WAVE PROPAGATION IN ONE-DIMENSIONAL RANDOM MEDIA

by

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1. Introduction

Random media have material properties with such complicated spatial variations that they can only be described statistically. When looking at waves propagating in these media, we can only expect in general a statistical description of the wave. But sometimes there exists a deterministic result. In this chapter we restrict ourselves to one-dimensional wave problems that arise naturally in many applications, in gravity waves in shallow channels, in layered elastic media such as the earth's mantle, in optical fiber transmission, etc. We shall address the propagation of linear and nonlinear waves. We focus here for simplicity and pedagogy on the Schrödinger equation, but the forthcoming results do apply to other situations. The problem for the linear case has been extensively studied. The results for the transmission problem will be recalled in the first part of the chapter. The main statement is that, for a given incident wave, the transmission coefficient for a system of finite length decays exponentially with the size of the system. This phenomenon is one of the manifestations of localization of waves in one-dimensional random media. We shall address both the stationary and the time-dependent problems.

The main aim of the chapter is to discuss the stability of localization with respect to nonlinearities. More exactly we want to know how the exponential decay of the transmission may be modified by a nonlinearity. The problem is much more difficult than for the linear case and the methods used to study the linear case seem to fail completely. Some stability of localization has been conjectured. In particular Fröhlich *et al.* have conjectured that solutions to the stationary nonlinear Schrödinger equation localize for sufficiently small initial data [**25**]. In case of a general nonlinearity, results can be found in the literature for the stationary problem. It is very different from the linear case since the transmission problem is no more uniquely defined. Indeed, because of the nonlinearity, the transmitted intensity is not a linear function of the incident intensity. This phenomenon called bistability means that there may exist more than one output state for a given input state depending on hysteresis. Effect of randomness on bistability was addressed in [**45**]. The problem with fixed output was also considered in [**22**]. The authors show that, for strong nonlinearity, the transmission coefficient cannot decay faster than a power law. These results show strong evidence that there exist delocalized transmission states. However since only the time-harmonic problem has been addressed, not all of these states are physical [**14**], so that a complete study with a time-dependent model is required to understand this issue. This is the topic addressed in the second part of this chapter where the propagation of a soliton through a slab of nonlinear and random medium is considered. Indeed some nonlinear dispersive systems such as the Nonlinear Schrödinger (NLS) equation have special solutions called solitons that can propagate without change of form or diminution of speed. Solitons are therefore candidates to test the stability of the exponential localization in nonlinear and random media. Physical [**63, 41**], numerical [**43**], and experimental works [**39, 36**] predict that, for a NLS soliton propagating in a random medium, there exist two distinct regimes of behavior which depend on the soliton parameters. Furthermore one of these regimes is expected to be very different from the localization regime in that the soliton retains its mass although it looses velocity. Using a perturbed version of the inverse scattering transform we shall give a proof of this conjecture.

The chapter is organized into two parts. In the first part we review some asymptotic methods for stochastic differential equations with a small parameter. We apply these methods to compute the localization length of a wave traveling through a slab of random medium. Localization is characterized by an exponential decay of the transmittivity as a function of the size of the slab. It appears as a universal feature in wave propagation in one-dimensional linear random media. In the second part we address the problem of the propagation of a soliton through a slab of nonlinear and random medium. Indeed some nonlinear dispersive systems such as the Nonlinear Schrödinger equation have special solutions called solitons that can propagate without change of form or diminution of speed. Solitons are therefore candidates to test the robustness of the exponential localization in nonlinear and random media. Using the inverse scattering transform we can exhibit several typical behaviors depending on the amplitude of the incoming soliton.

2. Linear propagation

2.1. Some generalities about waves in random media. — Wave propagation in linear random media has been studied for a long time by perturbation techniques when the random inhomogeneities are small. One finds that the mean amplitude decreases with distance traveled, since wave energy is converted to incoherent fluctuations. The fluctuating part of the field intensity is calculated approximatively from a transport equation, a linear radiative transport equation. This theory is well-known [**37**], although a complete mathematical theory is still lacking (for the most recent developments, see for instance [**23**]). However this theory is known to be false in one-dimensional random media. This was first noted by Anderson [**4**], who claimed that random inhomogeneities trap wave energy in a finite region and do not allow it to spread as it would normally. This is the so-called wave localization phenomenon. It was first proved mathematically in [**31**]. Extensions and generalizations follow these pioneer works so that the problem is now well understood [**17**]. The mathematical statement is that the spectrum of the reduced wave equation is pure point with exponentially decaying eigenfunctions. However the authors did not give quantitative information associated with the wave propagation as no exact solution is available. In this chapter we are not be interested in the study of the strongest form of Anderson localization. We actually address the simplest form of this problem: the wave transmission through a slab of random medium. It is now well-known that the transmission of the slab tends exponentially to zero as the length of the slab tends to infinity. Furstenberg first treated discrete versions of the transmission problem [**26**], and finally Kotani gave a proof of this result with minimal hypotheses [**48**]. The connection between the exponential decay of the transmission and the Anderson localization phenomenon is clarified in [**21**]. Once again, these works deal with qualitative properties. Quantitative information can be obtained only for some asymptotic limits: large or small wavenumbers, large or small variances of the fluctuations of the parameters of the medium, etc. A lot of work was devoted to the quantitative analysis of the transmission problem, in particular by Rytov, Tatarski, Klyatskin [**42**], and by Papanicolaou and its co-workers [**46**]. The tools for the quantitative analysis are limit theorems for stochastic equations developed by Khasminskii [**40**] and by Kushner [**49**].

There are three basic length scales in wave propagation phenomena: the typical wavelength λ, the typical propagation distance L, and the typical size of the inhomogeneities l_c. There is also a typical order of magnitude ε that characterizes the standard deviation of the dimensionless fluctuations of the parameters of the medium (index of refraction in optics). It is not always so easy to identify the scale l_c, but we may think of l_c as a typical correlation length. When the standard deviation of the fluctuations is small $\varepsilon \ll 1$, then the most effective interaction of the waves with the random medium will occur when $l_c \sim \lambda$, that is, the wavelength is comparable to the correlation length. And this interaction will be observable when the propagation distance L is large ($L \sim \lambda\varepsilon^{-2}$). This is the typical configuration in optical fibers. Indeed modern technology is able to produce fibers of very quality $\varepsilon \ll 1$. However engineers aim at using very long fibers; actually they always use fibers whose lengths

L are precisely of the order of the critical length at which the influence of randomness becomes of order 1. That is why the asymptotic framework $\varepsilon \ll 1$ and $L \sim \lambda\varepsilon^{-2}$ is the relevant one, and the one that will be addressed in this chapter.

2.2. Our model: the Schrödinger equation. — Throughout this chapter we shall consider the Schrödinger equation. In linear and homogeneous medium it reads:

$$iu_t + u_{xx} = 0. \tag{1}$$

This equation admits elementary solutions of the form:

$$u = a \exp i\left(kx - k^2 t\right),$$

where $k \in \mathbb{R}$ is the wavenumber. The phase of this monochromatic wave can be written as $k(x - kt)$, so that the phase velocity of this wave appears to be equal to k. Let us now consider the initial value problem that consists of the equation (1) together with an initial condition at $t = 0$: $u(t = 0, x) = u_0(x)$, where $u_0 \in L^2$. A solution procedure for this problem is by Fourier transform. One first performs a Direct Fourier Transform (DFT) to compute the spectral content of the initial condition:

$$\hat{u}(0, k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(0, x)e^{-ikx}dx.$$

The partial differential equation (1) is thus transformed into a set of uncoupled ordinary differential equations:

$$\hat{u}_t = -ik^2\hat{u} \quad \Longrightarrow \quad \hat{u}(t, k) = \hat{u}(0, k)e^{-ik^2 t}.$$

The solution at any time t can be obtained by applying the Inverse Fourier Transform (IFT):

$$u(t, x) = \int_{-\infty}^{\infty} \hat{u}(t, k)e^{ikx}dk.$$

Schematically, we have:

$$\begin{array}{ccc} u(0,x) & \overset{\text{DFT}}{\longrightarrow} & \hat{u}(0,k) \\ \text{Eq. (1)} \downarrow & & \downarrow \text{Explicit and uncoupled evolutions} \\ u(t,x) & \overset{\text{IFT}}{\longleftarrow} & \hat{u}(t,k) \end{array}$$

2.3. Propagation in a random slab of a monochromatic wave. — This subsection is devoted to the study of the propagation of monochromatic waves. This is the most natural approach since any wave can be described as the superposition of such elementary wavetrains by Fourier transform. Let $\hat{u}(x)$ be the amplitude at $x \in \mathbb{R}$ of a monochromatic wave $u(t, x) = \exp(-ik^2 t)\hat{u}(x)$ traveling in the one-dimensional medium described in Fig. 1. The medium is homogeneous outside the slab $[0, L]$ and the wave u obeys the Schrödinger equation $iu_t + u_{xx} = 0$. Accordingly $\hat{u}$ satisfies

$$\hat{u}_{xx} + k^2\hat{u} = 0$$

so that it has the form

$$\hat{u}(x) = e^{ikx} + R^\varepsilon e^{-ikx} \quad \text{for} \quad x \le 0$$

and

$$\hat{u}(x) = T^\varepsilon e^{ikx} \quad \text{for} \quad x \ge L.$$

The complex-valued random variables R^ε and T^ε are the reflection and transmission coefficients, respectively. They depend on the particular realization of εV, the wavenumber k and the slab width L.
Inside the slab $[0, L]$ the potential is nonzero. It is the realization of a random, stationary, ergodic, and zero-mean process V. The dimensionless parameter $\varepsilon > 0$ characterizes the amplitude of the random potential. The scalar field $\hat{u}$ satisfies, for $x \in [0, L]$:

$$\hat{u}_{xx} + (k^2 - \varepsilon V(x))\hat{u} = 0. \tag{2}$$

The continuity of $\hat{u}$ and $\hat{u}_x$ at $x = 0$ and $x = L$ implies that the solution $\hat{u}$ satisfies the two point boundary conditions:

$$ik\hat{u} + \hat{u}_x = 2ik \text{ at } x = 0, \qquad ik\hat{u} - \hat{u}_x = 0 \text{ at } x = L. \tag{3}$$

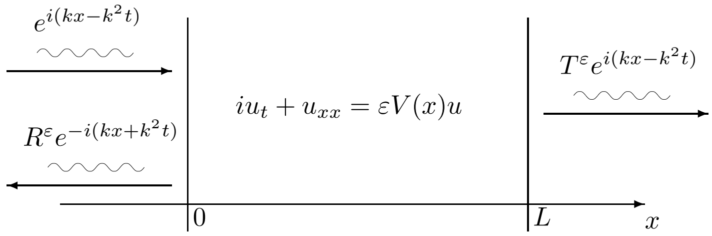

Fig. 1. Scattering of a monochromatic pulse.

The following statements hold true when the potential V is a stationary process that has finite moments of all orders and is rapidly mixing. We may think for instance that V is a Markov, stationary, ergodic process on a compact space satisfying the Fredholm alternative (see the preface for a brief review of the ergodic properties of Markov processes).

Proposition 2.1. — *There exists a length L_{loc}^ε such that, with probability one:*

$$\lim_{L\to\infty} \frac{1}{L} \ln |T^\varepsilon|^2(k, L) = -\frac{1}{L_{loc}^\varepsilon}. \tag{4}$$

This length can be expanded as powers of ε:

$$\frac{1}{L_{loc}^\varepsilon} = \frac{\alpha(k)}{2k^2}\varepsilon^2 + O(\varepsilon^3), \qquad \alpha(k) := \int_0^\infty du \mathbb{E}[V(0)V(u)] \cos(2ku). \tag{5}$$

Proof. The study of the exponential behavior of the transmittivity $|T^\varepsilon|^2$ can be divided into two steps. First the localization length is shown to be equal to the inverse of the Lyapunov exponent associated to the random oscillator $v_{xx}+(k^2-\varepsilon V(x))v=0$. Second the expansion of the Lyapunov exponent of the random oscillator is computed.

We shall first transform the boundary value problem (2)+(3) into an initial value problem, that is more tractable since the relevant quantities will then be adapted to the natural filtration of the random process V. Inside the perturbed slab we expand $\hat{u}$ in the form

$$\hat{u}(k,x) = A(k,x)e^{-ikx} + B(k,x)e^{ikx}, \tag{6}$$

where A and B are respectively backward (going to the left) and forward (going to the right) modes defined by:

$$A = \frac{ik\hat{u} - \hat{u}_x}{2ik} e^{ikx}, \qquad B = \frac{ik\hat{u} + \hat{u}_x}{2ik} e^{-ikx}.$$

The process (A,B) is solution of

$$\frac{d}{dx}\begin{pmatrix} A \\ B \end{pmatrix} = P(k,x)\begin{pmatrix} A \\ B \end{pmatrix}, \qquad P(k,x) = \frac{i\varepsilon}{2k}V(x)\begin{pmatrix} 1 & e^{2ikx} \\ -e^{-2ikx} & -1 \end{pmatrix}. \tag{7}$$

The boundary conditions (3) read in terms of A and B:

$$A(k,L) = 0, \quad B(k,0) = 1. \tag{8}$$

We introduce the propagator Y, i.e. the fundamental matrix solution of the linear system of differential equations: $Y_x = PY$, $Y(0) = I_d$. From symmetries in Eq. (7) it is apparent that if (a,b) is a solution of (7) with the initial conditions:

$$a(k,0) = 1, \quad b(k,0) = 0, \tag{9}$$

then (b^*, a^*) is another solution linearly independent of (a,b). Thus we can write

$$Y(k,x) = \begin{pmatrix} a(k,x) & b^*(k,x) \\ b(k,x) & a^*(k,x) \end{pmatrix}. \tag{10}$$

The modes A and B may be expressed in terms of the propagator:

$$\begin{pmatrix} A(k,x) \\ B(k,x) \end{pmatrix} = Y(k,x)\begin{pmatrix} A(k,0) \\ B(k,0) \end{pmatrix}. \tag{11}$$

From the identity (11) applied for $x = L$ and the boundary conditions (8) we can deduce that

$$R^\varepsilon(k,L) = -(b^*/a)(k,L), \qquad T^\varepsilon(k,L) = (1/a)(k,L). \tag{12}$$

Since the matrix P has trace zero, the determinant of the matrix Y is constant, i.e. $|a(k,x)|^2 - |b(k,x)|^2 = 1$, so that we get the energy conservation relation:

$$|R^\varepsilon(k,L)|^2 + |T^\varepsilon(k,L)|^2 = 1. \tag{13}$$

The transmittivity $|T^\varepsilon|^2$ is equal to $1/|a|^2(k,L)$. We introduce $v = ae^{-ikx} + be^{ikx}$. It satisfies the equation

$$v_{xx} + (k^2 - \varepsilon V)v = 0,$$

with the initial condition $v(0) = 1$, $v_x(0) = -ik$. If we denote by $\gamma^\varepsilon(k)$ the Lyapunov exponent that governs the exponential growth of the quantity $r(k,L) := |v|^2 + |v_x|^2/k^2$:

$$\gamma^\varepsilon(k) = \lim_{L\to\infty} \frac{1}{L} \ln r(k,L),$$

then $|T^\varepsilon|^2$ will decay as $\exp(-\gamma^\varepsilon(k)L)$ since $r(k,L) = 1 + 2|a|^2(k,L) = 1 + 2/|T^\varepsilon|^2$. In Subsection 2.5 the existence of the Lyapunov exponent $\gamma^\varepsilon(k)$ is proved, and its expansion with respect to ε is derived. □

Note that $\alpha(k)$ is a nonnegative real number since it is proportional to the power spectral density of the stationary random process V (Wiener-Khintchine theorem [**54**, p. 141]). The existence and positivity of the exponent $1/L^\varepsilon_{loc}$ can be obtained with minimal hypotheses. Kotani [**48**] established that a sufficient condition is that V is a stationary, ergodic process that is bounded with probability one and is nondeterministic. The expansion of the localization length requires some more hypotheses about the mixing properties of V. A discussion and sufficient hypotheses are proposed in Subsection 2.5.

Proposition 2.2. — *The square modulus of the transmission coefficient $|T^\varepsilon(k, L/\varepsilon^2)|^2$ weakly converges as a continuous process in L to the Markov process $W(L,k)$ whose infinitesimal generator is:*

$$\mathcal{L}_k = \frac{1}{2}\alpha(k)k^{-2}W^2(1-W)\frac{\partial^2}{\partial W^2} - \frac{1}{2}\alpha(k)k^{-2}W^2\frac{\partial}{\partial W}. \tag{14}$$

Proof. The square modulus of the transmission coefficient T^ε can be expressed in terms of a random variable that is the solution of a Ricatti equation. Indeed, as a byproduct of the proof of Proposition 2.1 we find that $|T^\varepsilon|^2 = 1 - |\Gamma^\varepsilon|^2$ where $\Gamma^\varepsilon(k,L) = b/a(k,L)$ and (a,b) are defined as the solutions of Eqs. (7)+(9). Differentiating b/a with respect to L yields that the coefficient Γ^ε satisfies a closed-form nonlinear equation:

$$\frac{d\Gamma^\varepsilon}{dL} = -\frac{i\varepsilon V(L)}{2k}\left(e^{-2ikL} + 2\Gamma^\varepsilon + e^{2ikL}\Gamma^{\varepsilon 2}\right), \quad \Gamma^\varepsilon(k,0) = 0. \tag{15}$$

One then consider the process $X^\varepsilon := (r^\varepsilon, \psi^\varepsilon) := (|\Gamma^\varepsilon|^2, \arg(\Gamma^\varepsilon))$ which satisfies:

$$\frac{dX^\varepsilon}{dL}(\frac{L}{\varepsilon^2}) = \frac{1}{\varepsilon}F\left(V(\frac{L}{\varepsilon^2}), X^\varepsilon(\frac{L}{\varepsilon^2}), \frac{L}{\varepsilon^2}\right),$$

where F is defined by:

$$F(V,r,\psi,l) = \frac{V}{2k}\begin{pmatrix} 2\sin(\psi+2kl)(r^{3/2} - r^{1/2}) \\ -2 - \cos(\psi+2kl)(r^{1/2} + r^{-1/2}) \end{pmatrix}$$

One then applies the diffusion-approximation theorem 2.6 (see Subsection 2.6) to the process $(r^\varepsilon, \psi^\varepsilon)$ which gives the result. □

In particular the expectation of the square modulus of the transmission coefficient converges to:

$$\bar{T}(L,k) = \frac{4}{\sqrt{\pi}} \exp\left(-\frac{\alpha(k)L}{8k^2}\right) \int_0^\infty dx \frac{x^2 e^{-x^2}}{\cosh(\sqrt{\alpha(k)k^{-2}L/2}x)}. \tag{16}$$

This shows that:

$$\frac{1}{L} \ln \bar{T}(L,k) \stackrel{L \gg 1}{\simeq} -\frac{\alpha(k)}{8k^2}$$

Note that the exponential behavior of the expectation of the transmittivity is very different from the sample behavior of the transmittivity. Apparently the "right" localization length is the "sample" one (5), in the sense that it is the one that is observed for a "typical" realization of the medium. Actually we shall see that this holds true only for purely monochromatic waves.

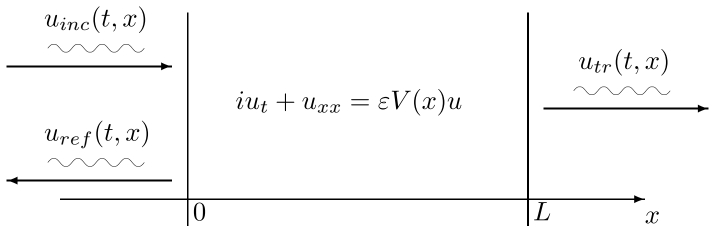

Fig. 2. Scattering of a pulse.

2.4. Propagation in a random slab of a pulse. — We consider an incoming wave from the left:

$$u_{inc}(t,x) = \frac{1}{2\pi} \int_0^\infty f(k) \exp i\left(kx - k^2 t\right) dk, \quad x \le 0, \tag{17}$$

where $f \in L^2$. The total field in the region $x \le 0$ thus consists of the superposition of the incoming wave u_{inc} and the reflected wave:

$$u_{ref}(t,x) = \frac{1}{2\pi} \int_0^\infty f(k) R^\varepsilon(k,L) \exp i\left(-kx - k^2 t\right) dk, \quad x \le 0,$$

where $R^\varepsilon(k,L)$ is the reflection coefficient. The field in the region $x \ge L$ consist only of the transmitted wave that is right going:

$$u_{tr}(t,x) = \frac{1}{2\pi} \int_0^\infty f(k) T^\varepsilon(k,L) \exp i\left(kx - k^2 t\right) dk, \quad x \ge L, \tag{18}$$

where $T^\varepsilon(k,L)$ is the transmission coefficient. Inside the slab the wave has the most general form:

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{u}(k,x) \exp\left(-ik^2 t\right) dk, \quad 0 \le x \le L.$$

The total transmitted energy is:

$$\mathcal{T}^\varepsilon(L) = \frac{1}{2\pi}\int_0^\infty |f(k)T^\varepsilon(k,L)|^2 dk.$$

The two-frequency correlation function $\mathbb{E}\left[|T^\varepsilon(k_1, L/\varepsilon^2)|^2 |T^\varepsilon(k_2, L/\varepsilon^2)|^2\right]$ will be required for the forthcoming statements. The following lemma is an extension of Proposition 2.2.

Lemma 2.3. — *Let $k_1 = k - h\varepsilon^a/2$ and $k_2 = k + h\varepsilon^a/2$.*
1. If $a = 0$, then the square moduli of the transmission coefficients $(|T^\varepsilon(k_1, L/\varepsilon^2)|^2, |T^\varepsilon(k_2, L/\varepsilon^2)|^2)$ weakly converge to $(W(L, k-h/2), W(L, k+h/2))$ where $W(L, k-h/2)$ and $W(L, k+h/2)$ are two independent Markov processes whose infinitesimal generators are respectively $\mathcal{L}_{k-h/2}$ and $\mathcal{L}_{k+h/2}$ defined by (14).
2. If $0 < a < 2$, then the square moduli of the transmission coefficients $(|T^\varepsilon(k_1, L/\varepsilon^2)|^2, |T^\varepsilon(k_2, L/\varepsilon^2)|^2)$ weakly converge to $(W_1(L), W_2(L))$ where W_1 and W_2 are two independent copies of the Markov process whose infinitesimal generator is $\mathcal{L}_k$.
3. If $a = 2$, then the square moduli of the transmission coefficients $(|T^\varepsilon(k_1, L/\varepsilon^2)|^2, |T^\varepsilon(k_2, L/\varepsilon^2)|^2)$ weakly converge to $(W_1(L), W_2(L))$ where $(W_1(L), W_2(L), \theta(L))$ is the Markov process whose infinitesimal generator is:

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{\alpha(k)}{2k^2} W_1^2(1-W_1)\frac{\partial^2}{\partial W_1^2} - \frac{\alpha(k)}{2k^2} W_1^2 \frac{\partial}{\partial W_1} \\
& + \frac{\alpha(k)}{2k^2} W_1^2(1-W_2)\frac{\partial^2}{\partial W_2^2} - \frac{\alpha(k)}{2k^2} W_2^2 \frac{\partial}{\partial W_2} \\
& + \frac{\alpha(k)}{k^2}\cos(\theta)\sqrt{(1-W_1)(1-W_2)}W_2 W_1 \frac{\partial^2}{\partial W_1 \partial W_2} \\
& - 2h\frac{\partial}{\partial \theta} \\
& + \frac{\alpha(k)}{4k^2}\left(\frac{(2-W_1)^2}{1-W_1} + \frac{(2-W_2)^2}{1-W_2} - 2\frac{(2-W_1)(2-W_2)}{\sqrt{(1-W_1)(1-W_2)}}\cos(\theta)\right)\frac{\partial^2}{\partial \theta^2} \\
& + \frac{\alpha(k)}{4k^2}\frac{\sqrt{1-W_1}W_1(2-W_2)}{\sqrt{1-W_2}}\sin(\theta)\frac{\partial^2}{\partial W_1 \partial \theta} \\
& + \frac{\alpha(k)}{4k^2}\frac{\sqrt{1-W_2}W_2(2-W_1)}{\sqrt{1-W_1}}\sin(\theta)\frac{\partial^2}{\partial W_2 \partial \theta}
\end{aligned} \tag{19}$$

starting from $W_1(0) = 1$, $W_2(0) = 1$, and $\theta(0) = 0$.
4. If $a > 2$, then the square moduli of the transmission coefficients $(|T^\varepsilon(k_1, L/\varepsilon^2)|^2, |T^\varepsilon(k_2, L/\varepsilon^2)|^2)$ weakly converge to $(W(L,k), W(L,k))$ where W is the Markov process whose infinitesimal generator is (14).

Proof. The most interesting case is $a = 2$, since this is the correct scaling which describes the correlation of the transmission coefficients at two nearby frequencies. Let us denote $|T(k_j, L/\varepsilon^2)|^2 = 1 - |\Gamma_j^\varepsilon|^2$ for $j = 1, 2$, where $\Gamma_j^\varepsilon(L) = b/a(k_j, L)$. We then introduce the four-dimensional process $X^\varepsilon := (r_1^\varepsilon, \psi_1^\varepsilon, r_2^\varepsilon, \psi_2^\varepsilon) := (|\Gamma_1^\varepsilon|^2, \arg(\Gamma_1^\varepsilon), |\Gamma_2^\varepsilon|^2, \arg(\Gamma_2^\varepsilon))$ which satisfies:

$$\frac{dX^\varepsilon}{dL}(\frac{L}{\varepsilon^2}) = \frac{1}{\varepsilon} F\left(V(\frac{L}{\varepsilon^2}), X^\varepsilon(\frac{L}{\varepsilon^2}), \frac{L}{\varepsilon^2}, L\right),$$

where F is defined by:

$$F(V, r_1, \psi_1, r_2, \psi_2, l, L) = \frac{V}{2k} \begin{pmatrix} 2\sin(\psi_1 + 2kl + hL)(r_1^{3/2} - r_1^{1/2}) \\ -2 - \cos(\psi_1 + 2kl + hL)(r_1^{1/2} + r_1^{-1/2}) \\ 2\sin(\psi_2 + 2kl - hL)(r_2^{3/2} - r_2^{1/2}) \\ -2 - \cos(\psi_2 + 2kl - hL)(r_2^{1/2} + r_2^{-1/2}) \end{pmatrix}$$

Applying the diffusion-approximation theorem 2.6 to the process X^ε establishes that X^ε converges to a non-homogeneous Markov process $X = (r_1, \psi_1, r_2, \psi_2)$ whose infinitesimal generator (that depends on L) can be computed explicitly. It then appears that, by introducing $\theta := \psi_1 - \psi_2 - 2hL$, the process (r_1, r_2, θ) is a homogeneous Markov process whose infinitesimal generator is given by (19). □

This lemma shows that the transmission coefficients corresponding to two nearby frequencies k_1 and k_2 are uncorrelated as soon as $|k_1 - k_2| \gg \varepsilon^2$. Once this result is known, it is easy to derive the asymptotic behavior of the transmittivity corresponding to the scattering of a pulse.

Proposition 2.4. — *The transmittivity $\mathcal{T}^\varepsilon(L/\varepsilon^2)$ converges in probability to $\mathcal{T}(L)$:*

$$\mathcal{T}(L) = \frac{1}{2\pi} \int_0^\infty |f(k)|^2 \bar{T}(k, L) dk,$$

where $\bar{T}(k, L)$ is the asymptotic value (16) of the expectation of the square modulus of the transmission coefficient $T^\varepsilon(k, L/\varepsilon^2)$.

Proof. The tightness (i.e. the relative compactness) is easy to establish since T^ε is bounded. Proposition 2.2 gives the limit value of the expectation of $|T^\varepsilon(k, L/\varepsilon^2)|$ for one frequency k, so that:

$$\mathbb{E}\left[\mathcal{T}^\varepsilon(\frac{L}{\varepsilon^2})\right] \stackrel{\varepsilon \to 0}{\longrightarrow} \frac{1}{2\pi} \int_0^\infty |f(k)|^2 |\bar{T}(k, L)|^2 dk.$$

Then one considers the second moment:

$$\mathbb{E}\left[\mathcal{T}^\varepsilon(\frac{L}{\varepsilon^2})^2\right] = \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty |f(k)|^2 |f(k')|^2 \mathbb{E}\left[|T^\varepsilon(k, \frac{L}{\varepsilon^2})|^2 |T^\varepsilon(k', \frac{L}{\varepsilon^2})|^2\right] dk dk'.$$

The computation of this moment requires to study the two-frequency process $(|T^\varepsilon(k, L/\varepsilon^2)|, |T^\varepsilon(k', L/\varepsilon^2)|)$ for $k \neq k'$. Applying Lemma 2.3 one finds that

$|T^\varepsilon(k, L/\varepsilon^2)|$ and $|T^\varepsilon(k', L/\varepsilon^2)|$ are asymptotically uncorrelated, so that

$$\mathbb{E}\left[\mathcal{T}^\varepsilon(\frac{L}{\varepsilon^2})^2\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \left(\frac{1}{2\pi}\int_0^\infty |f(k)^2|\bar{T}(k,L)|^2 dk\right)^2,$$

which proves the convergence of $\mathcal{T}^\varepsilon(L/\varepsilon^2)$ in L^2. □

Let us assume that the incoming wave is narrowband, that is to say that the spectral content f is concentrated around the carrier wavenumber k_0 and has narrow bandwidth (smaller than 1, but larger than ε^2). Then $\mathcal{T}(L)$ decays exponentially with the width of the slab as:

$$\frac{1}{L}\ln \mathcal{T}(L) \stackrel{L\gg 1}{\simeq} -\frac{\alpha(k_0)}{8k_0^2}$$

Note that this is the typical behavior of the *expected* value of the transmittivity of a monochromatic wave with wavenumber k_0. In the time domain the localization process is self-averaging ! This self-averaging is implied by the asymptotic decorrelation of the moduli of the transmission coefficients at different frequencies. Actually $T^\varepsilon(k, L/\varepsilon^2)$ and $T^\varepsilon(k', L/\varepsilon^2)$ are correlated only if $|k-k'| \leq \varepsilon^2$.

***Remark 2.5* (The O'Doherty-Anstey theory).** — This theory describes the deformation of a pulse traveling in a slab of random medium. It is well-known in geophysics literature [**56**]. It predicts that in proper conditions the transmitted pulse can be divided into two parts. The front part has a deterministic shape which is the result of a deterministic convolution of the initial pulse. Behind this front part emerges the "coda" which is incoherent, but may contains most of the transmitted energy. A rather convincing heuristic explanation can be found in [**16**, Section 2]. The theory is analyzed in detail in [**16**] for a special case of stepwise media, and further results can be found in [**15, 7**]. Let us assume that the frequency content of the initial pulse is concentrated around the carrier wavenumber k_0. If one analyzes the energy content of the front part of the wave in the framework introduced here above, then one finds that it decays exponentially with the size of the slab with a characteristic length which is the *sample* localization length $2k_0^2/\alpha(k_0)$. However, as shown here above, the total transmitted energy decays exponentially with a characteristic length which is the *mean* localization length $8k_0^2/\alpha(k_0)$.

2.5. The random harmonic oscillator. — The random harmonic oscillator:

$$y_{tt} + (1+\varepsilon\eta(t))y = 0 \tag{20}$$

with $\eta(t)$ a random process arises in many physical contexts such as solid state physics [**51, 24, 57**], vibrations in mechanical and electrical circuits [**60, 62**], and wave propagation in one-dimensional random media [**42, 7**]. The dimensionless parameter $\varepsilon > 0$ characterizes the amplitudes of the random fluctuations. The sample Lyapunov exponent governs the exponential growth of the modulation:

$$G := \lim_{t\to\infty} \frac{1}{t} \ln r(t), \qquad r(t) = \sqrt{|y(t)|^2 + |y_t(t)|^2}. \tag{21}$$

Note that G could be random since η is random. So it should be relevant to study the mean and fluctuations of the Lyapunov exponent. For this purpose we shall analyze the normalized Lyapunov exponent which governs the exponential growth of the p-th moment of the modulation:

$$G_p := \lim_{t\to\infty} \frac{1}{pt} \ln \mathbb{E}\left[r(t)^p\right], \tag{22}$$

where $\mathbb{E}$ stands for the expectation with respect to the distribution of the process η. If the process were deterministic, then we would have $G_p = G$ for every p. But due to randomness this may not hold true since we can not invert the nonlinear power function "$|.|^p$" and the linear statistical averaging "$\mathbb{E}[.]$". The random matrix products theory applies to the problem (20). For instance let us assume that the random process η is piecewise constant over intervals $[n, n+1)$ and take random values on the successive intervals. Under appropriate assumptions on the laws of the values taken by m, it is proved in Ref. [**8**, Theorem 4] that there exists an analytic function $g(p)$ such that:

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\left[|r(t)|^p\right] = g(p), \tag{23}$$

$$\lim_{t\to\infty} \frac{1}{t} \ln r(t) = g'(0) \text{ almost surely}, \tag{24}$$

$$\frac{\ln r(t) - tg'(0)}{\sqrt{t}} \xrightarrow{dist.} \mathcal{N}(0, g''(0)). \tag{25}$$

Moreover the convergence is uniform for $r(t=0)$ with unit modulus, and the function $p \mapsto g(p)/p$ is monotone increasing. This proves in particular that $G = g'(0)$ is non-random. In case of non-piecewise constant processes η, various versions of the above theorem exist which yield the same conclusion [**26, 61, 12, 5**]. Unfortunately the expression of $g(p)$ is very intricate, even for very simple random processes η. In the following subsubsections we shall derive closed form expressions for the Lyapunov exponents G and G_p in the framework where the noise level is low.

We assume from now on that the process $\eta(t) = f(m(t))$ where f is a smooth and bounded function and m is an ergodic Markov process with infinitesimal generator Q on a manifold M with invariant probability $\pi(dm)$. We also assume that $f(m)$ has zero-mean with respect to the invariant probability:

$$\int_M f(m)\pi(dm) = 0.$$

2.5.1. Expansion of the sample Lyapunov exponent. — Introducing polar coordinates $(r(t), \psi(t))$ as $y(t) = r(t)\cos(\psi(t))$ and $y_t(t) = r(t)\sin(\psi(t))$ the system (20) is equivalent to:

$$r(t) = r_0 \exp\left(\int_0^t q(\psi(s), m(s))ds\right), \tag{26}$$

$$\psi_t(t) = h(\psi(t), m(t)), \tag{27}$$

with $q(\psi, m) = q_0(\psi) + \varepsilon q_1(\psi, m)$ and $h(\psi, m) = h_0(\psi) + \varepsilon h_1(\psi, m)$:

$$q_0(\psi) = 0, \qquad q_1(\psi, m) = -f(m)\sin(\psi)\cos(\psi),$$
$$h_0(\psi) = -1, \qquad h_1(\psi, m) = -f(m)\cos^2(\psi).$$

From Eq. (27) (ψ, m) is a Markov process on the space $S^1 \times M$ where S^1 denotes the circumference of the unit circle with infinitesimal generator: $\mathcal{L} = Q + h(\psi, m)\frac{\partial}{\partial\psi}$ and with invariant measure $\bar{p}(\psi, m)d\psi\pi(dm)$ where $\bar{p}$ can be obtained as the solution of $\mathcal{L}^*\bar{p} = 0$. According to the theorem of Crauel [**20**] the long-time behavior of $r(t)$ can be expressed in terms of the Lyapunov exponent G which is given by:

$$G = \int_{S^1\times M} q(\psi, m)\bar{p}(\psi, m)d\psi\pi(dm). \tag{28}$$

This result and the following ones hold true in particular under condition *H1* [**58**] or *H2* [**6**]:

H1 M is a finite set and Q is a finite-dimensional matrix which generates a continuous parameter irreducible, time-reversible Markov chain.

H2 M is a compact manifold. Q is a self-adjoint elliptic diffusion operator on M with zero an isolated, simple eigenvalue.

Note that the result can be greatly generalized. For instance one can also work with the class of the ϕ-mixing processes with $\phi \in L^{1/2}$ (see [**49**, pp. 82-83]). The Lyapunov exponent G can be estimated in case of small noise using the technique introduced by Pinsky [**58**] under *H1* and Arnold et al. [**6**] under *H2*.

We shall assume from now on that $\varepsilon \ll 1$ and we look for an expansion of G with respect to $\varepsilon \ll 1$. The strategy follows closely the one developed in Ref. [**6**]. We first divide the generator $\mathcal{L}$ into the sum $\mathcal{L} = \mathcal{L}_0 + \varepsilon\mathcal{L}_1$ with:

$$\mathcal{L}_0 = Q + h_0(\psi)\frac{\partial}{\partial\psi}, \qquad \mathcal{L}_1 = h_1(\psi, m)\frac{\partial}{\partial\psi}.$$

As shown in [**6**] the probability density $\bar{p}$ can be expanded as $\bar{p} = \bar{p}_0 + \varepsilon\bar{p}_1 + \varepsilon^2\bar{p}_2 + ...$ where $\bar{p}_0$, $\bar{p}_1$, and p_2 satisfy $\mathcal{L}_0^*\bar{p}_0 = 0$ and $\mathcal{L}_0^*\bar{p}_1 + \mathcal{L}_1^*\bar{p}_0 = 0$, $\mathcal{L}_0^*\bar{p}_2 + \mathcal{L}_1^*\bar{p}_1 = 0$,... For once the expansion of $\bar{p}$ is known, it can be used in (28) to give the expansion of G at order 2 with respect to ε:

$$G = \int_{S^1\times M} \left(q_0\bar{p}_0 + \varepsilon(q_1\bar{p}_0 + q_0\bar{p}_1) + \varepsilon^2(q_1\bar{p}_1 + q_0\bar{p}_2)\right)(\psi, m)d\psi\pi(dm) + O(\varepsilon^3). \tag{29}$$

Since h_0 is constant $= -1$, $\bar{p}_0$ is the density of the uniform distribution on $S^1 \times M$: $\bar{p}_0 \equiv (2\pi)^{-1}$. Further $\bar{p}_1$ satisfies $\mathcal{L}_0^* \bar{p}_1 = -\mathcal{L}_1^* \bar{p}_0 = \partial_\psi (h_1 \bar{p}_0)$ and is consequently given by:

$$\bar{p}_1(\psi, m) = -\frac{1}{2\pi} \int_0^\infty ds \mathbb{E} \left[\sin(2\psi + 2s) f(m(s))/m(0) = m \right].$$

Substituting into (29) we obtain that $G = \varepsilon^2 \alpha_1 / 4 + O(\varepsilon^3)$, where α_1 is nonnegative and proportional to the power spectral density of the process $f(m)$ evaluated at 2-frequency:

$$\alpha_1 = \int_0^\infty ds \cos(2s) \mathbb{E}[f(m(0)) f(m(s))]. \tag{30}$$

2.5.2. Fluctuations of the sample exponent. — The sample Lyapunov exponent G is the limit of

$$G = \lim_{t \to \infty} \frac{1}{t} \int_0^t q(\psi(s), m(s)) ds$$

with probability one. It is interesting to obtain a central limit theorem for the corresponding fluctuations:

$$F(t) := \frac{1}{\sqrt{t}} \int_0^t q(\psi(s), m(s)) - G ds.$$

Let us denote $\bar{q}(\psi, m) := q(\psi, m) - G$. By definition of G the random variable $\bar{q}(\psi, m(0))$ has zero mean with respect to the invariant measure $\bar{p}\pi(dm)d\psi$. Thus the equation $\mathcal{L}v = \bar{q}$ has a solution v which is bounded and unique up to an additive constant. In the following we denote by v the solution which satisfies $\int v \bar{p}(\psi, m) \pi(dm) d\psi = 0$. The process M_v defined by:

$$M_v(t) := v(\psi(t), m(t)) - v(\psi, m(0)) - \int_0^t \mathcal{L}v(\psi(s), m(s)) ds$$

is a martingale whose increasing process is [**49**, Section 1.5]

$$\langle M_v, M_v \rangle_t = \int_0^t \left(\mathcal{L}v^2 - 2v\mathcal{L}v \right) (\psi(s), m(s)) ds.$$

Note that the process r reads in terms of the martingale M_v as:

$$r(t) = r_0 \exp \left(Gt - M_v(t) + v(\psi(t), m(t)) - v(\psi, m(0)) \right). \tag{31}$$

Applying Theorem 2.1 [**11**] we get that $\langle M_v, M_v \rangle$ satisfies:

$$\frac{\langle M_v, M_v \rangle_t}{t} \overset{t \to \infty}{\longrightarrow} V_\varepsilon := -2 \int_{S^1 \times M} \bar{q} v \bar{p}(\psi, m) d\psi \pi(dm), \tag{32}$$

and the normalized process $t^{-1/2} M_v(t)$ converges in distribution to a Gaussian random variable with 0 mean and variance V_ε. This proves the desired result that the normalized fluctuations $F(t)$ of the sample Lyapunov exponent obeys a Gaussian distribution with zero-mean and variance V_ε. In the remainder of this subsection we shall compute the expansion of V_ε at order 2 with respect to the noise level ε.

Since $\bar{q}$ expands as $\varepsilon\bar{q}_1 + \varepsilon^2\bar{q}_2 + ...$ with $\bar{q}_1 = q_1$, we can expand v in the form $\varepsilon v_1 + \varepsilon^2 v_2 + ...$, where v_1 is solution of $\mathcal{L}_0 v_1 = q_1$. Solving this equation we find that:

$$v_1(\psi, m) = \int_0^\infty ds \mathbb{E}\left[\sin(2\psi - 2s) f(m(s))/m(0) = m\right].$$

Collecting the terms of order ε^2 in (32) establishes that

$$V_\varepsilon = \varepsilon^2 V_2 + O(\varepsilon^3)$$

with

$$V_2 = -2 \int_{S^1 \times M} q_1 v_1 \bar{p}_0 d\pi(m) d\psi.$$

Substituting into (32) yields that $V_\varepsilon = \alpha_1 \varepsilon^2/2 + O(\varepsilon^3)$.

2.5.3. The mean exponents. — Once the sample exponent growth and its fluctuations are known it remains only a few technical points to handle before establishing a closed form expression for the mean exponent growth. First note that the identity (31) involves that the mean Lyapunov exponent G_p reads in terms of the martingale M_v as:

$$G_p = G + \lim_{t\to\infty} \frac{1}{pt} \ln \mathbb{E}\left[\exp(-pM_v(t))\right].$$

Second V_ε is nothing else than the expectation of $f := \mathcal{L}v^2 - 2v\mathcal{L}v$ with respect to the invariant measure $\bar{p}\pi(dm)d\psi$. Denoting $\bar{f}(\psi, m) := f(\psi, m) - V_\varepsilon$, the random variable $\bar{f}(\psi, m(0))$ has zero mean so that the equation $\mathcal{L}w = \bar{f}$ has a unique solution which satisfies $\int w\bar{p}(\psi, m)\pi(dm)d\psi = 0$. Accordingly the process M_w:

$$M_w := w(\psi(t), m(t)) - w(\psi, m(0)) - \int_0^t \mathcal{L}w(\psi(s), m(s)) ds$$

is a martingale whose increasing process is

$$\langle M_w, M_w \rangle_t = \int_0^t \left(\mathcal{L}w^2 - 2w\mathcal{L}w\right)(\psi(s), m(s)) ds.$$

Note that v is of order ε, so f and w are of order ε^2 and $\langle M_w, M_w \rangle_t \leq K\varepsilon^4 t$. Besides the increasing process of the martingale M_v reads as:

$$\langle M_v, M_v \rangle_t = V_\varepsilon t - M_w(t) + w(\psi(t), m(t)) - w(\psi, m(0)).$$

Thus, for any $p > 0$, the increasing process of the martingale $M_v + pM_w/2$ is:

$$\begin{aligned}
\left\langle M_v + \frac{p}{2} M_w, M_v + \frac{p}{2} M_w \right\rangle_t &= \langle M_v, M_v \rangle_t + \frac{p^2}{4} \langle M_w, M_w \rangle_t + p \langle M_v, M_w \rangle_t \\
&= V_\varepsilon t - M_w(t) + \xi_\varepsilon(t)
\end{aligned}$$

where $|\xi_\varepsilon(t)| \leq K(1+p\varepsilon^3 t + p^2\varepsilon^4 t)$. We substitute this identity into the expectation of the exponential martingale associated with $M_v + pM_w/2$ so that we get:

$$\begin{aligned} 1 &= \mathbb{E}\left[\exp\left(-p(M_v(t)+\frac{p}{2}M_w) - \frac{p^2}{2}\left\langle M_v + \frac{p}{2}M_w, M_v + \frac{p}{2}M_w\right\rangle_t\right)\right] \\ &= \mathbb{E}\left[\exp\left(-pM_v(t) - \frac{p^2}{2}V_\varepsilon t - \frac{p^2}{2}\xi_\varepsilon(t)\right)\right]. \end{aligned}$$

Applying the operation $(pt)^{-1}\ln\{.\}$ and taking the limit $t \to \infty$, we finally get the expansion at order 2 with respect to ε of G_p:

$$G_p = G + \frac{p}{2}V_\varepsilon + O(p^2\varepsilon^3) = \frac{1+p}{4}\alpha_1\varepsilon^2 + O(p^2\varepsilon^3). \tag{33}$$

2.6. Diffusion-approximation. — In this subsection we state and prove the diffusion-approximation theorem that is applied throughout this chapter.

Theorem 2.6. — *Let us consider the system:*

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon}F\left(X^\varepsilon(t), q(\frac{t}{\varepsilon^2}), \frac{t}{\varepsilon^2}\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

where $F(x,q,\phi)$ is periodic with respect to ϕ. Assume that q is a Markov, stationary, ergodic process on a compact space S with generator Q, satisfying the Fredholm alternative. F is periodic with respect to ϕ with period ϕ_0 and satisfies the centering condition: $\langle \mathbb{E}[F(x,q(0),\phi)]\rangle_\phi = 0$ where $\mathbb{E}[.]$ denotes the expectation with respect to the invariant probability measure $\mathbb{P}$ of q and $\langle . \rangle_\phi$ stands for an averaging over a period in ϕ. Instead of technical sharp conditions, assume also that F is smooth and has bounded partial derivatives in x. Then the continuous processes $(X^\varepsilon(t))_{t\geq 0}$ weakly converge to X with generator:

$$\mathcal{L}f(x) = \int_0^\infty du \left\langle \mathbb{E}\left[F(x,q(0),.).\nabla\left(F(x,q(u),.+u).\nabla f(x)\right)\right]\right\rangle_\phi.$$

Proof. For an extended version of the proof and sharp conditions we refer to **[49, 28]**. Let us introduce $\phi(t) := t\text{mod}\phi_0$. The process $\bar{X}^\varepsilon(.) := (X^\varepsilon(.), q(./\varepsilon^2), \phi(./\varepsilon^2))$ is Markov with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}\left(Q + \frac{\partial}{\partial\phi}\right) + \frac{1}{\varepsilon}F(x,q,\phi).\nabla.$$

This implies that, for any smooth function f, the process $f(\bar{X}^\varepsilon(t)) - f(\bar{X}^\varepsilon(s)) - \int_s^t \mathcal{L}^\varepsilon f(\bar{X}^\varepsilon(u))du$ is a martingale. The proof is based upon the convergence of the corresponding martingale problems.

Step 1. Perturbed function method. $\forall f \in \mathcal{C}_b^\infty$, $\forall K$ compact subset of $\mathbb{R}^d$, there exists a family f^ε such that:

$$\sup_{x\in K,q,\phi}|f^\varepsilon(x,q,\phi) - f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \qquad \sup_{x\in K,q,\phi}|\mathcal{L}^\varepsilon f^\varepsilon(x,q,\phi) - \mathcal{L}f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{34}$$

Define $f^\varepsilon(x,q) = f(x) + \varepsilon f_1(x,q) + \varepsilon^2 f_2(x,q)$. Applying $\mathcal{L}^\varepsilon$ to f^ε, one gets:

$$\mathcal{L}^\varepsilon f^\varepsilon = \frac{1}{\varepsilon}\left(\left(Q+\frac{\partial}{\partial\phi}\right)f_1 + F(x,q,\phi).\nabla f(x)\right) + \left(\left(Q+\frac{\partial}{\partial\phi}\right)f_2 + F.\nabla f_1(x,q,\phi)\right) + O(\varepsilon).$$

One then defines the corrections f_j as follows:

1. $f_1(x,q) = -\left(Q+\dfrac{\partial}{\partial\phi}\right)^{-1}(F(x,q).\nabla f(x))$. This function is well-defined although $\left(Q+\frac{\partial}{\partial\phi}\right)$ does not possess an inverse. Indeed (q,ϕ) is an ergodic process which satisfies the Fredholm alternative. Its invariant probability measure over $S\otimes[0,\phi_0]$ is $\mathbb{P}(dq)\times\frac{1}{\phi_0}\mathbf{1}_{[0,\phi_0]}d\phi$. As a consequence $\left(Q+\frac{\partial}{\partial\phi}\right)$ has an inverse on the subspace of functions which have zero-mean with respect to the invariant measure. This inverse admits the representation:

$$f_1(x,q,\phi) = \int_0^\infty du\mathbb{E}[F(x,q(u),\phi+u).\nabla f(x)/q(0)=q].$$

2. $f_2(x,q) = -\left(Q+\dfrac{\partial}{\partial\phi}\right)^{-1}\left(F.\nabla f_1(x,q) - \langle\mathbb{E}[F.\nabla f_1(x,q,\phi)]\rangle_\phi\right)$ is well defined since the argument of $\left(Q+\frac{\partial}{\partial\phi}\right)^{-1}$ has zero-mean. It thus remains: $\mathcal{L}^\varepsilon f^\varepsilon = \langle\mathbb{E}[F(x,q,\phi).\nabla f_1(x,q,\phi)]\rangle_\phi + O(\varepsilon)$ which proves (34).

Step 2. Convergence of martingale problems. One first establishes the tightness of the process X^ε in the space of the càd-làg functions equipped with the Skorohod topology by checking a standard criterion (see [**49**, Section 3.3]). Second one considers a subsequence $\varepsilon_p \to 0$ such that $X^{\varepsilon_p} \to X$. One takes $t_1 < ... < t_n < s < t$ and $h_1,...,h_n \in \mathcal{C}_b^\infty$:

$$\mathbb{E}\left[\left(f^\varepsilon(X^\varepsilon(t),q(\frac{t}{\varepsilon^2}),\phi(\frac{t}{\varepsilon^2})) - f^\varepsilon(X^\varepsilon(s),q(\frac{s}{\varepsilon^2}),\phi(\frac{s}{\varepsilon^2})) - \right.\right.$$
$$\left.\left. -\int_s^t \mathcal{L}^\varepsilon f^\varepsilon(X^\varepsilon(u),q(\frac{u}{\varepsilon^2}),\phi(\frac{u}{\varepsilon^2}))du\right)h_1(X^\varepsilon(t_1))...h_n(X^\varepsilon(t_n))\right] = 0$$

Taking the limit $\varepsilon_p \to 0$:

$$\mathbb{E}\left[\left(f(X(t)) - f(X(s)) - \int_s^t \mathcal{L}f(X(u))du\right)h_1(X(t_1))...h_n(X(t_n))\right] = 0$$

which shows that X is solution of the martingale problem associated to $\mathcal{L}$. This problem is well-posed, which proves the result. □

We refer to [**28**] for multi-scaled versions of this theorem.

3. Nonlinear propagation

3.1. Solitary waves and telecommunication. — A solitary wave is a wave that propagates without change of form or diminution of speed. The study of solitary waves began in 1838 with the observation by J. Scott Russel of such a water wave while riding on horseback along a channel. However no mathematical theory available at the time predicted a solitary wave. The problem was resolved in 1895 by Korteweg and de Vries who derived an equation (now known as the KdV equation) which governs small shallow-water waves [**47**]. Boussinesq in 1871 also derived a nonlinear wave equation governing such long waves [**13**]. Despite this early work no further application was discovered until the 1960's. In 1967 Gardner, Green, Kruskal, and Miura first discovered an original method of solution of KdV by applying an implicit linearization of the equation: the so-called inverse scattering transform [**27**]. Lax (1968) considerably generalized these ideas [**50**], and Zakharov and Shabat (1972) showed that the method worked for the nonlinear Schrödinger (NLS) equation [**67**]. At this time it was known that the NLS equation describes the propagation of short pulses in mono-mode optical fibers [**55**]. Hasegawa (1973) then claimed that the "soliton" was the ideal candidate to be the information bit for the next generation of optical fibers [**34**].

Indeed communication in optical fibers [**35**] consists in sending binary messages at very high rate. A sequence of "0" and "1" can be coded as a train of short pulses, where a "1" is represented by a pulse and a "0" by the absence of a pulse in the corresponding arrival time slot of the train. The success of this method is based on the fact that modern technology has succeeded in producing purified glass fiber with a very low level of attenuation. Unfortunately another phenomenon appears to be a limitation to the race towards higher and higher transmission rates. Indeed dispersion makes pulses spread out. However nonlinear effects such as self-focusing compete with dispersion. The nonlinear Schrödinger equation, which describes this competition to a good approximation, has a special solution, the so-called soliton, for which the nonlinear effects exactly counterbalance dispersion. It is therefore a good candidate to be the information bit for the next generation of optical fibers [**33**]. In order to confirm this hope, it is relevant to study the behavior of a soliton when it propagates through weakly perturbed media over very large distances.

3.2. Dispersion in wave propagation phenomena. — A linear dispersive system is any system which admits elementary solutions of the form:

$$u = a \exp i\,(kx - \omega t) \tag{35}$$

where the frequency ω is a definite real function of the wavenumber k and the so-called dispersion relation $\omega(k)$ is determined by the particular system. Any general solution is obtained by superposition of elementary wavetrains (35) to form Fourier integrals:

$$u = \int F(k) \exp i\,(kx - \omega(k)t)\, dk$$

where f is chosen to fit the boundary or initial conditions with use of the Fourier inversion theorem. The wavetrains travel with their own phase velocity $\omega(k)/k$. Dispersion is involved by the fact that the dispersion relation is usually not linear. As a consequence the phase velocity depends on the wavenumber k. As time evolves, the different component modes disperse and the pulse spreads out. A more quantitative analysis can be carried out in a few steps for the linear Schrödinger equation:

$$iu_t + u_{xx} = 0$$

whose dispersion relation is $\omega(k) = k^2$. If the initial condition $u(t = 0, x) = u_0(x)$ belongs to L^2, then the spectral content of the field is:

$$f(k) = \int_{-\infty}^{\infty} u_0(x) \exp(-ikx) dx$$

and the solution for any time t can be written as:

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk f(k) \exp i \left(kx - k^2 t\right)$$

If $f(k)$ is smooth enough so that the stationary phase method can be applied, then for $t \gg 1$:

$$u(t, x) \simeq \sqrt{\frac{\pi}{t}} f\left(\frac{x}{2t}\right) \exp i \left(\frac{x^2}{4t} - \frac{\pi}{4}\right)$$

which shows that the amplitude of the wave decays as $1/\sqrt{t}$ while its support increases as t.

3.3. Catastrophic collapse in nonlinear media. — The simplest equation describing nonlinear propagation effects is the Burgers equation:

$$u_t + uu_x = 0. \tag{36}$$

This equation can be solved analytically by the standard method of characteristics. Assume that the initial condition $u(t = 0, x) = u_0(x)$ is smooth. It is found that the solution remains smooth until time t_c:

$$u_x(t, x) = \frac{u_{0x}(x)}{1 + u_{0x}(x)t}$$

where t_c is defined by $t_c^{-1} := \max_x \{-u_{0x}(x)\}$. At time t_c the solution breaks up. An extensive study of this equation can be found for instance in [**64**].

3.4. An introduction to the inverse scattering transform. — The scattering transform aims at studying the solutions of nonlinear partial differential equations of the type $u_t = F(u)$ with rapidly decaying initial conditions. It can be applied in the case where the evolution equation is equivalent to an equality between linear operators:

$$\frac{\partial L(u)}{\partial t} + [L, A] = 0. \tag{37}$$

It is based on the fact that $u(t, .)$ can be characterized by some spectral data of the operator $L(u(t, .))$. The homogeneous nonlinear Schrödinger equation (NLS):

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{38}$$

can be expressed in the form (37) if we set

$$L(u) = iP\frac{\partial}{\partial x} + Q(u), \text{ with } P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Q(u) = \begin{pmatrix} 0 & u^* \\ -u & 0 \end{pmatrix}.$$

The operator A is of the type $-2iP\frac{\partial^2}{\partial x^2} + C(u)$, with $C(u) \to 0$ when $u \to 0$, $u_x \to 0$. The domain of $L(u)$ is the space $\mathbb{H}^1(\mathbb{R})$,

$$\mathbb{H}^1(\mathbb{R}) = \left\{\psi \text{ such that } \psi \in \mathbb{L}^2(\mathbb{R}), \psi_x \in \mathbb{L}^2(\mathbb{R})\right\},$$

which is a dense subset of the Hilbert space $\mathbb{L}^2(\mathbb{R})$:

$$\mathbb{L}^2(\mathbb{R}) = \left\{\psi = \psi_1 \mathbf{e_1} + \psi_2 \mathbf{e_2}, \psi_j \in L^2(\mathbb{R})\right\}, \quad \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

equipped with the scalar product:

$$\langle \psi, \phi \rangle = \int_{-\infty}^{+\infty} dx \psi_1^* \phi_1(x) + \psi_2^* \phi_2(x).$$

3.4.0.1. Operator $L(0)$.— $L(0)$ is self-adjoint. The real axis constitutes its essential spectrum. The eigenspace associated with the eigenvalue $\lambda \in \mathbb{R}$ has dimension 2 and admits the couple $\left(\mathbf{e_1} e^{-i\lambda x}, \mathbf{e_2} e^{i\lambda x}\right)$ as a base. Besides the point spectrum of $L(0)$ is empty, because the non-trivial solutions of $v_x = i\lambda v$ are not in $L^2(\mathbb{R})$.

3.4.0.2. Essential spectrum of the operator $L(u(t = t_0, .))$.— Let us consider the spectral problem associated with the operator $L(u) = L(0) + Q(u)$:

$$L(u(t,x))\psi(t,x) = \lambda(t)\psi(t,x), \quad \psi = \psi_1 \mathbf{e_1} + \psi_2 \mathbf{e_2}. \tag{39}$$

If $u(t = t_0, .) \in L^1(\mathbb{R})$, then $Q(u)$ is $L(0)$-compact. As a consequence of the Weyl theorem, the essential spectrum of $L(u)$ is equal to the real axis. Eq. (39) actually admits two linearly independent solutions when λ is real. We introduce the so-called Jost functions f and g, defined as the eigenfunctions of $L(u)$ associated with the real eigenvalue λ which satisfy the following boundary conditions:

$$f(x, \lambda) \overset{x \to +\infty}{\longrightarrow} \mathbf{e_2} e^{i\lambda x}, \quad g(x, \lambda) \overset{x \to -\infty}{\longrightarrow} \mathbf{e_1} e^{-i\lambda x}.$$

If we denote by $\bar{\psi}$ the vector $(\psi_2^*, -\psi_1^*)$ associated with a vector ψ solution of (39), then $\bar{\psi}$ is a solution of $L\bar{\psi} = \lambda^* \bar{\psi}$. In the case of a real eigenvalue, ψ and $\bar{\psi}$ are linearly independent and form a base of the space of the solutions of (39). It can then be proved that the Jost functions are related by:

$$g(x, \lambda) = a(\lambda)\bar{f}(x, \lambda) + b(\lambda) f(x, \lambda), \tag{40}$$

$$f(x, \lambda) = -a(\lambda)\bar{g}(x, \lambda) + b^*(\lambda) g(x, \lambda). \tag{41}$$

Substituting the second equality into the first one, we also exhibit the following conservation relation:

$$|a(\lambda)|^2 + |b(\lambda)|^2 = 1. \tag{42}$$

Using (39) we get two more conservation relations which concern the norms of the Jost functions f and g:

$$|f_1(x,\lambda)|^2 + |f_2(x,\lambda)|^2 = 1, \qquad |g_1(x,\lambda)|^2 + |g_2(x,\lambda)|^2 = 1.$$

Multiplying Eq. (40) by the vector $\bar{f}^*$, we get an explicit representation of the coefficient a as the Wronskian of f and g:

$$a(\lambda) = g_1(x,\lambda) f_2(x,\lambda) - g_2(x,\lambda) f_1(x,\lambda). \tag{43}$$

We are able to provide a more explicit representation of the Jost functions f and g. Denoting $\tilde{f}_1(x,\lambda) = e^{i\lambda x} f_1(x,\lambda)$ and $\tilde{f}_2(x,\lambda) = e^{-i\lambda x} f_2(x,\lambda)$, we can find from (39) that $\tilde{f}$ satisfies a system of integral equations. Besides $\tilde{f}_1$ can be eliminated from this system by substitution, so that we get a closed equation for $\tilde{f}_2$, whose solution is:

$$\tilde{f}_2(x,\lambda) = 1 + \int_x^\infty dy M(y,x,\lambda) \left(1 + \int_y^\infty dz M(z,x,\lambda)\,(\ldots)\right),$$

where $M(y,x,\lambda) = -u^*(y) \int_x^y dz u(z) e^{2i\lambda(y-z)}$. This expression holds true when $u \in L^1$, because the associated sequence absolutely converges. The function $\tilde{f}_1$ also admits a similar representation. Let us examine carefully the properties of $\tilde{f}$. If $y \mapsto |y|^n |u(y)| \in L^1$, then $\tilde{f}_1$ and $\tilde{f}_2$ are of class $\mathcal{C}^n$ over the real axis. Besides, if $u \in L^1$, then $\tilde{f}_1$ and $\tilde{f}_2$ can be analytically continued in the upper complex half-plane $\mathrm{Im}(\lambda) \geq 0$ where they have no singularity. Indeed, in view of the definition of M one can see that the exponential term has a norm equal to $e^{-2\mathrm{Im}\lambda(y-z)}$ (remember that we integrate over the domain $y - z > 0$) which decays faster than any polynomial term brought by the λ-derivatives.

3.4.0.3. Point spectrum of the operator $L(u(t = t_0, .))$.— From (43) we can define an analytic continuation of $a(\lambda)$ over the upper complex half-plane. A noticeable feature then appears. If λ_r is a zero of $a(\lambda)$, then f and g are linearly dependent, so there exists a coefficient ρ_r such that $g(x,\lambda_r) = \rho_r f(x,\lambda_r)$. The corresponding eigenfunction is bounded and decays exponentially as $x \to +\infty$ (because $|f| \sim e^{-\mathrm{Im}\lambda_r x}$) and as $x \to -\infty$ (because $|g| \sim e^{+\mathrm{Im}\lambda_r x}$). Thus λ_r is an element of the point spectrum of $L(u)$. Moreover we can compute from (39) and (43) the λ-derivative of a at $\lambda = \lambda_r$:

$$a'(\lambda_r) = -2i\rho_r \int_{-\infty}^{+\infty} dx f_1 f_2(x,\lambda_r). \tag{44}$$

It can then be proved that the set $(a(\lambda), b(\lambda), \lambda_r, \rho_r, a'(\lambda_r))$ characterizes the Jost functions f and g as well as the solution u. The inverse transform is essentially based on the resolution of the linear integro-differential Gelfand-Levitan-Marchenko equation, whose entries are constituted by the set $(a, b, \lambda_r, \rho_r, a'(\lambda_r))$:

$$\begin{aligned} & K_1(x,y) = \Phi^*(x+y) - \int_x^\infty K_1(x,y'') \int_x^\infty \Phi^*(y+y')\Phi(y'+y'')dy'dy'', \\ (45) \qquad & K_2(x,y) = -\int_x^\infty K_1^*(x,y')\Phi^*(y+y')dy', \\ & \text{where } \Phi(y) = -\sum_r \frac{i\rho_r}{a'(\lambda_r)} e^{i\lambda_r y} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{b(\lambda)}{a(\lambda)} e^{i\lambda y} d\lambda. \end{aligned}$$

We can get the eigenvector f from the kernel K solution of (45):

$$f(x,\lambda) = \mathbf{e_2} e^{i\lambda x} + \int_x^\infty K(x,y) e^{i\lambda y} dy. \tag{46}$$

We then obtain u by the formula $u(x) = -2iK_1^*(x,x)$. The study of the inverse problem associated with the operator $L(u)$ has not yet been completely achieved. In particular the precise characterization of the spectral data which lead to well-defined potentials u has not yet been completed. However, in the case where the initial condition u_0 is rapidly decaying so that it satisfies $x \mapsto |x|^n |u_0|(x) \in L^1$ for any n, the inverse scattering can be rigorously achieved [**2**].

The great advantage of the method is that the evolution equations of the scattering data are uncoupled:

$$a(t,\lambda) = a(t_0,\lambda), \quad b(t,\lambda) = b(t_0,\lambda)e^{-4i\lambda^2(t-t_0)}, \quad \rho_r(t) = \rho_r(t_0)e^{-4i\lambda_r^2(t-t_0)}.$$

To sum up, the scattering transform involves the following operations:

$$\begin{array}{ccc} u(t_0,x) & \overset{\text{direct scatt.}}{\longrightarrow} & (a,b,\lambda_r,\rho_r,a'(\lambda_r))\,(t_0) \\ \text{NLS} \downarrow & & \downarrow \text{uncoupled evolution equations} \\ u(t,x) & \overset{\text{inverse scatt.}}{\longleftarrow} & (a,b,\lambda_r,\rho_r,a'(\lambda_r))\,(t) \end{array}$$

What is striking is the remarkable analogy to Fourier analysis of the linear Schrödinger equation (see Subsection 2.2).

3.5. Conserved quantities. — There exists an infinite number of quantities which are preserved by the homogeneous nonlinear Schrödinger equation (38) [**53**]. They can be represented as functionals of the solution u or in terms of the scattering data. We shall present here only two of them which are of physical interest.

• The mass of the wave $N = \int |u|^2 dx$. Denoting $n(\lambda) = -\pi^{-1} \ln |a(\lambda)|^2$, the mass is also given by

$$N = \sum_r 2i(\lambda_r^* - \lambda_r) + \int n(\lambda) d\lambda. \tag{47}$$

• The Hamiltonian or energy $H = \int |u_x|^2 - |u|^4 dx$, which can also be expressed as

$$H = \sum_r \frac{8i}{3}({\lambda_r^*}^3 - {\lambda_r}^3) + 4\int \lambda^2 n(\lambda) d\lambda. \tag{48}$$

3.6. Soliton. — There exists a special solution with finite mass and energy of Eq. (38) that is called soliton:

$$u_0(t,x) = 2\nu_0 \frac{\exp i\left(2\mu_0(x-4\mu_0 t) + 4(\nu_0^2+\mu_0^2)t\right)}{\cosh\left(2\nu_0(x-4\mu_0 t)\right)}. \tag{49}$$

The mass and the velocity of the soliton are respectively $N_0 = 4\nu_0$ and $V_0 = 4\mu_0$. The width of the envelop of the soliton is conversely proportional to its mass. The soliton (49) is associated with the following scattering data:

$$a_0(\lambda) = \frac{\lambda - (\mu_0 + i\nu_0)}{\lambda - (\mu_0 - i\nu_0)}, \quad b_0(\lambda) = 0. \tag{50}$$

a_0 admits a unique zero in the upper complex half-plane denoted by $\lambda_0 = \mu_0 + i\nu_0$. The coefficient associated with the zero λ_0 is $\rho_0 = i\exp\left(-4i(\mu_0+i\nu_0)^2 t\right)$. Figure 3 plots two different solitons at time $t = 0$. Both have the same mass, and consequently the same envelop, but they have different velocities. Note that, in the case $\nu_0 \gg \mu_0$ (resp. $\nu_0 \ll \mu_0$), the soliton oscillates slowly (resp. quickly) within its envelop.

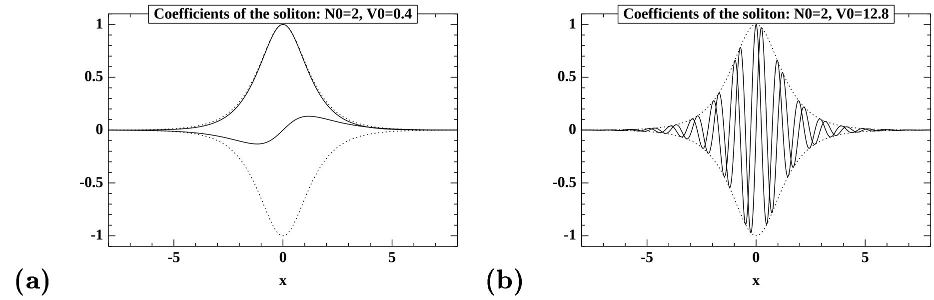

Fig. 3. Solitons at time $t = 0$. The dashed lines represent the envelops of the solitons, while the solid lines represent the real and imaginary parts. Picture a: $N_0 = 2$, $V_0 = 0.4$. Picture b: $N_0 = 2$, $V_0 = 12.8$.

3.7. Solitons in random media. —

3.7.1. Formulation. — We consider a perturbed Schrödinger equation with a non-zero right-hand side:

$$iu_t + u_{xx} + 2|u|^2 u = \varepsilon R(u)(t,x). \tag{51}$$

The small parameter $\varepsilon \in (0,1)$ characterizes the amplitude of the perturbation. The model of the perturbation is taken to be:

$$R(u)(t,x) = m_1(x)u(t,x) + m_2(x)|u|^2 u(t,x) + \left(m_3(x)u_x\right)_x.$$

where m_1, m_2, and m_3 are random, stationary, ergodic, zero-mean, and independent processes. We shall consider that the processes V_j are not only ergodic, but also

ϕ-mixing, i.e. that there exists a function $t \mapsto \phi(t)$ vanishing as $t \to +\infty$ such that

$$\sup_{s>0} \left\{ \mathbb{P}(B/A) - \mathbb{P}(B), A \in \mathcal{F}_0^s, B \in \mathcal{F}_{s+t}^\infty \right\} \leq \phi(t)$$

$$\text{where } \mathcal{F}_s^t = \sigma(V_j(x), s \leq x \leq t, j = 1, 2).$$

For technical reasons we shall assume that the function $t \mapsto \phi(t)$ decays at least as t^{-4}. This mixing condition is sufficient to prove all the convergence results that are necessary. However we believe that it can be weakened and we expect the condition $\phi \in L^{1/2}(\mathbb{R}^+)$ to be sufficient.

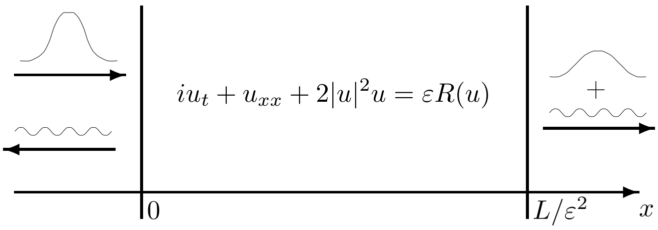

Fig. 4. Scattering of a soliton.

3.7.2. Main results. — Let $L > 0$. We denote by Ω_L^ε the measurable set of realizations of the process V such that the scattered wave consists of one soliton plus some radiation. In terms of the spectral data it means that the Jost coefficient a admits a unique zero in the upper complex half plane. We denote by ν^ε and μ^ε the re-scaled processes defined on Ω_L^ε by $\nu^\varepsilon(x) = \nu(x/\varepsilon^2)$ and $\mu^\varepsilon(x) = \mu(x/\varepsilon^2)$ (i.e. the coefficients of the transmitted soliton in position x/ε^2), and on ${\Omega_L^\varepsilon}^c$ by $\nu^\varepsilon(x) = 0$ and $\mu^\varepsilon(x) = 0$. We can now state our main convergence result.

Proposition 3.1. — *The following assertions hold true for any $L > 0$.*
1. $\liminf_{\varepsilon \to 0} \mathbb{P}\left(\Omega_L^\varepsilon\right) = 1.$
2. The $\mathbb{R}^2$-valued process $(\nu^\varepsilon(x), \mu^\varepsilon(x))_{x\in[0,L]}$ converges in probability in $\mathcal{C}^0$ to the $\mathbb{R}^2$-valued deterministic function $(\nu_l(x), \mu_l(x))_{x\in[0,L]}$ which satisfies the system of ordinary differential equations:

$$\left\{ \begin{aligned} \frac{d\nu_l}{dx} &= F(\nu_l, \mu_l), \quad \nu_l(0) = \nu_0 \\ \frac{d\mu_l}{dx} &= G(\nu_l, \mu_l), \quad \mu_l(0) = \mu_0, \end{aligned} \right. \tag{52}$$

where F and G are $\mathcal{C}^1$ functions where only the Fourier transforms of the autocorrelation functions appear:

$$F(\nu, \mu) = -\frac{1}{2\pi} \sum_{j=1}^{3} \int_{-\infty}^{\infty} |c_j|^2(\nu, \mu, \lambda) \alpha_j(k(\nu, \mu, \lambda)) d\lambda \tag{53}$$

$$G(\nu, \mu) = -\frac{1}{4\pi} \sum_{j=1}^{3} \int_{-\infty}^{\infty} \left(\frac{\lambda^2}{\mu\nu} + \frac{\nu}{\mu} - \frac{\mu}{\nu} \right) |c_j|^2(\nu, \mu, \lambda) \alpha_j(k(\nu, \mu, \lambda)) d\lambda. \tag{54}$$

The coefficients α_j *and* k *are defined by:*

$$\alpha_j(k) = \int_0^\infty \mathbb{E}[m_j(0)m_j(x)]\cos(kx)dx, \qquad k(\nu,\mu,\lambda) = \frac{(\lambda-\mu)^2+\nu^2}{\mu}. \tag{55}$$

The functions c_j *are written explicitly in* **[29, 1]**. *For consistency we write the full expression of* c_1*:*

$$c_1(\nu,\mu,\lambda) = \frac{\pi}{2^4\mu^3}\frac{(\lambda-\mu+i\nu)^2}{\cosh\left(\pi(\mu^2-\nu^2-\lambda^2)/(4\mu\nu)\right)}. \tag{56}$$

The first point means that the event "the scattered wave consists of one soliton plus some radiation" occurs with very high probability for small ε, while the second term gives the effective dynamics of the coefficients of the transmitted soliton in the asymptotic framework $\varepsilon \to 0$. The analysis of the effective system (52) puts into evidence that there exist two main regimes up to transitory regimes.

1) If the mass of the incoming soliton is small enough, then the velocity of the soliton is almost constant, while its mass decreases to 0:
- as $\exp(-L/L_{loc})$ (perturbation of the linear potential),
- as $L^{-1/4}$ (perturbation of the nonlinear coefficient),
- as $L^{-1/2}$, then as $\exp(-L/L'_{loc})$ (dispersive perturbation).

2) If the mass of the soliton is large enough, then the mass of the soliton is almost constant, while its velocity of the soliton slowly decreases to 0. The decay rate depends on the tail of the spectrum of the perturbation, but we can state in great generality that it is at most logarithmic.

It can be noted that, in the limit case $\nu_0/\mu_0 \to 0$, the incoming soliton can be approximated by a linear pulse:

$$u_0(t,x) \simeq \int_{-\infty}^{+\infty} dk f(k) e^{ikx-ik^2t}, \text{ with } f(k) = \frac{1}{2}\cosh^{-1}\left(\frac{\pi}{4}\left(\frac{k-2\mu_0}{\nu_0}\right)\right),$$

whose spectral content f is sharply peaked about the wavenumber $k_0 = 2\mu_0$. Furthermore the spectrum of the radiation is peaked about the wavenumber $-2\mu_0$ (there exists also a secondary peak about $+2\mu_0$ which is much weaker). These statements are in agreement with the linear approximation. The localization length L_{loc} corresponding to a perturbation of the linear potential can written in terms of the carrier wavenumber as $L_{loc} = 8k_0^2/\alpha_1(2k_0)$. It is equal to the localization length of a monochromatic wave with wavenumber k_0 scattered by a slab of linear random medium (see Proposition 2.1).

3.7.3. The main steps of the proof. — We now list the main steps of the proof **[29, 1]**.

a. A priori estimates.
The following quantities (mass and energy) are preserved by the perturbed

Schrödinger equation (51):

$$N_{tot} = \int |u|^2 dx, \tag{57}$$

$$E_{tot} = \int H_0(x)dx + \varepsilon \int H_1(x)dx, \tag{58}$$

$$H_1(x) = m_1(x)|u|^2 + \frac{1}{2}m_2(x)|u|^4 - m_3(x)|u_x|^2 \tag{59}$$

Assume that the m_j are bounded processes. Sobolev inequalities then prove that the H^1-norm, the L^4-norm and the L^∞-norm of $u(t,.)$ are uniformly bounded with respect to $t \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Furthermore $\int H_1(x)dx$ can be bounded uniformly with respect to $t \in \mathbb{R}$ by a constant that depends only on N_{tot} and E_{tot}.

b. Prove the stability of the zero of the Jost coefficient a.
The zero corresponds to the soliton. This part strongly relies on the analytical properties of a in the upper complex half plane. Basically we apply Rouché's theorem so as to prove that the number of zeros is constant. This method is efficient to prove that the zero is preserved, but it does not bring control on its precise location in the upper half plane. This step is not sufficient to compute the variations of the soliton parameters.

c. Compute the radiation.
The Jost coefficients a and b satisfy coupled equations [**38**]:

$$\left\{ \begin{array}{rcll} \dfrac{\partial a(\lambda,t)}{\partial t} & = & 0 & +\varepsilon \left(a(\lambda,t)\bar{\gamma}(\lambda,t) + b(\lambda,t)\gamma(\lambda,t) \right), \\ \dfrac{\partial b(\lambda,t)}{\partial t} & = & -4i\lambda^2 b(\lambda,t) & -\varepsilon \left(a(\lambda,t)\gamma^*(\lambda,t) + b(\lambda,t)\bar{\gamma}(\lambda,t) \right), \end{array} \right.$$

where $\gamma(\lambda,t) = -\int dx R(u) f_2^2 + R(u)^* f_1^2$ and $\bar{\gamma}(\lambda,t) = -\int dx R(u)^* f_1^* f_2 - R(u) f_1 f_2^*$. From these equations we can estimate the amount of radiation which is emitted during some time interval in terms of mass and energy thanks to (47) and (48). We are then able to deduce the evolution equations of the coefficients of the soliton by using the conservations of the total mass and energy. For times of order $O(1)$, since N_{tot} and E_{tot} are conserved, the variations $\Delta(..)$ of the relevant quantities are linked together by the relations:

$$\begin{aligned} 0 &= 4\Delta\nu + \int \Delta n(\lambda) d\lambda, \\ 0 &= 16\Delta\left(\nu\mu^2 - \nu^3/3\right) + 4\int \lambda^2 \Delta n(\lambda) d\lambda + \varepsilon\Delta\left(\int_{\mathbb{R}} H_1(x)dx\right). \end{aligned}$$

$\Delta n(\lambda)$ is of order ε^2, but the last term in the expression of the total energy is of order ε. Thus our strategy is not efficient for estimating the variations of the coefficients of the soliton for times of order $O(1)$. Let us now consider times of order $O(\varepsilon^{-2})$. $\Delta n(\lambda)$

is now of order 1, while the last term in the expression of the total energy is of order ε by the a priori estimates. Thus we can efficiently compute the long-time behavior of the coefficients of the soliton in the asymptotic framework $\varepsilon \to 0$, when the last term in the expression of the total energy is uniformly negligible. Applying probabilistic limit theorems (approximation-diffusion), we then find that the coefficients of the soliton converge in probability to non-random functions which satisfy the system (52).

d. Compute the form of the scattered wave.
Neglecting the terms of higher order, the total wave is given by the sum $u(t/\varepsilon^2, x) = u_S(t/\varepsilon^2, x) + u_L(t/\varepsilon^2, x)$, where u_S is a soliton of mass $4\nu(t/\varepsilon^2)$ and velocity $4\mu(t/\varepsilon^2)$:

$$u_S(\frac{t}{\varepsilon^2}, x) = -2i\nu \frac{\exp i\left(2\mu(x - x_s) + \phi_s\right)}{\cosh\left(2\nu(x - x_s)\right)}, \tag{60}$$

x_s and ϕ_s are respectively the position and the phase of the soliton at time t/ε^2:

$$x_s = \frac{1}{2\nu} \ln\left(\frac{1}{2\nu}\left|\frac{\rho_r(t/\varepsilon^2)}{a'(t/\varepsilon^2, \lambda_s)}\right|\right), \quad \phi_s = \arg\left(-i\frac{\rho_r(t/\varepsilon^2)}{a'(t/\varepsilon^2, \lambda_s)}\right) + 2\mu x_s, \tag{61}$$

$\lambda_s = \mu(t/\varepsilon^2) + i\nu(t/\varepsilon^2)$ and u_L admits the following expression:

$$\begin{aligned} u_L(\frac{t}{\varepsilon^2}, x) &= \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{b}{a}(\lambda) \frac{\left(\lambda - \mu + i\nu \tanh\left(2\nu(x - x_s)\right)\right)^2}{(\lambda - \mu + i\nu)^2} e^{2i\lambda x} d\lambda \\ &+ \frac{\nu^2}{i\pi} \frac{\exp 2i\left(2\mu(x - x_s) + \phi_s\right)}{\cosh^2\left(2\nu(x - x_s)\right)} \int_{-\infty}^{\infty} \frac{b^*}{a^*}(\lambda) \frac{1}{(\lambda - \mu - i\nu)^2} e^{-2i\lambda x} d\lambda. \end{aligned} \tag{62}$$

u_S is the soliton part of the total wave. The first component of u_L represents the radiation, with a correction in the neighborhood of the soliton $x \sim x_s(t/\varepsilon^2)$. The second component of u_L represents the interaction of the soliton and the radiation, which is only noticeable in the neighborhood of the soliton. This result is not surprising. Roughly speaking, the support of the radiation lies in an interval with length of order ε^{-2}. Since the L^2-norm is bounded by the conservation of the total mass, we can expect that the amplitude of the radiation is of order ε. More exactly it can be rigorously proved that the amplitude of the radiation can be bounded above by $K\varepsilon |\ln \varepsilon|$.

3.7.4. Numerical simulations. — The results in the previous subsections are theoretically valid in the limit case $\varepsilon \to 0$, where the amplitudes of the perturbations go to zero and the length of the random slab goes to infinity. In this subsection we aim at showing that the asymptotic behaviors of the soliton can be observed in numerical simulations in the case where ε is small, more precisely smaller than any other characteristic scale of the problem. We use a fourth-order split-step method to simulate the perturbed nonlinear Schrödinger equation (51). This numerical algorithm provides accurate and stable solutions **[43]**. For the sake of simplicity we only consider perturbations of the linear potential m_1 and take $m_2 = m_3 = 0$.

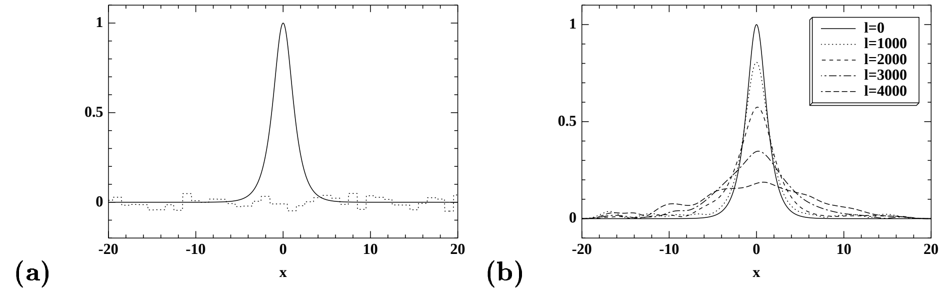

Fig 5. Picture a: Envelop of the initial soliton (solid line) with mass $N_0 = 2$ and velocity $V_0 = 1.6$. In dashed line is plotted a realization of the random potential εm_1 with $\varepsilon = 0.05$ and $l_c = 0.4$. Picture b: Envelops of the soliton when its center crosses different depth lines l for one of the realization of the random potential. The coordinate x is normalized around the depth line l.

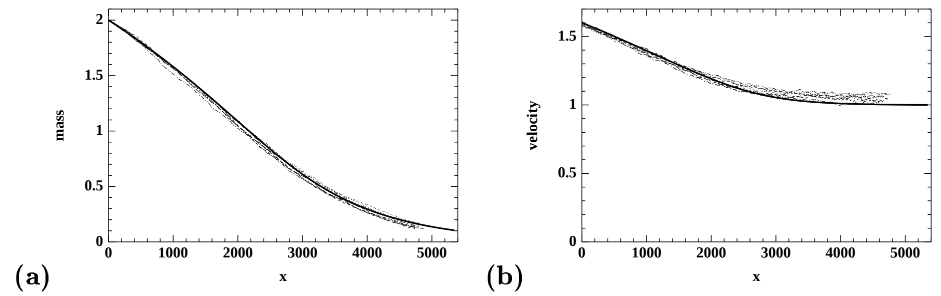

Fig. 6. Coefficients of the transmitted soliton whose initial coefficients are $N_0 = 2$, $V_0 = 1.6$ with a random potential whose amplitude is $\varepsilon = 0.05$ and correlation length $l_c = 0.4$. The picture a (resp. b) is devoted to the mass (resp. velocity). The thick solid lines represent the theoretical coefficients of the transmitted soliton. In thin dashed and dotted lines are plotted the simulated masses and velocities of the transmitted solitons for 7 different realizations of the random potential.

We assume in this subsection that the potential is constant over elementary intervals of length l_c and take independent random values over each interval which obey uniform distributions over $[-1, 1]$. We present simulations where the initial wave at time $t = 0$ is a soliton with mass $N_0 = 2$ and velocity $V_0 = 1.6$ centered at $x = 0$ (see Fig. 5a). The simulated evolutions of the coefficients of the soliton are presented in Fig. 6 for seven different realizations of the random potential with $\varepsilon = 0.05$ and $l_c = 0.4$. They are compared with the theoretical evolutions given by (52) in the scale x/ε^2. It thus appears that the numerical simulations are in very good agreement with the theoretical results. Fig. 5b plots the envelops of the solution at different depths corresponding to one of the simulations, which shows that the wave keeps the basic form of a soliton although it looses some mass. All these results confirm that system (52) describes with accuracy the transmission of a soliton through a random slab for small perturbations and long slab length.

3.8. Conclusion. — We have studied the propagation of a soliton in a nonlinear dispersive medium with spatially random perturbations by applying the inverse scattering transform to a random NLS equation. If the incoming soliton has small mass, then the mass of the soliton decays to zero exponentially or algebraically with the length of the system. In case of large mass, the mass of the soliton is almost constant. Furthermore the velocity is found to decrease at a slow rate (at most logarithmic) which depends on the high-frequency behavior of the power spectrum of the random perturbation.

We feel that it is of great interest to consider some other integrable systems with a different type of dispersion. For instance the Korteweg-de Vries equation, with a third order dispersion, is worth studying. Original results are derived that are very different compared to the randomly perturbed Nonlinear Schrödinger equation [**30**]. Indeed the scattering of the soliton generates not only radiation during its motion, but also a soliton gas, that is to say a collection of a very large number of solitons with very small masses, whose total mass is of order one. It should be interesting to get a classification of the integrable systems in terms of their respective behaviors with respect to random perturbations. Furthermore the interaction of solitons in random dispersive media represents also a great challenge for practical applications to telecommunications for instance.

References

[1] F. Kh. Abdullaev and J. Garnier, Solitons in media with random dispersive perturbations, Physica D **134**, 303-315 (1999).

[2] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981.

[3] G. P. Agrawal, *Nonlinear fiber optics, 2nd ed.*, Academic Press, New York, 1995.

[4] P. W. Anderson, Absences of diffusion in certain random lattices, Phys. Rev. **109**, 1492-1505 (1958).

[5] L. Arnold, A formula connecting sample and moment stability of linear stochastic systems, SIAM J. Appl. Math. **44**, 793-802 (1984).

[6] L. Arnold, G. Papanicolaou, and V. Wihstutz, Asymptotic analysis of the Lyapunov exponent and rotation number of the random oscillator and applications, SIAM J. Appl. Math. **46**, 427-450 (1986).

[7] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. White, Frequency content of randomly scattered signals, SIAM Rev. **33**, 519-625 (1991).

[8] P. H. Baxendale and R. Z. Khaminskii, Stability index for products of random transformations, Adv. Appl. Prob. **30**, 968-988 (1998).

[9] T. B. Benjamin and J. E. Feir, The disintregration of wave trains on deep water, J. Fluid Mech. **27**, 417-430 (1967).

[10] D. J. Benney and A. C. Newell, The propagation of nonlinear wave envelopes, J. Math. Phys. **46**, 133-139 (1967).

[11] R. N. Bhattacharya, On the functional central limit theorem and the law of the iterated logarithm for Markov process, Z. Wahrsch. Verw. Gebiete **60**, 185-201 (1982).

[12] P. Bougerol and J. Lacroix, *Products of random matrices with applications to Schrödinger operators*, Birkhaüser, Boston, 1985.

[13] J. Boussinesq, Théorie de l'intumescence liquide appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire, C. R. Acad. Sci. Paris **72** 755-759 (1871); **73** 256-260 (1871); **73** 1210-1212 (1871).

[14] J. C. Bronski, D. W. McLaughlin, and M. J. Shelley, On the stability of time-harmonic localized states in a disordered nonlinear medium, J. Statist. Phys. **88**, 1077-1115 (1997).

[15] R. Burridge and H. W. Chang, Multimode, one-dimensional wave propagation in a highly discontinuous medium, Wave Motion **11**, 231-249 (1989).

[16] R. Burridge, G. Papanicolaou, and B. White, One-dimensional wave propagation in a highly discontinuous medium, Wave Motion **10**, 19-44 (1988).

[17] R. Carmona, Random Schrödinger operators, in *École d'été de probabilités de Saint-Flour XIV*, pp. 1-124, Lecture Notes in Math., Vol. 1180, Springer, Berlin, 1986.

[18] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990.

[19] J. F. Clouet and J. P. Fouque, Spreading of a pulse travelling in random media, Ann. Appl. Prob. **4**, 1083-1097 (1994).

[20] H. Crauel, Lyapunov numbers of Markov solutions of linear stochastic systems, Stochastic **14**, 11-28 (1984).

[21] F. Delyon, Y. Levy, and B. Souillard, Approach à la Borland to multidimensional localization, Phys. Rev. Lett. **55**, 618-621 (1985).

[22] P. Desvillard and B. Souillard, Polynomially decaying transmission for the nonlinear Schrödinger equation in a random medium, J. Statist. Phys. **43**, 423-439 (1986).

[23] L. Erdös and H.-T. Yau, Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation, Comm. Pure Appl. Math. **53**, 667-735 (2000).

[24] H. Frisch and S. P. Lloyd, Electron levels in a one-dimensional random lattice, Phys. Rev. **120**, 1175-1189 (1960).

[25] J. Fröhlich, T. Spencer, and C. E. Wayne, Localization in disordered, nonlinear dynamical systems, J. Statist. Phys. **42**, 247-274 (1986).

[26] H. Furstenberg, Noncommuting random products, Trans. Amer. Math. Soc. **108**, 377-428 (1963).

[27] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett. **19**, 1095-1097 (1967).

[28] J. Garnier, A multi-scaled diffusion-approximation theorem. Applications to wave propagation in random media, ESAIM Probability & Statistics **1**, 183-206 (1997).

[29] J. Garnier, Asymptotic transmission of solitons through random media, SIAM J. Appl. Math. **58**, 1969-1995 (1998).

[30] J. Garnier, Long-time dynamics of Korteweg-de Vries solitons driven by random perturbations, J. Statist. Phys. **105**, 789-833 (2001).

[31] I. Goldsheid, S. Molchanov, and L. Pastur, A random homogeneous Schrödinger operator has a pure point spectrum, Functional Anal. Appl. **11**, 1-10 (1977).

[32] S. A. Gredeskul and Yu. U. Kivshar, Propagation and scattering of nonlinear waves in disordered systems, Phys. Rep. **216**, 1-61 (1992).

[33] A. Hasegawa, *Optical solitons in fibers*, Springer, Berlin, 1989.
[34] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers, I; Anomalous dispersion, Appl. Phys. Lett. **23**, 142-144 (1973).
[35] H. A. Haus and W. S. Wong, *Solitons in optical communications*, Rev. Mod. Phys. **68**, 423-444 (1996).
[36] V. A. Hopkins, J. Keat, G. D. Meegan, T. Zhang, and J. D. Maynard, Observation of the predicted behavior of nonlinear pulse propagation in disordered media, Phys. Rev. Lett. **76**, 1102-1104 (1996).
[37] A. Ishimaru, *Wave propagation and scattering in random media*, Academic Press, San Diego, 1978.
[38] V. I. Karpman, Soliton evolution in the presence of perturbations, Phys. Scr. **20**, 462-478 (1979).
[39] M. J. Kenna, R. L. Stanley, and J. D. Maynard, Effects of nonlinearity on Anderson localization, Phys. Rev. Lett. **69**, 1807-1810 (1992).
[40] R. Z. Khaminskii, A limit theorem for solutions of differential equations with random right-hand side, Theory Prob. Appl. **11**, 390-406 (1966).
[41] Yu. U. Kivshar, A. M. Kosevich, and O. A. Chubykalo, Resonant and non resonant soliton scattering by impurities, Phys. Lett. A **125**, 35-40 (1987).
[42] V. I. Klyatskin, *Stochastic equations and waves in random media*, Nauka, Moscow, 1980.
[43] R. Knapp, Transmission of solitons through random media, Physica D **85**, 496-508 (1995).
[44] R. Knapp, G. Papanicolaou, and B. White, Nonlinearity and localization in one dimensional random media, in *Disorder and nonlinearity*, A. R. Bishop, D. K. Campbell, and S. Pnevmaticos, eds., Springer, 1989, pp. 2-26.
[45] R. Knapp, G. Papanicolaou, and B. White, Transmission of waves by a nonlinear random medium, J. Statist. Phys. **63**, 567-583 (1991).
[46] W. Kohler and G. Papanicolaou, Power statistics for wave propagation in one dimension and comparison with transport theory, J. Math. Phys. **14**, 1733-1745 (1973); **15**, 2186-2197 (1974).
[47] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type long stationary waves, Philos. Mag. Ser. **39**, 422-443 (1895).
[48] S. Kotani, Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, in *Stochastic Analysis*, K. Ito ed., North Holland, 1984, pp. 225-247.
[49] H. J. Kushner, *Approximation and weak convergence methods for random processes*, MIT Press, Cambridge, 1984.
[50] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. **21**, 467-490 (1968).
[51] M. Lax and J. C. Phillips, One-dimensional impurity bands, Phys. Rev. **110**, 41-49 (1958) 41.
[52] I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, *Introduction to the theory of disordered systems*, Wiley, New York, 1988.
[53] S. V. Manakov, S. Novikov, J. P. Pitaevskii, and V. E. Zakharov, *Theory of solitons*, Consultants Bureau, New York, 1984.
[54] D. Middleton, *Introduction to statistical communication theory*, Mc Graw Hill Book Co., New York, 1960.

[55] A. C. Newell and J. V. Moloney, *Nonlinear optics*, Addison-Wesley, Redwood City, 1992.

[56] R. F. O'Doherty and N. A. Anstey, Reflections on amplitudes, Geophysical Prospecting **19**, 430-458 (1971).

[57] L. A. Pastur, Spectra of random self-adjoint operators, Russian Math. Surveys **28**, 3-64 (1973).

[58] M. A. Pinsky, Instability of the harmonic oscillator with small noise, SIAM J. Appl. Math. **46**, 451-463 (1986).

[59] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Springer, Berlin, 1991.

[60] R. L. Stratonovich, *Topics in the theory of random noise*, Gordon and Breach, New York, 1963.

[61] V. N. Tutubalin, On limit theorems for products of random matrices, Theory Prob. Appl. **10**, 15-27 (1965).

[62] N. G. Van Kampen, *Stochastic processes in physics and chemistry*, North Holland, Amsterdam, 1981.

[63] Yu. S. Kivshar, S. A. Gredeskul, A. Sánchez, and L. Vásquez, Localization decay induced by strong nonlinearity in disordered systems, Phys. Rev. Lett. **64**, 1693-1696 (1990).

[64] G. B. Whitham, *Linear and nonlinear waves*, Wiley, New York, 1974.

[65] N. J. Zabuski and M. D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. **15**, 240-243 (1965).

[66] V. E. Zakharov, Collapse of Langmuir waves, Sov. Phys. JETP **35**, 908-914 (1972).

[67] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34**, 62-69 (1972).

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