Bright Solitons in Bose-Einstein Condensates

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1 Introduction

It is a well-known fact that the interplay of nonlinearity and dispersion leads to the appearance of localized wavepackets moving without distortion. When the interaction between atoms is attractive (resp. repulsive), bright (resp. dark) solitons can be generated in BEC. Note that the term soliton is usually reserved to the localized solution of an integrable system [1]. Elastic scattering is a characteristic property of true solitons. However, many physical systems are described by non-integrable systems which have stationary or moving localized solutions. Collisions between these solitary waves are usually inelastic, which is a clear indication that they are not true solitons. However, in this chapter, we will use the term soliton for these waves as well.

The typical system under consideration in this chapter is a Bose gas with attractive interaction between atoms in different types of trap potentials. Without a trap, a free 2D or 3D BEC with attractive interaction can collapse. Indeed, the GP equation then takes the form of the cubic Nonlinear Schrödinger (NLS) equation. In the case of focusing nonlinearity (corresponding to negative scattering length) the solution can be singular or spreading [2]. In the 2D case, a necessary condition for collapse is that the number of atoms should exceed a critical one. In the 3D case collapse may, in principle, occur for any number of atoms. The simplest way to see this is to inspect the expressions for the energy E and the number of atoms N of the condensate in dimension D:

$$E = \int \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + V|\psi|^2 + \frac{g_D}{2} |\psi|^4 \right] d^D\mathbf{r}\,, \quad N = \int |\psi|^2 d^D\mathbf{r}\,, \tag{1}$$

and to perform a dimensional analysis by assuming that L is the typical size of the BEC. From the expression of N, we find that $[\psi] \sim 1/L^{D/2}$. From the expression of E we find that the kinetic energy behaves like $1/L^2$, the potential energy (for $V \sim r^n$) like L^n, and the interaction energy like $1/L^D$. In the following discussion,

we assume that g_D is negative (attractive interaction) and that the potential V is a symmetric harmonic trap ($V \sim r^2$).

In the one-dimensional case ($D = 1$) the estimate for the energy E is

$$E \sim \frac{c_{\rm kin}}{L^2} + c_{\rm pot} L^2 - \frac{c_{\rm int}}{L}. \tag{2}$$

The collapse is absent and a local minimum in L exists in the effective potential. This localized state is a bright matter-wave soliton. We will discuss its properties in the following sections.

For a two-dimensional ($D = 2$) condensate we obtain

$$E \sim \frac{c_{\rm kin} - c_{\rm int}}{L^2} + c_{\rm pot} L^2 \sim \frac{N(N_c - N)}{L^2} \text{ for } L \ll 1.$$

Thus for $N > N_c$ collapse can occur. The value of the critical number is $N_c = 1.862 \times (\pi\hbar^2)/(|g_2|m)$ [3, 4].

For a three-dimensional ($D = 3$) condensate we have the estimate

$$E \sim \frac{c_{\rm kin}}{L^2} + c_{\rm pot} L^2 - \frac{c_{\rm int}}{L^3}. \tag{3}$$

Here it is possible to generate a metastable state if the number of atoms N is small enough. The existence of such a state is due to the balance between the kinetic energy, the trap potential energy and the interaction energy. The criterion for parameters when this state disappears can be obtained from the requirement that the first and second derivatives of the energy $E(L)$ are equal to zero at the critical point. The energy can be calculated for example using the Gaussian ansatz for the wavefunction of condensate. Then the variational approach gives the value $N_c = k_{\rm VA} a_r/|a|$ for the critical number of atoms in the case of an attractive BEC in a spherical trap, with $a_r = \sqrt{\hbar/(m\omega_r)}$ and $k_{\rm VA} \approx 0.67$. Numerical simulations show that the exact value of the dimensionless constant is $k \approx 0.574$. The experimental value for this parameter deviates from about 20% [5]: If the longitudinal frequency is equal to zero $\omega_z = 0$, then the variational approach with the Gaussian transverse profile and *sech*-type longitudinal profile gives $k_{\rm VA} \approx 0.76$, while the numerical value is $k \approx 0.627$ [6]. In the case of the ^{7}Li condensate in the nearly symmetric trap with $a_r \approx 3\mu$m and $a \approx -1.45$nm used in the experiments at Rice University, the critical number is approximately $N_c \sim 1400$ [7].

In this chapter we describe the main properties of bright solitons in BECs. We focus our attention on the dynamics of solitons in elongated BEC, when the quasi 1D GP equation is valid. The interaction of solitons with linear and nonlinear impurities, transmission of solitons through barriers, the dynamics of localized states in nonlinear periodic potentials will be considered. The dynamics of 2D bright solitons under the temporal and spatial Feshbach resonance management as well as stable 2D solitons in dipolar BECs are also analyzed.

2 Bright solitons in quasi one-dimensional BEC

2.1 The 1D Gross-Pitaevskii equation

The GP equation for an elongated BEC with varying in space and time trap potential $V(z,t)$ and atomic scattering length $a(z,t)$ is [8, 9]

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\psi_{zz} - V(z,t)\psi - g(z,t)|\psi|^2\psi = 0, \tag{4}$$

with $g(z,t) = 2\hbar\omega_r a(z,t)$. In the case of a stationary harmonic trap we simply have $V(z,t) = (1/2)m\omega_z^2 z^2$. The validity of this equation is defined by the condition $(|a|N)/a_r \ll 1$ with $a_r = \sqrt{\hbar/(m\omega_r)}$. When $V = 0$ and the scattering length $a \equiv a_s$ is a negative constant, this equation admits the well-known bright soliton solution

$$\psi(z,t) = \frac{a_r}{\sqrt{2|a_s|}\kappa}\mathrm{sech}(\frac{z-vt}{\kappa})\exp\left[i\frac{mv}{\hbar}z - \frac{i}{\hbar}\left(\frac{mv^2}{2} - \frac{\hbar^2\kappa^2}{2m}\right)t\right], \tag{5}$$

where $\kappa = a_r^2/(|a_s|N)$ is the soliton width and v is the soliton velocity. Note that the condition $(|a_s|N)/a_r \ll 1$ for the validity of the 1D GP equation imposes $\kappa \gg a_r$. Bright solitons have been observed in experiments performed at Rice University [10] and at the Ecole Normale Supérieure in Paris [11]. The experiment at Rice University used ^{7}Li, the scattering length was $a_s = -3a_0$ (where $a_0 \simeq 0.053$nm is the Bohr radius), and the trap frequencies were $\omega_z = 2\pi \times 3.2$Hz and $\omega_r = 2\pi \times 640$Hz, which corresponds to $a_r \approx 1.5\mu$m. The number of atoms in the soliton was $N \approx 6000$, which corresponds to $\kappa \approx 2.5\mu$m, in agreement with the number allowed by the theory.

In this chapter we study the dynamics of bright solitons under different types of spatial and temporal modulations of the BEC and external potential parameters. The inhomogeneities of the potential $V(z,t)$ in (4) can be local (in the form of a well or a barrier) or extended (in the form of a periodic or random potential, generated e.g. by an optical lattice or an optical speckle pattern). Temporal and spatial variations of the scattering length $a(z,t)$ can be produced by using the Feshbach resonance method (described in detail in the next chapter), namely by the variation of the external magnetic field $B(z,t)$ near the resonant value B_0 [12]. Small variations of the field near the resonant value can induce large variations of the scattering length according to the formula

$$a(z,t) = a_b\left(1 - \frac{\Delta}{B_0 - B(z,t)}\right),$$

where a_b is the background value of the atomic scattering length and Δ is the resonance width. Optical methods for manipulating the value of the scattering length are also possible [13]. This type of potential is produced by the optically induced Feshbach resonance [14]. According to this technique, the periodic variation of the laser field intensity in the standing wave $I(z) = I_0\cos^2(kz)$ produces the periodic variation of the atomic scattering length a of the form

$$a(z) = a_0 + \alpha \frac{I(z)}{\delta + \alpha I(z)}, \tag{6}$$

where δ is the detuning from the Feshbach resonance. When the detuning is large we have $a(z) = a_0 + a_1 \cos^2(kz)$. As a result, the mean field nonlinear coefficient g (which is proportional to the scattering length a) in the GP equation has a spatial dependence [15, 16].

It should be noted that other forms of the 1D GP equation have been suggested.

1. *1D GP equation in the form of nonpolynomial NLS equation.* The exclusion of transverse degrees of freedom performed previously leads to the 1D GP equation. This derivation is valid only for a low density condensate. To obtain the equation for higher densities one can use the variational approach with the generalized ansatz for the wavefunction [17]

$$\Psi(x, y, z, t) = \frac{1}{\sqrt{\pi}\sigma(z, t)} \exp\left(-\frac{x^2 + y^2}{2\sigma^2(z, t)}\right) \psi(z, t).$$

The variational equation for the parameter σ has the form of a differential equation. With the hypothesis that the derivatives can be neglected, σ can be calculated from an algebraic equation, giving $\sigma^2 = \sigma^2(|\psi|^2) = a_r^2\sqrt{1 + 2a|\psi|^2}$. The resulting 1D GP equation has the form of the nonpolynomial NLS equation

$$i\hbar\psi_t = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{g_3|\psi|^2}{2\pi\sigma^2(|\psi|^2)} + \frac{\hbar\omega_r}{2}\left(\frac{\sigma^2(|\psi|^2)}{a_r^2} + \frac{a_r^2}{\sigma^2(|\psi|^2)}\right) + V(z)\right]\psi,$$

with $g_3 = 4\pi\hbar^2 a/m$. In the weak interaction limit $|a|N|\psi|^2 \ll 1$, we have $\sigma^2 = a_r^2$ and this equation reduces to the 1D NLS equation with the effective nonlinearity parameter $g_1 = g_3/(2\pi a_r^2)$. If $|a|N/a_r > 2/3$, then there is no soliton solution due to the condensate collapse. The latter prediction as well as the parameters of the ground states are in good agreement with the numerical simulations of the full GP equation.

2. A 1D description that does not neglect the derivatives in the equation for σ has been suggested in [18]. The ansatz used for the variational approach is of the form

$$\Psi(x, y, z, t) = \frac{1}{\sqrt{\pi}\sigma(z, t)} \exp\left(-\frac{x^2 + y^2}{2\sigma^2(z, t)} + i\frac{b(z, t)}{2}(x^2 + y^2)\right) \psi(z, t).$$

In the dimensionless variables $\tilde{\psi} = \psi\sqrt{a_r}$, $\tilde{\sigma} = \sigma/a_r$, $\tilde{b} = a_r^2 b$, $\tilde{t} = \omega_r t$, $\tilde{z} = z/a_r$ the evolution of the BEC is described by the system (where we have dropped the tildes)

$$i\psi_t + \frac{1}{2}\psi_{zz} - V(z)\psi - 3G\frac{|\psi|^2\psi}{\sigma^2} - \frac{1}{2}\left[\frac{2}{\sigma^2} + \frac{\sigma_{zz}}{\sigma} + \left(\frac{\sigma_z}{\sigma} - \frac{i}{2}b_z\sigma^2\right)\frac{(|\psi|^2)_z}{|\psi|^2} - \frac{1}{2}b_z^2\sigma^4 - \frac{i}{2}b_{zz}\sigma^2 - ib_z\sigma\sigma_z\right] = 0, \tag{7}$$

$$(\sigma^2|\psi|^2)_t + \left[\frac{i}{2}\sigma^2(\psi\psi_z^* - \psi^*\psi_z) + \sigma^4|\psi|^2 b_z\right]_z = 2b\sigma^2|\psi|^2, \tag{8}$$

$$b_t = \frac{\sigma_{zz}}{\sigma^3} + \frac{1-\sigma_z^2}{\sigma^4} + \frac{\sigma_z}{\sigma^3}\frac{(|\psi|^2)_z}{|\psi|^2} - 1 - b^2 - b_z^2\sigma^2 + \frac{i(\psi_z\psi^* - \psi\psi_z^*)}{2|\psi|^2}b_z + 2G\frac{|\psi|^2}{\sigma^4} \tag{9}$$

with $G = a/a_r$. Thus, the 3D GP equation has been transformed into a quasi-1D form for the case of the elongated cigar-shaped geometry when the radial distributions of the condensate density and its radial velocity can be approximated by simple Gaussian functions. The variables in Eqs. (7-9) depend only on one spatial coordinate, which is very convenient in numerical simulations. Besides, the analysis of the system (7-9) predicts important phenomena such as the existence in the repulsive BEC case (if $G\rho_0 > 3.91$, where ρ_0 is the condensate density) of small-amplitude bright solitons propagating with the velocity close to the speed of sound.

We finally introduce the dimensionless form of the GP equation:

$$i\psi_t + \psi_{zz} + 2|\psi|^2\psi = V_{nl}(z,t)|\psi|^2\psi + V_l(z,t)\psi. \tag{10}$$

This equation is obtained from (4) by the change of variables $\tilde{\psi} = \psi\sqrt{|a_s|}$, where $a_s < 0$ is the reference atomic scattering length, $\tilde{t} = \omega_r t$, $\tilde{z} = \sqrt{2}z/a_r$, and by dropping the tildes. Here $V_l(z,t)$ is the linear external potential and $V_{nl}(z,t) = 2 - 2a(z,t)/a_s$ is the nonlinear potential induced by the space-time variations of the scattering length a.

If $V_l = V_{nl} = 0$, then this equation can be reduced to the integrable NLS equation. The bright soliton solution is:

$$\psi(z,t) = 2\nu_0 \mathrm{sech}[2\nu_0(z - \zeta(t))]\exp[2i\mu_0(z - \zeta(t)) + i\phi(t)], \tag{11}$$

its amplitude is $2\nu_0$, its width is $1/(2\nu_0)$, and its velocity is $4\mu_0$. In terms of the physical parameters that appear in the dimensional expression (5), they are given by

$$2\nu_0 = \frac{N|a_s|}{\sqrt{2}a_r} = \frac{\sqrt{2}a_r}{\kappa}, \qquad 4\mu_0 = \frac{\sqrt{2}a_r m v}{\hbar}$$

The soliton center $\zeta(t)$ and phase $\phi(t)$ satisfy $\partial_t\zeta = 4\mu_0$ and $\partial_t\phi = 4(\nu_0^2 + \mu_0^2)$.

If V_l and/or V_{nl} have spatial and/or time variations, then the matter wave soliton experiences velocity and amplitude modulations and emits radiation.

2.2 Adiabatic soliton compression

Adiabatic spatial variations of the atomic scattering length can be used as an effective way for controlling the soliton parameters and for inducing changes in the solitons' shape, which could be useful for applications [15]. In contrast to abrupt variations, adiabatic changes make it possible to preserve the integrity of the soliton (no splitting occurs), thus leading for bright solitons to the compression of the pulse with the increase of the matter density. The possibility to compress

BEC solitons could be an experimental tool to investigate the range of validity of the 1D GP equation. Since the quasi 1D regime is valid for low densities, it would be indeed interesting to see how far one can compress a soliton in a real experiment by means of adiabatic changes of the scattering length.

To model the adiabatic variation of the atomic scattering length in 1D (cigar shaped) BEC we consider the normalized GP Eq. (10). We neglect the linear potential V_l by assuming that the range of variation of the trap potential is very large in comparison with the size of the BEC. We assume that the scattering length a is a negative-valued, slowly varying function of space and time. As a consequence $2 - V_{nl} = 2a/a_s$ is a positive function since the reference length $a_s < 0$. Although the analysis can be performed for a general smooth function $V_{nl}(z,t)$, we only discuss here the spatially varying case $V_{nl} \equiv V_{nl}(z)$. By the transform of field $\check{\psi} = \sqrt{1 - V_{nl}(z)/2}\psi$, we get from (10):

$$i\check{\psi}_t + \check{\psi}_{zz} + 2|\check{\psi}|^2\check{\psi} = R(\check{\psi}), \qquad R(\check{\psi}) = F(z)\check{\psi}_z, \qquad F(z) = [\ln(1 - V_{nl}(z)/2)]_z.$$

The term $R(\check{\psi})$ can be considered as a small perturbation when the soliton width is much smaller than the scale of variation of the scattering length. The soliton solution of the GP equation under small perturbation can be assumed to be of the form

$$\check{\psi}(z,t) = 2\nu(t)\mathrm{sech}[2\nu(t)(z - \zeta(t))] \exp\left[i2\mu(t)(z - \zeta(t)) + i\phi(t)\right]. \tag{12}$$

Using the adiabatic perturbation theory for solitons [15], we find the equations for the soliton parameters

$$\frac{d\nu}{dt} = 4\nu\mu F(\zeta), \qquad \frac{d\mu}{dt} = \frac{4}{3}\nu^2 F(\zeta), \qquad \frac{d\zeta}{dt} = 4\mu, \tag{13}$$

whose solutions satisfy the relations

$$\nu(\zeta) = \nu_0\left(1 - \frac{V_{nl}(\zeta)}{2}\right), \qquad \mu^2(\zeta) = \mu_0^2 + \frac{\nu_0^2}{3}\left[\left(1 - \frac{V_{nl}(\zeta)}{2}\right)^2 - 1\right] \tag{14}$$

These results can be compared to numerical simulations of Eq. (10) with the function $V_{nl}(z)$ given by

$$V_{nl}(z) = -V_{nl0}\left[\frac{1}{2} + \frac{1}{\pi}\arctan\left(s\pi(z - \frac{L_f}{2})\right)\right] \tag{15}$$

This function models, for small s, an adiabatic change of the scattering length from $a(-\infty) = a_s$ to $a(+\infty) = (1 + V_{nl0}/2)a_s$. Figure 1 shows the amplitude of a bright soliton as a function of the soliton center. The soliton, initially at rest, is sucked into the higher scattering length region, and reaches a constant velocity after passing the inhomogeneity.

If the nonlinear potential $V_{nl}(z) = -V_{nl0}f(z)$ is not slowly varying, then internal degrees of freedom can be excited. To take such effects into consideration

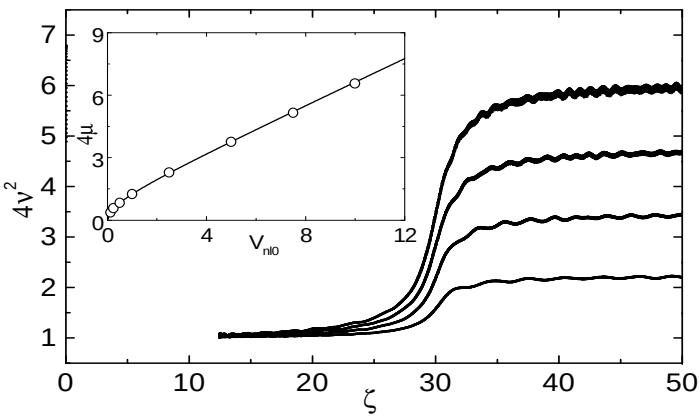

Fig. 1. The squared amplitude of a bright soliton vs soliton center ζ for different values of the amplitude V_{nl0} of a kink-like spatial nonlinear inhomogeneity (15) centered at $z = L_f/2$. The curves, from bottom to top, refer to $V_{nl0} = 2.5, 5.0, 7.5, 10.0$, respectively. The other parameters are $s = 0.2$, $L_f = 60$, $\mu_0 = 0$, and $\nu_0 = 0.5$. The soliton is initially at rest, placed at position $\zeta_0 = 12.5$. In the inset we show the soliton final velocity 4μ as a function of V_{nl0}. The open dots are numerical values while the solid line is obtained from (14) [15].

we assume that $|V_{nl0}| \ll 1$ and we consider that the solution has the form of a chirped *sech*-type profile

$$\psi(z,t) = A(t)\mathrm{sech}[2\nu(t)(z-\zeta(t))]\exp\left[iC(t)(z-\zeta(t)) + iB(t)(z-\zeta(t))^2 + i\phi(t)\right].$$

The variational equations for the parameters ν and ζ are [16]

$$\frac{d^2}{dt^2}\left(\frac{1}{\nu}\right) = \frac{256}{\pi^2}\nu^3 - \frac{16N}{\pi^2}\nu^2 - V_{nl0}\frac{24N}{\pi^2}\nu^2\frac{\partial F}{\partial \nu}, \qquad \frac{d^2\zeta}{dt^2} = V_{nl0}N\frac{\partial F}{\partial \zeta}, \quad (16)$$

where $N = A^2/\nu$ is a conserved quantity and

$$F(\nu,\zeta) = \nu^2\int_{-\infty}^{\infty} f(z)\mathrm{sech}^4[2\nu(z-\zeta)]dz.$$

If the inhomogeneity is Gaussian $f(z) = \exp(-z^2/l^2)$, with $\nu l \gg 1$, then we obtain $F(\nu,\zeta) = 2\nu f(\zeta)/3$. Thus there is the fixed point at $\zeta = 0$, given by $\nu_c = [N(1+2V_{nl0})]/16$. The soliton is trapped by the inhomogeneity for positive V_{nl0}. In the trapped regime, due to the nonzero chirp, the soliton width performs oscillations with the frequency $\omega_\nu = N^2(1+2V_{nl0})^2/(16\pi)$ and the soliton center also oscillates with the frequency $\omega_\zeta = N\sqrt{V_{nl0}(1+2V_{nl0})/12}/l$.

2.3 Transmission through nonlinear barriers and wells

When the matter wave soliton moves in the linear and nonlinear potentials V_l and V_{nl}, the perturbation theory based on the Inverse Scattering Transform (IST) of

the normalized GP Eq. (10) gives the evolutions of the localized soliton part of the matter wave and the delocalized radiation [19, 20]. A tractable perturbation analysis can be carried out by using series expansions with respect to the amplitudes V_{l0} and V_{nl0} of the linear and nonlinear potentials [21].

Quasi-particle approach

Applying the first-order perturbed IST theory [20] we obtain that the soliton center behaves like a quasi-particle moving in the effective potential W:

$$\frac{d^2\zeta}{dt^2} = -W'(\nu_0, \zeta(t)), \tag{17}$$

where the prime stands for a derivative with respect to ζ and $4\nu_0$ is the mass (number of atoms) of the soliton that is preserved. The effective potential has the form $W(\nu, \zeta) = W_l(\nu, \zeta) + W_{nl}(\nu, \zeta)$, with

$$W_l(\nu, \zeta) = K_{l,\nu} * V_l(\zeta), \qquad K_{l,\nu}(\zeta) = 2\nu \mathrm{sech}^2(2\nu\zeta), \tag{18}$$

$$W_{nl}(\nu, \zeta) = K_{nl,\nu} * V_{nl}(\zeta), \qquad K_{nl,\nu}(\zeta) = 4\nu^3 \mathrm{sech}^4(2\nu\zeta), \tag{19}$$

where $*$ stands for the convolution product: $K * V(\zeta) := \int_{-\infty}^{\infty} K(\zeta - z)V(z)dz$. In this first approximation terms of order V_{l0}^2 and V_{nl0}^2 are neglected. This approach gives the same result as the adiabatic perturbation theory for solitons that is a first-order method as well. This adiabatic perturbation theory was originally introduced for optical solitons [22] and it was recently applied to matter wave solitons [23]. Here it is the first step of the analysis as we will include second-order and radiation effects in the next section. The predictions of the quasi-particle approach are qualitatively the same ones for linear and nonlinear potentials, so V denotes either V_l or V_{nl} in the following discussion [21].

Barrier potential. Let us first examine the case where the potential V is a barrier, meaning that $V \geq 0$ and $\lim_{|z|\to\infty} V(z) = 0$. When the soliton reaches the barrier, it slows down, and it eventually goes through the barrier if the input energy is above the maximal energy barrier, meaning $8\mu_0^2 > W_{max}(\nu_0) := \max_z W(\nu_0, z)$. After passing through the barrier, the soliton recovers its initial mass and velocity. If, on the contrary, the velocity of the incoming soliton is such that $8\mu_0^2 < W_{max}(\nu_0)$, then the soliton is reflected by the barrier. After the interaction with the barrier, the soliton mass and velocity take the values $4\nu_0$ and $-4\mu_0$, respectively. However, we shall see in the next section that the interaction with the barrier generates radiation which plays a significant role especially when $8\mu_0^2 \sim W_{max}(\nu_0)$.

Potential well. We now examine the case where the potential is a well, meaning that $V \leq 0$ and $\lim_{|z|\to\infty} V(z) = 0$. When the soliton reaches the well, it speeds up, and it eventually goes through the well whatever its initial velocity is. However, we shall see in the next section that the interaction with the well generates radiation that reduces the soliton energy. As a result, the soliton cannot escape the well if its velocity is too small.

Radiation effects

Applying the first-order perturbation IST theory for the radiation emission [20], we obtain that the radiated mass density is of order V_{l0}^2 and V_{nl0}^2:

$$n(\lambda) = \frac{\pi[(\lambda-\mu)^2+\nu^2]^2}{256\mu^6\cosh^2\left(\frac{\pi}{4}\frac{\lambda^2+\nu^2-\mu^2}{\mu\nu}\right)} \times \left|\hat{V}_l\left(k(\lambda,\nu,\mu)\right)\right.$$
$$\left. + \frac{[(\lambda+\mu)^2+\nu^2][(\lambda-3\mu)^2+8\mu^2+\nu^2]}{12\mu^2}\hat{V}_{nl}\left(k(\lambda,\nu,\mu)\right)\right|^2 \qquad (20)$$

where $k(\lambda,\nu,\mu) = [(\lambda-\mu)^2+\nu^2]/\mu$ and $\hat{V}_{l,nl}(k) = \int V_{l,nl}(z)\exp(ikz)dz$. This formula is correct if the soliton goes through the potential and it allows us to capture the second-order evolution of the soliton parameters, as we show now. The mass (number of atoms) N and energy (Hamiltonian) E

$$N = \int_{-\infty}^{\infty}|\psi|^2 dz\,, \qquad E = \int_{-\infty}^{\infty}[|\psi_z|^2 - |\psi|^4 + V_l(z)|\psi|^2 + \frac{1}{2}V_{nl}(z)|\psi|^4]dz$$

can be expressed in terms of radiation and soliton contributions:

$$N = 4\nu + \int_{-\infty}^{\infty} n(\lambda)d\lambda, \qquad E = 2\nu\left[8\mu^2 - \frac{8\nu^2}{3} + W(\nu,\zeta)\right] + 4\int_{-\infty}^{\infty}\lambda^2 n(\lambda)d\lambda.$$

The total mass and energy are preserved by the perturbed NLS Eq. (10), so that it is possible to determine the decay of the soliton mass and energy from the expressions of the radiated mass and energy. This in turn gives the decay of the soliton parameters (ν,μ) which is proportional to V_{l0}^2 and V_{nl0}^2. The coefficients (ν_T,μ_T) of the transmitted soliton are [21]

$$\nu_T = \nu_0 - \frac{1}{4}\int_{-\infty}^{\infty} n(\lambda)d\lambda, \qquad \mu_T = \mu_0 - \frac{1}{8}\int_{-\infty}^{\infty}\frac{\lambda^2+\nu_0^2-\mu_0^2}{\mu_0\nu_0}n(\lambda)d\lambda.$$

Nonlinear barrier. If $V_{l0} = 0$ and $V_{nl0} > 0$, then the quasi-particle approach predicts that the soliton is fully transmitted if its velocity is large enough so that $8\mu_0^2 > W_{max}(\nu_0)$. Taking into account radiation yields that the transmission is not complete in the sense that the transmitted soliton mass is not equal to the incoming soliton mass.

Nonlinear well. If $V_{l0} = 0$ and $V_{nl0} < 0$ then the quasi-particle approach predicts full soliton transmission. The second-order analysis exhibits radiative mass and energy losses. As a result, if the initial velocity is not large enough, then the energy loss is too important and the soliton is trapped in the nonlinear potential.

Enhanced transmission by nonlinear modulation. A nonlinear positive potential V_{nl} can help a soliton going through a potential well V_l [23]. Indeed, the radiation emitted by the soliton due to the interaction with the linear well and with the nonlinear potential can cancel each other, resulting in an enhanced soliton transmittivity [21]. Similarly a nonlinear negative modulation can help the soliton going through a linear barrier [24].

2.4 Trapping by dynamically managed linear potentials

The purpose of this section is to show how it is possible to dynamically manage a potential well by a rapid time modulation of a barrier. The BEC wavefunction ψ in a quasi 1D geometry with fast moving potential is described by the GP Eq. [25]

$$i\psi_t + \psi_{zz} + 2|\psi|^2\psi = V_l\left(z - f(\frac{t}{\varepsilon}), t\right)\psi, \tag{21}$$

where $f(\tau)$ is a periodic function with period 1. The small parameter ε describes the fast oscillation period in the dimensionless variables. A standard multi-scale expansion yields the averaged GP equation

$$i\psi_t + \psi_{zz} + 2|\psi|^2\psi = V_{\text{eff}}(z,t)\psi, \qquad V_{\text{eff}}(z,t) = \int_0^1 V_l\left(z - f(\tau), t\right) d\tau. \tag{22}$$

If, for example, $f(\tau) = w\sin(2\pi\tau)$, then

$$V_{\text{eff}}(z,t) = K_{\text{eff}} * V_l(z,t), \qquad K_{\text{eff}}(z) = \frac{1}{\pi}\frac{1}{\sqrt{w^2 - z^2}}\mathbf{1}_{(-w,w)}(z), \tag{23}$$

where $\mathbf{1}_{(-w,w)}(z) = 1$ if $z \in (-w, w)$ and 0 otherwise. Let us assume that the potential V_l is time-independent. Applying the first-order perturbed IST the dynamics of the soliton center can be described as a quasi-particle (17) moving in the effective potential $W(\nu,\zeta) = K_{l,\nu} * V_{\text{eff}}(\zeta)$ where $K_{l,\nu}$ is given by (18). The quasi-particle potential is plotted in Fig. 2 for a Gaussian barrier potential V_l. The potential $W(\nu,\zeta)$ has a local minimum at $\zeta = 0$ between two global maxima that are close to $\zeta = \pm w$. The well amplitude is $\Delta W(\nu) = \max_z W(\nu,z) - W(\nu,0)$.

Let us consider an input soliton centered at $\zeta = 0$ with parameters (ν_0, μ_0). There is a critical value μ_c for the initial soliton velocity parameter μ_0 defined by $\mu_c^2 = \Delta W(\nu_0)/8$ that determines the type of motion:
- If $|\mu_0| < \mu_c$ then the soliton is trapped. Its motion is oscillatory between the positions $\pm\zeta_f$ defined by $W(\nu_0,\zeta_f) - W(\nu_0,0) = 8\mu_0^2$.
- If $|\mu_0| > \mu_c$, then the soliton motion is unbounded. It escapes the well and its velocity parameter becomes μ_a given by $\mu_a^2 = \mu_0^2 + W(\nu_0,0)/8$.

As can be seen in numerical simulations [25], if the initial soliton parameters are close to the critical case $|\mu_0| \sim \mu_c$, then radiation effects become non-negligible. The construction of an efficient trap requires to generate a barrier potential V_l that is high enough so that $\Delta W(\nu_0)$ is significantly larger than $8\mu_0^2$. Finally, if the potential V_l is not stationary but has an explicit time-dependence, then the equations derived in this section still hold true if the time-dependence is slow enough. If, for instance, $V_l(z,t)$ is a delta-like potential centered at the position $x_0(t)$, that moves slowly in time, then $V_{\text{eff}}(z,t) = K_{\text{eff}}(z - z_0(t))$ is a moving double-barrier potential that can be used to manage the soliton position (Fig. 2).

2.5 Controllable soliton emission by spatial variations of the scattering length

An interesting application of spatial variations of the scattering length has been proposed and discussed in [26], which shows the possibility of controllable emission

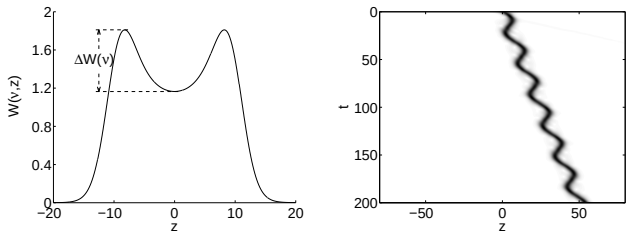

Fig. 2. Left picture: effective potential $W(\nu, z)$ for the quasi-particle motion (17). Here $\nu = 0.25$, $V_l(z) = 10\exp(-z^2/4)$, and $f(\tau) = 10\sin(2\pi\tau)$. Right picture: Evolution of a soliton with initial parameters $\nu_0 = \mu_0 = 0.25$ in the moving potential $V_l[z - 10\sin(50t) - 0.25t]$.

of bright solitons from BEC by using a spatial variation of the scattering length along the trapping axis. The principle is the following one. A magnetic or optical Feshbach resonance technique is applied to a cigar-shaped BEC to induce a sharp variation of the scattering length along the axis, which is changed from positive (or zero) to negative. If the region of negative scattering length is located close to the edge of the trap, overlapping with the wing of the cloud, then the tail of the condensate is able to form a single soliton which, because of its higher internal energy, is outcoupled from the cloud and thus emitted outward. When the condensate refills the gap left out by the outgoing pulse a new soliton is emitted. This process continues as long as there is a large enough remnant of atoms in the trap and leads to a soliton burst escaping from the BEC.

The single soliton emission can be studied by a variational approach based on the 1D GP equation with a Gaussian ansatz for the wavefunction and a stepwise varying scattering length. This analysis shows that the single soliton emission is possible if the scattering length is negative enough in the tail of the cloud. This prediction is confirmed by numerical simulations of the 1D GP equation. These simulations also show that it is possible to generate a train of several hundreds of solitons with physical parameters corresponding to realistic configurations with rubidium or cesium.

3 Bright solitons in nonlinear optical lattices

New solitonic phenomena occur in BECs when the potentials are periodic in space, which can be produced by counter propagating laser beams. Such potentials can be used to control the soliton parameters or to generate gap bright solitons. A lot of work has been devoted to the case of linear periodic potentials [27]. Below we consider the propagation of nonlinear matter wavepackets and the wave emission by solitons in the presence of new types of inhomogeneities, namely under *nonlinear periodic or random potentials*, produced by periodic or random modulations of the atomic scattering length. The latter variations can be realized in a 1D BEC close to a magnetic wire with small fluctuations of current.

3.1 Propagation through a weak nonlinear periodic potential

We consider an incoming soliton of the form (11) and we assume that the nonlinear potential is periodic $V_{nl}(z) = -V_{nl0}\cos(Kz)$. The soliton evolution can be analyzed using the first and the second-order perturbed IST theory when $0 < V_{nl0} \ll 1$. The first-order perturbed IST theory predicts that the soliton mass is preserved and its center ζ obeys the quasi-particle equation (17) with the effective potential

$$W(\nu,\zeta) = -W_{nl0}\cos(K\zeta), \qquad W_{nl0} = \frac{2\pi}{3}\frac{V_{nl0}\nu K}{\sinh(\frac{\pi K}{4\nu})}\left[1+\frac{K^2}{16\nu^2}\right],$$

where 2ν is the soliton amplitude. The soliton can be trapped at $\zeta = 2\pi n/K, n = 0, \pm 1, \pm 2...$, which correspond to the minima of the potential. The critical velocity for depinning the soliton starting from the minimum of the potential is $v_{dp} = 2\sqrt{W_{nl0}}$. From now on we assume that the initial soliton velocity $4\mu_0$ is larger than v_{dp}. The emission of matter waves by the soliton can be investigated by taking into account second-order effects. Two possible regimes are found [28]:
i) If the modulation wavenumber K is smaller that ν_0^2/μ_0, then radiation emission is negligible for times of order V_{nl0}^{-2}. The soliton parameters are almost constant in this regime.
ii) If K is larger than ν_0^2/μ_0, then the soliton emits a significant amount of radiation for times of order V_{nl0}^{-2} (see Fig 3). The soliton amplitude and velocity satisfy a system of ordinary differential equations (ODEs) given in [28]. The maximal radiative decay is obtained for K close to ν_0^2/μ_0. If μ_0 is large enough, then the soliton mass 4ν decays to 0. If μ_0 is not very large, then the soliton mass 4ν decays, but the velocity 4μ decays faster, so that the condition $K\mu = \nu^2$ is reached at finite time. After this time, radiation emission is not noticeable anymore and this state persists for long propagation times. This shows that a meta-stable soliton exists even in the case $K > \mu_0/\nu_0^2$, at the expense of the emission of a small amount of radiation to allow the soliton to reach this state.

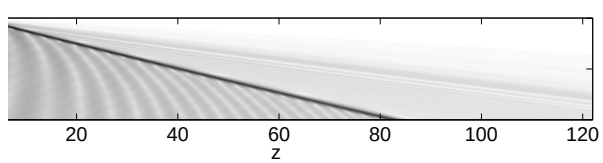

Fig. 3. Soliton profile $|\psi(z,t)|$. The initial soliton parameters are $\nu_0 = \mu_0 = 1$. Here $V_{nl0} = 0.4$ and $K = 2$, we can check that radiation emission is noticeable.

Bound states in the presence of nonlinear periodic potentials with moderate and strong modulations have been investigated by using an orbital stability analysis in [29] and the variational approach in [30]. It was shown that in the one-dimensional

case with cubic nonlinearity the wavepackets centered at a local maximum of the nonlinear optical lattice are stable, while wavepackets centered at a local minimum of the lattice are stable against symmetric perturbations only, while they are unstable against general perturbations. In the critical case (i.e. with quintic nonlinearity of the form $|\psi|^4\psi$) all localized solutions are practically unstable. Explicit solutions of the GP equation with spatially varying nonlinear potentials have been found in [31] in particular in the case of periodic or localized nonlinear potentials. It was shown that localized nonlinearities can support an infinite number of multi-soliton bound states.

3.2 Propagation through a weak random nonlinear potential

Here we assume that V_{nl} is a random zero-mean stationary process with autocorrelation function $B(z) = \langle V_{nl}(z)V_{nl}(0)\rangle$. For times of order $B(0)^{-1} \sim V_{nl0}^{-2}$ the soliton parameters satisfy a deterministic system of ODEs given in [28]. The analysis of this system gives the following results:
i) If $\mu_0 \gg \nu_0$, then the soliton velocity is almost constant, while the mass decays as a power law $\nu(t) \simeq \nu_0\,(1 + t/T_c)^{-1/4}$. The decay time $T_c = [3\mu_0]/[32\hat{B}(4\mu_0)\nu_0^4]$ is inversely proportional to the forth power of the soliton amplitude, which shows that this type of disorder intensively destroys large-amplitude solitons.
ii) If $\mu_0 \ll \nu_0$, then the soliton emits a very small amount of broadband radiation, its mass is almost constant, while the velocity decays very slowly, typically as a logarithm.
iii) If $\mu_0 \sim \nu_0$, then the mass and velocity both decay during the early steps of the propagation (fully described by the system of ODEs). After this transitory regime, either the velocity becomes constant and the mass decays exponentially, or the mass becomes constant and the velocity decays logarithmically.

4 Multidimensional bright solitons in BEC

4.1 2D bright solitons in BEC with time-varying scattering length

In 2D or 3D BECs with attractive interactions bright solitons are unstable. Different schemes have been suggested for soliton stabilization. One of them is to vary rapidly in time the scattering length, which can be realized by using the Feshbach resonance management technique [8, 32]. Here we illustrate the main ideas on the example of the 2D GP equation with nonlinearity management. The mechanism for dynamical stabilization of 2D bright solitons by a rapid periodic variation in time of the scattering length has been suggested in [33, 34]. The dimensionless GP equation in this case has the form

$$i\psi_t + \nabla_\perp^2\psi + [2 - V_{nl}(t)]|\psi|^2\psi = 0, \tag{24}$$

where $V_{nl}(t)$ is a periodic function and $\nabla_\perp = \partial_x^2 + \partial_y^2$. Using the variational approach or the moment method, the equation for the soliton width $w^2 = \int(x^2 + y^2)|\psi|^2 dxdy$ can be derived:

$$\frac{d^2w}{dt^2} = \frac{p(t)}{w^3}, \quad p(t) = Q_1 + Q_2 V_{nl}(t) \tag{25}$$

where Q_1 and Q_2 are invariants of the NLS equation and $p(t)$ is a periodic function that can be parameterized as $p(t) = \alpha + \beta\tilde{p}(t)$, with $\langle\tilde{p}\rangle = 0$ and $\max\tilde{p} = 1$. Using the theory of ODEs with periodic coefficients [35] it is found that a necessary condition for the existence of bound states is that the average value of p should be negative, which implies that $\alpha < 0$. Besides, the function p should change sign otherwise the solution would collapse, which implies $\alpha + |\beta| > 0$. The second condition is well satisfied in the strong nonlinearity management regime [36]. The parameter α is proportional to $N_c - N$, roughly speaking, where N_c is the critical mass equal to the mass of the Townes soliton. The numerical simulations performed in Ref. [35] for the 2D NLS equation show that it is possible to obtain a stable solution in the form of a Townes soliton with modulated parameters. When the initial data is taken in the form of the Gaussian function, the ODE (25) does not describe correctly the region of existence of the bound state. Numerical simulations show that the initial Gaussian pulse ejects a significant amount of radiation. The remaining part of the wavepacket has the form of the Townes soliton with time-varying parameters. Results based on the averaging of the NLS equation with strong nonlinearity management and the analysis of the averaged equation have been obtained for the critical 1D NLS equation with quintic nonlinearity [37] and the NLS equation in the higher dimensional case [38, 39].

4.2 2D bright solitons in BEC with spatially-varying scattering length

Standing bright solitons in inhomogeneous condensed media have been studied in two dimensions in radial geometry. In the presence of a stepwise nonlinear inhomogeneity and harmonic trapping, solutions of the form $\psi(\rho, t) = \exp(i\lambda t)\phi(\rho)$, $\rho = \sqrt{x^2 + y^2}$, can be exhibited. Numerical simulations of the 2D GP equation have shown that the number of atoms of the standing bright soliton is strongly affected by the magnitude of the nonlinear inhomogeneity [40]. Radial nonlinear modulations $\gamma(\rho) = 2 - V_{nl}(\rho)$ in the form of a disk, annulus or narrow ring have been considered in [41], where it is shown that two-dimensional axisymmetric solitons are supported by such systems. In particular, the exact form of the solution can be found for an infinitely narrow ring.

If the nonlinearity coefficient is slowly varying, then the condition for the existence of 2D bright solitons can be obtained [42]. Let $f(\rho)$ be a smooth, bounded, and positive function such that its first four derivatives do not grow faster than $C\exp(\rho)$ for some constant C, and

$$f(0)f^{(4)}(0) < C_2[f''(0)]^2, \qquad C_2 \approx -1.6723.$$

If the nonlinearity coefficient is of the form $\gamma(\rho) = f(\epsilon\rho)$, then the ground state of the GP equation exists, is unique and stable for $\lambda > 0$ and for ϵ small enough. This means in particular that a necessary condition for the existence and stability of 2D bright solitons is that $\gamma^{(4)}(0)$ should be negative.

4.3 2D bright solitons in dipolar BEC

As we mentioned before in 2D BEC with short-range attractive potential the solutions either become singular or spread out if the number of atoms exceeds the critical value. However if the interatomic interactions have long-range characters, stable 2D bright solitons can exist. Recently a dipolar BEC of chromium atoms has been realized [43]. In distinction from the GP equation with local cubic nonlinear term, a dipolar condensate is described by the 3D GP equation with nonlocal nonlinearity

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g|\Psi|^2 + \int V_d(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d\mathbf{r}'\right]\Psi, \qquad (26)$$

where $V_d(\mathbf{r}) = g_d(1-3\cos^2\theta)/r^3$ is the dipole-dipole potential, $g_d = \alpha d^2/(4\pi\epsilon_0)$ with ϵ_0 the vacuum permittivity and d the electric dipole, θ the angle formed by the vector joining the interacting particles and the dipole direction, and $-1/2 \le \alpha \le 1$ a tunable parameter by means of rotating orienting fields. Below we denote by $N = \int |\Psi|^2 d\mathbf{r}$ the number of atoms, we consider a trap potential $V(\mathbf{r}) = m\omega_z^2 z^2/2$ with no trapping in the xy plane, and we assume that the short-range interaction is repulsive $g = 4\pi\hbar^2 a/m > 0$. Applying the variational approach with a Gaussian ansatz [44], it is found that a stable solution can be obtained if

$$\frac{\tilde{g}_d}{3} < \sqrt{2\pi} + \frac{\tilde{g}}{4\pi} < -\frac{2\tilde{g}_d}{3}, \qquad (27)$$

where $\tilde{g} = 8\pi N a/a_z$, $\tilde{g}_d = 2g_d N/(\hbar\omega_z a_z^3)$ and $a_z = \sqrt{\hbar/(m\omega_z)}$. This condition is valid if $\tilde{g}_d < 0$ and $|\tilde{g}_d|/\tilde{g} > 0.12$. Experimental realization of the chromium condensate, where $|\tilde{g}_d|/\tilde{g} \sim 0.03$, requires the additional reduction of short-range interactions via the Feshbach resonance. Numerical simulations show that the interactions between 2D bright solitons are inelastic.

4.4 3D bright solitons in anisotropic trap

As was pointed out at the beginning of this chapter, metastable states can exist in 3D BEC with trap potential if the number of atoms is below a critical one. In anisotropic traps long-living solitonic states can be formed. A recent experiment with ^{85}Rb condensate in a 3D magnetic trap confirms this prediction [45]. The bright matter-wave soliton has been observed during the collapse of ^{85}Rb condensate, occurring when the scattering length was switched from a positive value to a negative one. A system of highly robust solitons was generated. The observed interaction between solitons in the attractive condensate was repulsive. However, the solitons did not strongly overlap, so that they could survive for a few seconds and could be stable during around 40 collisions with each other. The observation of local spikes in the condensate density supports the idea of the formation of bright solitons train by a modulational instability of the condensate wavefunction [46, 47, 48]. Numerical simulations of the 3D GP equation predict the existence of a lower cut-off for the number of atoms below which no stable soliton can be found, in addition to the upper critical number [49].

5 Future Challenges

In conclusion we would like mention open problems that in our opinion could be of interest in the near future. First, one should study solitons in systems with long range interactions, such as cold dipolar gases, in different types of trap potentials, especially in optical lattices. Second, one should look for localized states in different types of linear-nonlinear periodic potentials, including ring-type configurations. Third, solitons and systems of solitons should be investigated in Fermi-Bose mixtures [50, 51, 52, 53]. The manipulations of the relative ratio of bosons and fermions and the parameters of their interaction opens new possibilities for the existence of stable multidimensional bright solitons. Finally, properties of 3D solitons and soliton trains recenlty observed [45] require further consideration.

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