 \int_0^\infty du \Big( \int_0^{T-t} ds \mathcal{K}^\varepsilon(s+u) \Big) \Big( \int_{t'-t}^{T'-t} ds' \mathcal{K}^\varepsilon(s'+u) \Big), \\
V_2^\varepsilon &= \Big( \int_0^{T-t} ds (\sigma^\varepsilon_{0,s} - \sigma_{\rm ou}) \Big) \Big( \int_{t'-t}^{T'-t} ds' (\sigma^\varepsilon_{0,s'} - \sigma_{\rm ou}) \Big).
\end{aligned}
$$

Using again (B.20) we find that

$$
V_2^\varepsilon \le C \varepsilon^{4-4H},
$$

while

$$V_1^\varepsilon = \frac{\sigma_{\rm ou}^2}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((T-t+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big) \\ \times\big((T'-t+u)^{H-\frac{1}{2}} - (u+t'-t)^{H-\frac{1}{2}}\big) du\,\varepsilon^{2-2H} \\ +o(\varepsilon^{2-2H}).$$

Applying the change of variable

$$u \to (T-t)^{\frac{1}{2}}(T'-t')^{\frac{1}{2}} u$$

gives the first and second items of the lemma with

$$\sigma_\phi^2 = \frac{\sigma_{\rm ou}^2 \left\langle G'\right\rangle^2}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big)^2 du,$$

which is equivalent to (4.10).

In order to prove the third item we introduce

$$\check\phi^\varepsilon_{t,T} = \mathbb{E}\Big[\int_t^T Z^\varepsilon_s ds \big|\mathcal{F}_t\Big], \tag{B.21}$$

which is a Gaussian random process with mean zero and covariance, for $t\in[0,T]$, $t'\in[0,T']$, with $t\le t'$:

$$\begin{aligned}
{\rm Cov}\big(\check\phi^\varepsilon_{t,T},\check\phi^\varepsilon_{t',T'}\big) &= \int_t^T ds\int_{t'}^{T'} ds' \mathbb{E}\Big[\mathbb{E}[Z^\varepsilon_s|\mathcal{F}_t]\mathbb{E}[Z^\varepsilon_s|\mathcal{F}_{t'}]\Big]\\
&= \int_t^T ds\int_{t'}^{T'} ds' \mathbb{E}\Big[\mathbb{E}[Z^\varepsilon_s|\mathcal{F}_t]\mathbb{E}[Z^\varepsilon_s|\mathcal{F}_{t}]\Big]\\
&= \sigma_{\rm ou}^2\int_0^{T-t} ds\int_{t'-t}^{T'-t} ds' \mathbb{E}\Big[\Big(\int_{-\infty}^0 \mathcal{K}^\varepsilon(s-u)dW_u\Big)\Big(\int_{-\infty}^0 \mathcal{K}^\varepsilon(s'-u)dW_u\Big)\Big]\\
&= \sigma_{\rm ou}^2\int_0^\infty du\Big(\int_0^{T-t} ds\mathcal{K}^\varepsilon(s+u)\Big)\Big(\int_{t'-t}^{T'-t} ds'\mathcal{K}^\varepsilon(s'+u)\Big).
\end{aligned}$$

Therefore, for $t_j\in[0,T_j]$, with $t_1\le\cdots\le t_n$, the random vector $(\varepsilon^{H-1}\left\langle G'\right\rangle\check\phi^\varepsilon_{t_1,T_1},\dots,\varepsilon^{H-1}\left\langle G'\right\rangle\check\phi^\varepsilon_{t_n,T_n})$ converges to a Gaussian random vector with mean 0 and covariance matrix $(\sigma_\phi^2(T_j-t_j)^H(T_l-t_l)^H\mathcal{C}_\phi(t_j,t_l;T_j,T_l))_{j,l=1}^n$. In other words, the random process $\varepsilon^{H-1}\left\langle G'\right\rangle\check\phi^\varepsilon_{t,T}$, $t\le T$, converges in the sense of finite-dimensional distributions to a Gaussian process $\phi_{t,T}$, $t\le T$, with mean 0 and covariance function $\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2(T-t)^H(T'-t')^H\mathcal{C}_\phi(t,t';T,T')$, for $t\in[0,T]$, $t'\in[0,T']$, with $t\le t'$.

Moreover, we have

$${\rm Var}\big(\check\phi^\varepsilon_{t,T}\big) = \frac{\sigma_{\rm ou}^2}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big)^2 du\,(T-t)^{2H}\varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

Similarly,

$$\mathbb{E}\big[\check\phi^\varepsilon_{t,T}\phi^\varepsilon_{t,T}\big] = \frac{\sigma_{\rm ou}^2\left\langle G'\right\rangle}{\Gamma(H+\frac{1}{2})^2}\int_0^\infty \big((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\big)^2 du\,(T-t)^{2H}\varepsilon^{2-2H} + o(\varepsilon^{2-2H}).$$

As a result,

$$\varepsilon^{2H-2}\mathbb{E}\big[(\phi^\varepsilon_{t,T} - \langle G'\rangle\,\check{\phi}^\varepsilon_{t,T})^2\big] \stackrel{\varepsilon\to 0}{\longrightarrow} 0,$$

and the random process $\varepsilon^{H-1}\langle G'\rangle\,\check{\phi}^\varepsilon_{t,T}$, $t \le T$, converges in the sense of finite-dimensional distributions to a Gaussian process $\phi_{t,T}$, $t \le T$, with mean 0 and covariance function $\mathbb{E}[\phi_{t,T}\phi_{t',T'}] = \sigma_\phi^2 (T-t)^H (T'-t')^H \mathcal{C}_\phi(t,t';T,T')$ for $t \in [0,T]$, $t' \in [0,T']$, with $t \le t'$. This gives the third item of the lemma.

To prove the fourth item of the lemma, we note that

$$\phi^\varepsilon_{t,T} = \frac{1}{2}\mathbb{E}\big[I^\varepsilon_T|\mathcal{F}_t\big] - \frac{1}{2}I^\varepsilon_t,$$

where $I^\varepsilon_t$ is defined by (A.6). Therefore

$$\sup_{t\in[0,T]} \mathbb{E}\big[(\phi^\varepsilon_{t,T})^4\big] \le \sup_{t\in[0,T]} \mathbb{E}\big[(I^\varepsilon_t)^4\big],$$

and the result follows from Lemma A.1, Eq. (A.7). □

Lemma B.4. *Let us define for any $t \in [0,T]$:*

$$\gamma^\varepsilon_t = \frac{1}{2}\int_0^t \big((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\big)\phi^\varepsilon_s ds, \tag{B.22}$$

*as in (4.20). We have*

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\gamma^\varepsilon_t)^2\big]^{1/2} = 0. \tag{B.23}$$

*Proof.* Let us define for any $t \in [0,T]$:

$$\Gamma^\varepsilon_t = \int_t^T \big((\sigma^\varepsilon_s)^2 - \overline{\sigma}^2\big)\phi^\varepsilon_s ds. \tag{B.24}$$

By the definition (4.12) of $\phi^\varepsilon_s$, we have

$$\Gamma^\varepsilon_t = 2\int_t^T ds \int_s^T du \mathbb{E}\big[G(Z^\varepsilon_s)G(Z^\varepsilon_u)|\mathcal{F}_s\big].$$

Therefore

$$\begin{aligned}
\mathbb{E}\Big[(\Gamma^\varepsilon_t)^2\Big] &= 2\int_t^T ds \int_s^T du \int_s^T ds' \int_{s'}^T du' \mathbb{E}\Big[\mathbb{E}\big[G(Z^\varepsilon_s)G(Z^\varepsilon_u)|\mathcal{F}_s\big]\mathbb{E}\big[G(Z^\varepsilon_{s'})G(Z^\varepsilon_{u'})|\mathcal{F}_{s'}\big]\Big] \\
&= 2\int_t^T ds \int_s^T du \int_s^T ds' \int_{s'}^T du' \mathbb{E}\Big[G(Z^\varepsilon_s)G(Z^\varepsilon_u)\mathbb{E}\big[G(Z^\varepsilon_{s'})G(Z^\varepsilon_{u'})|\mathcal{F}_s\big]\Big] \\
&= \int_t^T ds \int_s^T du \mathbb{E}\Big[G(Z^\varepsilon_s)G(Z^\varepsilon_u)\mathbb{E}\big[\big(\int_s^T G(Z^\varepsilon_{s'})ds'\big)^2\big|\mathcal{F}_s\big]\Big] \\
&= \int_t^T ds \mathbb{E}\Big[G(Z^\varepsilon_s)\mathbb{E}\big[\int_s^T G(Z^\varepsilon_u)du\big|\mathcal{F}_s\big]\mathbb{E}\big[\big(\int_s^T G(Z^\varepsilon_{s'})ds'\big)^2\big|\mathcal{F}_s\big]\Big] \\
&\le \|G\|_\infty \int_t^T ds \mathbb{E}\Big[\Big|\mathbb{E}\big[\big(\int_s^T G(Z^\varepsilon_{s'})ds'\big)^2\big|\mathcal{F}_s\big]\Big|^{3/2}\Big] \\
&\le \|G\|_\infty \int_t^T ds \mathbb{E}\Big[\Big|\int_s^T G(Z^\varepsilon_{s'})ds'\Big|^3\Big] \\
&\le \|G\|_\infty \int_t^T ds \mathbb{E}\Big[\Big(\int_s^T G(Z^\varepsilon_{s'})ds'\Big)^4\Big]^{3/4},
\end{aligned}$$

which is smaller than $\varepsilon^{3-3H}$ by Lemma A.1. This proves

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\Gamma_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.25}$$

Note that $\gamma_t^\varepsilon$ defined by (4.20) is related to $\Gamma_t^\varepsilon$ through

$$\gamma_t^\varepsilon = 2\left(\Gamma_0^\varepsilon - \Gamma_t^\varepsilon\right),$$

therefore Eq. (B.25) also implies

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\gamma_t^\varepsilon)^2\big]^{1/2} = 0,$$

which is the desired result. □

LEMMA B.5. *Let us define for any $t \in [0,T]$:*

$$\eta_t^\varepsilon = \varepsilon^{1-H} \int_0^t \big(\sigma_s^\varepsilon - \widetilde{\sigma}\big) ds, \tag{B.26}$$

*as in (4.22). We have*

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\eta_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.27}$$

*Proof.* By Lemma 3.1,

$$\begin{aligned}
\mathbb{E}\big[(\eta_t^\varepsilon)^2\big] &= \varepsilon^{2-2H} \mathbb{E}\Big[\Big(\int_0^t \big(\sigma_s^\varepsilon - \widetilde{\sigma}\big) ds\Big)^2\Big] \\
&= \varepsilon^{2-2H} \int_0^t \int_0^t \mathrm{Cov}\big(F(Z_s^\varepsilon), F(Z_{s'}^\varepsilon)\big) ds ds' \\
&= \varepsilon^{2-2H} \big( \langle F^2 \rangle - \langle F \rangle^2 \big) \int_0^t \int_0^t \mathcal{C}_\sigma\big(\frac{s-s'}{\varepsilon}\big) ds ds' \\
&\leq K \varepsilon^{2-2H} \int_0^T \int_0^T \big(\frac{|s-s'|}{\varepsilon}\big)^{2H-2} ds ds' \\
&\leq K' \varepsilon^{4-4H},
\end{aligned}$$

for some constants $K, K'$, because $s^{2H-2}$ in integrable over $(0,T)$, which gives the desired result. □

LEMMA B.6. *Let us define for any $t \in [0,T]$:*

$$\kappa_t^\varepsilon = \frac{\varepsilon^{1-H}}{2} \int_0^t \big((\sigma_s^\varepsilon)^2 - \overline{\sigma}^2\big) ds, \tag{B.28}$$

*as in (4.21). We have*

$$\limsup_{\varepsilon\to 0} \varepsilon^{H-1} \sup_{t\in[0,T]} \mathbb{E}\big[(\kappa_t^\varepsilon)^2\big]^{1/2} = 0. \tag{B.29}$$

*Proof.* The proof is similar to the one of Lemma B.5. □

**Appendix C.**

E. Alòs, J. A. León, and J. Vives, On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility, Finance Stoch. **11** (2007), pp. 571–589. 4

E. Alòs and Y. Yang, A closed-form option pricing approximation formula for a fractional Heston model, http://repositori.upf.edu/bitstream/handle/10230/22737/1446.pdf?sequence=1. 4

J.-M. Bardet, G. Lang, G. Oppenheim, A. Philippe, and M. S. Taqqu, Generators of the long-range dependence processes: a survey, in *Theory and applications of long-range dependence*, P. Doukhan, G. Oppenheim, and M. S. Taqqu, eds., Birkhauser (2003), pp. 579–624. 9

D. S. Bates, Jumps and stochastic volatility: exchange rate processes implicit in Deutsche Mark options, Rev. Finance Stud., **9** (1996), pp. 69–107. 4

F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer, London, 2008. 5

T. Bjork and H. Hult, A note on Wick products and the fractional Black-Scholes model, Fin. and Stochastics **9** (2005), pp. 197–209. 4

F. Black, Studies in Stock Price Volatility Changes, *Proceedings of the 1976 Business Meeting of the Business and Economic Statistics Section*, American Statistical Association, 1976, pp. 177–181. 5

T. Bollerslev and H.O. Mikkelsen, Long-term equity anticipation securities and stock market volatility dynamics, Journal of Econometrics **92** (1999), pp. 75–99. 3

T. Bollerslev, D. Osterrieder, N. Sizova, and G. Tauchen, Risk and return: Long-run relations, fractional cointegration, and return predictability, Journal of Financial Economics **108** (2013), pp. 409–424. 1, 3

F. J. Breidt, N. Crato, and P. De Lima, The detection and estimation of long-memory in stochastic volatility, Journal of Econometrics **83** (1998), pp. 325–348. 1, 2

P. Carr and L. Wu, What type of processes underlies options? A simple robust test, Journal of Finance **58** (2003), pp. 2581–2610. 3

C. Cheridito, H. Kawaguchi, and M. Maejima, Fractional Ornstein-Uhlenbeck processes, Electronic Journal of Probability **8** (2003), pp. 1–14. 5

A. Chronopoulou and F. G. Viens, Estimation and pricing under long-memory stochastic volatility, Annals of Finance **8** (2012), pp. 379–403. 1, 3

A. Chronopoulou and F. G. Viens, Stochastic volatility models with long-memory in discrete and continuous time, Quantitative Finance **12** (2012), pp. 635–649. 3

F. Comte, L. Coutin, and E. Renault, Affine fractional stochastic volatility models, Annals of Finance **8** (2012), pp. 337–378. 3

F. Comte and E. Renault, Long memory in continuous-time stochastic volatility models, Mathematical Finance **8** (1998), pp. 291–323. 3

R. Cont, Empirical properties of asset returns: Stylized facts and statistical issues, Quant. Finance **1** (2001), pp. 1–14. 1, 2

R. Cont, Long range dependence in financial markets, in Fractals in Engineering, edited by J. Lévy Véhel and E. Lutton, Springer, London, 2005, pp. 159–179. 1

S. Corley, J. Lebovits and J. Lévy Véhel, Multifractional Stochastic volatility models, Mathematical Finance **24** (2014), pp. 364–402. 4

L. Coutin, An introduction to (stochastic) calculus with respect to fractional Brownian motion, in Séminaire de Probabilités XL, Lecture Notes in Mathematics, Vol. 1899, Springer, 2007, pp. 3–65. 5

P. Doukhan, G. Oppenheim, and M. S. Taqqu, *Theory and Applications of Long-Range Dependence*, Birkhäuser, Boston, 2003. 5

R. F. Engle, and A. J. Patton, What good is a volatility model?, Quantitative Finance **1** (2001), pp. 237–245. 1, 2

J. E. Figueroa-López and S. Olafsson, Short-time expansions for close-to-the-money options under a Lévy jump model with stochastic volatility, arXiv:1404.0601.

M. Forde and H. Zhang, Asymptotics for rough stochastic volatility and Lévy models, preprint available at http://www.mth.kcl.ac.uk/~fordem/. 4

J. P. Fouque, G. Papanicolaou, and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, 2000. 7

J. P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Sølna, *Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives*, Cambridge University Press, Cambridge, 2011. 1, 2, 23

J. P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Sølna, Maturity cycles in implied volatility, Finance and Stochastics **8** (2004), pp. 451–477. 2

J. P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Sølna, Short time scales in S&P500 volatility, The Journal of Computational Finance **6** (2003), p. 1-24. 2, 4

M. Fukasawa, Asymptotic analysis for stochastic volatility: martingale expansion, Finance and Stochastics **15** (2011), pp. 635–654. 4

J. Garnier and K. Solna, Correction to Black-Scholes formula due to fractional stochastic volatility, arXiv:1509.01175. 1, 2, 4, 7

J. Gatheral, *The Volatility Surface: a Practitioner's Guide*, Wiley, Hoboken, 2006. 1

E. Ghysels, A. Harvey, and E. Renaud, Stochastic volatility, in Handbook of Statistics, Vol. 14, Statistical Methods in Finance, edited by G. S. Maddala and C. R. Rao, North Holland, Amsterdam, 1995, pp. 119–191. 1

H. Guennoun, A. Jacquier, and P. Roome, Aymptotic behaviour of the fractional Heston model, arXiv:1411.7653. 3

A. Gulisashvili, *Analytically Tractable Stochastic Stock Price Models*, Springer, Berlin, 2012. 1

A. Gulisashvili, F. Viens, and X. Zhang, Small-time asymptotics for Gaussian self-similar stochastic volatility models, arXiv:1505.05256. 3

P. Henry-Labordére, *Analysis, Geometry, and Modeling in Finance: Advanced Methods in Option Pricing*, Chapman & Hall, Boca Raton, 2009. 1

T. Kaarakka and P. Salminen, On fractional Ornstein-Uhlenbeck processes, Communications on Stochastic Analysis **5** (2011), pp. 121–133. 5

A. Lewis, *Option Valuation under Stochastic Volatility*, Finance Press, Newport Beach, 2000.

B. B. Mandelbrot and J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, SIAM Review **10** (1968), pp. 422–437. 3, 5, 8

R. V. Mendes, M. J. Oliveira, and A. M. Rodrigues, The fractional volatility model: no-arbitrage, leverage and completeness, Physica A : Statistical Mechanics and its Applications **419** (2015), pp. 470–478. 4

A. Mijatovic and P. Tankov, A new look at short-term implied volatility in asset price models with jumps, Mathematical Finance, **26:1** (2016), pp. 149–183. 3

G. Oh, S. Kim, and C. Eom, Long-term memory and volatility clustering in high-frequency price changes, Physica A: Statistical Mechanics and its Applications, **387** (2008), pp. 1247–1254. 1

H. Omre, K. Sølna, and H. Tjelmeland, Simulation of random functions on large lattices, In Geostatistics, edited by A. Soares, Kluwer Academic Publishers, 1993, pp. 179–199. 9

R. Rebonato, *Volatility and Correlation: The Perfect Hedger and the Fox*, John Wiley

& Sons, Chichester, 2004. 1

L. C. G. Rogers, Arbitrage with fractional Brownian motion, Mathematical Finance **7** (1997), pp. 95–105. 4

A. Shiryaev, On arbitrage and replication for fractal models, Research report 20, MaPhySto, Department of Mathematical Sciences, University of Aarhus, Denmark, 1998. 4

S. Sundaresan, Continuous-time methods in Finance: a review and an assessment, Journal of Finance **4** (2000), pp. 1569–1622. 3

M. S. Taqqu, A representation for self-similar processes, Stochastic Processes and their Applications **7** (1978), pp. 55–64. 29