Stochastic Modelling and Applied Probability

(Formerly:
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56

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Wave Propagation and Time Reversal in Randomly Layered Media

Springer

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To our families

Preface

Our motivation for writing this book is twofold: First, the theory of waves propagating in randomly layered media has been studied extensively during the last thirty years but the results are scattered in many different papers. This theory is now in a mature state, especially in the very interesting regime of separation of scales as introduced by G. Papanicolaou and his coauthors and described in [8], which is a building block for this book. Second, we were motivated by the time-reversal experiments of M. Fink and his group in Paris. They were done with ultrasonic waves and have attracted considerable attention because of the surprising effects of enhanced spatial focusing and time compression in random media. An exposition of this work and its applications is presented in [56]. Time reversal experiments were also carried out with sonar arrays in shallow water by W. Kuperman [113] and his group in San Diego. The enhanced spatial focusing and time compression of signals in time reversal in random media have many diverse applications in detection and in focused energy delivery on small targets as, for example, in the destruction of kidney stones. Enhanced spatial focusing is also useful in sonar and wireless communications for reducing interference. Time reversal ideas have played an important role in the development of new methods for array imaging in random media as presented in [19]. A quantitative mathematical analysis is crucial in the understanding of these phenomena and for the development of new applications. In a series of recent papers by the authors and their coauthors, starting with [40] in the one-dimensional case and [16] in the multidimensional case, a complete analysis of time reversal in random media has been proposed in the two extreme cases of strongly scattering layered media, and weak fluctuations in the parabolic approximation regime. These results are important in the understanding of the intermediate situations and will contribute to future applications of time reversal.

Wave propagation in three-dimensional random media has been studied mostly by perturbation techniques when the random inhomogeneities are small. The main results are that the amplitude of the mean waves decreases with distance traveled, because coherent wave energy is converted into incoherent fluctuations, while the mean energy propagates diffusively or by radiative transport. These phenomena are analyzed extensively from a physical and engineering point of view in the book of Ishimaru [90]. It was first noted by Anderson [5] that for electronic waves in strongly disordered materials there is wave localization. This means that wave energy does not propagate, because the random inhomogeneities trap it in finite regions. What is different and special in one-dimensional random media is that wave localization always occurs, even when the inhomogeneities are weak. This means that there is never a diffusive or transport regime in one-dimensional random media. This was first proved by Goldsheid, Molchanov, and Pastur in [79]. It is therefore natural that the analysis of waves in one-dimensional or strongly anisotropic layered media presented in this book should rely on methods and techniques that are different from those used in general, multidimensional random media.

The content of this book is multidisciplinary and presents many new physically interesting results about waves propagating in randomly layered media as well as applications in time reversal. It uses mathematical tools from probability and stochastic processes, partial differential equations, and asymptotic analysis, combined with the physics of wave propagation and modeling of time-reversal experiments. It addresses an interdisciplinary audience of students and researchers interested in the intriguing phenomena related to waves propagating in random media. We have tried to gradually bring together ideas and tools from all these areas so that no special background is required. The book can also be used as a textbook for advanced topics courses in which random media and related homogenization, averaging, and diffusion approximation methods are involved. The analytical results discussed here are proved in detail, but we have chosen to present them with a series of explanatory and motivating steps instead of a "theorem-proof" format. Most of the results in the book are illustrated with numerical simulations that are carefully calibrated to be in the regimes of the corresponding asymptotic analysis. At the end of each chapter we give references and additional comments related to the various results that are presented.

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Contents

1 Introduction and Overview of the Book 1

2 Waves in Homogeneous Media 9
- 2.1 Acoustic Wave Equations 9
  - 2.1.1 Conservation Equations in Fluid Dynamics 9
  - 2.1.2 Linearization 10
  - 2.1.3 Hyperbolicity 11
  - 2.1.4 The One-Dimensional Wave Equation 12
  - 2.1.5 Solution of the Three-Dimensional Wave Equation by Spherical Means 14
  - 2.1.6 The Three-Dimensional Wave Equation With Source . . . 17
  - 2.1.7 Green's Function for the Acoustic Wave Equations 19
  - 2.1.8 Energy Density and Energy Flux 21
- 2.2 Wave Decompositions in Three-Dimensional Media 22
  - 2.2.1 Time Harmonic Waves 22
  - 2.2.2 Plane Waves 23
  - 2.2.3 Spherical Waves 24
  - 2.2.4 Weyl's Representation of Spherical Waves 25
  - 2.2.5 The Acoustic Wave Generated by a Point Source 27
- 2.3 Appendix 29
  - 2.3.1 Gauss–Green Theorem 29
  - 2.3.2 Energy Conservation Equation 30

3 Waves in Layered Media 33
- 3.1 Reduction to a One-Dimensional System 33
- 3.2 Right- and Left-Going Waves 34
- 3.3 Scattering by a Single Interface 36
- 3.4 Single-Layer Case 39
  - 3.4.1 Mathematical Setup 39
  - 3.4.2 Reflection and Transmission Coefficient for a Single Layer 41

3.4.3 Frequency-Dependent Reflectivity and Antireflection Layer . . . 43
3.4.4 Scattering by a Single Layer in the Time Domain . . . 44
3.4.5 Propagator and Scattering Matrices . . . 47
3.5 Multilayer Piecewise-Constant Media . . . 48
3.5.1 Propagation Equations . . . 48
3.5.2 Reflected and Transmitted Waves . . . 51
3.5.3 Reflectivity Pattern and Bragg Mirror for Periodic Layers . . . 54
3.5.4 Goupillaud Medium . . . 57

4 Effective Properties of Randomly Layered Media . . . 61
4.1 Finely Layered Piecewise-Constant Media . . . 62
4.1.1 Periodic Case . . . 63
4.1.2 Random Case . . . 65
4.1.3 Conclusion . . . 68
4.2 Random Media Varying on a Fine Scale . . . 68
4.3 Boundary Conditions and Equations for Right- and Left-Going Modes . . . 70
4.3.1 Modes Along Local Characteristics . . . 72
4.3.2 Modes Along Constant Characteristics . . . 73
4.4 Centering the Modes and Propagator Equations . . . 75
4.4.1 Characteristic Lines . . . 75
4.4.2 Modes in the Fourier Domain . . . 76
4.4.3 Propagator . . . 77
4.4.4 The Riccati Equation for the Local Reflection Coefficient 79
4.4.5 Reflection and Transmission in the Time Domain . . . 81
4.4.6 Matched Medium . . . 81
4.5 Homogenization and the Law of Large Numbers . . . 82
4.5.1 A Simple Discrete Random Medium . . . 82
4.5.2 Random Differential Equations . . . 85
4.5.3 The Effective Medium . . . 88

5 Scaling Limits . . . 91
5.1 Identification of the Scaling Regimes . . . 92
5.1.1 Modeling of the Medium Fluctuations . . . 92
5.1.2 Modeling of the Source Term . . . 94
5.1.3 The Dimensionless Wave Equations . . . 95
5.1.4 Scaling Limits . . . 96
5.1.5 Right- and Left-Going Waves . . . 98
5.1.6 Propagator and Reflection and Transmission Coefficients 100
5.2 Diffusion Scaling . . . 102
5.2.1 White-Noise Regime and Brownian Motion . . . 103
5.2.2 Diffusion Approximation . . . 104

5.2.3 Finite-Dimensional Distributions of the Transmitted Wave . . . 106

6 Asymptotics for Random Ordinary Differential Equations . 109
6.1 Markov Processes . . . 110
6.1.1 Semigroups . . . 110
6.1.2 Infinitesimal Generators . . . 111
6.1.3 Martingales and Martingale Problems . . . 111
6.1.4 Kolmogorov Backward and Forward Equations . . . 113
6.1.5 Ergodicity . . . 115
6.2 Markovian Models of Random Media . . . 116
6.2.1 Two-Component Composite Media . . . 116
6.2.2 Multicomponent Composite Media . . . 118
6.2.3 A Continuous Random Medium . . . 120
6.3 Diffusion Approximation Without Fast Oscillation . . . 122
6.3.1 Markov Property . . . 123
6.3.2 Perturbed Test Functions . . . 124
6.3.3 The Poisson Equation and the Fredholm Alternative . . . 124
6.3.4 Limiting Infinitesimal Generator . . . 126
6.3.5 Relative Compactness of the Laws of the Processes . . . 131
6.3.6 The Multiplicative-Noise Case . . . 134
6.4 The Averaging and Fluctuation Theorems . . . 135
6.4.1 Averaging . . . 135
6.4.2 Fluctuation Theory . . . 136
6.5 Diffusion Approximation with Fast Oscillations . . . 139
6.5.1 Semifast Oscillations . . . 139
6.5.2 Fast Oscillations . . . 142
6.6 Stochastic Calculus . . . 145
6.6.1 Stochastic Integrals . . . 147
6.6.2 Itô's Formula . . . 150
6.6.3 Stochastic Differential Equations . . . 152
6.6.4 Diffusions and Partial Differential Equations . . . 153
6.6.5 Feynman–Kac Representation Formula . . . 155
6.7 Limits of Random Equations and Stochastic Equations . . . 156
6.7.1 Itô Form of the Limit Process . . . 156
6.7.2 Stratonovich Stochastic Integrals . . . 158
6.7.3 Limits of Random Matrix Equations . . . 160
6.8 Lyapunov Exponent for Linear Random Differential Equations 161
6.8.1 Lyapunov Exponent of the Random Differential Equation . . . 162
6.8.2 Lyapunov Exponent of the Limit Diffusion . . . 169
6.9 Appendix . . . 172
6.9.1 Quadratic Variation of a Continuous Martingale . . . 172

7 Transmission of Energy Through a Slab of Random Medium.... 175
7.1 Transmission of Monochromatic Waves 176
7.1.1 The Diffusion Limit for the Propagator............... 177
7.1.2 Polar Coordinates for the Propagator 180
7.1.3 Martingale Representation of the Transmission Coefficient 183
7.1.4 The Localization Length $L_{\mathrm{loc}}(\omega)$ 185
7.1.5 Mean and Fluctuations of the Power Transmission Coefficient 187
7.1.6 The Strongly Fluctuating Character of the Power Transmission Coefficient 188
7.2 Exponential Decay of the Transmitted Energy for a Pulse 190
7.2.1 Transmission of a Pulse Through a Slab of Random Medium 190
7.2.2 Self-Averaging Property of the Transmitted Energy 191
7.2.3 The Diffusion Limit for the Two-Frequency Propagator 193
7.3 Wave Localization in the Weakly Heterogeneous Regime...... 196
7.3.1 Determination of the Power Transmission Coefficient from a Random Harmonic Oscillator 196
7.3.2 Comparisons of Decay Rates 198
7.4 Wave Localization in the Strongly Heterogeneous White-Noise Regime... 199
7.5 The Random Harmonic Oscillator 201
7.5.1 The Lyapunov Exponent of the Random Harmonic Oscillator 202
7.5.2 Expansion of the Lyapunov Exponent in the Strongly Heterogeneous Regime 203
7.5.3 Expansion of the Lyapunov Exponent in the Weakly Heterogeneous Regime 208
7.6 Appendix. Statistics of the Power Transmission Coefficient.... 209
7.6.1 The Probability Density of the Power Transmission Coefficient 209
7.6.2 Moments of the Power Transmission Coefficient 211

8 Wave-Front Propagation 215
8.1 The Transmitted Wave Front in the Weakly Heterogeneous Regime... 216
8.1.1 Stabilization of the Transmitted Wave Front 217
8.1.2 The Integral Equation for the Transmitted Field 220
8.1.3 Asymptotic Analysis of the Transmitted Wave Front ... 222
8.2 The Transmitted Wave Front in the Strongly Heterogeneous Regime... 225
8.2.1 Asymptotic Representation of the Transmitted Wave Front 226
8.2.2 The Energy of the Transmitted Wave 229

8.2.3 Numerical Illustration of Pulse Spreading 230
8.2.4 The Diffusion Limit for the Multifrequency Propagators 230
8.2.5 Martingale Representation of the Multifrequency Transmission Coefficient 233
8.2.6 Identification of the Limit Wave Front 234
8.2.7 Asymptotic Analysis of Travel Times 236
8.3 The Reflected Front in Presence of an Interface 238
8.3.1 Integral Representation of the Reflected Pulse 238
8.3.2 The Limit for the Reflected Front 242
8.4 Appendix. Proof of the Averaging Theorem 245

9 Statistics of Incoherent Waves 249
9.1 The Reflected Wave 249
9.1.1 Reformulation of the Reflection and Transmission Problem 249
9.1.2 The Riccati Equation for the Reflection Coefficient 252
9.1.3 Representation of the Reflected Field 253
9.2 Statistics of the Reflected Wave in the Frequency Domain 254
9.2.1 Moments of the Reflection Coefficient 254
9.2.2 Probabilistic Representation of the Transport Equations 258
9.2.3 Explicit Solution for a Random Half-Space 261
9.2.4 Multifrequency Moments 262
9.3 Statistics of the Reflected Wave in the Time Domain 266
9.3.1 Mean Amplitude 266
9.3.2 Mean Intensity 266
9.3.3 Autocorrelation and Time-Domain Localization 267
9.3.4 Gaussian Statistics 269
9.4 The Transmitted Wave 272
9.4.1 Autocorrelation Function of the Transmission Coefficient 272
9.4.2 Probabilistic Representation of the Transport Equations 274
9.4.3 Statistics of the Transmitted Wave in the Time Domain 277

10 Time Reversal in Reflection and Spectral Estimation 281
10.1 Time Reversal in Reflection 283
10.1.1 Time-Reversal Setup 283
10.1.2 Time-Reversal Refocusing 285
10.1.3 The Limiting Refocused Pulse 286
10.1.4 Time-Reversal Mirror Versus Standard Mirror 290
10.2 Time Reversal Versus Cross Correlations 291
10.2.1 The Empirical Correlation Function 292
10.2.2 Measuring the Spectral Density 293
10.2.3 Signal-to-Noise Ratio Comparison 294
10.3 Calibrating the Initial Pulse 302

11 Applications to Detection 305
11.1 Detection of a Weak Reflector 306
11.2 Detection of an Interface Between Media 311
11.3 Waves in One-Dimensional Dissipative Random Media 313
11.3.1 The Acoustic Model with Random Dissipation 313
11.3.2 Propagator Formulation 314
11.3.3 Transmitted Wave Front 317
11.3.4 The Refocused Pulse for Time Reversal in Reflection 317
11.4 Application to the Detection of a Dissipative Layer 320
11.4.1 Constant Mean Dissipation 321
11.4.2 Thin Dissipative Layer 321
11.4.3 Thick Dissipative Layer 324

12 Time Reversal in Transmission 327
12.1 Time Reversal of the Stable Front 328
12.1.1 Time-Reversal Experiment 329
12.1.2 The Refocused Pulse 331
12.2 Time Reversal with Coda Waves 333
12.2.1 Time-Reversal Experiment 333
12.2.2 Decomposition of the Refocusing Kernel 335
12.2.3 Midband Filtering by the Medium 336
12.2.4 Low-Pass Filtering 337
12.3 Discussion and Numerical Simulations 339

13 Application to Communications 343
13.1 Review of Basic Communications Schemes 344
13.1.1 Nyquist Pulse 344
13.1.2 Signal-to-Interference Ratio 345
13.1.3 Modulated Nyquist Pulse 346
13.2 Communications in Random Media Using Nyquist Pulses 347
13.2.1 Direct Transmission 350
13.2.2 Communications Using Time Reversal 351
13.2.3 SIRs for Coherent Pulses 353
13.2.4 Influence of the Incoherent Waves 355
13.2.5 Numerical Simulations 357
13.3 Communications in Random Media Using Modulated Nyquist Pulses 358
13.3.1 SIRs of Modulated Nyquist Pulses 359
13.3.2 Numerical Simulations 362
13.3.3 Discussion 363

14 Scattering by a Three-Dimensional Randomly Layered Medium 365
14.1 Acoustic Waves in Three Dimensions 366
14.1.1 Homogenization Regime 366

14.1.2 The Diffusion Approximation Regime 368
14.1.3 Plane-Wave Fourier Transform 369
14.1.4 One-Dimensional Mode Problems 370
14.1.5 Transmitted-Pressure Integral Representation 374
14.2 The Transmitted Wave Front 374
14.2.1 Characterization of Moments 374
14.2.2 Stationary-Phase Point 376
14.2.3 Characterization of the Transmitted Wave Front 378
14.3 The Mean Reflected Intensity Generated by a Point Source . . . 380
14.3.1 Reflected-Pressure Integral Representation 380
14.3.2 Autocorrelation Function of the Reflection Coefficient at Two Nearby Slownesses and Frequencies 381
14.3.3 Asymptotics of the Mean Intensity 385
14.4 Appendix: Stationary-Phase Method 389

15 Time Reversal in a Three-Dimensional Layered Medium . . . 393
15.1 The Embedded-Source Problem 393
15.2 Time Reversal with Embedded Source 395
15.2.1 Emission from a Point Source 395
15.2.2 Recording, Time Reversal, and Reemission 401
15.2.3 The Time-Reversed Wave Field 403
15.3 Homogeneous Medium 405
15.3.1 The Field Recorded at the Surface 406
15.3.2 The Time-Reversed Field 407
15.4 Complete Description of the Time-Reversed Field in a Random Medium 411
15.4.1 Expectation of the Refocused Pulse 413
15.4.2 Refocusing of the Pulse 414
15.5 Refocusing Properties in a Random Medium 416
15.5.1 The Case $|z_s| \ll L_{\mathrm{loc}}$ 416
15.5.2 Time Reversal of the Front 417
15.5.3 Time Reversal of the Incoherent Waves with Offset 417
15.5.4 Time Reversal of the Incoherent Waves Without Offset . 422
15.5.5 Record of the Pressure Signal 424
15.6 Appendix A: Moments of the Reflection and Transmission Coefficients 424
15.6.1 Autocorrelation Function of the Transmission Coefficient at Two Nearby Slownesses and Frequencies . 424
15.6.2 Shift Properties 425
15.6.3 Generalized Coefficients 426
15.7 Appendix B: A Priori Estimates for the Generalized Coefficients 428
15.8 Appendix C: Derivation of (15.74) 430

16 Application to Echo-Mode Time Reversal . . . 435
16.1 The Born Approximation for an Embedded Scatterer . . . 435
16.1.1 Integral Expressions for the Wave Fields . . . 437
16.2 Asymptotic Theory for the Scattered Field . . . 439
16.2.1 The Primary Field . . . 439
16.2.2 The Secondary Field . . . 440
16.3 Time Reversal of the Recorded Wave . . . 442
16.3.1 Integral Representation of the Time-Reversed Field . . . 442
16.3.2 Refocusing in the Homogeneous Case . . . 444
16.3.3 Refocusing of the Secondary Field in the Random Case 446
16.3.4 Contributions of the Other Wave Components . . . 451
16.4 Time-Reversal Superresolution with a Passive Scatterer . . . 451
16.4.1 The Refocused Pulse Shape . . . 451
16.4.2 Superresolution with a Random Medium . . . 453

17 Other Layered Media . . . 457
17.1 Nonmatched Effective Medium . . . 457
17.1.1 Boundary and Jump Conditions . . . 458
17.1.2 Transmission of a Pulse through a Nonmatched Random Slab . . . 459
17.1.3 Reflection by a Nonmatched Random Half-Space . . . 464
17.2 General Background . . . 466
17.2.1 Mode Decomposition . . . 467
17.2.2 Transport Equations . . . 469
17.2.3 Applications . . . 471
17.3 Medium with Random Density Fluctuations . . . 472
17.3.1 The Coupled-Propagator White-Noise Model . . . 474
17.3.2 The Transmitted Field . . . 479
17.3.3 Transport Equations . . . 482
17.3.4 Reflection by a Random Half-Space . . . 484

18 Other Regimes of Propagation . . . 487
18.1 The Weakly Heterogeneous Regime in Randomly Layered Media . . . 487
18.1.1 Mode Decomposition . . . 488
18.1.2 Transport Equations . . . 490
18.1.3 Applications . . . 490
18.2 Dispersive Media . . . 492
18.2.1 The Terrain-Following Boussinesq Model . . . 493
18.2.2 The Propagating Modes of the Boussinesq Equation . . . 494
18.2.3 Mode Propagation in a Dispersive Random Medium . . . 495
18.2.4 Transport Equations . . . 497
18.2.5 Time Reversal . . . 498
18.3 Nonlinear Media . . . 499
18.3.1 Shallow-Water Waves with Random Depth . . . 500

18.3.2 The Linear Hyperbolic Approximation . . . 502
18.3.3 The Effective Equation for the Nonlinear Front Pulse . . 504
18.3.4 Analysis of the Pseudospectral Operator . . . 508
18.3.5 Time Reversal . . . 509
18.4 Time Reversal with Changing Media . . . 510
18.4.1 The Experiment . . . 510
18.4.2 Convergence of the Finite-Dimensional Distributions . . . 511
18.4.3 Convergence of the Refocused Pulse . . . 515

19 The Random Schrödinger Model . . . 519
19.1 Linear Regime . . . 519
19.1.1 The Linear Schrödinger Equation . . . 519
19.1.2 Transmission of a Monochromatic Wave . . . 521
19.1.3 Transmission of a Pulse . . . 526
19.2 Nonlinear Regime . . . 528
19.2.1 Waves Called Solitons . . . 528
19.2.2 Dispersion and Nonlinearity . . . 531
19.2.3 The Nonlinear Schrödinger Equation . . . 532
19.2.4 Soliton Propagation in Random Media . . . 536
19.2.5 Reduction of Wave Localization by Nonlinearity . . . 540

20 Propagation in Random Waveguides . . . 545
20.1 Propagation in Homogeneous Waveguides . . . 547
20.1.1 Modeling of the Waveguide . . . 547
20.1.2 The Propagating and Evanescent Modes . . . 548
20.1.3 Excitation Conditions for an Incoming Wave . . . 550
20.1.4 Excitation Conditions for a Source . . . 550
20.2 Mode Coupling in Random Waveguides . . . 551
20.2.1 Coupled Amplitude Equations . . . 553
20.2.2 Conservation of Energy Flux . . . 554
20.2.3 Evanescent Modes in Terms of Propagating Modes . . . 556
20.2.4 Propagating-Mode-Amplitude Equations . . . 557
20.2.5 Propagator Matrices . . . 558
20.2.6 The Forward-Scattering Approximation . . . 561
20.3 The Time-Harmonic Problem . . . 562
20.3.1 The Coupled Mode Diffusion Process . . . 562
20.3.2 Mean Mode Amplitudes . . . 564
20.3.3 Coupled Power Equations . . . 564
20.3.4 Fluctuations Theory . . . 566
20.4 Broadband Pulse Propagation in Waveguides . . . 567
20.4.1 Integral Representation of the Transmitted Field . . . 567
20.4.2 Broadband Pulse Propagation in a Homogeneous Waveguide . . . 569
20.4.3 The Stable Wave Field in a Random Waveguide . . . 569
20.5 Time Reversal for a Broadband Pulse . . . 571

20.5.1 Time Reversal in Waveguides 571
20.5.2 Integral Representation of the Broadband Refocused Field 573
20.5.3 Refocusing in a Homogeneous Waveguide 574
20.5.4 Refocusing in a Random Waveguide 575
20.6 Statistics of the Transmission Coefficients at Two Nearby Frequencies 579
20.6.1 Transport Equations for the Autocorrelation Function of the Transfer Matrix 579
20.6.2 Probabilistic Representation of the Transport Equations 582
20.7 Incoherent Wave Fluctuations in the Broadband Case 584
20.8 Narrowband Pulse Propagation in Waveguides 587
20.8.1 Narrowband Pulse Propagation in a Homogeneous Waveguide 588
20.8.2 The Mean Field in a Random Waveguide 588
20.8.3 The Mean Intensity in a Random Waveguide 589
20.9 Time Reversal for a Narrowband Pulse 590
20.9.1 Refocusing in a Homogeneous Waveguide 591
20.9.2 The Mean Refocused Field in a Random Waveguide ... 591
20.9.3 Statistical Stability of the Refocused Field 592
20.9.4 Numerical Illustration of Spatial Focusing and Statistical Stability in Narrowband Time Reversal 594

References 599

Index 609