Laplacian growth of parallel needles: A Fokker-Planck equation approach

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Using a conformal transformation to set up the iterative nonlinear equations, we study analytically the kinetics of growth of parallel needles. We establish a discrete Fokker-Planck equation for the probability of finding at time t a given distribution of needle lengths. In the linear regime, it shows a short-wavelength Laplacian instability which we investigate in detail. From the crossover of the solutions to the nonlinear regime, we deduce analytically the general scale invariance of the two-dimensional models.

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Among the growth processes, Laplacian growth plays a crucial role by its ubiquity. It is underlying electrodeposition, viscous fingering, the start of dendritic growth of crystalline structures during solidification when the diffusion length is large, dielectric breakdown, or the growth of bacterial colonies. Diffusion limited aggregation (DLA) [1] was the first proposed stochastic model belonging to the class of Laplacian growth. The Laplacian field takes here its origin in the diffusion of individual particles in front of the growing structures. The Laplacian field may also be the electric field as in electrodeposition or in dielectric breakdown. For these reasons the DLA model has been the subject of a considerable amount of work. Unfortunately, in spite of the apparent simplicity of the model, our analytic understanding of DLA remains very unsatisfactory. Therefore a more modest but useful approach consists of a better understanding of DLA growth in the absence of branching [2,3]. The simpler problem of growing needles has then been considered: they may be either radial or parallel, in the presence of reflection (model R) or of absorption (model A [4]) of particles as shown in Fig. 1 (see, for instance, Krug [5] for a review). In this paper we will concentrate our attention on systems of parallel absorbing needles which is of more fundamental interest. In two dimensions, conformal mapping allows us to obtain analytical results as shown by Derrida and Hakim [6] for radial needles.

Considering the growth of two-dimensional structures, we will follow here the same approach of conformal mapping (for details the reader is referred to Shraiman and Bensimon [7], Szép and Lugosi [8], Peterson and Ferry [9], Kurtze [10], Derrida and Hakim [6], and Ref. [11] for the dynamics). But contrary to previous works, we shall focus our attention on the establishment of a Fokker-Planck equation to describe the growth kinetics. This is because DLA is in general not a deterministic continuous nonlinear problem, but basically a stochastic phenomenon discretized in space and time. Competition between the size of the sticking particles and the relative position of growing branches play a crucial role. Consequently the Fokker-Planck (or equivalently the Langevin) approach appears as very appropriate.

Only the case of two needles can be solved exactly. So, as a first step we will study the Laplacian instability, in the linear regime, starting from an initial distribution of needles with equal lengths. The corresponding behavior is somewhat analogous to the ''Mullins-Sekerka'' instability [12] in dendritic growth from a flat interface. In a recent experiment, Losert *et al.* [13] showed the spatial period-doubling instability of dendritic arrays in directional solidification as suggested theoretically by Warren and Langer [14]. We will prove analytically the existence of a similar behavior in needles growth.

In Laplacian growth, two ingredients are needed: on the one hand the Laplacian behavior determines the long-range interaction characteristic of DLA growth, but on the other hand the inherent Laplacian instability could not operate without the presence of local noise. This noise can be introduced in the initial state as in Refs. [6] and [11]. But in real systems it comes from the finite size δl of the diffusing particles, sticking to the needles at discrete time intervals.

The classical way to parametrize in a convenient manner Laplace's equation with a zero potential boundary condition on a set of n parallel needles is to introduce first a mapping of the unit circle in the complex plane z onto an n-branched star in the complex plane ω:

$$\omega = f(z) = Az \prod_{j=0}^{n-1} (1 - e^{i\theta_j}/z)^{\alpha_j}. \tag{1}$$

In this transformation, $\pi\alpha_j$ is the angle between two successive needles $\{j-1,j\}$ in the initial plane ω, the sum of the angles being 2π $(\Sigma_{j=0}^{n-1}\alpha_j = 2)$. The angles θ_j fix the lengths l_i of the needles, and A is a known coefficient. The tip positions in plane z are parametrized by the angles ϕ_i:

$$z_i = \exp(i\phi_i), \quad 0 \leq i \leq n-1. \tag{2}$$

A second transformation $\Omega = \ln \omega$ maps the star in ω into a set of parallel needles in a complex plane Ω and the initial

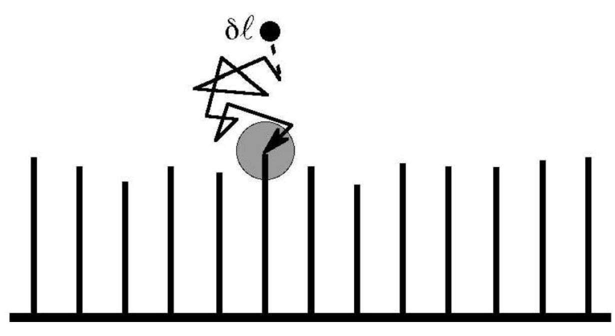

FIG. 1. Diffusion limited aggregation of parallel needles.

angles $\pi\alpha_j$ between needles becomes now a distance between successive parallel needles, and this pattern of needles is spatially repeated with a periodic length 2π. The Laplacian field is then

$$\Phi(\Omega)=\mathrm{Re}\{\ln[f^{-1}(\exp\Omega)]\} \tag{3}$$

in the plane Ω. The lengths of the needles are then given by $|\ln[f(z)]|$ at the points $z=z_i$ $(0\leqslant i\leqslant n-1)$:

$$l_i=\log(4A)+\sum_{j=0}^{n-1}\alpha_j\ln|\sin[(\phi_i-\theta_j)/2]|. \tag{4}$$

n additional constraints must be imposed to the angles ϕ_i and θ_j to takes into account the fact that the needle tips maximize $|f(z)|$ at $z=z_i(\forall 0\leqslant i\leqslant n-1)$

$$\sum_{j=0}^{n-1}\alpha_j\mathrm{cotan}[(\phi_i-\theta_j)/2]=0, \tag{5}$$

or if $\mathbf{C}$ is the matrix of $c_{ij}=\mathrm{cotan}[(\phi_i-\theta_j)/2]$, and $\vec{\alpha}$ the vector $\{\alpha_i\}$,

$$\mathbf{C}\cdot\vec{\alpha}=\vec{0}. \tag{6}$$

Since the number of unknown quantities is larger than the number of parameters by one, we put the average position of the needles at the origin of the l coordinate (this fixes A),

$$\tilde{l}_0=\overline{l}\equiv\frac{1}{n}\sum_{i=0}^{n-1}l_i=0. \tag{7}$$

The growth rate is now supposed to be proportional to the potential gradient along the needles, and the growth to be restricted to the tips while the needles remain at zero potential (model A). Therefore, following Refs. [6] and [11], the growth rate of the needles is

$$\frac{dl_i}{dt}\propto\left[\sum_{j=0}^{n-1}\alpha_j\{1+\cot^2[(\phi_i-\theta_j)/2]\}\right]^{-1/2}, \tag{8}$$

from which we will determine the growth probability p_i of the needles.

The general two-needle case $(n=2)$ can be exactly solved [15]. With condition (7),

$$\tilde{l}_0=(l_0+l_1)/2\equiv 0,\quad \tilde{l}_1=(l_0-l_1)/2. \tag{9}$$

The problem is completely defined by the knowledge of the probability $P(\tilde{l}_1,t)$ to find at time t a length gap $\tilde{l}_1$. The growth probabilities of both needles are

$$p_0(\tilde{l}_1)=u/(1+u),\quad p_1(\tilde{l}_1)=1/(1+u), \tag{10}$$

with

$$\exp[\tilde{l}_1]=u\left(\frac{\alpha_0+\alpha_1u^2}{\alpha_1+\alpha_0u^2}\right)^{(\alpha_0-\alpha_1)/4}. \tag{11}$$

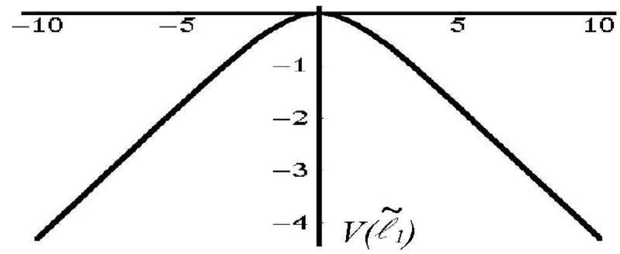

FIG. 2. Graph of the effective potential $V(\tilde{l}_1)$ when $\alpha_0=\alpha_1$.

Now we add at each time interval δt with probabilities $p_i(\tilde{l}_1)$ (note that consequently, $\delta\tilde{l}_1=\delta l/2$), a particle of size δl on needle i,

$$P(\tilde{l}_1,t+\delta t)=p_0(\tilde{l}_1-\delta\tilde{l}_1)P(\tilde{l}_1-\delta\tilde{l}_1,t)+p_1(\tilde{l}_1+\delta\tilde{l}_1)P(\tilde{l}_1+\delta\tilde{l}_1,t). \tag{12}$$

The evolution equation (12) can be expanded to second order in δl, leading to the Fokker-Planck equation

$$\frac{\partial P(\tilde{l}_1,t)}{\partial t}=D_2\partial^2_{\tilde{l}_1^2}P(\tilde{l}_1,t)-\frac{v}{2}\partial_{\tilde{l}_1}(P(\tilde{l}_1,t)\mathcal{U}(\tilde{l}_1)), \tag{13}$$

where the following constants have been introduced:

$$v=\delta l/\delta t \quad\text{and}\quad D_n=\delta l^2/(2n^2\delta t), \tag{14}$$

$v/2$ $(-v/2)$ is the relative growth velocity of needle 0 (1), and D_2 is the ''diffusion'' coefficient of the sticking between the two needles. At short time the particles stick at random on both needle tips, up to the moment when one needle gives way to the other. The function $\mathcal{U}$ characterizes the screening effect,

$$\mathcal{U}(\tilde{l}_1)=[u(\tilde{l}_1)-1]/[u(\tilde{l}_1)+1], \tag{15}$$

where $u(\tilde{l}_1)$ is implicitly defined by Eq. (11). Equations (13) represent the diffusion of a ''particle'' with coordinate $\tilde{l}_1$ in a potential

$$V(\tilde{l}_1)=-\frac{v}{2}\int\mathcal{U}(\tilde{l}_1)d\tilde{l}_1. \tag{16}$$

If the needles are equidistant, $\alpha_0=\alpha_1=1$ and $u(\tilde{l}_1)=\exp\tilde{l}_1$, we have explicitly $\mathcal{U}(\tilde{l}_1)=\tanh(\tilde{l}_1/2)$. The potential (shown in Fig. 2) is then explicitly

$$V(\tilde{l}_1)=-\frac{v}{2}\ln[\cosh(\tilde{l}_1/2)]. \tag{17}$$

The time evolution of the solution of Eq. (13) for needles of equal length $(\alpha_0=\alpha_1=1)$ at $t=0$, is shown in Fig. 3.

Fokker-Planck equation for the growth of a comb of needles. We consider now a comb of equidistant needles $(\alpha_i\equiv 2/n$ for all $i)$. In this case it is convenient to use the Fourier transform of the distribution of lengths. To avoid the problems due to the zero eigenvalue of $\mathbf{C}$ (6), the completely

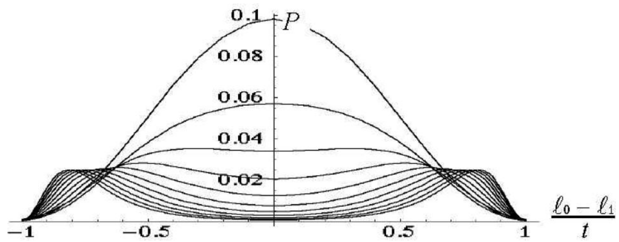

FIG. 3. Probability distribution of $P(\tilde{l}_1,t)$ at times $t/\delta t = 20,30,\ldots,120$.

symmetrical components $k=0$ (corresponding to $\tilde{l}_0$) are excluded and we use an index r to specify these reduced vectors,

$$\vec{\tilde{V}}_r = \mathbf{F}_r \cdot \vec{V} \quad \text{with}$$

$$\mathbf{F}_r = \{\mathbf{F}_{\tilde{k}h} = (1/n)\exp(2\pi i\tilde{k}h/n)\}_{1\leq\tilde{k}\leq n-1, 0\leq h\leq n-1}. \tag{18}$$

In particular, we note $\vec{L}=\{l_i\}_{0\leq i\leq n-1}$, $\overrightarrow{\delta L}[h]=\delta l\{\delta_{ih} - 1/n\}_{0\leq i\leq n-1}$ [δ_{ih} is the Kronecker symbol, and $\overrightarrow{\delta L}[h]$ satisfies condition (7)]. $P(\vec{\tilde{L}}_r,t)$ is the probability to find a given vector of Fourier modes $\vec{\tilde{L}}_r=\{\tilde{l}_i\}_{1\leq i\leq n-1}=\mathbf{F}_r\cdot\vec{L}$, and we have to build the Fokker-Planck equation for $P(\vec{\tilde{L}}_r,t)$. With probability $p_h(\vec{\tilde{L}}_r)$ we add a particle of size δl on needle h. This probability may be expanded on the basis of modes $\tilde{k}$,

$$p_h(\vec{\tilde{L}}_r) = \sum_k (\mathbf{F}_r^{-1})_{h\tilde{k}} p_{\tilde{k}}(\vec{\tilde{L}}_r), \tag{19}$$

$p_{\tilde{k}}(\vec{\tilde{L}}_r)$ being the probability to add $\overrightarrow{\delta \tilde{L}}_r[\tilde{k}]=\Sigma_h(\mathbf{F}_r)_{\tilde{k}h} \overrightarrow{\delta L}[h]$ to $\vec{\tilde{L}}_r$. This leads to the following Fokker-Planck equation generalizing Eq. (13):

$$\frac{\partial P(\vec{\tilde{L}}_r,t)}{\partial t} = -\frac{\delta l}{\delta t}\sum_{\tilde{k}=1}^{n-1}\frac{\partial}{\partial \tilde{l}_k}[p_{\tilde{k}}(\vec{\tilde{L}}_r)P(\vec{\tilde{L}}_r,t)] + \frac{\delta l^2}{2n^2\delta t}\sum_{k'=1}^{n-1}\frac{\partial^2 P(\vec{\tilde{L}}_r,t)}{\partial\tilde{l}_{k'}\partial\tilde{l}_{-k'}} \tag{20}$$

(note that $\tilde{l}_{-k}\equiv\tilde{l}_{n-k}=\tilde{l}_k^*$, k being defined modulo n). All the difficulty is now in the determination of $p_{\tilde{k}}(\vec{\tilde{L}}_r)$. For $n>2$, the simplest approach is to consider the linear approximation which already provides the *Laplacian* instability behavior of the initial growth.

The initial conditions: when the needles have an equal length at $t=0$, the initial angles ϕ_i and θ_j ($0\leq i,j\leq n-1$) are also regularly spaced,

$$\phi_i(0)=2\pi i/n \quad \text{and} \quad \theta_j(0)=\pi(2j-1)/n. \tag{21}$$

We first calculate $\overrightarrow{\delta\theta}(0)$ and $\overrightarrow{\delta\varphi}(0)$ corresponding to a variation $\overrightarrow{\delta L}(0)$, for instance the addition of a particle on needle h. Then the sticking probability $p_i[h]$ of a new particle on needle i, while a particle of size δl was already stuck on needle h, and finally the sticking probability $p\tilde{k}[\tilde{h}]$ of a particle in mode $\tilde{k}$, while a particle of size δl was already stuck in mode $\tilde{h}$. We find

$$p_{\tilde{k}}[\tilde{h}] = \frac{\delta_{k0}}{n} + \delta_{kh}\lambda_k\delta l, \quad \lambda_k = \frac{k}{n}\left(1-\frac{k}{n}\right). \tag{22}$$

Linearized kinetic equation of a comb: Let $P_{lin}(\vec{L},t)$ be the linearized probability to find a set of needles of size $\vec{L}$. Linearization supposes that the lengths l_i are not too different in such a way that no needle can be completely screened by the others. The sticking probabilities on $\vec{L}$ are then the superposition of the individual probabilities $p^i[l_h]$, to add a new particle on needle i while needle h has already a length l_h and is deduced from $p_i[h]$ by replacing δl by l_h (superposition rule).

The Fokker-Planck equation in the linear regime is then [with the notations of Eq. (14)],

$$\frac{\partial P_{lin}(\vec{\tilde{L}}_r,t)}{\partial t} = -v\sum_{k=1}^{n-1}\frac{\partial}{\partial\tilde{l}_k}[\lambda_k\tilde{l}_k P_{lin}(\vec{\tilde{L}}_r,t)] + D_n\sum_{k'=1}^{n-1}\frac{\partial^2 P_{lin}(\vec{\tilde{L}}_r,t)}{\partial\tilde{l}_{k'}\partial\tilde{l}_{-k'}}. \tag{23}$$

When $n=2$, we recover Eq. (13) linearized with $\alpha_0=\alpha_1=1$, and $\mathcal{U}(\tilde{l}_1)\sim\tilde{l}_1/2$ from Eqs. (11) and (15).

Correlation between modes and fluctuation of a mode q in the initial regime: The correlation between modes,

$$\langle\tilde{l}_{q_1}\tilde{l}_{q_2}\rangle(t) = \int_{-\infty}^{+\infty}\tilde{l}_{q_1}\tilde{l}_{q_2}P(\vec{\tilde{L}}_r,t)d\vec{\tilde{L}}_r \tag{24}$$

can be explicitly obtained from Eq. (23) in the linear regime,

$$\langle\tilde{l}_{q_1}\tilde{l}_{q_2}\rangle_{lin} = \frac{D_n\delta_{q_1+q_2,n}}{v_{\text{eff}}(q_1,q_2)}\{\exp[v_{\text{eff}}(q_1,q_2)t]-1\} \tag{25}$$

with an effective velocity $v_{\text{eff}}(q_1,q_2)=(\delta l/\delta t)(\lambda_{q_1}+\lambda_{q_2})$, while the fluctuation of a mode q is in the linear regime,

$$\langle\tilde{l}_q^2\rangle_{lin} = \frac{2D_n\delta_{q,n/2}}{v_{\text{eff}}(q,q)}\{\exp[v_{\text{eff}}(q,q)t]-1\}. \tag{26}$$

Screening time $t^*(n,\delta l)$, crossover to nonlinearity and scaling: Starting from a periodic array of two needles with an initial distribution [from (9)], $\tilde{l}_1(t=0)=\Delta_0/2$, Krug *et al.* [4] suggested using scaling arguments that $t^*\sim a\ln(a/\Delta_0)$, where a is the needle spacing. In our case $a=2\pi/n$ and all scaling functions will only depend on the ratio $\delta l/a = n\delta l/(2\pi)$. To determine this screening time from the above considerations, we choose some arbitrary positive real number l_{crit} which is expected to be of order of the distance $2\pi/n$ between the needles, and we introduce the "stopping time" t_{crit}

$$t_{crit}=\inf\left\{t\geqslant 0,\left|\vec{L}_r\right|=l_{crit}=\frac{\alpha}{n}\right\}, \tag{27}$$

where $|\cdot|$ is the standard Euclidean norm in $\mathbb{C}^{n-1}$: $|\vec{L}_r|^2=\Sigma_{k=1}^{n-1}|\tilde{l}_k|^2$. If n is even, then we introduce $X_k=\mathrm{Re}(\tilde{l}_k)_{k\leqslant n/2}, X_{n-k}=\mathrm{Im}(\tilde{l}_k)_{k<n/2}$. In the linear regime the process $\vec{X}$ satisfies an equation similar to Eq. (23), or equivalently a system of independent stochastic differential equations (Langevin equations),

$$dX_k(t)=v\lambda_k X_k(t)dt+\sqrt{D_n}dW_k(t), \tag{28}$$

where $\vec{W}$ is a $(n-1)$-dimensional Brownian motion. When $n\gg 1$, $n\delta l\ll 1$, we have

$$t_{crit}=\frac{\delta t}{\delta l}[2\ln M+\ln(\ln M)-\ln\pi+o(1)], \tag{29}$$

where $M=\alpha^2/(n\delta l)$. We get the distribution of $\vec{L}$ at time t_{crit} from the Fourier transform of Eq. (25) ($\overline{l}=0$):

$$\langle l_j l_{j+j'}\rangle_{lin}=\frac{l_{crit}^2}{n^2}(-1)^{j'}\exp\left(-\frac{\pi^2 j'^2}{4\ln n}\right). \tag{30}$$

A periodic modulation with a doubled period is growing, which is characterized by a factor $(-1)^{j'}$ to which a slow random modulation of width $\sqrt{\ln n}$ is superimposed. So, t_{crit} can be considered as the crossover time $t^*(n)$ between the linear and nonlinear regime. If we consider that the density of needles $\rho(y)$ at distance y from the initial distribution is divided by a factor 2 at each critical time t^*, after p period doubling,

$$y\simeq\frac{\delta l}{\delta t}\left[\frac{1}{n}t^*(n)+\frac{2}{n}t^*\left(\frac{n}{2}\right)+\cdots+\frac{2^p}{n}t^*\left(\frac{n}{2^p}\right)\right]$$

for which $\rho(y)=n/2^p$. From Eq. (29) $y(\rho)\sim(1/\rho)[\ln\rho+\frac{1}{2}\ln(\ln\rho)]$ for $\rho\ll n$, so

$$\rho(y)\sim\frac{1}{y}[\ln y-\tfrac{1}{2}\ln(\ln y)]. \tag{31}$$

This scaling, derived analytically from crossover considerations between linear and nonlinear regimes, agrees with the one suggested by Krug *et al.* [4], using numerical and heuristic considerations.

Mode coupling. To make progress in the nonlinear regime, a second step can be the determination of the Fokker-Planck equation when the dominant mode $k=n/2$ in equation (23) has reached this nonlinear regime. The Fokker-Planck equation coupling nonlinearly the modes n and $n/2$ is (b_k and c_k have been explicitly determined),

$$\begin{aligned}\frac{\partial P(\tilde{\vec{L}}_r,t)}{\partial t}=-\frac{\delta l}{\delta t}\Bigg\{&\frac{\partial}{\partial\tilde{l}_{n/2}}\left[\tanh\left(\frac{n\tilde{l}_{n/2}}{4}\right)P(\tilde{\vec{L}}_r,t)\right]\\&-\sum_{k\neq n/2}\frac{\partial}{\partial\tilde{l}_k}\{[b_k(\tilde{l}_{n/2})\tilde{l}_k\\&+c_k(\tilde{l}_{n/2})\tilde{l}_{k+n/2}]P(\tilde{\vec{L}}_r,t)\}\\&+\frac{\delta l^2}{2n^2\delta t}\sum_{k'=1}^{n-1}\frac{\partial^2 P(\tilde{\vec{L}}_r,t)}{\partial\tilde{l}_{k'}\partial\tilde{l}_{-k'}}\Bigg\}.\end{aligned} \tag{32}$$

In the limit $\tilde{l}_{n/2}\rightarrow 0$, $b_k[0]=\lambda_k, c_k[0]=0$, we recover Eq. (23) while in the limit $\tilde{l}_{n/2}\rightarrow\infty$ it reduces to four equivalent equations, similar to Eq. (23), but now for a problem with $n/2$ needles. When $n=2$, it is identical with Eq. (13) when $\alpha_0=\alpha_1$. Equation (32) may give an idea of the expected structure of the Fokker-Planck equation in the nonlinear regime. This will be discussed in a more detailed paper.

In conclusion, we have shown that a Fokker-Planck approach, a way which had never been previously exploited, could provide an interesting enlightenment of diffusion-limited aggregation at least in the simpler case of parallel needles. We hope it could open new prospects in the field of Laplacian growth.

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