THE ROLE OF EVANESCENT MODES IN RANDOMLY PERTURBED SINGLE-MODE WAVEGUIDES

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Abstract. Pulse propagation in randomly perturbed single-mode waveguides
is considered. By an asymptotic analysis the pulse front propagation is reduced
to an effective equation with diffusion and dispersion. Apart from a random
time shift due to a random total travel time, two main phenomena can be
distinguished. First, coupling and energy conversion between forward- and
backward-propagating modes is responsible for an effective diffusion of the
pulse front. This attenuation and spreading is somewhat similar to the one-
dimensional case addressed by the O'Doherty-Anstey theory. Second, coupling
between the forward-propagating mode and the evanescent modes results in
an effective dispersion. In the case of small-scale random fluctuations we show
that the second mechanism is dominant.

1. **Introduction.** In one-dimensional random media, pulse propagation is described
by the O'Doherty Anstey (ODA) theory. This theory, originally proposed by geo-
physicists [15, 18], predicts that the pulse front is modified in two ways. First
the pulse front shape spreads out in a deterministic way due to multiple scatter-
ing. Second the wave itself is not deterministic anymore but it is shifted by a
random time delay. The mathematical analysis of the random coupling between
the forward-going mode and the backward-going mode has confirmed these predic-
tions in one-dimensional random media, first in the case of weak fluctuations [3],
and then in presence of strong media fluctuations [5, 11]. Furthermore, it has been
shown that the ODA theory is still valid in three-dimensional randomly layered me-
dia [1, 4], even in the presence of slow variations in the transverse spatial directions
[19].

In randomly perturbed multi-mode waveguides, the coupled mode theory is usu-
ally applied. This theory describes the transfer of energy between the propagating
modes. It was described for applications to underwater acoustics [8, 6], fiber optics
[12, 10], and quantum mechanics [7]. It is efficient to compute pulse spreading in
the regime where the coupling between the forward-propagating modes is the dom-
inant phenomenon. In this theory the evanescent modes are neglected, with the
argument that these modes do not propagate energy over distances larger than the
typical wavelength.

In this paper we consider wave propagation in an acoustic waveguide whose bulk modulus is a three-dimensional random function. We shall see that the coupling with evanescent modes does not remove energy from the propagating modes, as expected, but it introduces frequency-dependent phases. The overall effect for the propagating modes is dispersion that can be strong enough to prevail over other effects. This result is obtained from first principles by using an asymptotic theory in the case where the waveguide is single-mode, i.e., the waveguide supports only one pair of forward-backward propagating modes and an infinite set of evanescent modes. The main tool is a diffusion-approximation theorem [16, 17] for the random differential equation satisfied by the propagator matrix. We take into account the coupling mechanisms between forward, backward, and evanescent modes. As far as the pulse front propagation is concerned, the final result is an extension of the ODA theory where a dispersion is added to take into account the coupling with the evanescent modes.

The results obtained in this paper are especially relevant for small-scale transverse fluctuations, as follows from the following simple argument. Let us suppose that the random perturbations are characterized by a transverse correlation length r_c. If the propagation takes place in a waveguide of constant cross section whose maximum diameter is smaller than r_c, then the dependence of the medium inside the waveguide on the transverse coordinates can be neglected, and the problem can be reduced to the one-dimensional case [2]. However, if r_c is smaller than the diameter of the waveguide, then a more detailed study is necessary, which is addressed in this paper. In Section 2 we describe the propagating and evanescent modes of an acoustic waveguide. We describe the mode coupling induced by the random medium in Section 3, with special emphasis on the evanescent modes. In Section 4 we describe and discuss the pulse front propagation in a random waveguide.

2. **Propagation in homogeneous waveguides.** In this section we study wave propagation in a homogeneous acoustic waveguide that supports a finite number of propagating modes. In an ideal waveguide the geometric structure and the medium parameters can have a general form in the transverse directions but they must be homogeneous along the waveguide axis. There are two general types of ideal waveguides, the ones that surround a homogeneous region with a confining boundary and the ones in which the confinement is achieved with a transversely varying index of refraction. In the next sections we will present the analysis of the effects of random perturbations on waveguides of the first type and we will illustrate specific results with a planar waveguide.

2.1. **Acoustic waveguide.** We consider linear acoustic waves propagating in three spatial dimensions modeled by the system of wave equations

$$\rho(\mathbf{r})\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F}\,, \qquad \frac{1}{K(\mathbf{r})}\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0\,, \tag{1}$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ is the density of the medium and K the bulk modulus. The source is modeled by the forcing term $\mathbf{F}(t, \mathbf{r})$. We assume that the transverse profile of the waveguide is a simply connected region $\mathcal{D}$ in two dimensions. The direction of propagation along the waveguide axis is z and the transverse coordinates are denoted by $\mathbf{x} \in \mathcal{D}$. In the interior of the waveguide the medium parameters are homogeneous

$$\rho(z, \mathbf{x}) = \bar{\rho}, \qquad K(z, \mathbf{x}) = \bar{K}, \qquad \text{for } \mathbf{x} \in \mathcal{D} \text{ and } z \in \mathbb{R}\,.$$

By differentiating with respect to time the second equation of (1) and substituting the first equation into it, we get the standard wave equation for the pressure field

$$\Delta p - \frac{1}{\bar{c}^2}\frac{\partial^2 p}{\partial t^2} = \nabla \cdot \mathbf{F}\,, \tag{2}$$

where $\Delta = \Delta_\perp + \partial_z^2$ and $\Delta_\perp$ is the transverse Laplacian. The sound speed is $\bar{c} = \sqrt{K/\bar{\rho}}$. We must now prescribe boundary conditions on the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$. If the boundary of the waveguide is a rigid wall, then the normal velocity vanishes on $\partial\mathcal{D}$. By (1) we obtain Neumann boundary conditions for the pressure:

$$\mathbf{n}\cdot\nabla_\perp p(t,\mathbf{x},z) = 0 \qquad \text{for } \mathbf{x}\in\partial\mathcal{D} \text{ and } z\in\mathbb{R}\,. \tag{3}$$

2.2. The propagating and evanescent modes.
A waveguide mode is a monochromatic wave $p(t,\mathbf{x},z) = \hat{p}(\omega,\mathbf{x},z)e^{-i\omega t}$ with frequency ω, where $\hat{p}(\omega,\mathbf{x},z)$ satisfies the time-harmonic form of the wave equation (2) without a source term

$$\partial_z^2\hat{p}(\omega,\mathbf{x},z) + \Delta_\perp\hat{p}(\omega,\mathbf{x},z) + k^2(\omega)\hat{p}(\omega,\mathbf{x},z) = 0\,. \tag{4}$$

Here $k = \omega/\bar{c}$ is the wavenumber and we have Neumann boundary conditions on $\partial\mathcal{D}$. The transverse Laplacian in $\mathcal{D}$ with Neumann boundary conditions on $\partial\mathcal{D}$ is self-adjoint in $L^2(\mathcal{D})$. Its spectrum is an infinite number of discrete eigenvalues

$$-\Delta_\perp\phi_j(\mathbf{x}) = \lambda_j\phi_j(\mathbf{x}),\ \mathbf{x}\in\mathcal{D}, \qquad \mathbf{n}\cdot\nabla_\perp\phi_j(\mathbf{x}) = 0,\ \mathbf{x}\in\partial\mathcal{D},$$

for $j = 0,1,2,\dots$. The smallest eigenvalue is $\lambda_0 = 0$ and the corresponding eigenvector is the constant mode $\phi_0(\mathbf{x}) \equiv \phi_0$, $\phi_0 = 1/\sqrt{|\mathcal{D}|}$. The other eigenvalues are nonnegative, and we assume for simplicity that they are simple so we have $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. The eigenmodes are real and form an orthonormal set

$$\int_\mathcal{D}\phi_j(\mathbf{x})\phi_l(\mathbf{x})d\mathbf{x} = \delta_{jl}\,.$$

Let $N(\omega)$ be the integer such that $\bar{c}\lambda_{N-1} \le \omega < \bar{c}\lambda_N$.

The modal wavenumbers $\beta_j(\omega)$ are defined by

$$\beta_j(\omega) = \sqrt{\frac{\omega^2}{\bar{c}^2} - \lambda_j}\,, \quad j = 0,\dots,N(\omega)-1\,. \tag{5}$$

The solutions

$$\hat{p}_j(\omega,\mathbf{x},z) = \phi_j(\mathbf{x})e^{\pm i\beta_j(\omega)z}\,, \quad j = 0,\dots,N(\omega)-1\,,$$

of the wave equation (4) are the propagating waveguide modes.

For $j \ge N(\omega)$ we define the modal wavenumbers by

$$\beta_j(\omega) = \sqrt{\lambda_j - \frac{\omega^2}{\bar{c}^2}}\,, \quad j \ge N(\omega)\,, \tag{6}$$

and the corresponding solutions

$$\hat{q}_j(\omega,\mathbf{x},z) = \phi_j(\mathbf{x})e^{\pm\beta_j(\omega)z}\,, \quad j \ge N(\omega)\,,$$

of the wave equation (4) are the evanescent waveguide modes.

The planar waveguide. This is the special case where $\mathcal{D}$ is $(0,d)\times\mathbb{R}$ and we consider only solutions that depend on $x \in (0,d)$. In this case

$$\lambda_j = \frac{\pi^2 j^2}{d^2}\,, \quad \phi_j(x) = \frac{\sqrt{2}}{\sqrt{d}}\cos\left(\frac{\pi j x}{d}\right)\,, \quad j \ge 1\,, \quad \phi_0(x) = \frac{1}{\sqrt{d}}\,,$$

and the number of propagating modes is

$$N(\omega) = 1 + \left[\frac{\omega d}{\pi \bar{c}}\right],$$

where $[x]$ is the integer part of x.

2.3. **Excitation conditions for a source.** We consider a source located in the plane $z = 0$ that emits a signal with orientation in the z-direction

$$\mathbf{F}(t, \mathbf{x}, z) = f(t, \mathbf{x})\delta(z)\mathbf{e}_z. \tag{7}$$

Here $\mathbf{e}_z$ is the unit vector pointing in the z-direction. We may think for instance at a point-like source located at $(\mathbf{x}_0, z = 0)$, so that $f(t, \mathbf{x}) = f(t)\delta(\mathbf{x} - \mathbf{x}_0)$. By the first equation of (1), this source term implies that the pressure satisfies the following jump conditions across the plane $z = 0$

$$\hat{p}(\omega, \mathbf{x}, z = 0^+) - \hat{p}(\omega, \mathbf{x}, z = 0^-) = \hat{f}(\omega, \mathbf{x}),$$

while the second equation of (1) implies that there is no jump in the longitudinal velocity so that the pressure field also satisfies

$$\partial_z \hat{p}(\omega, \mathbf{x}, z = 0^+) - \partial_z \hat{p}(\omega, \mathbf{x}, z = 0^-) = 0.$$

Here $\hat{f}$ is the Fourier transform of f with respect to time:

$$\hat{f}(\omega, \mathbf{x}) = \int f(t, \mathbf{x})e^{i\omega t}dt, \qquad f(t, \mathbf{x}) = \frac{1}{2\pi}\int \hat{f}(\omega, \mathbf{x})e^{-i\omega t}d\omega.$$

The pressure field can be written as a superposition of the complete set of modes

$$
\begin{aligned}
\hat{p}(\omega, \mathbf{x}, z) &= \left[\sum_{j=0}^{N-1}\frac{\hat{a}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{i\beta_j z}\phi_j(\mathbf{x}) + \sum_{j=N}^{\infty}\frac{\hat{c}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{-\beta_j z}\phi_j(\mathbf{x})\right]\mathbf{1}_{(0,\infty)}(z) \\
&\quad + \left[\sum_{j=0}^{N-1}\frac{\hat{b}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{-i\beta_j z}\phi_j(\mathbf{x}) + \sum_{j=N}^{\infty}\frac{\hat{d}_j(\omega)}{\sqrt{\beta_j(\omega)}}e^{\beta_j z}\phi_j(\mathbf{x})\right]\mathbf{1}_{(-\infty,0)}(z),
\end{aligned}
$$

where $\hat{a}_j$ is the amplitude of the jth right-going mode propagating in the right half-space $z > 0$, $\hat{b}_j$ is the amplitude of the jth left-going mode propagating in the left half-space $z < 0$, and $\hat{c}_j$ (resp. $\hat{d}_j$) is the amplitude of the jth right-going (resp. left-going) evanescent mode. Substituting this expansion into the jump conditions, multiplying by $\phi_j(\mathbf{x})$, integrating with respect to $\mathbf{x}$ over $\mathcal{D}$, and using the orthogonality of the modes, we can express the mode amplitudes in terms of the source

$$\hat{a}_j(\omega) = -\hat{b}_j(\omega) = \frac{\sqrt{\beta_j(\omega)}}{2}\int_{\mathcal{D}}\hat{f}(\omega, \mathbf{x})\phi_j(\mathbf{x})d\mathbf{x}, \qquad 0 \leq j \leq N(\omega) - 1, \tag{8}$$

$$\hat{c}_j(\omega) = -\hat{d}_j(\omega) = -\frac{\sqrt{\beta_j(\omega)}}{2}\int_{\mathcal{D}}\hat{f}(\omega, \mathbf{x})\phi_j(\mathbf{x})d\mathbf{x}, \qquad j \geq N(\omega). \tag{9}$$

3. **Mode coupling in random waveguides.** We consider a randomly perturbed waveguide section occupying the region $z \in [0, L/\varepsilon^2]$, with two homogeneous waveguides occupying the two half-spaces $z < 0$ and $z > L/\varepsilon^2$. The bulk modulus and the density have the form

$$\frac{1}{K(\mathbf{x}, z)} = \begin{cases} \frac{1}{\bar{K}}(1 + \varepsilon \nu(\mathbf{x}, z)) & \text{for } \mathbf{x} \in \mathcal{D}, \quad z \in [0, L/\varepsilon^2], \\ \frac{1}{\bar{K}} & \text{for } \mathbf{x} \in \mathcal{D}, \quad z \in (-\infty, 0) \cup (L/\varepsilon^2, \infty), \end{cases}$$

$$\rho(\mathbf{x}, z) = \bar{\rho} \text{ for } \mathbf{x} \in \mathcal{D}, \quad z \in (-\infty, \infty),$$

where ε is a small dimensionless parameter that characterizes the typical amplitude of the random fluctuations. The function $(\mathbf{x}, z) \mapsto \nu(\mathbf{x}, z)$ is a bounded zero-mean random process. It is stationary and ergodic with respect to the axis coordinate z. Moreover, it is supposed to possess enough decorrelation, more exactly that it fulfills the mixing condition "ν is ϕ-mixing, with $\phi \in L^{1/2}(\mathbb{R}^+)$" (see [9, Section 4-6-2]). The weak fluctuations of the medium parameters induce a coupling between the propagating modes, as well as between propagating and evanescent modes, which build up and become of order one after a propagation distance of order ε^{-2}, as we shall see below.

The perturbed wave equation satisfied by the pressure field is

$$\Delta p - \frac{1 + \varepsilon \nu(\mathbf{x}, z)}{\bar{c}^2} \frac{\partial^2 p}{\partial t^2} = \nabla \cdot \mathbf{F}, \tag{10}$$

where the average sound speed is $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$. The pressure field also satisfies the Neumann boundary conditions (3). We consider a source located in the plane $z = 0$ modeled by the forcing term (7) and we denote by $\hat{a}_j^{(0)}(\omega)$ the initial mode amplitudes given by (8). Our strategy follows the asymptotic analysis carried out in [8], but we pay special attention to the role of the evanescent modes and we do not apply the forward scattering approximation, that is, we dot not neglect the coupling between the right- and left-going modes.

3.1. **Coupled amplitude equations.** We fix the frequency ω and we expand the field $\hat{p}$ in terms of the transverse eigenmodes

$$\hat{p}(\mathbf{x}, z) = \sum_{j=0}^{N-1} \hat{p}_j(z) \phi_j(\mathbf{x}) + \sum_{j=N}^{\infty} \hat{q}_j(z) \phi_j(\mathbf{x}), \tag{11}$$

where $\hat{p}_j$ is the amplitude of the jth propagating mode, $\hat{q}_j$ is the amplitude of the jth evanescent mode.

For $j \leq N - 1$ we introduce the right-going and left-going propagating mode amplitudes $\hat{a}_j$ and $\hat{b}_j$ defined by

$$\hat{p}_j(z) = \frac{1}{\sqrt{\beta_j}}\left(\hat{a}_j(z)e^{i\beta_j z} + \hat{b}_j(z)e^{-i\beta_j z}\right), \tag{12}$$

$$\frac{d\hat{p}_j(z)}{dz} = i\sqrt{\beta_j}\left(\hat{a}_j(z)e^{i\beta_j z} - \hat{b}_j(z)e^{-i\beta_j z}\right). \tag{13}$$

Then it follows that $\hat{a}_j$ and $\hat{b}_j$ have the form

$$\hat{a}_j(z) = \frac{i\beta_j \hat{p}_j + \frac{d\hat{p}_j}{dz}}{2i\sqrt{\beta_j}}e^{-i\beta_j z}, \qquad \hat{b}_j(z) = \frac{i\beta_j \hat{p}_j - \frac{d\hat{p}_j}{dz}}{2i\sqrt{\beta_j}}e^{i\beta_j z}, \qquad j \leq N - 1.$$

We finally introduce the normalized amplitude of the evanescent modes:

$$\hat{c}_j(z) = \sqrt{\beta_j}\hat{q}_j(z) \qquad j \geq N.$$

The total field $\hat{p}$ satisfies the time-harmonic wave equation in the perturbed waveguide

$$\Delta\hat{p}(\omega,\mathbf{x},z) + k^2(1 + \varepsilon\nu(\mathbf{x},z))\hat{p}(\omega,\mathbf{x},z) = 0\,. \tag{14}$$

Using (11) in this equation, multiplying it by $\phi_j(\mathbf{x})$, and integrating over $\mathbf{x} \in \mathcal{D}$, we deduce from the orthogonality of the eigenmodes $(\phi_j)_{j\geq 0}$ the following system of coupled differential equations for the mode amplitudes

$$\frac{d\hat{a}_j}{dz} = i\varepsilon \sum_{l=0}^{N-1} C_{jl}(z)\left(\hat{a}_l e^{i(\beta_l - \beta_j)z} + \hat{b}_l e^{-i(\beta_l + \beta_j)z}\right)$$

$$+ i\varepsilon \sum_{l=N}^{\infty} C_{jl}(z)\hat{c}_l(z)e^{-i\beta_j z}, \qquad 0 \leq j \leq N-1\,, \tag{15}$$

$$\frac{d\hat{b}_j}{dz} = -i\varepsilon \sum_{l=0}^{N-1} C_{jl}(z)\left(\hat{a}_l e^{i(\beta_l + \beta_j)z} + \hat{b}_l e^{i(-\beta_l + \beta_j)z}\right)$$

$$- i\varepsilon \sum_{l=N}^{\infty} C_{jl}(z)\hat{c}_l(z)e^{i\beta_j z}, \qquad 0 \leq j \leq N-1\,, \tag{16}$$

$$\frac{d^2\hat{c}_j}{dz^2} - \beta_j^2\hat{c}_j + 2\varepsilon\beta_j\hat{g}_j(z) = 0, \qquad j \geq N\,. \tag{17}$$

Here the coupling coefficients are given by

$$C_{jl}(z) = \frac{k^2}{2\sqrt{\beta_j\beta_l}} \int_{\mathcal{D}} \phi_j(\mathbf{x})\phi_l(\mathbf{x})\nu(\mathbf{x},z)d\mathbf{x}\,, \qquad j,l \geq 0\,, \tag{18}$$

$$\hat{g}_j(z) = \sum_{l=N}^{\infty} C_{jl}(z)\hat{c}_l + \sum_{l=0}^{N-1} C_{jl}(z)\left(\hat{a}_l e^{i\beta_l z} + \hat{b}_l e^{-i\beta_l z}\right)\,, \qquad j \geq N\,. \tag{19}$$

Note that $\hat{g}_j$ is a linear function of $\hat{a}$, $\hat{b}$, and $\hat{c}$. The system (15-17) is complemented with the boundary conditions

$$\hat{a}_j(0) = \hat{a}_{j,0}, \qquad \hat{b}_j\left(\frac{L}{\varepsilon^2}\right) = 0, \qquad 0 \leq j \leq N-1\,, \tag{20}$$

for the propagating modes. The second conditions at $z = L/\varepsilon^2$ indicate that no wave is incoming from the right. We also use the radiation conditions

$$\lim_{z\to\pm\infty} \hat{c}_j(z) = 0, \qquad j \geq N\,,$$

which imply that the evanescent modes are decaying. The solution of (17) that satisfies the radiation conditions is

$$\hat{c}_j(z) = \varepsilon \int_{-\infty}^{\infty} \hat{g}_j(z+s)e^{-\beta_j|s|}ds\,. \tag{21}$$

3.2. **Conservation of energy flux.** Energy flux conservation in the one-dimensional case implies a conservation relation for the right- and left-propagating wave amplitudes that has the form $|\hat{a}|^2 - |\hat{b}|^2 = \text{constant}$ [3]. We now generalize this relation to waveguides. Let us denote $|\hat{a}|^2 = \sum_{j=0}^{N-1} |\hat{a}_j|^2$ and $|\hat{b}|^2 = \sum_{j=0}^{N-1} |\hat{b}_j|^2$. Using (15-16) we have

$$\frac{d}{dz}(|\hat{a}|^2 - |\hat{b}|^2) = -2\varepsilon\text{Im}\left[\sum_{j=0}^{N-1}\sum_{l\geq N} C_{jl}(\overline{\hat{a}_j}e^{-i\beta_j z} + \overline{\hat{b}_j}e^{i\beta_j z})\hat{c}_l\right]\,.$$

By (19) the right-hand side can be rewritten as

$$\frac{d}{dz}(|\hat{a}|^2 - |\hat{b}|^2) = -2\varepsilon\mathrm{Im}\left[\sum_{l\geq N}(\overline{\hat{g}_l} - \sum_{l'\geq N}C_{ll'}\overline{\hat{c}_{l'}})\hat{c}_l\right] = -2\varepsilon\mathrm{Im}\left[\sum_{l\geq N}\overline{\hat{g}_l}\hat{c}_l\right]$$

Note that $\hat{g}_l(z)$ is vanishing for $z < 0$ because the waveguide in homogeneous for $z < 0$. This shows that $(|\hat{a}|^2 - |\hat{b}|^2)(z) = (|\hat{a}|^2 - |\hat{b}|^2)(0)$ for any $z < 0$. Next, we use the integral representation (21) and we integrate over z to get

$$(|\hat{a}|^2 - |\hat{b}|^2)(z) - (|\hat{a}|^2 - |\hat{b}|^2)(0) = -2\varepsilon^2 \sum_{l\geq N}\hat{G}_l(z)\,, \qquad (22)$$

$$\hat{G}_l(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty}\mathrm{Im}\left[\overline{\hat{g}_l}(y)\hat{g}_l(y+s)\right]e^{-\beta_l|s|}dsdy\,.$$

The quantity $\hat{G}_l(z)$ can be written as

$$2i\hat{G}_l(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty}\overline{\hat{g}_l}(y)\hat{g}_l(y+s)e^{-\beta_l|s|}dsdy - \int_{-\infty}^{z}\int_{-\infty}^{\infty}\hat{g}_l(y)\overline{\hat{g}_l}(y+s)e^{-\beta_l|s|}dsdy\,.$$

We transform the second term as follows

$$\int_{-\infty}^{z}\int_{-\infty}^{\infty}\hat{g}_l(y)\overline{\hat{g}_l}(y+s)e^{-\beta_l|s|}dsdy \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{z+s}\hat{g}_l(y-s)\overline{\hat{g}_l}(y)dye^{-\beta_l|s|}ds$$

$$=\; \int_{-\infty}^{\infty}\int_{-\infty}^{z-s}\hat{g}_l(y+s)\overline{\hat{g}_l}(y)dye^{-\beta_l|s|}ds\,,$$

so that we get

$$\hat{G}_l(z) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{z-s}^{z}\mathrm{Im}[\overline{\hat{g}_l}(y)\hat{g}_l(y+s)]dye^{-\beta_l|s|}ds\,.$$

The waveguide is homogeneous in the section $(L/\varepsilon^2, \infty)$, which implies that $\hat{g}_l(y) = 0$ if $y > L/\varepsilon^2$. In the previous double integral, we can notice that, if $s < 0$, then $y > z$, while if $s > 0$, then $y + s > z$. As a consequence, if $z \geq L/\varepsilon^2$, then either y or $y + s$ is larger than L/ε^2, so that the product $\overline{\hat{g}_l}(y)\hat{g}_l(y+s)$ is always 0. This shows that $\hat{G}_l(z) = 0$ if $z \geq L/\varepsilon^2$. By (22) we thus obtain

$$|\hat{a}|^2(L/\varepsilon^2) - |\hat{b}|^2(L/\varepsilon^2) = |\hat{a}|^2(0) - |\hat{b}|^2(0)\,.$$

Using the boundary conditions (20) at 0 and L/ε^2, we get the conservation of energy flux

$$|\hat{a}|^2(L/\varepsilon^2) + |\hat{b}|^2(0) = |\hat{a}^{(0)}|^2\,, \qquad (23)$$

which is exact for any ε.

3.3. Evanescent modes in terms of propagating modes.

In this and the next subsection we show how to obtain asymptotically a closed system of equations for the propagating mode amplitudes which takes into account the coupling with the evanescent modes.

We first substitute (19) into (21) and rewrite it in the vector-matrix form

$$(I_d - \varepsilon\Psi)\hat{c} = \varepsilon\widetilde{c}\,. \qquad (24)$$

Here the operator Ψ is defined by

$$(\Psi\hat{c})_j(z) = \sum_{l\geq N}\int_{-\infty}^{\infty}C_{jl}(z+s)\hat{c}_l(z+s)e^{-\beta_j|s|}ds, \quad j \geq N\,,$$

and the vector-valued function $\widetilde{c}$ is given by

$$\widetilde{c}_j(z) = \sum_{l=0}^{N-1} \int_{-\infty}^{\infty} C_{jl}(z+s)$$
$$\times \left(\hat{a}_l(z+s)e^{i\beta_l(z+s)} + \hat{b}_l(z+s)e^{-i\beta_l(z+s)} \right) e^{-\beta_j|s|}ds, \quad j \geq N. \quad (25)$$

Introducing the norm $\|\hat{c}\| = \sum_{j \geq N} \beta_j^{-1/2} \sup_z |\hat{c}_j(z)|$, we have

$$\|\Psi \hat{c}\| \leq \sum_{j \geq N} \beta_j^{-1/2} \sup_z \sum_{l \geq N} \int_{-\infty}^{\infty} |C_{jl}(z+s)||\hat{c}_l(z+s)|e^{-\beta_j|s|}ds.$$

Note that

$$\sup_z |C_{jl}(z)| \leq \frac{k^2}{2\sqrt{\beta_j\beta_l}} \sup_z \int_{\mathcal{D}} \phi_j\phi_l\nu(\mathbf{x},z)d\mathbf{x} \leq \frac{k^2}{2\sqrt{\beta_j\beta_l}}\|\nu\|_\infty,$$

where we have used the normalizing condition $\int \phi_j^2 d\mathbf{x} = 1$. This implies

$$\|\Psi\hat{c}\| \leq \frac{\|\nu\|_\infty k^2}{2} \sum_{j \geq N} \beta_j^{-1} \sum_{l \geq N} \beta_l^{-1/2} \sup_z |\hat{c}_l(z)| \int_{-\infty}^{\infty} e^{-\beta_j|s|}ds$$
$$\leq \|\nu\|_\infty k^2 \sum_{j \geq N} \beta_j^{-2} \sum_{l \geq N} \beta_l^{-1/2} \sup_z |\hat{c}_l(z)| = \|\nu\|_\infty K_\beta k^2\|\hat{c}\|,$$

where $K_\beta = \sum_{j \geq N} \beta_j^{-2}$. Note that K_β is indeed finite for a broad class of waveguides. This is, for example, the case in the planar waveguide because we have $\beta_j = \sqrt{\pi^2 j^2/d^2 - \omega^2/\bar{c}^2} \sim \pi j/d$ for $j \gg N$. Thus Ψ is a bounded operator and so for ε small enough, $I_d - \varepsilon\Psi$ is invertible and can be approximated by $I_d + \varepsilon\Psi + O(\varepsilon^2)$. The vector $\widetilde{c}$ defined by (25) has finite $\|\cdot\|$ norm:

$$\|\widetilde{c}\| \leq K_\beta k^2\|\nu\|_\infty \sum_{l=0}^{N-1} \sup(|\hat{a}_l|, |\hat{b}_l|).$$

By inverting (24) we can therefore write for $j \geq N$

$$\hat{c}_j(z) = \varepsilon \sum_{l=0}^{N-1} \int_{-\infty}^{\infty} C_{jl}(z+s)$$
$$\times \left(\hat{a}_l(z+s)e^{i\beta_l(z+s)} + \hat{b}_l(z+s)e^{-i\beta_l(z+s)} \right) e^{-\beta_j|s|}ds + O(\varepsilon^2).$$

Furthermore, over a distance of order 1, the variation of $\hat{a}$ and $\hat{b}$ is at most of order ε, so we can substitute $\hat{a}_l(z)$ and $\hat{b}_l(z)$ for $\hat{a}_l(z+s)$ and $\hat{b}_l(z+s)$ up to an error of order ε and we obtain, for $j \geq N$,

$$\hat{c}_j(z) = \varepsilon \sum_{l=0}^{N-1} \int_{-\infty}^{\infty} C_{jl}(z+s) \left(\hat{a}_l(z)e^{i\beta_l(s+z)} + \hat{b}_l(z)e^{-i\beta_l(s+z)} \right) e^{-\beta_j|s|}ds + O(\varepsilon^2).$$
$$(26)$$

3.4. Propagating mode amplitude equations.
We introduce the rescaled processes $\hat{a}_j^\varepsilon$, $\hat{b}_j^\varepsilon$, $j = 0, \ldots, N-1$ given by

$$\hat{a}_j^\varepsilon(z) = \hat{a}_j\left(\frac{z}{\varepsilon^2}\right), \quad \hat{b}_j^\varepsilon(z) = \hat{b}_j\left(\frac{z}{\varepsilon^2}\right). \quad (27)$$

Substituting (26) into (15-16) we have the following system of differential equations for the propagating mode amplitudes

$$\frac{d\hat{a}^\varepsilon}{dz} = \left[\frac{1}{\varepsilon}\mathbf{H}^{(a)}\left(\frac{z}{\varepsilon^2}\right) + \mathbf{G}^{(a)}\left(\frac{z}{\varepsilon^2}\right)\right]\hat{a}^\varepsilon + \left[\frac{1}{\varepsilon}\overline{\mathbf{H}^{(b)}}\left(\frac{z}{\varepsilon^2}\right) + \overline{\mathbf{G}^{(b)}}\left(\frac{z}{\varepsilon^2}\right)\right]\hat{b}^\varepsilon, (28)$$

$$\frac{d\hat{b}^\varepsilon}{dz} = \left[\frac{1}{\varepsilon}\mathbf{H}^{(b)}\left(\frac{z}{\varepsilon^2}\right) + \mathbf{G}^{(b)}\left(\frac{z}{\varepsilon^2}\right)\right]\hat{a}^\varepsilon + \left[\frac{1}{\varepsilon}\overline{\mathbf{H}^{(a)}}\left(\frac{z}{\varepsilon^2}\right) + \overline{\mathbf{G}^{(a)}}\left(\frac{z}{\varepsilon^2}\right)\right]\hat{b}^\varepsilon. (29)$$

Here the matrices $\mathbf{G}^{(\cdot)}(z)$ and $\mathbf{H}^{(\cdot)}(z)$ are given by

$$H_{jl}^{(a)}(z) = iC_{jl}(z)e^{i(\beta_l - \beta_j)z}, \tag{30}$$

$$H_{jl}^{(b)}(z) = -iC_{jl}(z)e^{i(\beta_l + \beta_j)z}, \tag{31}$$

$$G_{jl}^{(a)}(z) = i\sum_{l' \geq N}\int_{-\infty}^{\infty} C_{jl'}(z)C_{ll'}(z+s)e^{i\beta_l(z+s) - i\beta_j z - \beta_{l'}|s|}ds, \tag{32}$$

$$G_{jl}^{(b)}(z) = -i\sum_{l' \geq N}\int_{-\infty}^{\infty} C_{jl'}(z)C_{ll'}(z+s)e^{i\beta_l(z+s) + i\beta_j z - \beta_{l'}|s|}ds. \tag{33}$$

The system (28-29) and the boundary conditions

$$\hat{a}_j^\varepsilon(0) = \hat{a}_j^{(0)}, \quad \hat{b}_j^\varepsilon(L) = 0 \tag{34}$$

determine the propagating mode amplitudes. Even though this system involves only the propagating mode amplitudes, it takes into account the effect of the evanescent modes through the matrices $\mathbf{G}^{(\cdot)}$. If one is interested in determining the evanescent modes, one should first integrate the system (28-29) and then substitute the solutions into the integral representations (26) of the evanescent modes.

3.5. Propagator matrices. The two-point linear boundary value problem for Eqs. (28-29) can be solved using propagator matrices. The system (28-29) can be put into full vector-matrix form

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon}\mathbf{H}\left(\frac{z}{\varepsilon^2}\right)X^\varepsilon + \mathbf{G}\left(\frac{z}{\varepsilon^2}\right)X^\varepsilon, \tag{35}$$

where X^ε is the $2N$-vector obtained by concatenating the N-vectors $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$

$$X^\varepsilon(z) = \left[\begin{array}{c} \hat{a}^\varepsilon(z) \\ \hat{b}^\varepsilon(z) \end{array}\right],$$

while $\mathbf{H}$ and $\mathbf{G}$ are the $2N \times 2N$ matrices:

$$\mathbf{H}(z) = \left[\begin{array}{cc} \mathbf{H}^{(a)}(z) & \overline{\mathbf{H}^{(b)}}(z) \\ \mathbf{H}^{(b)}(z) & \overline{\mathbf{H}^{(a)}}(z) \end{array}\right], \qquad \mathbf{G}(z) = \left[\begin{array}{cc} \mathbf{G}^{(a)}(z) & \overline{\mathbf{G}^{(b)}}(z) \\ \mathbf{G}^{(b)}(z) & \overline{\mathbf{G}^{(a)}}(z) \end{array}\right].$$

The $2N \times 2N$ propagator matrices $\mathbf{P}^\varepsilon(z)$ are the solution of the initial value problem

$$\frac{d\mathbf{P}^\varepsilon}{dz} = \left[\frac{1}{\varepsilon}\mathbf{H}\left(\frac{z}{\varepsilon^2}\right) + \mathbf{G}\left(\frac{z}{\varepsilon^2}\right)\right]\mathbf{P}^\varepsilon,$$

with the initial condition $\mathbf{P}^\varepsilon(z = 0) = \mathbf{I}$. The general solution of (28-29) satisfies for any $0 \leq z \leq L$

$$\left[\begin{array}{c} \hat{a}^\varepsilon(z) \\ \hat{b}^\varepsilon(z) \end{array}\right] = \mathbf{P}^\varepsilon(z)\left[\begin{array}{c} \hat{a}^\varepsilon(0) \\ \hat{b}^\varepsilon(0) \end{array}\right].$$

When we specialize this relation to $z = L$ and use the boundary conditions (34), with $\hat{a}^{(0)}$ successively the unit vectors, we get

$$\left[\begin{array}{c} \mathbf{T}^\varepsilon(L) \\ \mathbf{0} \end{array} \right] = \mathbf{P}^\varepsilon(L) \left[\begin{array}{c} \mathbf{I} \\ \mathbf{R}^\varepsilon(L) \end{array} \right]. \tag{36}$$

Here we have defined the $N \times N$ random reflection and transmission matrices $\mathbf{R}^\varepsilon(L)$ and $\mathbf{T}^\varepsilon(L)$.

We can check that, if $(\hat{a}^\varepsilon(z), \hat{b}^\varepsilon(z))^T$ is a solution of (35), then $(\overline{\hat{b}^\varepsilon(z)}, \overline{\hat{a}^\varepsilon(z)})^T$ is also a solution. This imposes that the propagator has the form

$$\mathbf{P}^\varepsilon(z) = \left[\begin{array}{cc} \mathbf{P}_a^\varepsilon(z) & \overline{\mathbf{P}_b^\varepsilon(z)} \\ \mathbf{P}_b^\varepsilon(z) & \overline{\mathbf{P}_a^\varepsilon(z)} \end{array} \right]. \tag{37}$$

Note that the matrix $\mathbf{P}_a^\varepsilon$ describes the coupling between different right-going modes, while $\mathbf{P}_b^\varepsilon$ describes the coupling between right-going and left-going modes.

In addition to the general symmetry or reciprocity relation that gives the form (37) for the propagator matrices, we also have the global flux conservation relation (23). When this relation is used in conjunction with (36) we get the N-mode generalization of the reflection-transmission conservation relation

$$\mathbf{R}^{\varepsilon\dagger}\mathbf{R}^\varepsilon(L) + \mathbf{T}^{\varepsilon\dagger}\mathbf{T}^\varepsilon(L) = \mathbf{I}, \tag{38}$$

where the sign $\dagger$ stands for the conjugate transpose. This relation holds in general, with the evanescent modes taken into consideration and for any $\varepsilon > 0$.

3.6. **Convergence of the propagator matrix in the single-mode case.** Remember that $\lambda_1 > 0$ is the first nonzero eigenvalue of the Laplacian operator $\Delta_\perp$ over $\mathcal{D}$ with Neumann boundary conditions at $\partial\mathcal{D}$. The critical frequency $\omega_c = \sqrt{\lambda_1}\bar{c}$ is such that the number of propagating modes $N(\omega) = 1$ for all $\omega < \omega_c$. In this section, we consider a frequency $\omega < \omega_c$ such that $N(\omega) = 1$. This means that the waveguide supports only one propagating mode, with modal wavenumber $\beta_0(\omega) = \omega/\bar{c}$ and eigenmode $\phi_0(\mathbf{x}) = 1/\sqrt{|\mathcal{D}|}$. The propagator matrix is a 2×2 matrix:

$$\mathbf{P}^\varepsilon(z) = \left[\begin{array}{cc} \alpha_\omega^\varepsilon(z) & \overline{\beta_\omega^\varepsilon(z)} \\ \beta_\omega^\varepsilon(z) & \overline{\alpha_\omega^\varepsilon(z)} \end{array} \right],$$

where $(\alpha_\omega^\varepsilon, \beta_\omega^\varepsilon)$ is solution of the coupled random differential equations

$$\frac{d\alpha_\omega^\varepsilon}{dz} = \left[\frac{1}{\varepsilon} H^{(a)}\left(\frac{z}{\varepsilon^2}\right) + G^{(a)}\left(\frac{z}{\varepsilon^2}\right) \right] \alpha_\omega^\varepsilon + \left[\frac{1}{\varepsilon}\overline{H^{(b)}}\left(\frac{z}{\varepsilon^2}\right) + \overline{G^{(b)}}\left(\frac{z}{\varepsilon^2}\right) \right] \beta_\omega^\varepsilon, \tag{39}$$

$$\frac{d\beta_\omega^\varepsilon}{dz} = \left[\frac{1}{\varepsilon} H^{(b)}\left(\frac{z}{\varepsilon^2}\right) + G^{(b)}\left(\frac{z}{\varepsilon^2}\right) \right] \alpha_\omega^\varepsilon + \left[\frac{1}{\varepsilon}\overline{H^{(a)}}\left(\frac{z}{\varepsilon^2}\right) + \overline{G^{(a)}}\left(\frac{z}{\varepsilon^2}\right) \right] \beta_\omega^\varepsilon, \tag{40}$$

starting from $\alpha_\omega^\varepsilon(z = 0) = 1$ and $\beta_\omega^\varepsilon(z = 0) = 0$. Here the functions $G^{(\cdot)}(z)$ and $H^{(\cdot)}(z)$ are given by

$$\begin{aligned} H^{(a)}(z) &= iC_{00}(z), \\ H^{(b)}(z) &= -iC_{00}(z)e^{2i\frac{\omega}{\bar{c}}z}, \\ G^{(a)}(z) &= i\sum_{l\geq 1}\int_{-\infty}^\infty C_{0l}(z)C_{0l}(z+s)e^{i\frac{\omega}{\bar{c}}s - \beta_l|s|}ds, \\ G^{(b)}(z) &= -i\sum_{l\geq 1}\int_{-\infty}^\infty C_{0l}(z)C_{0l}(z+s)e^{i\frac{\omega}{\bar{c}}(2z+s) - \beta_l|s|}ds. \end{aligned}$$

By (38) the transmission and reflection coefficients satisfy the conservation of energy relation

$$|R^\varepsilon(\omega, L)|^2 + |T^\varepsilon(\omega, L)|^2 = 1\,. \tag{41}$$

By (36) they are given in terms of the entries of the propagation matrix by

$$R^\varepsilon(\omega, L) = -\frac{\beta_\omega^\varepsilon(L)}{\overline{\alpha_\omega^\varepsilon}(L)}, \qquad T^\varepsilon(\omega, L) = \frac{1}{\overline{\alpha_\omega^\varepsilon}(L)}\,. \tag{42}$$

The following proposition is an application of Theorem 2-7 [17] to the system (39-40) satisfied by $(\alpha_\omega^\varepsilon, \beta_\omega^\varepsilon)$. The infinitesimal generator of the limit process $(\alpha_\omega, \beta_\omega)$ has a simple form provided we write it in terms of α_ω, β_ω, $\overline{\alpha_\omega}$, and $\overline{\beta_\omega}$, rather than in terms of the real and imaginary parts of α_ω and β_ω. This generator will be used in the next section to compute the expectation of the pulse transmitted through a randomly perturbed single-mode waveguide.

Proposition 1. *The processes $(\alpha_\omega^\varepsilon(z), \beta_\omega^\varepsilon(z))_{z\geq 0}$ converge in distribution as $\varepsilon \to 0$ to the diffusion process $(\alpha_\omega(z), \beta_\omega(z))_{z\geq 0}$ whose infinitesimal generator is*

$$\mathcal{L} = \frac{\gamma^{(c)}(\omega)\omega^2}{8\bar{c}^2}\left(A_{\alpha_\omega \beta_\omega}\overline{A_{\alpha_\omega \beta_\omega}} + \overline{A_{\alpha_\omega \beta_\omega}}A_{\alpha_\omega \beta_\omega}\right) + \frac{i\gamma^{(s)}(\omega)\omega^2}{8\bar{c}^2}(A_{\beta_\omega \beta_\omega} - A_{\alpha_\omega \alpha_\omega})$$

$$+ \frac{\gamma^{(1)}\omega^2}{8\bar{c}^2}(A_{\alpha_\omega \alpha_\omega} + A_{\beta_\omega \beta_\omega})(\overline{A_{\alpha_\omega \alpha_\omega}} + \overline{A_{\beta_\omega \beta_\omega}}) + i\kappa(\omega)(A_{\alpha_\omega \alpha_\omega} - A_{\beta_\omega \beta_\omega}) \tag{43}$$

$$A_{ab} = a\frac{\partial}{\partial b} - \bar{b}\frac{\partial}{\partial \bar{a}} = -\overline{A_{ab}}\,.$$

Here we have defined the complex derivatives in the standard way: if $a = x + iy$, then $\partial_a = (1/2)(\partial_x - i\partial_y)$ and $\partial_{\bar{a}} = (1/2)(\partial_x + i\partial_y)$. The coefficients $\gamma^{(c)}$, $\gamma^{(s)}$, and $\gamma^{(1)}$ are given by

$$\gamma^{(c)}(\omega) = 2\int_0^\infty \cos\left(2\frac{\omega}{\bar{c}}z\right)\mathbb{E}[\nu_0(0)\nu_0(z)]dz\,, \tag{44}$$

$$\gamma^{(s)}(\omega) = 2\int_0^\infty \sin\left(2\frac{\omega}{\bar{c}}z\right)\mathbb{E}[\nu_0(0)\nu_0(z)]dz\,, \tag{45}$$

$$\gamma^{(1)} = 2\int_0^\infty \mathbb{E}[\nu_0(0)\nu_0(z)]dz\,, \tag{46}$$

$$\kappa(\omega) = \sum_{l=1}^\infty \frac{\omega^3}{2\bar{c}^3\beta_l(\omega)}\int_0^\infty \mathbb{E}[\nu_l(0)\nu_l(z)]\cos\left(\frac{\omega}{\bar{c}}z\right)e^{-\beta_l(\omega)z}dz\,, \tag{47}$$

$$\nu_l(z) = \int_{\mathcal{D}}\nu(\mathbf{x}, z)\phi_l(\mathbf{x})\phi_0(\mathbf{x})d\mathbf{x}\,. \tag{48}$$

4. Pulse propagation in random waveguides.

4.1. **Integral representation of the transmitted field.** We assume that a source is located in the plane $z = 0$ and modeled by the forcing term (7). This source generates evanescent modes, left-going propagating modes that we do not need to consider as they propagate in a homogeneous half-space, and right-going modes that we analyze. As shown in Subsection 2.3 the interface conditions at $z = 0$ and $z = L$ have the form

$$\hat{a}_j^\varepsilon(\omega, 0) = \frac{1}{2}\sqrt{\beta_j(\omega)}\int_{\mathcal{D}}\hat{f}(\omega, \mathbf{x})\phi_j(\mathbf{x})d\mathbf{x}, \qquad \hat{b}_j^\varepsilon(\omega, L) = 0\,, \qquad 0 \leq j \leq N(\omega) - 1\,.$$

The transmitted field observed at time t is therefore

$$p\left(t, \mathbf{x}, \frac{L}{\varepsilon^2}\right) = \frac{1}{2\pi} \int \sum_{j,l=0}^{N(\omega)-1} \frac{1}{\sqrt{\beta_j(\omega)}} \phi_j(\mathbf{x}) T_{jl}^\varepsilon(\omega, L) \hat{a}_l^\varepsilon(\omega, 0) e^{i\beta_j(\omega)\frac{L}{\varepsilon^2} - i\omega t} d\omega,$$

where $\mathbf{T}^\varepsilon$ is the transmission matrix defined by (36). We now assume that the support of the pulse spectrum $\hat{f}$ is contained in $(-\omega_c, \omega_c)$ so that $N(\omega) = 1$ for all frequencies in the integral representation of the transmitted field. In this case, the transmitted pulse is

$$p\left(t, \mathbf{x}, \frac{L}{\varepsilon^2}\right) = \frac{1}{4\pi|\mathcal{D}|} \int \hat{f}(\omega) T^\varepsilon(\omega, L) e^{i\frac{\omega}{\bar{c}}\frac{L}{\varepsilon^2} - i\omega t} d\omega,$$

where T^ε is the transmission coefficient given by (42) and

$$\hat{f}(\omega) = \int_{\mathcal{D}} \int f(t, \mathbf{x}) e^{i\omega t} dt d\mathbf{x}.$$

We observe the field in a time window of order 1, which is comparable to the pulse width, and centered at time $L/(\bar{c}\varepsilon^2)$, which is the average travel time to go from $z = 0$ to $z = L/\varepsilon^2$:

$$p^\varepsilon(t, L) \quad := \quad p\left(\frac{L}{\bar{c}\varepsilon^2} + t, \mathbf{x}, \frac{L}{\varepsilon^2}\right),$$

$$p^\varepsilon(t, L) \quad = \quad \frac{1}{4\pi|\mathcal{D}|} \int \hat{f}(\omega) T^\varepsilon(\omega, L) e^{-i\omega t} d\omega. \tag{49}$$

4.2. Tightness.

Lemma 1. *The transmitted pulse $((p^\varepsilon(t, L))_{-\infty < t < \infty})_{\varepsilon > 0}$ is a tight (i.e. weakly compact) family in the space of continuous trajectories equipped with the sup norm.*

Proof. We must show that, for any $\delta > 0$, there exists a compact subset K of the space of continuous bounded functions such that:

$$\sup_{\varepsilon > 0} \mathbb{P}(p^\varepsilon(\cdot, L) \in K) \geq 1 - \delta.$$

On the one hand (41) yields that $p^\varepsilon(t, L)$ is uniformly bounded by:

$$|p^\varepsilon(t, L)| \leq \frac{1}{4\pi|\mathcal{D}|} \int |\hat{f}(\omega)| d\omega. \tag{50}$$

On the other hand the modulus of continuity

$$M^\varepsilon(\delta) = \sup_{|t_1 - t_2| \leq \delta} |p^\varepsilon(t_1, L) - p^\varepsilon(t_2, L)|$$

is bounded by

$$M^\varepsilon(\delta) \leq \frac{1}{4\pi|\mathcal{D}|} \int \sup_{|t_1 - t_2| \leq \delta} |1 - \exp(i\omega(t_1 - t_2))| |\hat{f}(\omega)| d\omega,$$

which goes to zero as δ goes to zero uniformly with respect to ε. $\square$

The uniform bound (50) also shows that the finite-dimensional distributions are characterized by the moments

$$\mathbb{E}[p^\varepsilon(t_1, L)^{q_1} \cdots p^\varepsilon(t_k, L)^{q_k}] \tag{51}$$

for every real numbers $t_1 < \cdots < t_k$ and every integers $q_1, \ldots, q_k$.

4.3. **First moment.** Let us first address the first moment. Using the representation (49) the expectation of $p^\varepsilon(t, L)$ is

$$\mathbb{E}[p^\varepsilon(t, L)] = \frac{1}{4\pi|\mathcal{D}|} \int e^{-i\omega t} \hat{f}(\omega) \mathbb{E}[T^\varepsilon(\omega, L)] d\omega \,.$$

We denote $\mathcal{T}_\omega^\varepsilon(z) = \mathbb{E}[T^\varepsilon(\omega, z)]$. By (42) we have $\mathcal{T}_\omega^\varepsilon(z) = \mathbb{E}[\overline{\alpha_\omega^\varepsilon}(z)^{-1}]$. By application of Proposition 1, $\mathcal{T}_\omega^\varepsilon(z)$ converges as $\varepsilon \to 0$ to the limit $\mathcal{T}_\omega(z) = \mathbb{E}[\overline{\alpha_\omega}^{-1}]$. Here α_ω is the first coordinate of the diffusion process whose infinitesimal generator $\mathcal{L}$ is defined by (43). By application of the infinitesimal generator $\mathcal{L}$, we have

$$\mathcal{L}(\overline{\alpha_\omega}^{-1}) = -\frac{\omega^2}{8\bar{c}^2} \left(\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega) + \gamma^{(1)} \right) \overline{\alpha_\omega}^{-1} + i\kappa(\omega)\overline{\alpha_\omega}^{-1} \,.$$

Thus, the limit $\mathcal{T}_\omega$ satisfies the equation

$$\frac{d\mathcal{T}_\omega}{dz} = \xi(\omega)\mathcal{T}_\omega, \qquad \text{where} \qquad \xi(\omega) = -\frac{\omega^2}{8\bar{c}^2} \left(\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega) + \gamma^{(1)} \right) + i\kappa(\omega) \,,$$

starting from $\mathcal{T}_\omega(0) = 1$. The solution of this ordinary differential equation is: $\mathcal{T}_\omega(L) = \exp(\xi(\omega)L)$. The expectation of $T^\varepsilon(\omega, L)$ thus converges to $\mathcal{T}_\omega(L)$. Using the fact that $|T^\varepsilon| \leq 1$ and applying Lebesgue's theorem, we finally obtain that the expectation of $p^\varepsilon(t, L)$ converges to:

$$\mathbb{E}[p^\varepsilon(t, L)] \xrightarrow{\varepsilon \to 0} \frac{1}{4\pi|\mathcal{D}|} \int e^{-i\omega t} \hat{f}(\omega) \exp(\xi(\omega)L) d\omega \,.$$

4.4. **Higher moments.** Let us now consider the general moment (51). Using the representation (49) for each factor p^ε, these moments can be written as multiple integrals over $n = \sum_{j=1}^{k} q_j$ frequencies:

$$\mathbb{E}\left[p^\varepsilon(t_1, L)^{q_1} \cdots p^\varepsilon(t_k, L)^{q_k} \right] = \frac{1}{(4\pi|\mathcal{D}|)^n} \int \cdots \int \prod_{\substack{1 \leq j \leq k \\ 1 \leq l \leq q_j}} \hat{f}(\omega_{j,l}) e^{-i\omega_{j,l} t_j}$$

$$\times \mathbb{E}\left[\prod_{\substack{1 \leq j \leq k \\ 1 \leq l \leq q_j}} T^\varepsilon(\omega_{j,l}, L) \right] \prod_{\substack{1 \leq j \leq k \\ 1 \leq l \leq q_j}} d\omega_{j,l} \,. \quad (52)$$

The dependency in ε and in the randomness only appears through the quantity $\mathbb{E}[T^\varepsilon(\omega_1, L) \cdots T^\varepsilon(\omega_n, L)]$. Our problem is now to find the limit, as ε goes to 0, of these moments for n distinct frequencies. In other words we want to study the limit in distribution of $(T^\varepsilon(\omega_1, L), \ldots, T^\varepsilon(\omega_n, L))$ which results once again from the application of a diffusion-approximation theorem (Theorem 2-7 [17]).

Proposition 2. *The multi-frequency processes*

$$(\alpha_{\omega_1}^\varepsilon(z), \beta_{\omega_1}^\varepsilon(z), \ldots, \alpha_{\omega_n}^\varepsilon(z), \beta_{\omega_n}^\varepsilon(z))_{z \geq 0}$$

converge in distribution as $\varepsilon \to 0$ to the diffusion process

$$(\alpha_{\omega_1}(z), \beta_{\omega_1}(z), \ldots, \alpha_{\omega_n}(z), \beta_{\omega_n}(z))_{z \geq 0}$$

whose infinitesimal generator is

$$\mathcal{L} = \sum_{j=1}^{n} \frac{\gamma^{(c)}(\omega_j)\omega_j^2}{8\bar{c}^2}\left(A_{\alpha_{\omega_j}\beta_{\omega_j}}\overline{A_{\alpha_{\omega_j}\beta_{\omega_j}}} + \overline{A_{\alpha_{\omega_j}\beta_{\omega_j}}}A_{\alpha_{\omega_j}\beta_{\omega_j}}\right)$$

$$+ \sum_{j=1}^{n} \frac{\gamma^{(s)}(\omega_j)\omega_j^2}{8\bar{c}^2}(A_{\beta_{\omega_j}\beta_{\omega_j}} - A_{\alpha_{\omega_j}\alpha_{\omega_j}})$$

$$+ \sum_{j,l=1}^{n} \frac{\gamma^{(1)}\omega_j\omega_l}{4\bar{c}^2}(A_{\alpha_{\omega_j}\alpha_{\omega_j}} + A_{\beta_{\omega_j}\beta_{\omega_j}})(\overline{A_{\alpha_{\omega_l}\alpha_{\omega_l}}} + \overline{A_{\beta_{\omega_l}\beta_{\omega_l}}})$$

$$+ i\sum_{j=1}^{n} \kappa(\omega_j)(A_{\alpha_{\omega_j}\alpha_{\omega_j}} - A_{\beta_{\omega_j}\beta_{\omega_j}}). \qquad (53)$$

The quantity of interest $\mathbb{E}[T^\varepsilon(z,\omega_1)\cdots T^\varepsilon(z,\omega_n)]$ is denoted by $\mathcal{T}^\varepsilon(z)$. Applying the infinitesimal generator (53)

$$\mathcal{L}\left(\prod_{j=1}^{n}\overline{\alpha_{\omega_j}}^{-1}\right) = -\frac{1}{8\bar{c}^2}\left[\sum_{j=1}^{n}\omega_j^2\left(\gamma^{(c)}(\omega_j) + i\gamma^{(s)}(\omega_j)\right) + \sum_{j,l=1}^{n}\omega_j\omega_l\gamma^{(1)}\right]\prod_{j=1}^{n}\overline{\alpha_{\omega_j}}^{-1}$$

$$+ i\left[\sum_{j=1}^{n}\kappa(\omega_j)\right]\prod_{j=1}^{n}\overline{\alpha_{\omega_j}}^{-1}$$

gives the following equation for $\mathcal{T}(z) = \lim_{\varepsilon\to 0}\mathcal{T}^\varepsilon(z)$:

$$\frac{d\mathcal{T}(z)}{dz} = \left[-\sum_{j=1}^{n}\frac{\omega_j^2}{8\bar{c}^2}\left(\gamma^{(c)}(\omega_j) + i\gamma^{(s)}(\omega_j)\right) - \sum_{j,l=1}^{n}\frac{\omega_j\omega_l}{8\bar{c}^2}\gamma^{(1)} + i\sum_{j=1}^{n}\kappa(\omega_j)\right]\mathcal{T}(z),$$

with the initial condition $\mathcal{T}(0) = 1$. This linear equation has a unique solution but instead of solving it and computing explicitly the moments one can easily see that it is also satisfied by $\widetilde{\mathcal{T}}(z) = \mathbb{E}\left[\prod_{j=1}^{n}\widetilde{T}(z,\omega_j)\right]$ where

$$\widetilde{T}(z,\omega) = \exp\left(i\frac{\omega\sqrt{\gamma^{(1)}}}{2\bar{c}}B_z - \frac{\omega^2(\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega))}{8\bar{c}^2}z + i\kappa(\omega)z\right),$$

and $(B_z)_{z\geq 0}$ is a standard Brownian motion. Therefore $\mathcal{T}(L) = \widetilde{\mathcal{T}}(L)$ and the limit of the moment (52) is

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[p^\varepsilon(t_1,L)^{q_1}\cdots p^\varepsilon(t_k,L)^{q_k}\right] = \frac{1}{(4\pi|\mathcal{D}|)^n}\int\cdots\int\prod_{\substack{1\leq j\leq k \\ 1\leq l\leq q_j}}\hat{f}(\omega_{j,l})e^{-i\omega_{j,l}t_j}$$

$$\times\mathbb{E}\left[\prod_{\substack{1\leq j\leq k \\ 1\leq l\leq q_j}}\widetilde{T}(\omega_{j,l},L)\right]\prod_{\substack{1\leq j\leq k \\ 1\leq l\leq q_j}}d\omega_{j,l}, \quad (54)$$

which can be factorized as

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[p^\varepsilon(t_1,L)^{q_1}\cdots p^\varepsilon(t_k,L)^{q_k}\right] = \mathbb{E}\left[\prod_{j=1}^{k}\left(\frac{1}{4\pi|\mathcal{D}|}\int\hat{f}(\omega)e^{-i\omega t_j}\widetilde{T}(\omega,L)d\omega\right)^{q_j}\right].$$

This shows that the finite-dimensional distributions of the continuous processes $t \mapsto p^\varepsilon(t, L)$ converge as $\varepsilon \to 0$ to those of $t \mapsto (4\pi|\mathcal{D}|)^{-1} \int e^{-i\omega t} \hat{f}(\omega) \widetilde{T}(\omega, L) d\omega$.

4.5. **Pulse front stabilization.** We have seen that the processes $(t \mapsto p^\varepsilon(t, L))_{\varepsilon > 0}$ are tight in the space of the continuous functions, and that their finite-dimensional distributions converge. This proves the main result of this section.

Proposition 3. *As $\varepsilon \to 0$, the processes $(p^\varepsilon(t, L))_{t \in (-\infty, \infty)}$ converge in distribution in the space of the continuous functions to $(\widetilde{p}(t, L))_{t \in (-\infty, \infty)}$ given by*

$$\widetilde{p}(t, L) = f * G_L \left(t - \frac{\sqrt{\gamma^{(1)}}}{2\bar{c}} B_L \right), \tag{55}$$

where B_L is a standard Brownian motion and the Fourier transform of the convolution kernel G_L is

$$\hat{G}_L(\omega) = \frac{1}{2|\mathcal{D}|} \exp \left(i\kappa(\omega)L - \frac{\omega^2(\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega))}{8\bar{c}^2} L \right). \tag{56}$$

The coefficients κ, $\gamma^{(c)}$, $\gamma^{(s)}$, and $\gamma^{(1)}$ are given by (44-48).

This proposition shows that the transmitted pulse experiences a random time shift described by the Brownian motion B_L and a deterministic deformation described by the convolution kernel G_L. Apart from the random time shift, the limit pulse shape $P(L, t) = G_L * f(t)$ is solution of the equation

$$\frac{\partial P}{\partial L} = \mathcal{L}^{(1)} P + \mathcal{L}^{(2)} P, \quad P(t, 0) = \frac{1}{2|\mathcal{D}|} f(t), \tag{57}$$

where the operators $\mathcal{L}^{(j)}$ can be written explicitly in the Fourier domain as

$$\int_{-\infty}^{\infty} \mathcal{L}^{(1)} P(t) e^{i\omega t} dt = -\frac{\omega^2(\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega))}{8\bar{c}^2} \int_{-\infty}^{\infty} P(t) e^{i\omega t} dt, \tag{58}$$

$$\int_{-\infty}^{\infty} \mathcal{L}^{(2)} P(t) e^{i\omega t} dt = i\kappa(\omega) \int_{-\infty}^{\infty} P(t) e^{i\omega t} dt. \tag{59}$$

In the next two subsections we discuss in more detail the properties of these operators.

4.6. **Role of the coupling between propagating modes.** $\mathcal{L}^{(1)}$ is a pseudo-differential operator that models the deterministic pulse deformation due to the coupling between the right- and the left-going propagating modes. This operator depends only on the statistics of ν_0 given by (48), that is the transversely-averaged random perturbation ν. The first qualitative property satisfied by the pseudo-differential operator $\mathcal{L}^{(1)}$ is that it preserves the hyperbolic nature of the original equation. Indeed, in the time domain, we can write

$$\mathcal{L}^{(1)} P(t) = \left[\frac{1}{8\bar{c}} R_0 \left(\frac{\bar{c}t}{2} \right) \mathbf{1}_{[0,\infty)}(t) \right] * \left[\frac{\partial^2 P}{\partial t^2}(t) \right] = \frac{1}{8\bar{c}} \int_0^\infty R_0 \left(\frac{\bar{c}s}{2} \right) \frac{\partial^2 P}{\partial t^2}(t - s) ds,$$

where $R_0(z) = \mathbb{E}[\nu_0(0)\nu_0(z)]$. The indicator function $\mathbf{1}_{[0,\infty)}$ is essential to interpret correctly the convolution. If t_0 is a time such that P is vanishing for $t < t_0$, then $\mathcal{L}^{(1)} P$ is also vanishing for $t < t_0$. This means that the coupling between right-going and left-going modes cannot diffuse the wave in the right direction (ahead the front), but only in the left direction (behind the front).

The pseudo-differential operator $\mathcal{L}^{(1)}$ can be divided into two parts

$$
\mathcal{L}^{(1)} = \mathcal{L}_r^{(1)} + \mathcal{L}_i^{(1)}, \tag{60}
$$

$$
\int_{-\infty}^{\infty} \mathcal{L}_r^{(1)} P(t) e^{i\omega t} dt = -\frac{\gamma^{(c)}(\omega)\omega^2}{8\bar{c}^2} \int_{-\infty}^{\infty} P(t) e^{i\omega t} dt, \tag{61}
$$

$$
\int_{-\infty}^{\infty} \mathcal{L}_i^{(1)} P(t) e^{i\omega t} dt = -\frac{i\gamma^{(s)}(\omega)\omega^2}{8\bar{c}^2} \int_{-\infty}^{\infty} P(t) e^{i\omega t} dt. \tag{62}
$$

By the Wiener-Khintchine theorem [14], $\gamma^{(c)}$ is proportional to the power spectral density of the random stationary process ν_0. As a result, $\gamma^{(c)}$ is nonnegative and $\mathcal{L}_r^{(1)}$ can be interpreted as an effective diffusion operator. More precisely, for small frequencies, $\mathcal{L}_r^{(1)}$ behaves like a second-order diffusion. Let us denote by r_c the correlation length of the process ν_0 and by ω the typical pulse frequency. If $\omega r_c/\bar{c} \ll 1$, then $\gamma^{(c)}(\omega) \simeq 2\mu_0$ where $\mu_0 := \int_0^{\infty} R_0(z) dz$, and

$$
\mathcal{L}_r^{(1)} \simeq \frac{\mu_0}{4\bar{c}^2} \frac{\partial^2}{\partial t^2} .
$$

$\mathcal{L}_i^{(1)}$ is an effective dispersion operator, it preserves the energy. It behaves like a third-order dispersion for small frequencies. Indeed, if $\omega r_c/\bar{c} \ll 1$, then $\gamma^{(s)}(\omega) \simeq 4\omega\mu_1/\bar{c}$ where $\mu_1 := \int_0^{\infty} z R_0(z) dz$, and

$$
\mathcal{L}_i^{(1)} \simeq -\frac{\mu_1}{2\bar{c}^3} \frac{\partial^3}{\partial t^3} .
$$

It is interesting to determine which operator, $\mathcal{L}_r^{(1)}$ or $\mathcal{L}_i^{(1)}$, is the most important one. By scaling arguments, we get that $\omega^3\mu_1/\bar{c}^3$ is of order $(\omega r_c/\bar{c}) \times \mu_0(\omega/\bar{c})^2$ which is smaller than $\mu_0(\omega/\bar{c})^2$ if $\omega r_c/\bar{c} \ll 1$. As a result, the effective dispersion is usually smaller than the effective diffusion.

Note finally that the regime $\omega r_c/\bar{c} \ll 1$ is typically the one that is encountered in randomly perturbed single-mode waveguides. Indeed, it is natural to assume that r_c is much smaller than the typical diameter d of the waveguide, so that transverse random modulations of the medium can be observed. Besides, it is necessary for ω to be smaller than the cutoff frequency of the waveguide to ensure single-mode propagation. Qualitatively, this means that $\omega d/\bar{c}$ should be smaller than 1, and thus $\omega r_c/\bar{c}$ should be much smaller than 1.

4.7. **Role of the coupling with the evanescent modes.** $\mathcal{L}^{(2)}$ is a pseudo-differential operator that models the deterministic pulse deformation due to the coupling between the right-going propagating mode and the evanescent modes. This operator depends only on the statistics of ν_l, $l \geq 1$, that is to say it depends only on the statistics of the transverse fluctuations $\nu(\mathbf{x}, z) - \nu_0(z)$. Note that this is in contrast with the operator $\mathcal{L}^{(1)}$ studied in the previous section, that depends only on the transversely-averaged process ν_0.

The operator $\mathcal{L}^{(2)}$ is an effective dispersion operator, it does not remove energy from the pulse front. It is a pseudo-differential operator with a complicated symbol, which makes it difficult to infer the dispersion order. However $\mathcal{L}^{(2)}$ can be reduced to a much simpler operator with a supplementary assumption. Remember that $\omega_c = \bar{c}\sqrt{\lambda_1}$ is the cutoff frequency and that we have assumed the single-mode condition $\omega < \omega_c$. If, moreover, $\omega \ll \omega_c$, then the propagation modal wavenumbers

are $\beta_l(\omega) \simeq \sqrt{\lambda_l}$ and

$$\kappa(\omega) \simeq \frac{\mu_2 \omega^3}{2\bar{c}^3}, \qquad \text{with} \qquad \mu_2 = \sum_{l=1}^{\infty} \frac{1}{\sqrt{\lambda_l}} \int_0^{\infty} \mathbb{E}[\nu_l(0)\nu_l(z)]e^{-\sqrt{\lambda_l}z}\, dz\,.$$

This shows that the operator $\mathcal{L}^{(2)}$ is a third-order dispersion in the regime where the typical pulse frequency ω satisfies $\omega \ll \omega_c$

$$\mathcal{L}^{(2)} \simeq \frac{\mu_2}{2\bar{c}^3} \frac{\partial^3}{\partial t^3}\,.$$

4.8. **The planar waveguide.** We now address a particular example. We consider the case of a planar waveguide in which the random perturbation ν is a stationary zero-mean random process with Gaussian autocorrelation function:

$$\mathbb{E}[\nu(x', z')\nu(x' + x, z' + z)] = \sigma^2 \exp\left(-\frac{x^2 + z^2}{2r_c^2}\right)\,,$$

where σ is the standard deviation of the random fluctuations and r_c is the correlation length, that is assumed to be much smaller than d. This is achieved if $\nu(x, z) = \sum_{l=0}^{\infty} \nu_l(z)\phi_l(x)$ where the ν_l are independent, stationary, zero-mean processes with autocorrelation function

$$\mathbb{E}[\nu_l(z')\nu_l(z' + z)] = \frac{\sigma^2 r_c}{\sqrt{2\pi}d} \exp\left(-\frac{z^2}{2r_c^2} - \frac{\pi^2 r_c^2 l^2}{2d^2}\right)\,.$$

The typical pulse frequency ω is smaller than the cutoff frequency $\omega_c = \pi\bar{c}/d$ to ensure single-mode propagation. We have

$$\mu_0 = \frac{\sigma^2 r_c^2}{2d}, \qquad \mu_1 = \frac{\sigma^2 r_c^3}{\sqrt{2\pi}d}, \qquad \mu_2 = \frac{\sigma^2 r_c^2}{2\pi} \sum_{l \geq 1} \text{erfc}\left(\frac{\pi r_c l}{\sqrt{2}d}\right)\,,$$

where $\text{erfc}(x) = 1 - \text{erf}(x) = (2/\sqrt{\pi})\int_x^{\infty} \exp(-s^2)ds$. As pointed out above, it is also natural in this study to assume that $r_c \ll d$. As a result, we have $\omega r_c/\bar{c} \ll 1$. This assumption allows us to give estimates for the different terms that appear in the diffusion-dispersion equation (57). On the one hand, the typical orders of magnitude of the two terms of the operator $\mathcal{L}^{(1)}P$ are

$$\frac{\mu_0 \omega^2}{4\bar{c}^2} \sim \frac{\sigma^2 r_c^2 \omega^2}{d\bar{c}^2}, \qquad \frac{\mu_1 \omega^3}{2\bar{c}^3} \sim \frac{\sigma^2 r_c^3 \omega^3}{d\bar{c}^3} \sim \frac{\sigma^2 r_c^2 \omega^2}{d\bar{c}^2} \times \frac{\omega r_c}{\bar{c}}\,,$$

which shows that the effective second-order diffusion prevails over the effective third-order dispersion. On the other hand, the typical order of magnitude of the term $\mathcal{L}^{(2)}P$ is

$$\frac{\mu_2 \omega^3}{2\bar{c}^3} \sim \frac{\sigma^2 r_c^2 \omega^3}{\bar{c}^3} \ln\left(\frac{d}{\pi r_c}\right) \sim \frac{\mu_0 \omega^2}{4\bar{c}^2} \times \frac{\omega d}{\bar{c}} \ln\left(\frac{d}{\pi r_c}\right)\,.$$

In the regime where $\omega \sim \bar{c}/d$ and $d \gg r_c$, this formula shows that the operator $\mathcal{L}^{(2)}$ plays a dominant role. In this case the coupling of the propagating mode with the evanescent modes can be responsible for the main pulse distortion in the form of strong dispersion.

5. **Conclusion.** In this paper we have analyzed the role of evanescent modes in pulse propagation in randomly perturbed waveguides. We have shown that the random coupling between the propagating and evanescent modes induces dispersion. The analysis is based on separation of scales when the propagation distance is large compared to the size of the inhomogeneities. The asymptotic analysis is carried out with single-mode waveguides. However, it is natural to conjecture that it could be extended to multi-mode waveguides, in the case in which the number of propagating modes is finite and fixed. This generalization deserves further investigation.

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