Effective fractional acoustic wave equations in random multiscale media

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Wave propagation in a one-dimensional random medium with short- or long-range correlations is analyzed. Multiple scattering is studied in the regime where the fluctuations of the medium parameters are small and the propagation distance is large. In this regime pulse propagation is characterized by a random time shift described in terms of a standard or fractional Brownian motion and a deterministic deformation described by a pseudo-differential operator. This operator is characterized by a frequency-dependent attenuation that obeys a power law with an exponent ranging from 0 to 2. The exponent is between 1 and 2 for a long-wavelength pulse and it is determined by the power decay rate at infinity of the autocorrelation function of the random medium parameters. The exponent is between 0 and 1 for a short-wavelength pulse and it is determined by the power decay rate at zero of the autocorrelation function of the random medium parameters. This frequency-dependent attenuation is associated with a frequency-dependent phase responsible for dispersion, which ensures causality and that the Kramers-Kronig relation is satisfied. In the time domain the effective wave equation has the form of a linear integro-differential equation with a fractional derivative.

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I. INTRODUCTION

Frequency-dependent attenuation has been observed in a wide range of applications in acoustics[4,34], and also in other domains, such as seismic wave propagation[7,8,35]. Experimental observations show that the attenuation of acoustic waves has a frequency dependence of the form $E = E_0 \exp(-\alpha(\omega)z)$, where E denote the amplitude of an acoustic variable such as velocity or pressure and ω is the frequency. The damping coefficient has been seen to obey the empirical power law

$$\alpha(\omega) = \alpha_0 |\omega|^y, \tag{1}$$

where $\alpha_0 \in (0,\infty)$ and $y \in (0,2)$ are parameters that are characteristic of the medium and obtained through a fitting of measured data[1,19,22,23]. Different wave equation models have been proposed to reproduce such a power law[9,10,18,21,33,34,36]. One of the problems discussed in these papers is to obtain a causal wave equation in the space-time domain that reproduces the power law. Another problem is to relate such an equation to first principles in physics. In our paper we propose a derivation from first principles of an effective equation that exhibits a frequency-dependent attenuation with a power law, and we show that this attenuation is accompanied by a frequency-dependent phase that ensures the causality of the associated approximation and the automatic satisfaction of the Kramers-Kronig relations. The physical model is a one-dimensional acoustic wave equation in a random medium that exhibits short- or long-range correlations. The basic phenomenon is multiple scattering responsible for an effective attenuation and dispersion. The relation between the microscopic statistics of the random medium and the parameters of the frequency-dependent attenuation power law is given and discussed. The effective wave equation has the form of a partial differential equation with special fractional derivatives.

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II. ACOUSTIC WAVE PROPAGATION IN RANDOM MEDIA

A. Acoustic Wave Equations

We develop an asymptotic probabilistic theory for the acoustic wave equations in the presence of random fluctuations of the medium with short- or long-range correlations. The one-dimensional acoustic wave equations are given by

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0\,, \tag{2}$$
$$\frac{1}{K(z)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} = 0\,, \tag{3}$$

where p is the pressure and u is the velocity. For simplicity we assume that the density of the medium ρ is a constant equal to ρ_0. The bulk modulus of the medium K is assumed to be randomly varying in the region $z \in [0, L]$ and we consider the weakly heterogeneous regime[15,26], in which the fluctuations of the bulk modulus are small and rapid (compared to the propagation distance):

$$\frac{1}{K(z)} = \begin{cases} \frac{1}{K_0}\big(1+\varepsilon\nu\big(\frac{z}{\varepsilon^2}\big)\big), & z \in [0, L]\,, \\ \frac{1}{K_0}, & z \in (-\infty, 0) \cup (L, \infty)\,, \end{cases}$$
$$\rho(z) = \rho_0 \quad \text{for all} \quad z\,.$$

Here, the dimensionless parameter ε is small and it characterizes the separation of scales. The effective impedance and speed of sound are $\zeta_0 = \sqrt{K_0\rho_0}$ and $c_0 = \sqrt{K_0/\rho_0}$, respectively. The source located at $z_0 < 0$ emits a pulse at time z_0/c_0. This pulse is impinging on the section $[0, L]$ and hits the boundary at 0 at time 0.

The random process ν is assumed to be stationary and to have mean zero. Its autocorrelation function is denoted by

$$\phi(z) := \mathbb{E}[\nu(x)\nu(x+z)]\,, \tag{4}$$

where $\mathbb{E}$ stands for the expectation with respect to the distribution of the random medium. The function $\phi(z)$ is bounded, even, and maximal at zero ($\phi(0)$ is the variance of the fluctuations). Its local properties at zero and its asymptotic behavior at infinity characterize the short- and long-range correlations of the random medium, as we discuss in the next subsection.

B. Random Medium Properties

Wave propagation in random media is usually studied with a driving process ν that has mixing properties. This means that the random values $\nu(x+z)$ and $\nu(x)$ taken at two points separated by the distance z become rapidly uncorrelated when $z \to \infty$. In other words the autocorrelation function $\phi(z)$ decays rapidly to zero as $z \to \infty$. More precisely we say that the random process ν is mixing if its autocorrelation function decays fast enough at infinity so that it is absolutely integrable:

$$\int_0^\infty |\phi(z)| dz < \infty\,. \tag{5}$$

This is the usual assumption for random media, under which the theory is well established. In this case the correlation length can be defined as $l_c = 2\int_0^\infty \phi(z)dz/\phi(0)$. The standard O'Doherty-Anstey (ODA) theory describes the propagating pulse in this regime. The effective equation for the wave front has been obtained by several authors[5,6,11,15,27,32]. The pulse propagation is characterized by a random time shift and a deterministic spreading. The random time shift is described in terms of a standard Brownian motion, while the deterministic spreading is described by a pseudo-differential operator that we will describe in Section III. If, additionally, the correlation length of the medium is smaller than the typical wavelength, then the pseudo-differential operator can be reduced to a second-order diffusion[15].

Wave propagation in multiscale and rough media, with short- or long-range fluctuations, has recently attracted a lot of attention, as more and more data collected in real environments confirm that this situation can be encountered in many different contexts, such as in geophysics[12] or in laser beam propagation through the atmosphere[13,16,31].

Qualitatively, the long-range correlation property means that the random process has long memory (in contrast with a mixing process). This means that the correlation degree between the random values $\nu(x+z)$ and $\nu(x)$ taken at two points separated by the distance z is not completely negligible when $z \to \infty$. It corresponds to the fact that the autocorrelation function has a slow decay at infinity. More precisely we say that the random process ν has the H-long-range correlation property if its autocorrelation function satisfies:

$$\phi(z) \overset{|z|\to\infty}{\simeq} r_H \Big|\frac{z}{l_c}\Big|^{2H-2}\,, \tag{6}$$

where $r_H > 0$ and $H \in (1/2, 1)$. Here the correlation length l_c is the critical length scale beyond which the power law behavior (6) is valid. Note that the autocorrelation function is not integrable since $2H-2 \in (-1,0)$, which means that a random process with the H-long-range correlation property is not mixing.

Qualitatively the short-range correlation property means that the random process is rough at small scales. This means that the correlation degree between the random values $\nu(x+z)$ and $\nu(x)$ taken at two points separated by the distance z has a sharp decay at zero. It corresponds to the fact that the autocorrelation function decays faster than an affine function at zero. More precisely we say that the random process ν has the H-short-range correlation property if its autocorrelation function satisfies:

$$\phi(z) \overset{|z|\to 0}{\simeq} \phi(0)\Big(1 - d_H\Big|\frac{z}{l_c}\Big|^{2H} + O\Big(\Big|\frac{z}{l_c}\Big|\Big)\Big)\,, \tag{7}$$

where $d_H > 0$ and $H \in (0, 1/2)$. Here the correlation length l_c is the critical length scale below which the power law behavior (7) is valid. For technical reasons we also assume that $\phi(z)$ is continuously differentiable and piecewise twice continuously differentiable on $(0,\infty)$ (note that $\phi'(z)$ blows up at 0^+); that $\phi(z)$, $\phi'(z)$ and $\phi''(z)$ are absolutely integrable at infinity, say on (l_c,∞); and that $\phi''(z)+2H(2H-1)\phi(0)d_H|z/l_c|^{2H-2}/l_c^2$ is absolutely integrable at zero, say on $(0,l_c)$. Note that the expansion (7) is the key hypothesis for the H-short-range correlation property and that the additional technical hypotheses allow us to get the desired result in a general context. However, these technical hypotheses are not absolutely necessary, and we will show that the desired result can be obtained without them on a particular example.

C. Random Medium Models with Short- or Long-range Correlations

In this section we present random processes ν that satisfy the conditions that we have imposed on the medium fluctuations. The first model we may think of is based on the fractional Brownian motion, that is known to have special properties in terms of long-range dependence and in terms of roughness of its trajectories. We remind the reader that the fractional Brownian motion $W_H(z)$ with Hurst index $H \in (0,1)$ is the Gaussian process with mean zero and covariance:

$$\mathbb{E}[W_H(x)W_H(x+z)] = \frac{1}{2}\big(|x+z|^{2H} + |x|^{2H} - |z|^{2H}\big)\,. \tag{8}$$

It is a self-similar process:

$$W_H(az) \overset{dist.}{\sim} a^H W_H(z) \quad \text{for all } a>0\,, \tag{9}$$

with stationary increments:

$$\mathbb{E}\big[(W_H(x+z)-W_H(x))^2\big] = |z|^{2H}\,. \tag{10}$$

For $H = 1/2$ it is the standard Brownian motion which has independent increments and Hölder continuous trajectories of any order strictly less than $1/2$.
For $H < 1/2$ it is a random process with negatively-correlated increments (which is a type of short-range correlation property). The realizations are Hölder continuous trajectories of any order strictly less than H, which means that they are more irregular than the trajectories of a Brownian motion.
For $H > 1/2$ it is a random process with positively-correlated increments (which is a type of long-range correlation property). The realizations are Hölder continuous trajectories of any order strictly less than H, which means that they are more regular than the trajectories of a Brownian motion.
However, the fractional Brownian motion is not stationary. We will first introduce two random processes that have local properties similar to the ones of the fractional Brownian motion but are stationary. We will also introduce two binary medium models that have either the long- or short-range correlation property.

Fractional Ornstein Uhlenbeck medium. The fractional Ornstein Uhlenbeck (OU) process $\nu(z)$ is defined by

$$\nu(z) := \frac{\sigma}{\sqrt{H\Gamma(2H)}l_c^H}\Big[W_H(z) - \frac{1}{l_c}\int_{-\infty}^{z} e^{\frac{x-z}{l_c}} W_H(x)dx\Big]\,, \tag{11}$$

where W_H is a fractional Brownian motion with Hurst index $H \in (0,1)$. The fractional OU process is in fact a fractional Brownian motion with a restoring force towards zero. The fractional OU process is a zero-mean, stationary, Gaussian process. Its variance is σ^2 and its autocorrelation function is given by

$$\phi(z) = \frac{\sigma^2}{H\Gamma(2H)l_c^{2H}}\Big[\frac{1}{4l_c}\int_{-\infty}^{\infty} e^{-\frac{|x|}{l_c}}|z+x|^{2H}dx - \frac{1}{2}|z|^{2H}\Big]\,. \tag{12}$$

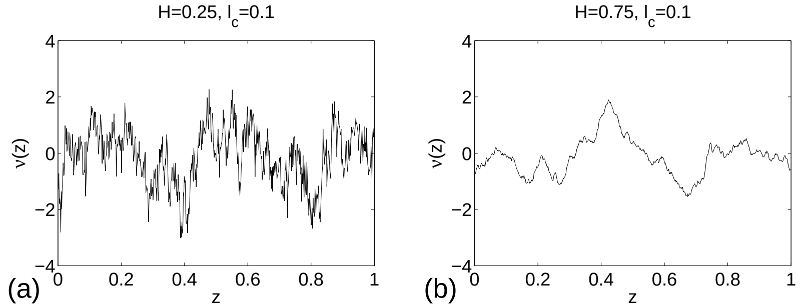

FIG. 1. Realizations of the fractional OU process with Hurst index H and correlation length l_c. The trajectories are more regular when H is larger.

If $H = 1/2$, then the standard OU process (synthesized with a standard Brownian motion) is a stationary Gaussian Markov process. Its autocorrelation function is

$$\phi(z) = \sigma^2 \exp\Big(- \frac{|z|}{l_c}\Big) , \tag{13}$$

which shows that it is a mixing process.

If $H \in (1/2, 1)$ then the fractional OU process has the H-long-range correlation property since its autocorrelation function $\phi(z)$ satisfies (6) with

$$r_H = \frac{\sigma^2 (2H-1)}{\Gamma(2H)} . \tag{14}$$

If $H \in (0, 1/2)$, then the fractional OU process has the H-short-range correlation property since its autocorrelation function $\phi(z)$ satisfies (7) with

$$d_H = \frac{1}{2H\Gamma(2H)} . \tag{15}$$

Moreover, $\phi(z)$ is infinitely differentiable over $(0,\infty)$; $\phi(z)$, resp. $\phi'(z)$, $\phi''(z)$, decays as $|z|^{2H-2}$, resp. $|z|^{2H-3}$, $|z|^{2H-4}$, at infinity and is absolutely integrable at infinity; $\phi''(z) + \phi(0) d_H 2H(2H-1)|z/l_c|^{2H-2}/l_c^2$ converges to $\phi(0)/l_c^2$ as $z \to 0$, so it is absolutely integrable at 0.

It is possible to simulate paths of the fractional OU process using the Cholesky method (see Figure 1) or other well-known methods[3].

Fractional white noise medium. As a second example we consider the model

$$\nu(z) := \frac{\sigma}{l_c^H} \Big[W_H(z) - W_H(z+l_c) \Big] , \tag{16}$$

where W_H is a fractional Brownian motion with Hurst index $H \in (0, 1)$. The fractional white noise is a zero-mean, stationary, Gaussian process. Its variance is σ^2 and its autocorrelation function is given by

$$\phi(z) = \frac{\sigma^2}{2 l_c^{2H}} \big(|z+l_c|^{2H} + |z-l_c|^{2H} - 2|z|^{2H} \big) . \tag{17}$$

If $H = 1/2$, then the standard white noise process (synthesized with a standard Brownian motion) is a stationary Gaussian process. Its autocorrelation function is

$$\phi(z) = \sigma^2 \Big(1 - \frac{|z|}{l_c}\Big) \mathbf{1}_{[0,l_c]}(|z|) , \tag{18}$$

which shows that it is a mixing process.

If $H \in (1/2, 1)$ then the fractional white noise has the H-long-range correlation property since its autocorrelation function $\phi(z)$ satisfies (6) with

$$r_H = \sigma^2 H(2H-1) . \tag{19}$$

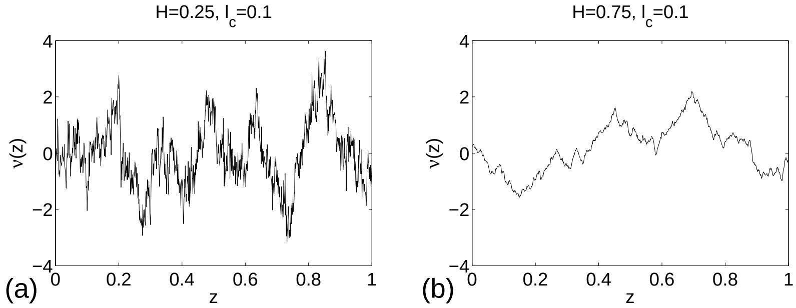

FIG. 2. Realizations of the fractional white noise with Hurst index H and correlation length l_c. The trajectories are more regular when H is larger.

If $H \in (0, 1/2)$, then the fractional white noise "almost" has the H-short-range correlation property. Indeed its autocorrelation function $\phi(z)$ satisfies (7) with

$$d_H = 1\,, \tag{20}$$

but the function $\phi(z)$ is not continuously differentiable over $(0, \infty)$ since $\phi'(z)$ has a singularity at $z = l_c$. This gives rise to a slightly different result than the general result stated in Subsection III.F, and we will analyze it in detail in Appendix C.

Note that the typical trajectories of the fractional white noise are very similar to the ones of the fractional OU process (compare Figures 1 and 2).

Binary medium with mixing property. Here we construct a process corresponding to a binary medium so that the process ν is stepwise constant and takes values $\pm\sigma$ over intervals with random lengths. We denote by $(l_j)_{j\geq 0}$ the lengths of these intervals and by $(n_j)_{j\geq 0}$ the values taken by the process over each elementary interval. The process $\nu(z)$ is defined by

$$\nu(z) := n_{N_z} \text{ where } N_z = \sup\{n \geq 0,\, L_n \leq z\}\,, \tag{21}$$

where $L_0 = 0$ and $L_{n+1} = L_n + l_n$. The random variables n_j are independent and identically distributed with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2}\,. \tag{22}$$

The random variables l_j are independent and identically distributed with the exponential distribution whose probability density function (pdf) is

$$p_{l_1}(z) = \frac{1}{l_c}\exp\Big(-\frac{z}{l_c}\Big)\mathbf{1}_{[0,\infty)}(z)\,. \tag{23}$$

Note that it is very easy to simulate the random variable l_1, since $-l_c \ln U$ has the pdf (23) if U is uniformly distributed over $[0, 1]$. The random process $\nu(z)$ is a stationary jump Markov process and its autocorrelation function is

$$\phi(z) = \sigma^2 \exp\Big(-\frac{|z|}{l_c}\Big)\,, \tag{24}$$

which shows that it is a mixing process. Note that the binary medium process has the same autocorrelation function as the standard OU process, although these two random processes are very different (the first one is a jump process that takes only two values, the second one is a continuous process). It is of course known that the autocorrelation function is not sufficient to characterize the statistics of a random medium, but we will see that it is sufficient to characterize wave propagation in a random medium.

Binary medium with long-range correlations. The long-range correlation property for a binary medium corresponds to the existence of intervals much longer than the average interval length.

We again consider the process (21) corresponding to a binary medium where the random variables n_j are independent and identically distributed with the distribution (22) and the random variables l_j are independent and identically distributed with the distribution with the pdf

$$p_{l_1}(z) = (3-2H)\frac{l_c^{3-2H}}{z^{4-2H}}\mathbf{1}_{[l_c,\infty)}(z)\,, \tag{25}$$

where $H \in (1/2, 1)$. Note that it is very easy to simulate the random variable l_1, since $l_c U^{-1/(3-2H)}$ has the pdf (25) if U is uniformly distributed over $[0, 1]$. The average length of the random interval is

$$\mathbb{E}[l_1] = \frac{3-2H}{2-2H} l_c \,, \tag{26}$$

while the variance of l_1 is infinite. A salient aspect of this model is that very long intervals (i.e. much longer than $\mathbb{E}[l_1]$) can be generated, which are responsible for the infinite variance of the length of the interval and for the long-range correlation property of the random medium. The process ν is bounded, it has mean zero and variance σ^2, but it is not stationary. However, using renewal theory[14], the distribution of the process $(\nu(x+z))_{z\geq 0}$ converges to a stationary distribution when $x \to \infty$ and the autocorrelation function of ν satisfies

$$\mathbb{E}[\nu(x)\nu(x+z)] \stackrel{x\to\infty}{\longrightarrow} \phi(z) = \sigma^2 \int_z^\infty \frac{\mathbb{P}(l_1 > s)}{\mathbb{E}[l_1]} ds \,,$$

The autocorrelation function

$$\phi(z) = \sigma^2 \Big[\frac{1}{3-2H} \frac{l_c^{2-2H}}{|z|^{2-2H}} \mathbf{1}_{[l_c,\infty)}(|z|) + \Big(1 - \frac{2-2H}{3-2H} \frac{|z|}{l_c} \Big) \mathbf{1}_{[0,l_c)}(|z|) \Big] \,, \tag{27}$$

satisfies the H-long-range correlation property (6) with $r_H = \sigma^2/(3-2H)$.

It is also possible to make the process stationary by simply modifying the statistical distribution of the length of the first interval: If the random lengths $(l_j)_{j\geq 1}$ are independent and identically distributed according to the distribution with the pdf (25), and if l_0 is independent of the $(l_j)_{j\geq 1}$ and has the distribution with the pdf:

$$\begin{aligned} p_{l_0}(z) &= \frac{\mathbb{P}(l_1 > z)}{\mathbb{E}[l_1]} \\ &= \frac{2-2H}{3-2H} \frac{1}{l_c} \mathbf{1}_{[0,l_c)}(z) + \frac{2-2H}{3-2H} \frac{l_c^{2-2H}}{z^{3-2H}} \mathbf{1}_{[l_c,\infty)}(z) \,, \end{aligned} \tag{28}$$

then the process ν is bounded, zero-mean and stationary, and it has the H-long-range correlation property since its autocorrelation function $\mathbb{E}[\nu(x)\nu(x+z)]$ is (27) for any x.

Binary medium with short-range correlations. The short-range correlation property for a binary medium corresponds to the accumulation of intervals with lengths much smaller than the average length.

We again consider the process (21) corresponding to a binary medium where the random variables n_j are independent and identically distributed with the distribution (22) and the random variables l_j are independent and identically distributed with the distribution with the pdf

$$p_{l_1}(z) = \frac{1-2H}{(l_i/l_c)^{2H-1} - 1} \frac{l_c^{1-2H}}{z^{2-2H}} \mathbf{1}_{[l_i,l_c]}(z) \,, \tag{29}$$

where $H \in (0, 1/2)$ and $0 < l_i < l_c$. Here the inner scale l_i is introduced in order to obtain a well-defined and normalized pdf and it will be taken to be much smaller than the correlation length l_c. Note that it is very easy to simulate the random variable l_1, since $l_i[1-(1-(l_i/l_c)^{1-2H})U]^{-1/(1-2H)}$ has the pdf (29) if U is uniformly distributed over $[0, 1]$. The average length of the random interval is

$$\mathbb{E}[l_1] = \frac{1-2H}{2H} \frac{l_c^{2H} - l_i^{2H}}{l_i^{2H-1} - l_c^{2H-1}} \,, \tag{30}$$

and its variance is finite. A salient aspect of this model is that it exhibits an accumulation of very small intervals (i.e. much smaller than $\mathbb{E}[l_1]$) which corresponds to very rapid changes in the medium properties. Using renewal theory the distribution of the process $(\nu(x+z))_{z\geq 0}$ converges to a stationary distribution when $x \to \infty$ and the autocorrelation function of ν satisfies

$$\mathbb{E}[\nu(x)\nu(x+z)] \stackrel{x\to\infty}{\longrightarrow} \phi(z) \,, \tag{31}$$

where

$$\begin{aligned} \phi(z) &= \frac{\sigma^2}{1-(l_i/l_c)^{2H}} \Big(1 - \frac{1}{1-2H} \frac{|z|^{2H}}{l_c^{2H}} + \frac{2H}{1-2H} \frac{|z|}{l_c} \Big) \mathbf{1}_{(l_i,l_c]}(|z|) \\ &\quad + \sigma^2 \Big(1 - \frac{2H}{1-2H} \frac{\big((l_i/l_c)^{2H} - (l_i/l_c)\big)}{1-(l_i/l_c)^{2H}} \frac{|z|}{l_i} \Big) \mathbf{1}_{[0,l_i]}(|z|) \,. \end{aligned} \tag{32}$$

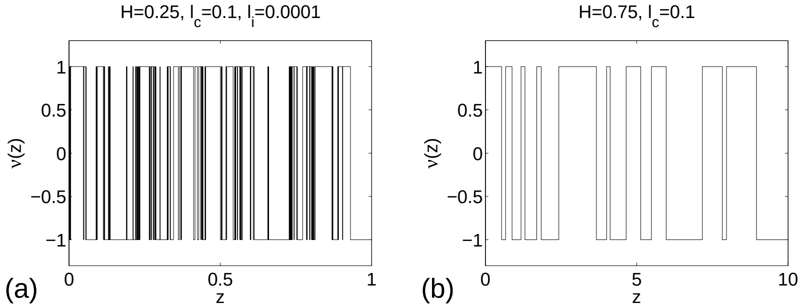

FIG. 3. Realizations of a binary medium with the index H. When $H < 1/2$ we can observe an accumulation of very small intervals responsible for the H-short-range correlation property (a). When $H > 1/2$ we can observe the existence of some intervals longer than the average responsible for the H-long-range correlation property (b).

If $l_i \ll l_c$, then

$$\phi(z) \simeq \sigma^2 \Big(1 - \frac{1}{1-2H} \frac{|z|^{2H}}{l_c^{2H}} + \frac{2H}{1-2H} \frac{|z|}{l_c} \Big) \mathbf{1}_{(0,l_c]}(|z|) \,, \tag{33}$$

which satisfies the H-short-range correlation property (7) with $d_H = 1/(1-2H)$. Note that ϕ is continuous and continuously differentiable over $(0,\infty)$, even at $z = l_c$ since $\phi(l_c^-) = 0 = \phi(l_c^+)$ and $\phi'(l_c^-) = 0 = \phi'(l_c^+)$, ϕ'' is piecewise twice continuously differentiable (there is a jump of ϕ'' at l_c) and ϕ, ϕ', and ϕ'' are absolutely integrable at infinity and $\phi''(z) + \phi(0) d_H 2H(2H-1)|z/l_c|^{2H-2}/l_c^2 = 0$ close to 0 so it is absolutely integrable at 0.

Note that the trajectories of the process $\nu(z)$ for a binary medium (see Figure 3) are very different from the ones of the fractional OU process or the fractional white noise. However, we will see that wave propagation in these random media can be described in the same effective way.

III. ANALYSIS OF THE WAVE FRONT

A. Mode Decomposition

We consider the right- and left-going waves defined in terms of the local impedance and moving with the local sound speed:

$$\begin{bmatrix} A^\varepsilon(t,z) \\ B^\varepsilon(t,z) \end{bmatrix} := \begin{bmatrix} \zeta^{\varepsilon\,-1/2}(z) p(t,z) + \zeta^{\varepsilon\,1/2}(z) u(t,z) \\ -\zeta^{\varepsilon\,-1/2}(z) p(t,z) + \zeta^{\varepsilon\,1/2}(z) u(t,z) \end{bmatrix} . \tag{34}$$

The local impedance and sound speed are

$$\zeta^\varepsilon(z) := \sqrt{K(z)\rho(z)} = \frac{\zeta_0}{\sqrt{1+\varepsilon\nu(z/\varepsilon^2)}} \,, \tag{35}$$

$$c^\varepsilon(z) := \sqrt{K(z)/\rho(z)} = \frac{c_0}{\sqrt{1+\varepsilon\nu(z/\varepsilon^2)}} \,. \tag{36}$$

The mode amplitudes satisfy

$$\frac{\partial}{\partial z} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\zeta^{\varepsilon\prime}(z)}{2\zeta^\varepsilon(z)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} . \tag{37}$$

This system is completed with an initial condition corresponding to a right-going pulse wave that is incoming from the homogeneous half-space $z < 0$ and is impinging on the random medium in $[0, L]$,

$$A^\varepsilon(t,z) = f\Big(\frac{t - z/c_0}{\varepsilon^2}\Big) \,, \qquad B^\varepsilon(t,z) = 0 \,, \qquad t < 0 \,. \tag{38}$$

Equation (37) clearly exhibits the two important aspects of the propagation mechanisms. The first term on the right describes transport along the random characteristics with the local sound speed $c^\varepsilon(z)$. The second term on the right describes coupling between the right- and left-going modes, which is proportional to the derivative of the logarithmic impedance. The asymptotic analysis of this system is given in the mixing case in Ref. 15 and in the long-range case in Ref. 17. The main steps are the following ones. We first perform a series of transformations to rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode. We then obtain an upper-triangular system that can be integrated more easily. In a second step we apply a limit theorem to this system to establish an effective equation for the wave front.

B. Wave Front Transmission

We now state the fundamental results that characterize the wave front transmitted through a random medium. Let us introduce the random travel time

$$\tau_0^\varepsilon(z) := \frac{z}{c_0} + \frac{\varepsilon}{2c_0}\int_0^z \nu\Big(\frac{x}{\varepsilon^2}\Big)\,dx\,. \tag{39}$$

1. If the random process is mixing or if it satisfies the H-long-range correlation property, $H \in (1/2, 1)$, then the wave front observed in the random frame moving with the random travel time

$$A^\varepsilon\left(\tau_0^\varepsilon(z) + \varepsilon^2\tau, z\right)\,, \qquad z > 0\,, \tag{40}$$

converges in distribution as $\varepsilon \to 0$ to the deterministic profile

$$a(\tau, z) := \frac{1}{2\pi}\int \exp\Big(-i\omega\tau - \frac{\gamma_c(\omega)\omega^2}{8c_0^2}z - i\frac{\gamma_s(\omega)\omega^2}{8c_0^2}z\Big)\hat{f}(\omega)d\omega\,, \tag{41}$$

where $\hat{f}(\omega)$ is the Fourier transform of the initial pulse and

$$\gamma_c(\omega) := 2\int_0^\infty \phi(z)\cos\Big(\frac{2\omega z}{c_0}\Big)dz\,, \tag{42}$$

$$\gamma_s(\omega) := 2\int_0^\infty \phi(z)\sin\Big(\frac{2\omega z}{c_0}\Big)dz\,. \tag{43}$$

2. If the random process is mixing then the expectation of the random travel time $\tau_0^\varepsilon(z)$ is z/c_0 and its variance is

$$\mathrm{Var}\big(\tau_0^\varepsilon(z)\big) = \frac{\varepsilon^4}{c_0^2}\frac{\gamma_c(0)}{4}z + o(\varepsilon^4)\,, \tag{44}$$

as $\varepsilon \to 0$. The random travel time $\tau_0^\varepsilon(z)$ has the distribution of

$$\frac{z}{c_0} + \frac{\varepsilon^2}{c_0}\frac{\sqrt{\gamma_c(0)}}{2}W(z) + o(\varepsilon^2)\,, \tag{45}$$

as $\varepsilon \to 0$, where $W(z)$ is a standard Brownian motion.

3. Under the H-long-range correlation property, $H \in (1/2, 1)$, the expectation of the random travel time $\tau_0^\varepsilon(z)$ is z/c_0 and its variance is

$$\mathrm{Var}\big(\tau_0^\varepsilon(z)\big) = \frac{\varepsilon^{2(3-2H)}l_c^{2-2H}}{c_0^2}\frac{r_H}{4H(2H-1)}z^{2H} + o(\varepsilon^{2(3-2H)})\,, \tag{46}$$

as $\varepsilon \to 0$. The random travel time $\tau_0^\varepsilon(z)$ has the distribution of

$$\frac{z}{c_0} + \frac{\varepsilon^{3-2H}l_c^{1-H}}{c_0}\sqrt{\frac{r_H}{4H(2H-1)}}W_H(z) + o(\varepsilon^{3-2H})\,, \tag{47}$$

as $\varepsilon \to 0$, where $W_H(z)$ is a fractional Brownian motion with Hurst index H.

The first point in the mixing case and the second point are classical results known as the O'Doherty-Anstey theory[28] and they can be found in Refs. 2, 6, 15. The third point was established in Ref. 26 for a certain class of subordinated Gaussian processes and in Ref. 17 for the processes under consideration in this paper. The first point in the long-range case was proved in Ref. 17.

We see from the first point that the frequency-dependent attenuation

$$\frac{\gamma_c(\omega)\omega^2}{8c_0^2}, \tag{48}$$

of the wave front in (41) is always nonnegative because $\gamma_c(\omega)$ is the power spectral density of the stationary fluctuations $\nu(z)$ of the random medium.

The term $\exp[-i\gamma_s(\omega)\omega^2 z/(8c_0^2)]$ in (41) is a frequency-dependent phase modulation and $\gamma_s(\omega)$ is conjugate to $\gamma_c(\omega)$. This shows that the transmitted wave front when centered with respect to the random travel time correction propagates in a dispersive effective medium with the frequency-dependent wavenumber given by

$$k(\omega) = \frac{\omega}{c_0} - \varepsilon^2 \frac{\gamma_s(\omega)\omega^2}{8c_0^2}, \tag{49}$$

up to smaller terms in ε.

The fundamental results stated above show that the transmitted wave front in the random medium is modified in two ways compared to propagation in a homogeneous one.

First, its arrival time at the end of the slab $z = L$ has a small random component. In the usual case in which ϕ is integrable, the random time shift is of order ε^2 and its statistical distribution is described in terms of a standard Brownian motion. In the long-range correlation case, the random time shift is of order ε^{3-2H} and its statistical distribution is described in terms of a fractional Brownian motion[26]. Remember, however, that the pulse width is of order ε^2, which means that the random time delay is large compared to the pulse width, moreover, it becomes relatively larger as H is closer to 1.

Second, if we observe the wave front near its random arrival time, then we see a pulse profile that, to leading order, is deterministic and is the original pulse shape convolved with a deterministic kernel that depends on the second-order statistics of the medium through the autocorrelation function of ν:

$$a(\tau, z) = [\mathcal{H}(\cdot, z) * f](\tau). \tag{50}$$

The convolution kernel is given by

$$\mathcal{H}(\tau, z) = \frac{1}{2\pi} \int \exp\Big(- i\omega\tau - \frac{\gamma_c(\omega)\omega^2}{8c_0^2} z - i\frac{\gamma_s(\omega)\omega^2}{8c_0^2} z\Big) d\omega. \tag{51}$$

We describe the effective pulse attenuation and dispersion in the next subsections.

C. Deterministic Pulse Deformation

In this section we analyze the main properties of the effective equation for the wave front: The important function affecting the dynamics is the Fourier transform (42-43) of the positive lag part of the autocorrelation function of the random fluctuations of the medium. We have stated that $A^\varepsilon(\tau_0^\varepsilon(z) + \varepsilon^2\tau, z)$ converges to a given by (41). By taking an inverse Fourier transform, it is possible to identify the partial differential equation (PDE) satisfied by a:

$$\frac{\partial a}{\partial z} = \mathcal{L}a, \tag{52}$$

where $\mathcal{L}$ is a pseudo-differential operator that describes the deterministic pulse deformation:

$$\mathcal{L} = \mathcal{L}_c + \mathcal{L}_s, \tag{53}$$

$$\int_{-\infty}^{\infty} \mathcal{L}_c a(\tau) e^{i\omega\tau} d\tau = -\frac{\gamma_c(\omega)\omega^2}{8c_0^2} \int_{-\infty}^{\infty} a(\tau) e^{i\omega\tau} d\tau, \tag{54}$$

$$\int_{-\infty}^{\infty} \mathcal{L}_s a(\tau) e^{i\omega\tau} d\tau = -\frac{i\gamma_s(\omega)\omega^2}{8c_0^2} \int_{-\infty}^{\infty} a(\tau) e^{i\omega\tau} d\tau. \tag{55}$$

The PDE (52) is completed with the initial condition $a(\tau, z = 0) = f(\tau)$.

The first qualitative property satisfied by the pseudo-differential operator $\mathcal{L}$ is that it preserves the causality. Indeed, in the time domain, we can write

$$\begin{aligned}\mathcal{L}a(\tau) &= \left[\frac{1}{8c_0}\phi\left(\frac{c_0\tau}{2}\right)\mathbf{1}_{[0,\infty)}(\tau)\right] * \left[\frac{\partial^2 a}{\partial \tau^2}(\tau)\right] \\ &= \frac{1}{8c_0}\int_0^\infty \phi\left(\frac{c_0 s}{2}\right)\frac{\partial^2 a}{\partial \tau^2}(\tau - s)ds.\end{aligned} \tag{56}$$

The indicator function $\mathbf{1}_{[0,\infty)}$ is essential to interpret correctly the convolution. If a is vanishing for $\tau < 0$, then $\mathcal{L}a$ is also vanishing for $\tau < 0$, which is a manifestation of causality.

The pseudo-differential operator also satisfies the Kramers-Kronig relations[24,25]. Indeed the function $\omega \to \omega^2[\gamma_c(\omega) + i\gamma_s(\omega)]$ is analytic in the upper complex half plane and vanishes as $|\omega| \to \infty$. Note that our effective equation is derived from first principles in physics, so that there is no surprise that causality and Kramers-Kronig relations are satisfied.

Equation (52) gives the effective evolution of the front wave $a(\tau, z)$ in the frame moving with the random travel time $\tau_0^\varepsilon(z)$. If we ignore the small random time shift and focus our attention to the deterministic pulse deformation, then we can write the effective equation in the form of a wave equation for the pressure in the original frame and at the scale $z = \varepsilon^2\xi$ and $t = \varepsilon^2\tau$ as:

$$\frac{\partial p}{\partial \xi} + \frac{1}{c_0}\frac{\partial p}{\partial \tau} = \varepsilon^2 \mathcal{L} p\,, \tag{57}$$

$$p(\tau, \xi = 0) = \zeta_0^{1/2} f(\tau)\,.$$

This PDE is valid up to $\xi = L/\varepsilon^2$ in the asymptotic framework $\varepsilon \to 0$ and it is equivalent to the following effective wave equation

$$\frac{\partial^2 p}{\partial \xi^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial \tau^2} = \frac{2\varepsilon^2}{c_0}\big(\partial_\tau \mathcal{L}\big)p\,, \tag{58}$$

$$p(\tau, \xi = 0) = \zeta_0^{1/2} f(\tau)\,, \quad \partial_\xi p(\tau, \xi = 0) = -\frac{\zeta_0^{1/2}}{c_0} f'(\tau)\,.$$

The pseudo-spectral operator $\mathcal{L}$ can be divided into two parts as (53). The first component $\mathcal{L}_c$, as pointed out in Subsection III.B after (48), is a frequency-dependent attenuation which can be interpreted as an effective diffusion operator. The second component $\mathcal{L}_s$ is an effective dispersion operator, since it preserves the energy.

D. Mixing Random Media

In this subsection we consider the case of a mixing random medium. We first assume that the typical wavenumber ω/c_0 of the input pulse is such that

$$\frac{|\omega| l_c}{c_0} \ll 1\,.$$

This condition means that the typical wavelength is longer than l_c and in this case we find that

$$\frac{\gamma_c(\omega)\omega^2}{c_0^2} = \frac{\gamma_c(0)\omega^2}{c_0^2}\,, \tag{59}$$

$$\frac{\gamma_s(\omega)\omega^2}{c_0^2} = 0\,, \tag{60}$$

which shows that we have an effective second-order diffusion (i.e. a quadratic frequency dependence of the attenuation of the form (1) with $y = 2$) and no effective dispersion.

We next consider the case of a mixing random medium with the linear decay behavior at zero:

$$\phi(z) = \phi(0)\Big(1 - d_{1/2}\frac{|z|}{l_c} + o\Big(\frac{|z|}{l_c}\Big)\Big)\,, \tag{61}$$

with $d_{1/2} \geq 0$. We also assume that ϕ is continuously differentiable and piecewise twice continuously differentiable, and that ϕ, ϕ', and ϕ'' are absolutely integrable over $(0, \infty)$. The linear decay of the autocorrelation function is typical of a Markov process, such as the standard OU process synthesized with a standard Brownian motion or the binary medium process in the case in which the lengths of the intervals have an exponential distribution. This behavior is in fact more general. For instance the linear decay rate (61) holds for the white noise process synthesized with a standard Brownian motion or the binary medium process in the case in which the lengths of the intervals have an arbitrary distribution with positive finite expectation. If we assume that the typical wavenumber ω/c_0 of the input pulse is such that

$$\frac{|\omega| l_c}{c_0} \gg 1\,,$$

which means that the typical wavelength is smaller than l_c, then we have (see Appendix A)

$$\frac{\gamma_c(\omega)\omega^2}{c_0^2} = \frac{\phi(0)d_{1/2}}{2l_c}, \tag{62}$$

$$\frac{\gamma_s(\omega)\omega^2}{c_0^2} = \frac{\phi(0)\omega}{c_0}, \tag{63}$$

which shows that we have an effective constant attenuation (of the form (1) with $y = 0$) and no effective dispersion.

These two cases (quadratic and constant attenuations) are the standard cases observed with standard models of random media, that are mixing and not rough. As we will see in the next subsections, the picture becomes more interesting when non-mixing or rough random media are considered.

E. Random Media with Long-range Correlations

This is an interesting regime that leads to explicit formula. This is the regime in which the random medium has the H-long-range correlation property, $H \in (1/2, 1)$ and the typical wavenumber ω/c_0 of the input pulse is such that

$$\frac{|\omega| l_c}{c_0} \ll 1 .$$

This second condition means that the typical wavelength is longer than l_c and therefore the pulse probes the long-range properties of the medium. In this case we find by using (6) and formula 3.761 in Ref. 20 that

$$\frac{\gamma_c(\omega)\omega^2}{c_0^2} = r_H \frac{\Gamma(2H-1)}{2^{2H-2}} \cos\Big((H-\frac{1}{2})\pi\Big)\frac{1}{l_c}\Big(\frac{|\omega| l_c}{c_0}\Big)^{3-2H}, \tag{64}$$

$$\frac{\gamma_s(\omega)\omega^2}{c_0^2} = r_H \frac{\Gamma(2H-1)}{2^{2H-2}} \sin\Big((H-\frac{1}{2})\pi\Big)\Big(\frac{|\omega| l_c}{c_0}\Big)^{2-2H}\frac{\omega}{c_0}. \tag{65}$$

This shows that the wave propagation in random media with long-range correlations exhibits frequency-dependent attenuation that is characterized by a power law of the form (1) with the exponent $y = 3 - 2H$ ranging from 1 to 2. This exponent is related to the power decay rate at infinity of the autocorrelation function of the medium fluctuations. The frequency-dependent attenuation is associated with a frequency-dependent phase. This ensures that causality is respected and Kramers-Kronig relations are satisfied. Moreover the ratio of the effective dispersion coefficient over the effective diffusion coefficient is

$$R_{\text{disp./diff.}}(H) = \frac{\sin\big((H-\frac{1}{2})\pi\big)}{\cos\big((H-\frac{1}{2})\pi\big)}, \tag{66}$$

which shows that the effective dispersion is stronger than the effective diffusion when H is close to 1 and that it is weaker when H is close to 1/2 (see Figure 4). Note also that we recover the standard mixing case (formulas (59-60)) when $H \to 1/2$.

In the time domain, we can write

$$\mathcal{L}a(\tau) = \frac{r_H l_c^{2-2H}}{2^{1+2H} c_0^{3-2H}} \int_0^\infty \frac{1}{s^{2-2H}} \frac{\partial^2 a}{\partial \tau^2}(\tau - s) ds . \tag{67}$$

If we go back to the original frame and substitute the expression (67) of the pseudo-differential operator into (58) we obtain the effective fractional wave equation

$$\frac{\partial^2 p}{\partial \xi^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial \tau^2} = \frac{r_H l_c^{2-2H} \varepsilon^2}{2^{2H} c_0^{4-2H}} \int_0^\infty \frac{1}{s^{2-2H}} \frac{\partial^3 p}{\partial \tau^3}(\tau - s) ds , \tag{68}$$

which is of the form of the one proposed in Ref. 34, but not exactly. Indeed, in Ref. 34 Szabo proposes to use the Riemann-Liouville fractional derivative[29] and writes the fractional wave equation in the form

$$\frac{\partial^2 p}{\partial \xi^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial \tau^2} = c_{\varepsilon,H} \int_0^\infty \frac{1}{s^{5-2H}} p(\tau - s) ds . \tag{69}$$

As noted in Refs. 7–9 the Riemann-Liouville fractional derivative has disadvantages, for instance the derivative of a constant is not zero so that attenuation does not vanish for a system at equilibrium[30]. The Caputo derivative has been introduced to overcome this drawback and has been discussed in Ref. 9. Our derivation from first principles allows us to get the correct form (68) of the effective wave equation.

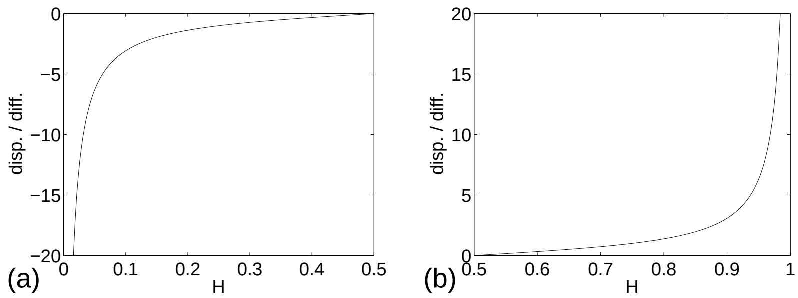

FIG. 4. Ratios of the effective dispersion coefficient over the effective diffusion coefficient. Note the symmetry of the ratio with respect to $H = 1/2$.

F. Random Media with Short-range Correlations

This is the regime in which the random medium possesses the H-short-range correlation property, $H \in (0, 1/2)$, and the typical wavenumber ω/c_0 of the input pulse is such that

$$\frac{|\omega| l_c}{c_0} \gg 1 \,.$$

This second condition means that the typical wavelength is smaller than l_c and therefore the pulse probes the short-range properties of the medium. In this case we find by using (7) that

$$\frac{\gamma_c(\omega)\omega^2}{c_0^2} = \phi(0) d_H \frac{\Gamma(1+2H)}{2^{2H}} \sin(H\pi) \frac{1}{l_c} \Big(\frac{|\omega| l_c}{c_0} \Big)^{1-2H} , \tag{70}$$

$$\frac{\gamma_s(\omega)\omega^2}{c_0^2} = \phi(0) \frac{\omega}{c_0} - \phi(0) d_H \frac{\Gamma(1+2H)}{2^{2H}} \cos(H\pi) \Big(\frac{|\omega| l_c}{c_0} \Big)^{-2H} \frac{\omega}{c_0} \,. \tag{71}$$

This result is proved in the Appendix B and it shows that wave propagation in random media with short-range correlations exhibits frequency-dependent attenuation that is characterized by a power law of the form (1) with the exponent $y = 1 - 2H$ ranging from 0 to 1. This exponent is related to the power decay rate at zero of the autocorrelation function of the medium fluctuations. Moreover the ratio of the effective dispersion coefficient over the effective diffusion coefficient is

$$R_{\text{disp./diff.}}(H) = -\frac{\cos(H\pi)}{\sin(H\pi)} , \tag{72}$$

which shows that the effective dispersion is stronger than the effective diffusion when H is close to 0 and that it is weaker when H is close to 1/2 (see Figure 4). Note also that we recover the standard mixing case (formulas (62-63)) when $H \to 1/2$.

In the time domain, we can write

$$\begin{aligned} \mathcal{L}a(\tau) &= \frac{\phi(0)}{8c_0} \frac{\partial a}{\partial \tau} - \frac{\phi(0) d_H}{2^{3+2H} c_0^{1-2H} l_c^{2H}} \int_0^\infty s^{2H} \frac{\partial^2 a}{\partial \tau^2}(\tau - s) ds \\ &= \frac{\phi(0)}{8c_0} \frac{\partial a}{\partial \tau} + \frac{\phi(0) H d_H}{2^{2+2H} c_0^{1-2H} l_c^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial a}{\partial \tau}(\tau - s) ds \,. \end{aligned} \tag{73}$$

If we go back to the original frame, neglect the random time shift, and substitute the expression (73) of the pseudo-differential operator into (57), we obtain exactly the form of the integro-differential wave equation proposed by Hanyga[21]:

$$\frac{\partial p}{\partial \xi} + \frac{1}{c_0} \frac{\partial p}{\partial \tau} = \frac{\varepsilon^2 \phi(0)}{8c_0} \frac{\partial p}{\partial \tau} + \frac{\phi(0) H d_H \varepsilon^2}{2^{2+2H} c_0^{1-2H} l_c^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial p}{\partial \tau}(\tau - s) ds \,. \tag{74}$$

If we substitute the expression (73) into (58) we obtain the effective fractional wave equation

$$\frac{\partial^2 p}{\partial \xi^2} - \frac{1}{c_0^2} \Big(1 + \frac{\varepsilon^2 \phi(0)}{4} \Big) \frac{\partial^2 p}{\partial \tau^2} = \frac{\phi(0) H d_H \varepsilon^2}{2^{1+2H} c_0^{2-2H} l_c^{2H}} \int_0^\infty \frac{1}{s^{1-2H}} \frac{\partial^2 p}{\partial \tau^2}(\tau - s) ds \,. \tag{75}$$

IV. CONCLUSION

In this paper we have clarified the relation between the effective attenuation/dispersion of a wave propagating through a random medium and the statistics of the random medium, in particular its short- or long-range correlation properties. We have given explicit formulas between the power decay rate of the autocorrelation function of the random medium and the exponents of the power law frequency-dependences of the effective attenuation and dispersion. The main two results are the following ones.

When a long-wavelength pulse propagates in a random medium with an autocorrelation function that decays at infinity as $|z|^{2H-2}$, $H \in (1/2, 1)$, then the attenuation has a power law frequency-dependence of the form $\alpha(\omega) = \alpha_0 |\omega|^y$ with $y = 3 - 2H \in (1, 2)$.

When a short-wavelength pulse propagates in a random medium with an autocorrelation function that behaves at zero like $1 - d_H |z|^{2H}$, $H \in (0, 1/2)$, then the attenuation has a power law frequency-dependence of the form $\alpha(\omega) = \alpha_0 |\omega|^y$ with $y = 1 - 2H \in (0, 1)$.

In both cases a frequency-dependent phase responsible for dispersion is associated to the frequency-dependent attenuation and it ensures that causality and Kramers-Kronig relations are respected. Effective fractional wave equations can be written that have the form of equations studied in the literature[9,21,34].

These results were derived in the case of one-dimensional wave equations and the mathematical tools that have been used are restricted to this case. The generalization of the theory to three-dimensional random media remains an open question.

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APPENDIX A: PROOF OF (62-63)

Let us assume that the process is mixing and the autocorrelation function $\phi(z)$ satisfies the linear decay (61). The function $\phi(z)$ can then be written as

$$\phi(z) = \phi(0) \Big[\phi_{1/2}\Big(\frac{d_{1/2}}{2} \frac{|z|}{l_c} \Big) + \widetilde{\phi}\Big(\frac{|z|}{l_c} \Big) \Big] , \tag{A1}$$

where $\phi_{1/2}$ is defined by

$$\phi_{1/2}(u) = (1-u)^2 \mathbf{1}_{[0,1]}(u) . \tag{A2}$$

The function $\widetilde{\phi}$ is continuously differentiable and piecewise twice differentiable over $(0, \infty)$, because ϕ and $\phi_{1/2}$ are continuously differentiable and piecewise twice differentiable over $(0, \infty)$ (For $\phi_{1/2}$, one can check that $\phi_{1/2}(1^-) = 0 = \phi_{1/2}(1^+)$, $\phi'_{1/2}(1^-) = 0 = \phi'_{1/2}(1^+)$, and $\phi''_{1/2}(1^-) = 2$, $\phi''_{1/2}(1^+) = 0$). Moreover, $\widetilde{\phi}$, $\widetilde{\phi}'$, and $\widetilde{\phi}''$ are absolutely integrable over $(0, \infty)$, $\widetilde{\phi}(0) = 0$, and $\widetilde{\phi}'(0^+) = 0$. By performing a double integration by parts, we obtain

$$\int_0^\infty \widetilde{\phi}(u) e^{iNu} du = -\frac{1}{N^2} \int_0^\infty \widetilde{\phi}''(u) e^{iNu} du . \tag{A3}$$

Riemann-Lebesgue's lemma then gives

$$\int_0^\infty \widetilde{\phi}(u) e^{iNu} du \overset{N \to \infty}{\simeq} o\Big(\frac{1}{N^2} \Big) . \tag{A4}$$

Moreover, by performing a double integration by parts, we obtain

$$\begin{aligned} \int_0^\infty \phi_{1/2}(u) e^{iNu} du &= \frac{i}{N} + \frac{2}{N^2} + \frac{2i}{N^3}(1 - e^{iN}) \\ &\overset{N \to \infty}{\simeq} \frac{i}{N} + \frac{2}{N^2} + o\Big(\frac{1}{N^2} \Big) . \end{aligned} \tag{A5}$$

Using (A4) and (A5) in (A1) gives formulas (62-63).

APPENDIX B: PROOF OF (70-71)

Let us assume that the autocorrelation function $\phi(z)$ satisfies the H-short-range property, $H \in (0,1/2)$. The function $\phi(z)$ can then be written as

$$\phi(z) = \phi(0)\Big[\phi_H\Big(\big[d_H(1-2H)\big]^{1/(2H)}\frac{|z|}{l_c}\Big) + \widetilde{\phi}\Big(\frac{|z|}{l_c}\Big)\Big], \tag{B1}$$

where ϕ_H is defined by

$$\phi_H(u) = \Big(1 - \frac{1}{1-2H}u^{2H} + \frac{2H}{1-2H}u\Big)\mathbf{1}_{[0,1]}(u). \tag{B2}$$

The function $\widetilde{\phi}$ is continuously differentiable and piecewise twice differentiable over $(0,\infty)$, because ϕ and ϕ_H are continuously differentiable and piecewise twice differentiable over $(0,\infty)$ (For ϕ_H, one can check that $\phi_H(1^-) = 0 = \phi_H(1^+)$, $\phi'_H(1^-) = 0 = \phi'_H(1^+)$, and $\phi''_H(1^-) = 2H$, $\phi''_H(1^+) = 0$). Moreover, $\widetilde{\phi}$, $\widetilde{\phi}'$, and $\widetilde{\phi}''$ are absolutely integrable over $(0,\infty)$, $\widetilde{\phi}(0) = 0$, and $\widetilde{\phi}'(0^+)$ is well defined (since $\widetilde{\phi}''$ is absolutely integrable at 0). Consequently a double integration by parts yields:

$$\begin{aligned}\int_0^\infty \widetilde{\phi}(u)e^{iNu}du &= -\frac{\widetilde{\phi}'(0^+)}{N^2} - \frac{1}{N^2}\int_0^\infty \widetilde{\phi}''(u)e^{iNu}du \\ &\overset{N\to\infty}{=} O\Big(\frac{1}{N^2}\Big).\end{aligned} \tag{B3}$$

Moreover, by performing an integration by parts step, we obtain

$$\int_0^\infty \phi_H(u)e^{iNu}du \overset{N\to\infty}{\simeq} \frac{i}{N} - i\frac{2H}{1-2H}\frac{1}{N^{1+2H}}\int_0^\infty v^{2H-1}e^{iv}dv + o\Big(\frac{1}{N^{1+2H}}\Big). \tag{B4}$$

The computation of the definite integral (see formula 3.761 in Ref. 20)

$$\int_0^\infty v^{2H-1}e^{iv}dv = \Gamma(2H)e^{iH\pi} \tag{B5}$$

gives

$$\begin{aligned}\int_0^\infty \phi_H(u)e^{iNu}du \overset{N\to\infty}{\simeq} &\ \frac{i}{N} - \frac{i}{N^{1+2H}}\frac{\Gamma(1+2H)}{1-2H}\cos(H\pi) \\ &+\frac{1}{N^{1+2H}}\frac{\Gamma(1+2H)}{1-2H}\sin(H\pi) + o\Big(\frac{1}{N^{1+2H}}\Big).\end{aligned} \tag{B6}$$

Using (B3) and (B6) in (B1) gives (70-71).

APPENDIX C: THE FRACTIONAL WHITE NOISE WITH $H \in (0,1/2)$

As we already noticed the fractional white noise with the Hurst index $H \in (0,1/2)$ "almost" satisfies the H-short-range property, but fails to meet one of the technical hypotheses. Indeed its autocorrelation function $\phi(z)$ is not differentiable at $z = l_c$. However we show in this appendix that this model has "almost" the effective attenuation and dispersion predicted in the general case in the H-short-range property.

Here $\phi(z)$ is explicit so we can carry out the computation and we find that, if $|\omega|l_c/c_0 \gg 1$, then

$$\frac{\gamma_c(\omega)\omega^2}{c_0^2} = \sigma^2\frac{\Gamma(1+2H)}{2^{2H}}\sin(H\pi)\Big[1-\cos\Big(\frac{2\omega l_c}{c_0}\Big)\Big]\frac{1}{l_c}\Big(\frac{|\omega|l_c}{c_0}\Big)^{1-2H}, \tag{C1}$$

$$\frac{\gamma_s(\omega)\omega^2}{c_0^2} = \sigma^2\frac{\omega}{c_0} - \sigma^2\frac{\Gamma(1+2H)}{2^{2H}}\Big[\cos(H\pi)+\sin(H\pi)\sin\Big(\frac{2|\omega|l_c}{c_0}\Big)\Big]\Big(\frac{|\omega|l_c}{c_0}\Big)^{-2H}\frac{\omega}{c_0}. \tag{C2}$$

The singularity at l_c is responsible for the additional terms that have a fast periodic modulation in ω. In this paper we are interested in pulse propagation, and as shown by (41) we integrate over ω. Using the identity (formula 3.936 in Ref. 20)

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\big(x\cos(\theta)+ix\sin(\theta)\big)d\theta = 1, \quad \forall x \in \mathbb{R}, \tag{C3}$$

we find that the additional modulated terms vanish when the expressions (C1-C2) are substituted into (41). Therefore, we can consider that the effective attenuation and dispersion terms for the fractional white noise model are given by the general formulas (70-71).

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