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Long-Time Dynamics of Korteweg–de Vries Solitons Driven by Random Perturbations

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This paper deals with the transmission of a soliton in a random medium described by a randomly perturbed Korteweg–de Vries equation. Different kinds of perturbations are addressed, depending on their specific time or position dependences, with or without damping. We derive effective evolution equations for the soliton parameter by applying a perturbation theory of the inverse scattering transform and limit theorems of stochastic calculus. Original results are derived that are very different compared to a randomly perturbed Nonlinear Schrödinger equation. First the emission of a soliton gas is proved to be a very general feature. Second some perturbations are shown to involve a speeding-up of the soliton, instead of the decay that is usually observed in random media.

KEY WORDS: Korteweg–de Vries equation; inverse scattering transform; solitons; random media

1. INTRODUCTION

The stability properties of the solitons were discovered by Zabusky and Kruskal in their well-known numerical study,[(1)] and were confirmed by the theory developed in the work by Gardner *et al.*[(2)] They show that the Korteweg–de Vries (KdV) solitons preserve their identities during soliton–soliton interactions. Lax considerably generalized these ideas,[(3)] and Zakharov and Shabat showed that the method worked for the nonlinear Schrödinger (NLS) equation.[(4)] Following this pioneer work solitons driven by various types of perturbations were studied. A lot of work has been

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devoted to the transmission of a soliton through a slab of nonlinear and random medium, especially in the case of the KdV equation and of the NLS equation.[5–7] The main results have been obtained for the randomly perturbed NLS equation. Kivshar *et al.*[8] obtained results in the case of a random medium consisting of pure point impurities with very low density which affect only the potential. In such conditions it is shown that there is a threshold below which the pulses decay quickly. This analysis was later extended by Bronski[9] and by the author.[10] Numerical modeling of the NLS equation with random linear potential confirms this theory.[11, 12] In ref. 10 we consider the NLS equation and we assume that inhomogeneities affect the potential and the nonlinear coefficient. Using the inverse scattering transform, we exhibit several typical behaviors. The mass of the transmitted soliton may tend to zero exponentially (as a function of the size of the slab) or following an algebraic decay; or else the soliton may keep its mass, while its velocity decreases at a very slow rate.

There exist also experimental investigations of the propagation of nonlinear waves in random media,[13] describing the KdV type wave propagation. Indeed for both historical, physical, and theoretical reasons the KdV equation is a paradigm for the study of nonlinear wave propagation. As a consequence there can be found in the literature a lot of papers devoted to the study of the influence of various types of random perturbations on the KdV soliton propagation. In ref. 14 the evolution of randomly modulated initial solitons in the non-random KdV equation is investigated. Other authors have examined the propagation of an initially deterministic wave controlled by a randomly perturbed KdV equation. In some very special configurations the solution of the randomly perturbed KdV equation can be computed explicitly. By applying the exact inverse scattering transform in a suitably moving reference frame, Wadati obtained the exact solution in case of an additive time-dependent noise. This solution is a single-soliton whose position is shifted by a random process. He was then able to study the average of this solution in case of a white noise,[15] even by taking into account some dissipation;[16] finally this work was generalized by Iizuka to the case of a noise with long-range correlation,[17] and confirmed by extensive numerical simulations.[18] Other authors have applied the collective variable approximation or the averaged Lagrangian approach, where the solution is sought in a soliton-like form with time-dependent parameters.[19, 5] This ansatz is substituted into the Lagrangian of the system, so that a finite-dimensional system of ordinary differential equations is obtained for the set of soliton parameters. The collective variable approach is efficient when dealing with a slowly varying perturbation in the sense that it is almost constant at the scale of the soliton width. The most significant drawback of this method is that it neglects radiation effects. The generation of radiation by a soliton during its motion in a randomly inhomogeneous medium is studied in ref. 5, but the analysis does not take into account the interplay between the generated radiation and the soliton parameter, and do not consider the generation of new solitons during the motion, although the production of a secondary soliton when the KdV equation undergoes a small perturbation that satisfies an appropriate sign condition has been predicted.[20] We shall consider more general types of perturbations and proceed under a different asymptotic framework. Our main contribution is that we use the Inverse Scattering Transform so as to take into account both the variations of the soliton part and the radiative part of the wave. Both effects and their interplay are important, especially when the correlation length of the perturbation is of the same order as the soliton width. The interaction of different length scales are an important issue in localization theory for wave propagation in linear media[21, 22] so the relationship of the width of the soliton and the correlation length of the perturbation will clearly have a fundamental effect on the questions we are trying to answer. We shall consider the influence of small random perturbations and aim at reporting the possible asymptotic behaviors when the amplitudes of the random fluctuations go to zero and the size of the system goes to infinity. We shall put into evidence several novel and interesting features concerning the dynamics of the input soliton, the emission of radiation, and the generation of a soliton gas.

The paper is organized as follows. Section 2 is devoted to a short review of the Korteweg–de Vries equation and the inverse scattering. We introduce exact traveling solutions (soliton solutions) of the integrable system, and we also present basic results that are required for our study. In the following sections we address different random problems: perturbations with time-dependent coefficients, perturbations with position-dependent coefficients that preserve different integrals of motion. By applying a modified version of the inverse scattering transform we study the interaction of the main soliton, the radiation, and the soliton gas, and we derive an effective system that governs the evolution of the soliton parameter. We also compare the theoretical results with full numerical simulations of the KdV equation.

2. HOMOGENEOUS KORTEWEG–DE VRIES EQUATION

The equation we consider in this paper is the Korteweg–de Vries equation:

$$u_t + u_{xxx} - 6uu_x = 0, \tag{1}$$

that governs the evolution of a real field u. The subscripts x and t stand for partial derivatives with respect to position and time, respectively. This integrable equation supports moving nonlinear localized excitations in the form of solitons, so we can study the effects of various perturbations with the known analytic behavior of the unperturbed dynamics. We shall begin by a short review of the Inverse Scattering Transform applied to the KdV equation which was originally introduced by Gardner *et al.*[2] Details of the rigorous theory can be found in ref. 23.

2.1. Direct Transform: The Scattering Problem

The scattering problem associated with the KdV equation is the Schrödinger spectral problem:

$$\psi_{xx} + (k^2 - u(x))\,\psi = 0, \tag{2}$$

where $k \in \mathbb{R}$ is the spectral parameter. Let us first assume that $u \equiv 0$. In such conditions, there exists no solution in $L^2(\mathbb{R})$ of (2) whatever k, which means that the discrete spectrum is empty. The continuous spectrum consists of the real axis; the associated eigenspace is of dimension 2, and the pair of functions (e^{-ikx}, e^{ikx}) is a basis of the eigenspace associated with the parameter k (eigenvalue k^2 of Eq. (2)). From now on we assume $u \not\equiv 0$.

Continuous Spectrum. The so-called Jost functions f and g are the eigenfunctions which are associated with the spectral parameter k (eigenvalue k^2) and which satisfy the following boundary conditions:

$$f(x, k) \simeq e^{ikx}, \qquad x \to \infty, \tag{3}$$

$$g(x, k) \simeq e^{-ikx}, \qquad x \to -\infty. \tag{4}$$

If u decays sufficiently rapidly as $|x| \to \infty$ (more exactly for $u \in L^1$), then $f(x)\, e^{-ikx}$ and $g(x)\, e^{ikx}$ are well-defined for any $k \in \mathbb{R}$ and can be analytically continued for $\mathrm{Im}(k) > 0$. The function $f^*(x, k) = f(x, -k)$ (resp. $g^*(x, k) = g(x, -k)$) associated with the function f (resp. g) solution of (2) is also a solution of (2) for the same parameter $k \in \mathbb{R}$. We thus consider also the eigenfunctions f^* and g^* which can be defined either as the complex conjugates of f and g respectively, or as the eigenfunctions which are associated with the spectral parameter k and which satisfy the following boundary conditions:

$$f^*(x, k) \simeq e^{-ikx}, \qquad x \to \infty, \tag{5}$$

$$g^*(x, k) \simeq e^{ikx}, \qquad x \to -\infty. \tag{6}$$

Furthermore the Jost functions $f(x,k)$ and $f^*(x,k)$ are linearly independent because their Wronskian

$$W(f, f^*) := f f_x^* - f_x f^* = -2ik \tag{7}$$

is nonzero as soon as $k \neq 0$ (compute the Wronskian at $x \to \infty$). They form a base of the space of the solutions of (2), so that we have the decomposition:

$$g(x,k) = a(k)\, f^*(x,k) + b(k)\, f(x,k). \tag{8}$$

where a and b are the so-called Jost coefficients. Similarly, $W(g, g^*) = 2ik$ (compute the Wronskian at $x \to -\infty$). Equation (8) gives another form for this Wronskian: $W(g, g^*) = 2ik(|a|^2 - |b|^2)$. This implies that:

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{9}$$

Multiplying (8) by $-b^*$, and summing with (8) complex-conjugated and multiplied by a we get that:

$$f(x,k) = a(k)\, g^*(x,k) - b^*(k)\, g(x,k). \tag{10}$$

If follows from (8)–(10) in particular that:

$$a(k) = \frac{i}{2k} W(f, g). \tag{11}$$

From this definition we can see that a is well-defined over the real axis and can be analytically continued in the upper complex half-plane $\mathrm{Im}(k) > 0$. b is well-defined over $\mathbb{R}$, but there is no reason to believe that it could be continued out of the real axis, except if u is exponentially decaying. Furthermore it can be shown by symmetry arguments that $a(-k) = a^*(k)$ and $b(-k) = b^*(k)$. From (2) and (11) it follows also that:

$$a(k) \to 1 \quad \text{as} \quad |k| \to \infty, \qquad \mathrm{Im}(k) \geqslant 0. \tag{12}$$

Discrete Spectrum. On the one hand we can see from (12) that a can only have a finite number of zeros in the upper complex half plane. On the other hand Eq. (9) shows that a cannot have any zero on the real axis (except maybe in $x = 0$). Let us denote by k_j, $j = 1,\ldots, J$ the zeros of a in the upper complex half plane. For any j, the functions $f(.,k_j)$ and $g(.,k_j)$ are linearly dependent, i.e., there exists ρ_j such that $g(.,k_j) = \rho_j f(.,k_j)$. Accordingly f and g are exponentially decaying as $|x| \to \infty$ which implies that they are eigenfunctions of the discrete spectrum. Thus $k_j^2 \in \mathbb{R}^-$, or else

$k_j = i\kappa_j$, $\kappa_j > 0$, $f(.,k_j)$ is real-valued and the discrete spectrum of Eq. (2) is $\lambda = -\kappa_j^2$. Finally note that we can deduce from (2) and (11) that:

$$\left.\frac{da}{dk}\right|_{k=k_j} = -i\rho_j \int f^2(x,k_j)\,dx$$

This relation shows that $a'(k_j) \neq 0$, that is to say the zeros of a are simple.

In summary the set of quantities $\{a(k),\ b(k),\ k \in \mathbb{R};\ k_j,\ \rho_j,\ a'(k_j),\ j = 1,..., J\}$ is the scattering data for the spectral problem (2). The interpretation of these data is well-known and gives the name of the method. The Jost function f describes a wave incident from the left and scattered by the potential u. According to Eqs. (8), (10), (3), and (4) the reflection and transmission coefficients are $b^*/a(k)$ and $1/a(k)$, respectively, and the conservation of the flux gives: $|b^*/a(k)|^2 + |1/a(k)|^2 = 1$ which also reads as (9).

2.2. Time Evolutions of the Scattering Data

The time equations for the scattering data are:

$$a(t,k) = a(t_0,k), \qquad k \in \mathbb{R}, \tag{13}$$

$$b(t,k) = b(t_0,k)\exp(i\omega(k)(t-t_0)), \qquad k \in \mathbb{R}, \tag{14}$$

$$\rho_j(t) = \rho_j(t_0)\exp(i\omega(k_j)(t-t_0)), \qquad j = 1,..., J, \tag{15}$$

where $\omega(k) = 8k^3$. Note that $\omega(k)$ is the dispersion relation of the linearized KdV equation . Indeed the linear form of Eq. (1) is $u_t + u_{xxx} = 0$, whose dispersion relation is obtained by letting $u(t,x) = \exp(2ikx + i\omega t)$.

2.3. Inverse Transform

Given the set of scattering data, we define:

$$F(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{b}{a}(k)\, e^{ikx}\,dk - i\sum_{j=1}^{J} \frac{\rho_j}{a'_j}\, e^{-\kappa_j x}, \tag{16}$$

Then we compute the kernel K as the solution of the Gel'fand–Levitan–Marchenko equation:

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,z)\,F(z+y)\,dz = 0. \tag{17}$$

In such conditions, it can be proved[2] that:

$$u(x) = -2\,\frac{dK(x, x)}{dx}. \tag{18}$$

2.4. Conserved Quantities

Conserved quantities can be worked out as in any integrable system.[24] The mass N, the energy E, and the Hamiltonian H:

$$N := \int_{\mathbb{R}} u\,dx, \qquad E := \int_{\mathbb{R}} u^2\,dx, \qquad H := \int_{\mathbb{R}} 2u^3 + u_x^2\,dx \tag{19}$$

are three of the infinite number of conserved quantities for the KdV equation. It will be necessary below to express the mass and energy in terms of scattering data. Let us define

$$n(k) := \frac{1}{\pi}\log(|a|^2\,(k))$$

for $k \in \mathbb{R}$. The mass, energy, and Hamiltonian can be decomposed into the sums of continuous parts and discrete parts:[25]

$$N = \int_{\mathbb{R}} n(k)\,dk - 2\sum_{j=1}^{J}(2\kappa_j), \tag{20}$$

$$E = \int_{\mathbb{R}} (2k)^2\,n(k)\,dk + \frac{2}{3}\sum_{j=1}^{J}(2\kappa_j)^3, \tag{21}$$

$$H = \int_{\mathbb{R}} (2k)^4\,n(k)\,dk - \frac{2}{5}\sum_{j=1}^{J}(2\kappa_j)^5. \tag{22}$$

2.5. Soliton

Equation (1) possesses soliton solutions, that is to say waves that propagate at constant velocities with constant envelopes. These solutions are of the form:

$$u_s(t, x) = -2\kappa^2\,\mathrm{sech}^2(\kappa(x - x_s(t))), \tag{23}$$

where

$$x_s(t) = x_0 + 4\kappa^2 t \tag{24}$$

is the center of the soliton. The mass, the velocity, the energy, and the Hamiltonian of the soliton are respectively:

$$N_s = -4\kappa, \qquad U_s = 4\kappa^2, \qquad E_s = \tfrac{16}{3}\kappa^3, \qquad H_s = -\tfrac{64}{5}\kappa^5. \tag{25}$$

The width of the envelope of the soliton is inversely proportional to its mass. Note also that the bandwidth of the time spectrum of the soliton (i.e., the Fourier transform of u_s with respect to t) is of the order of $8\kappa^3$. In the following we shall refer to $8\kappa^3$ as the typical frequency of the soliton. In the spatial domain, the bandwidth of the spatial spectrum of the soliton (i.e., the Fourier transform of u_s with respect to x) is of the order of 2κ. In the following we shall refer to 2κ as the typical wavenumber of the soliton. The soliton solution (23) is associated with the following scattering data:

$$a_s(k) = \frac{k - i\kappa}{k + i\kappa}, \qquad b_s(k) = 0. \tag{26}$$

a_s admits a unique zero in the upper complex half plane which is $i\kappa$. The corresponding Jost functions are:

$$f_s(x,k) = e^{ikx}\,\frac{k + i\kappa\tanh(\kappa(x - x_s))}{k + i\kappa}, \tag{27}$$

$$g_s(x,k) = e^{-ikx}\,\frac{k - i\kappa\tanh(\kappa(x - x_s))}{k + i\kappa}. \tag{28}$$

3. TIME PERTURBATIONS OF THE KDV EQUATION

3.1. The Perturbed Model

We consider a perturbed Korteweg–de Vries equation with a non-zero right-hand side:

$$u_t + u_{xxx} - 6uu_x = \varepsilon R(u)(t,x). \tag{29}$$

The small parameter $\varepsilon \in (0,1)$ characterizes the amplitude of the perturbation, and R may be a combination of the following three terms:

$$R_1(u) = V_1(t)\,u_x, \tag{30}$$

$$R_2(u) = 6V_2(t)\,uu_x, \tag{31}$$

$$R_3(u) = V_3(t)\,u_{xxx}, \tag{32}$$

where V_1 represents a time-dependent perturbation of the velocity, V_2 a time-dependent perturbation of the nonlinear coefficient, and V_3 a time-dependent perturbation of the dispersion. Such a random KdV equation may be derived for instance from the plasma fluid equations that describe the propagation of an ion-acoustic soliton in the presence of noise.[26] We shall refer to ref. 5 and references therein for further physical motivations. Note that we could consider other kinds of perturbations, as long as they satisfy the condition:

$$\int uR(u)\,dx = 0$$

which implies that the energy of the solution is preserved:

$$\int u^2(t,x)\,dx = \int u^2(t=0,x)\,dx$$

Note also that the perturbations R_j also preserve the total mass of the system since $\int R_j(u)\,dx = 0$. The mass conservation is not necessary for the derivations of the results contained in the forthcoming Proposition 3.1, while the energy conservation is essential. However the mass conservation will allow us to establish a complementary result about the mass of the generated soliton gas. The perturbations V_j are assumed to be zero-mean, stationary and ergodic processes. The exact technical condition is that the V_j should be "ϕ-mixing" (in the sense of ref. 27, Section 4-6-2) with $\phi \in L^{1/2}$. For the sake of simplicity we also assume that they are independent processes, but this hypothesis could be removed at the expense of computing crossed terms that do not affect qualitatively the forthcoming statements. The autocorrelation function of V_j is denoted by:

$$\gamma_j(t) := \mathbb{E}[V_j(0)\,V(t)] = \mathbb{E}[V_j(t')\,V_j(t'+t)], \tag{33}$$

where $\mathbb{E}$ stands for the statistical average with respect to the stationary distribution of V_j. The ϕ-mixing condition with $\phi \in L^{1/2}$ means in particular that the perturbation has enough decorrelation properties so that the integral $\int |\gamma_j(t)|^{1/2}\,dt$ is finite. We can then introduce the Fourier transform of the autocorrelation function of the perturbation V_j:

$$\hat{\gamma}_j(\omega) := \int_{-\infty}^{\infty} \gamma_j(t)\cos(\omega t)\,dt, \tag{34}$$

which is nonnegative real-valued since it is proportional to the power spectral density by the Wiener–Khintchine theorem.[28] For instance, if the perturbation V_j is a white noise $\gamma_j(t) = \sigma_j^2 \delta(t)$, then the spectrum of the perturbation is flat and given by $\hat{\gamma}_j(\omega) = \sigma_j^2$ for any ω.

Our method is based on the Inverse Scattering Transform. The random perturbation induces variations of the spectral data. Calculating these changes we are able to find the effective evolution of the field and calculate the characteristic parameters of the wave. We shall be interested in the effective dynamics of the soliton propagating over long times T/ε^2. The total energy is conserved but the discrete and continuous components evolve during the propagation. The evolution of the continuous component corresponding to radiation will be found from the evolution equations of the Jost coefficients. The evolutions of the soliton parameter will then be derived from the conservation of the total energy. However we shall see that this approach turns out to be a little more tricky than expected because of the generation of new solitons.

We now describe the evolutions of the Jost coefficients a and b during the propagation. They satisfy the following exact equations:[19]

$$\frac{\partial a}{\partial t} = \frac{i\varepsilon}{2k} (a\alpha + b\beta), \tag{35}$$

$$\frac{\partial b}{\partial t} = 8ik^3 b - \frac{i\varepsilon}{2k} (a\beta^* + b\alpha), \tag{36}$$

where

$$\alpha(k, t) = \int |f|^2 (x, k, t) \, R(u)(x, t) \, dx,$$

$$\beta(k, t) = \int f^2(x, k, t) \, R(u)(x, t) \, dx.$$

These equations will be needed below when calculating the radiative mass and energy generated by the moving soliton. They express the fact that the evolutions of the scattering data are coupled due to the perturbation R through the coefficients α and β. We would like to point out a remarkable feature of Eqs. (35)–(36). The coupling coefficients α and β are typically of order 1 uniformly with respect to k and ε, but the actual coupling terms in the right-hand sides of Eqs. (35)–(36) are not of order ε uniformly with respect to k because of the factor $i\varepsilon/(2k)$. If $|k|$ is of order ε, then the perturbed term could be of order 1. Consider for a while the early steps of the propagation, and assume that the wave is still very close to the original soliton u_s. If $k = i\varepsilon k'$, then Eq. (35) can be approximated by $a_t = \alpha_s/(2k')$ where $\alpha_s = \int R(u_s) \, dx$. Two different behaviors are possible that depend whether the total mass is preserved. If the total mass is not preserved, then $\alpha_s \neq 0$ in general so that a new soliton with parameter of order ε is generated if $\alpha_s > 0$. If the total mass is preserved, then $\alpha_s = 0$ and $a_t \sim \varepsilon^2/k$ so that a new soliton with parameter of order ε^2 may be generated.

3.2. Effective Regime

We can now state the main result of this section. For this we need to define the concept of soliton gas in our framework. A soliton gas is a collection of solitons whose total energy goes to zero as $\varepsilon \to 0$ while the sum of their masses is non-zero.

Proposition 3.1. The following event has a probability which tends to 1 as $\varepsilon \to 0$: the scattered wave at time t/ε^2 consists of one main soliton with parameter $\kappa^\varepsilon(t)$, a soliton gas, and radiation. The process $(\kappa^\varepsilon(t))_{t\in[0,T]}$ converges in probability to the deterministic function $(\kappa_l(t))_{t\in[0,T]}$ which satisfies the ordinary differential equation:

$$\frac{d\kappa_l}{dt} = F(\kappa_l), \qquad \kappa_l(0) = \kappa_0, \tag{37}$$

The function F is equal to:

$$F(\kappa) = -\frac{1}{16\kappa^2} \sum_{j=1}^{3} \int_{\mathbb{R}} (2k)^2 \, C_j(\kappa, k) \, dk, \tag{38}$$

where the functions C_j are the mass density scattered by the soliton with parameter κ per unit time due to the perturbation R_j. The exact expressions of the C_j's are the following:

$$C_1(\kappa, k) = 0, \tag{39}$$

$$C_2(\kappa, k) = C(\kappa, k)\, \hat{\gamma}_2(\Omega(\kappa, k)), \tag{40}$$

$$C_3(\kappa, k) = C(\kappa, k)\, \hat{\gamma}_3(\Omega(\kappa, k)), \tag{41}$$

where the functions C and Ω are defined by:

$$C(\kappa, k) = \frac{256\pi k^2 \kappa^4 \left(1 + \frac{k^2}{\kappa^2}\right)^2}{9 \sinh^2\left(\pi \frac{k}{\kappa}\right)}, \qquad \Omega(\kappa, k) = 8k(k^2 + \kappa^2). \tag{42}$$

The first assertion of the proposition means that the event "the transmitted wave consists of one soliton plus some other small amplitude wave" occurs with very high probability for small ε, while the second assertion gives the effective evolution equation of the parameter of the transmitted soliton in the asymptotic framework $\varepsilon \to 0$. $C(\kappa, k)$ is actually the spectral mass density scattered by the soliton with parameter κ per unit time due to a perturbation $R = 6V(t)\,uu_x$ or $R = V(t)\,u_{xxx}$ with a random process $V(t)$ which is white-noise distributed with autocorrelation function $\mathbb{E}[V(t')\,V(t+t')] = \delta(t)$. Note that the function C satisfies the scaling relationship $C(\kappa, k) = \kappa^6 C(1, k/\kappa)$. In case of a colored noise the power spectral density of this noise should be taken into account according to Eqs. (40)–(41). Since the function $k \mapsto k^2 C(\kappa, k)$ is maximal around $\pm\kappa$ (see Fig. 1b), the spectral band of the power spectral density of V that imposes the value of the function F lies around $\Omega(\kappa, \kappa) = 16\kappa^3$ which is twice the typical frequency of the soliton.

The scattered wave consists of one main soliton (with parameter κ_l of order 1), a soliton gas (with zero energy but non-zero mass), and radiation (associated with the continuous spectrum). The mass and energy of the radiation are:

$$N_r = \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}} C_j(\kappa_l(s), k)\, dk\, ds, \tag{43}$$

$$E_r = \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}} 4k^2 C_j(\kappa_l(s), k)\, dk\, ds, \tag{44}$$

respectively. We can compute also the mass N_g of the soliton gas since the total mass is preserved by the perturbation R, so:

$$N_g = 4(\kappa_l(t) - \kappa_0) - N_r.$$

The soliton gas actually consists of about ε^{-2} solitons with parameters of order $\kappa_j^\varepsilon \sim \varepsilon^2$. That is why the mass (equal to $-4\sum_j \kappa_j^\varepsilon$) is of order 1, while the energy (equal to $(16/3)\sum_j \kappa_j^{\varepsilon\,3}$) is of order ε^4 and hence asymptotically zero.

Before giving the proof of Proposition 3.1, we would like to give a short comment on the result concerning the perturbation V_1 of the velocity. Proposition 3.1 claims that a time-perturbation V_1 of the velocity does not induce any modification of the dynamics of the soliton parameter with respect to the homogeneous configuration. This was expected, since Eq. (29) with $R(u) = V_1(t)\,u_x$ also reads:[16]

$$u_T + u_{XXX} - 6uu_X = 0$$

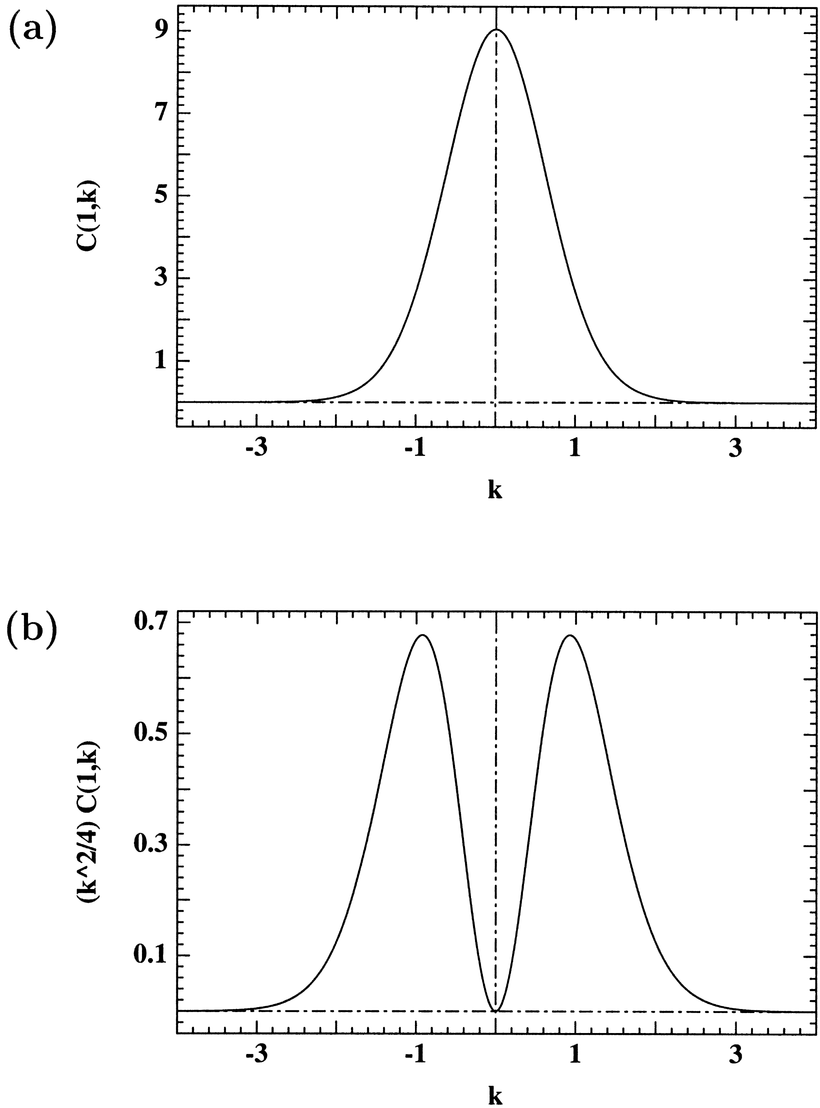


Fig. 1. (a) Spectral mass density $C(\kappa, k)$ scattered by the soliton with parameter κ per unit time due to a perturbation $R = 6V(t)\, uu_x$ or $R = V(t)\, u_{xxx}$ with a random white-distributed process $V(t)$. Here we take $\kappa = 1$. (b) Spectral energy density $k^2 C(\kappa, k)/4$ in the same configuration. The spectral energy density is maximal at $k = \pm 0.9244\kappa$.

with the change of variables $X = x + \varepsilon \int_0^t V_1(s)\, ds$ and $T = t$. Starting from a pure incoming soliton, the scattered wave can be computed explicitly. It is still a soliton, which the same mass as the incoming soliton, with a random shift of the center:

$$u(t, x) = -2\kappa^2 \operatorname{sech}^2\left[\kappa\left(x + \varepsilon \int_0^t V_1(s)\, ds - 4\kappa^2 t\right)\right].$$

Finally, note that a perturbation V_2 of the nonlinear coefficient or a perturbation V_3 of the dispersion gives rise to the very same effective regime. This is in some sense expected, since a soliton is a special solution of the KdV equation for which the nonlinear effects exactly counterbalance dispersion.

3.3. Sketch of Proof

In this section we outline the main steps of the proof of Proposition 3.1 which follows closely the strategy developed in ref. 10 in the NLS framework and we shall underline the key-points.

In a first time we carry out the analysis under the so-called adiabatic hypothesis. The adiabatic approximation consists in assuming a priori that, while the soliton exists, its evolution and the other components of the wave do not interact. More precisely, we assume that the time evolutions of the Jost coefficients a and b given by Eqs. (35)–(36) depend only on the components of the functions α and β which are associated with the soliton. We then carry out calculations under this approximation. It reduces the analysis to a infinite-dimensional set of ordinary differential equations with random coefficients, and eventually it provides an expression of the solution u. A posteriori we check for consistency that this approximation is actually justified in the asymptotic framework $\varepsilon \to 0$. More exactly we show that the components of the functions α and β which correspond to the interplay between the computed radiation (including the soliton gas) and the soliton, or else which originate from the sole effect of the radiation, can be considered as negligible terms for the soliton evolution.

3.3.1. Prove the Stability of the Zero of the Jost Coefficient a

The zero corresponds to the soliton. This part strongly relies on the analytical properties of a in the upper complex half-plane. For $|k| \gg 1$ and for $k = iK\varepsilon^2 + k'$, $k' \in \mathbb{R}$, $K \gg 1$. we can derive estimates of $a(k)$ by Eqs. (35)–(36). We then apply Rouché's theorem so as to prove that the number of zeros is constant inside a half disk whose basis is parallel to the real axis, but at distance $K\varepsilon^2$ from the real axis. This method is efficient to prove that the main zero (corresponding to the input soliton) is preserved, but it does not bring control on its precise location in the upper complex half-plane. This step is not sufficient to compute the variations of the soliton parameter. Furthermore it is also possible (and it actually holds true) that new solitons with parameters of order ε^2 are generated. Note however that the number $J^\varepsilon(t/\varepsilon^2)$ of such new solitons can be bounded above by ε^{-2}. Indeed we shall see in the next paragraph that the amount N_r of radiated mass is of order 1 for propagation time of order ε^{-2}. The conservation of the total mass then implies that the mass $N_g = -4\sum_{j=1}^{J^\varepsilon} \kappa_j^\varepsilon$ of the soliton gas is bounded above by 0 and below by $-N_r - 4\kappa_0$.

3.3.2. *Compute the Amount of Radiation and then the Variations of the Soliton Parameters*

Under the adiabatic approximation, we solve the evolution equations (35)–(36) which then read as:

$$\frac{\partial a}{\partial t} = \frac{i\varepsilon}{2k\kappa(k+i\kappa)^2} \int R(u_s)(k^2+\kappa^2 \tanh^2(z))\, dz, \tag{45}$$

$$\frac{\partial b}{\partial t} = 8ik^3 b - \frac{i\varepsilon \exp(-2ikx(t))}{2k\kappa(k^2+\kappa^2)} \int R(u_s)(k-i\kappa \tanh(z))^2 \exp\left(-2i\frac{kz}{\kappa}\right) dz. \tag{46}$$

The perturbation is of the form $R_j(u)(t, x) = m_j(t)\, r_j(u)$, where r_j is a polynomial of u and its partial derivatives with respect to x. Up to a phase term, the increment of $\bar{b}(k, t) := b(k, t)/a(k, t) \exp(-8ik^3 t)$ is:

$$\Delta\bar{b}\left(\frac{\Delta T}{\varepsilon^2}, k\right) = -iG_j(\kappa, k)\,\frac{1}{\varepsilon}\int_{T/\varepsilon^2}^{(T+\Delta T)/\varepsilon^2} m_j(s) \exp(-i8k(k^2+\kappa^2)\, s)\, ds,$$

where ($z = \kappa(x - x(t))$):

$$G_j(\kappa, k) = \int r_j(u_s)(z)\,\frac{(k-i\kappa \tanh(z))^2}{2\kappa k(k-i\kappa)^2}\, e^{-2ikz/\kappa}\, dz.$$

In the case of a perturbation of the velocity, $r_1(u_s)(z) = 4\kappa^3 \sinh(z)/\cosh(z)^3$. Computations based on tabulated formulas [ref. 29, Eq. (3.985)] show that $G_1(\kappa, k) = 0$ for any κ and k, while:

$$G_3^2(\kappa, k) = G_2^2(\kappa, k) = C(\kappa, k).$$

The density of radiation in terms of mass, energy, and Hamiltonian are $n(k, (T+\Delta T)/\varepsilon^2) = n(k, T/\varepsilon^2) + \pi^{-1} \int dk\, |\Delta \bar{b}|^2\, (k, \Delta T/\varepsilon^2)$ times 1, $(2k)^2$, and $(2k)^4$, respectively. The time perturbations preserve the total energy, so that we can deduce from the radiated energy the decay of the energy due to the soliton part:

$$E_0 = \frac{16}{3} \kappa^3 \left(\frac{t}{\varepsilon^2} \right) + \underbrace{\frac{16}{3} \sum_{j=1}^{J^\varepsilon} \kappa_j^{\varepsilon\, 3}}_{\varepsilon^{-2} \times \varepsilon^6 \sim \varepsilon^4} + \int (2k)^2\, n(k)\, dk$$

Since the new generation of solitons consists of about ε^{-2} solitons whose energies are of order ε^6, only the discrete energy of the main soliton and the continuous energy of the radiation are of order 1 in the balance of the total energy. This establishes the formula (38).

3.3.3. Compute the Form of the Scattered Wave and Check the Adiabatic Hypothesis

Given the scattering data, we can reconstruct the wave by Inverse Scattering Transform. The procedure is given for instance in ref. 19 for a general type of perturbed KdV equation. We get the first two terms of the expansion of the kernel $K = K_s + K_r$, where K_s corresponds to the soliton and K_r corresponds to the radiation:

$$K_r(x, y, t) = -\frac{1}{2\pi} \int \frac{b}{a}(k, t)\, e^{ik(x+y)}\, \frac{k + i\kappa \tanh(z)}{k + i\kappa} \times \frac{k - i\kappa + i\kappa(1 + \tanh(z)\, e^{-(\kappa + ik)(y-x)})}{k + i\kappa}\, dk$$

where $z = \kappa(x - x_s(t))$ and $x_s(t)$ is the position of the center of the soliton. By solving the Gel'fand–Levitan–Marchenko equation we can deduce the form of the radiation in the vicinity of the soliton:

$$u_r(t, x) = \frac{2i}{\pi} \int \frac{b}{a}(k, t)\, e^{2ikx}\, \frac{\kappa^2(1 - \tanh^2(z)) + k^2 + ik\kappa \tanh(z)}{k + i\kappa} \times \frac{k + i\kappa \tanh(z)}{k + i\kappa}\, dk$$

as well as the corresponding Jost functions f_s and f_r. Substituting $f = f_s + f_r$ and $u = u_s + u_r$ into Eqs. (35)–(36) allows us to derive the second-order correction to the Jost coefficient a (of order ε^2/k). This step puts into evidence that new solitons with parameters of order ε^2 are generated.

The final part of the proof consists in checking a posteriori the adiabatic hypothesis, that is to say proving that the radiated wavepacket which has been determined here above has actually no noticeable influence on the evolutions (35)–(36) of the Jost coefficients a and b. We must estimate the components of the functions α and β which have been neglected until now and which are related to the interplay of the main soliton, the soliton gas, and the radiation. These are technical calculations which are based upon the mixing properties of the process V. Some of the estimates are qualitatively similar to the ones that are presented in ref. 10 for the randomly perturbed nonlinear Schrödinger equation. However there are noticeable differences. The most important difference concerns the existence of a soliton gas. A qualitative study shows that the gas has low amplitude of order ε^3. It does not have an influence onto the main soliton for propagation times of order ε^{-2}. The second difference is concerned with the asymmetric dispersion relation of the linear KdV equation (with a third-order dispersion). The dispersion relation is such that the radiation has very small amplitude in the right-half line of the point where it has been emitted (If radiation is emitted at point $x=0$ at time $t=0$, then it will decay as $(xt)^{-1/4}\exp(-2x^{3/2}/(3t^{1/2}))$ for $x \gg t^{1/3} \gg 1$). As a consequence, after a time of order $|\ln \varepsilon|$ after the emission of some radiation, the main soliton and the generated radiation have no noticeable interaction so that the interaction process has actually a very short memory. This allows us to derive sufficient estimates for the components of the coupling coefficients α and β that originate from the interplay of the main soliton, the soliton gas, and the radiation. We shall not give the detailed derivations of these estimates because they consist of lengthy calculations that are specific to the perturbation that is under consideration. This means that the rigorous way requires to deal with the three types of perturbations of the model (30)–(32) separately. Furthermore, the first steps of the proof of Proposition 3.1 are common with the proofs of the forthcoming propositions that are devoted to various sorts of perturbations, but we should also check the adiabatic hypothesis for each new type of perturbations. Since this work was performed in ref. 10 for a randomly perturbed NLS equation, and the technical estimates are essentially similar although the details are different, we thought it better to refer the interested reader to this paper for an example of the technical estimates that are necessary for the check of the adiabatic hypothesis, while we provide here the reader with the main steps of the proof of Proposition 3.1 as well as we point out the main differences with the detailed proof of the analogous results for the random NLS equation studied in ref. 10.

3.4. Effective Dynamics of the Soliton Parameter

We shall now study the evolution of the parameter of the transmitted soliton. By Proposition 3.1 this evolution is given by (37). We aim at exhibiting the relevant characteristics of this deterministic ordinary differential equation.

Short-Range Correlation. Let us first assume that the perturbation has short range correlation in the sense that frequencies of order $8\kappa^3$ are smaller than the bandwidth of the perturbation. Accordingly we can substitute $\sigma_j^2 t_c := \hat{\gamma}_j(0)$ for $\hat{\gamma}_j(\omega)$ in the expressions of the C_j's:

$$\frac{d\kappa_l}{dt} = -\frac{512}{315}\left(\sigma_2^2+\sigma_3^2\right) t_c \kappa_l^7, \tag{47}$$

with the initial condition imposed by the incoming soliton: $\kappa_l(0) = \kappa_0$. It thus appears that the decay of the parameter κ is algebraic:

$$\kappa_l(t) = \kappa_0 \left(1+\frac{t}{T_1}\right)^{-1/6}, \qquad T_1 = \frac{105}{1024\left(\sigma_2^2+\sigma_3^2\right) t_c \kappa_0^6}. \tag{48}$$

We can be more precise about the other components of the scattered wave. The spectrum of the radiation is centered around 0. By the conservation of the total mass and energy of the wave we can establish the three following identities. The mass (resp. energy) of the main soliton is:

$$N_s(t) = -4\kappa_0 \left(1+\frac{t}{T_1}\right)^{-1/6}, \qquad E_s(t) = \frac{16\kappa_0^3}{3}\left(1+\frac{t}{T_1}\right)^{-1/2}.$$

The mass (resp. energy) of the radiation is:

$$N_r(t) = \frac{9\kappa_0}{2}\left(1-\left(1+\frac{t}{T_1}\right)^{-1/6}\right), \qquad E_r(t) = \frac{16\kappa_0^3}{3}\left(1-\left(1+\frac{t}{T_1}\right)^{-1/2}\right).$$

The mass (resp. energy) of the soliton gas is:

$$N_g(t) = -\frac{17\kappa_0}{2}\left(1-\left(1+\frac{t}{T_1}\right)^{-1/6}\right), \; E_g(t) = 0.$$

For $t \gg T_1$, we get that N_g tends to the limit value $N_{g,\,\mathrm{lim}} = -17\kappa_0/2$. For $t \gg T_1$, the incoming soliton has almost vanished. Radiation has been emitted with mass $N_{r,\,\mathrm{lim}} = 9\kappa_0/2$. A soliton gas with mass $N_{g,\,\mathrm{lim}}$ has been created. Note that the mass of the soliton gas is larger (in absolute value) than the mass of the incoming soliton, and that the balance $-4\kappa_0 = N_{r,\,\lim} + N_{g,\,\lim}$ holds true.

Long-Range Correlation. Let us now assume that the perturbation has long-range correlation in the sense that frequencies of order $8\kappa^3$ lie in the tail of the power spectral density of the perturbation. The decay of the parameter then depends on the decay of the tail of the power spectrum of the perturbation. Let us assume for instance that the spectrum decays as:

$$\hat{\gamma}(\omega) \simeq c_p t_c^{1-p} \, |\omega|^{-p}, \qquad \text{for} \quad |\omega| \, t_c \gg 1.$$

Note that $p > 1$ since $\mathbb{E}[V^2(0)] < \infty$. Then κ_l obeys the equation:

$$\frac{d\kappa_l}{dt} = -\bar{c}_p t_c^{1-p} \kappa_l^{7-3p}, \qquad \bar{c}_p = \frac{2^{6-3p}\pi}{9} \, c_p \int \frac{s^{4-p}(1+s^2)^{2-p}}{\sinh^2(\pi s)} \, ds.$$

If $p \in (1, 2)$, then the soliton parameter κ_l decays as $t^{-1/(6-3p)}$:

$$\kappa_l(t) = \kappa_0 (1 + \bar{c}_p \kappa_0^{6-3p} (6-3p) \, t_c^{1-p} t)^{-1/(6-3p)}.$$

If $p = 2$, the decay is exponential:

$$\kappa_l(t) = \kappa_0 \, \exp\left(-\bar{c}_2 \, \frac{t}{t_c} \right), \tag{49}$$

where $\bar{c}_2 = c_2/54$. This case is especially relevant, since a ω^{-2} decay for the power spectral density is very common. For instance the two following classical models belong to this class. Both of them are step-wise constant processes that are constant over elementary time intervals $[t_j, t_{j+1})$ and take values v_j over the jth interval. The v_j's are assumed to be independent random variables with zero-means and variances σ^2. First assume that the t_j are deterministic and that all intervals have the same duration t_c: $t_j = jt_c$. Then the autocorrelation function is:

$$\gamma(t) = \sigma^2 \left(1 - \frac{|t|}{t_c} \right) \mathbf{1}_{|t| \leqslant t_c}, \tag{50}$$

so that the power spectral density is:

$$\hat{\gamma}(\omega) = \sigma^2 t_c \, \operatorname{sinc}\left(\frac{\omega t_c}{2} \right)^2,$$

where $\mathrm{sinc}(s) = \sin(s)/s$. Second assume that the t_j are random and have independent increments $\tau_j := t_j - t_{j-1}$ that obey exponential distribution with mean t_c. Then the autocorrelation function is:

$$\gamma(t) = \sigma^2 \exp\left(-\frac{|t|}{t_c}\right),$$

so that the power spectral density is:

$$\hat{\gamma}(\omega) = \sigma^2 \frac{2t_c}{1+\omega^2 t_c^2}.$$

If $p > 2$, then the decay of κ_l is so dramatic that it apparently leads to the disintegration of the soliton at finite time T_p:

$$\kappa_l(t) = \kappa_0 \left(1 - \frac{t}{T_p}\right)^{1/(3p-6)}, \qquad T_p = \frac{\kappa_0^{3p-6} t_c^{p-1}}{(3p-6)\,\bar{c}_p}.$$

However, as κ decreases, the typical frequency $8\kappa^3$ also decays and finally enters into the bandwidth of the power spectrum of the perturbation. We then get back the regime in $t^{-1/6}$, whatever p.

Integration of the Effective Ordinary Differential Equation. In the above paragraphs we have reported two remarkable domains when the typical frequency $8\kappa^3$ of the incoming soliton lie either in the tail or at the top of the power spectral density of the perturbation. The intermediate case (when the product of the typical frequency $8\kappa^3$ times the coherence time t_c of the perturbation is of order 1) is also worth studying. In order to study this configuration we are going to solve numerically the ordinary differential equation (37) for different incoming solitons. Any software like Maple or Matlab can do the job accurately.

Let us first consider the step-wise constant process described here above whose autocorrelation function is given by Eq. (50). Figure 2 plots the evolutions of the energy of the soliton, defined as $16\kappa_l^3(t)/3$. The initial parameter κ_0 is chosen to be equal to 0.5, while the correlation time t_c varies from 0.2 to 250. These figures put into evidence the following features. If t_c is small (i.e., $8\kappa_0^3 t_c$ is smaller than 1), then the energy decays as $t^{-1/2}$ as described by formula (48). If t_c is large, then we first observe an exponential decay as proposed by Eq. (49), which is only transitory and is replaced by the $t^{-1/2}$ decay when the product $8\kappa_l(t)^3 t_c$ becomes smaller than 1.

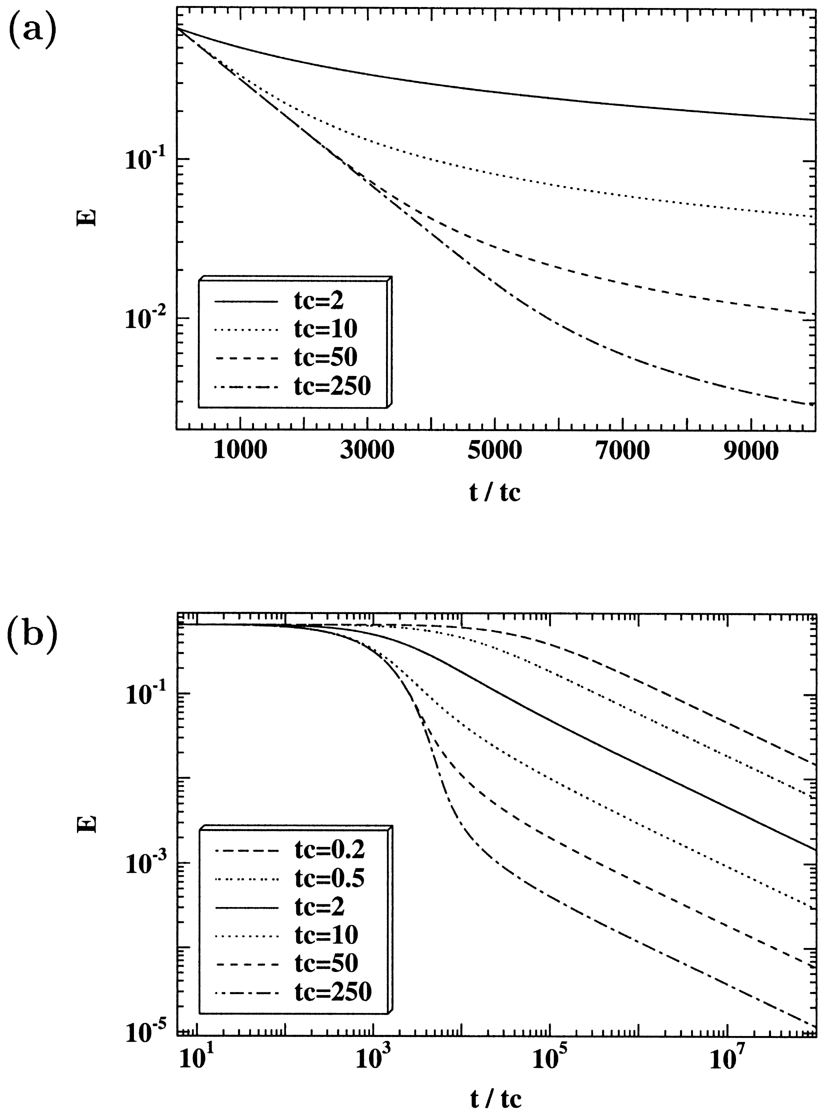


Fig. 2. Energy of the soliton during the propagation. The lines correspond to the theoretical values computed from the system (37). We assume that $\sigma^2 = 10^{-2}$. We take different values for the correlation time t_c. (a) (in lin-log scale) the energy is shown to decay exponentially if $8\kappa_0^3 t_c \geqslant 1$. (b) (in log-log scale), it is shown on the one hand that, if $8\kappa_0^3 t_c \geqslant 1$, then the behavior of the energy for long times switch from the exponential decay to the algebraic decay $t^{-1/2}$. On the other hand, if $8\kappa_0^3 t_c \leqslant 1$, then no exponential decay of the energy can be observed, but the algebraic decay $t^{-1/2}$ is noticeable.

We consider the case of a perturbation of the nonlinear coefficient and take a autocorrelation function with Gaussian shape:

$$\gamma_2(t) = \sigma^2 \exp\left(-\frac{t^2}{2t_c^2}\right)$$

with correlation time t_c. The power spectral density is then:

$$\hat{\gamma}_2(\omega) = \sigma^2 \sqrt{2\pi}\, t_c \exp\left(-\frac{\omega^2 t_c^2}{2}\right).$$

Figure 3 plots the evolutions of the energy of the soliton as a function of t. The parameter κ_0 is chosen at some fixed value for all lines, equal to 0.5, but the correlation time t_c varies from 0.05 to 50.

The fastest initial decay is obtained for $1/t_c$ of the same order of the typical frequency $8\kappa^3$ (see Fig. 3a). If $8\kappa^3 t_c \ll 1$, then the decay is algebraic (48), and it is all the slower as t_c is smaller. If $8\kappa^3 t_c \gg 1$, then the decay is initially all the slower as t_c is larger, since $F(\kappa) \simeq -\pi^2/(3\times 2^6 t_c^4 \kappa^8)$. However, as κ decreases, the typical frequency $8\kappa^3$ also decays, and exits the tail of the power spectral density. Then the function $F(\kappa) \simeq -(2^{19/2}\pi^{1/2}\kappa^7 t_c)/(9\times 35)$ which is all the larger as t_c is larger. Thus the energy decay is finally all the faster as t_c is larger (see Fig. 3b).

3.5. Numerical Simulations

The results in the previous sections are theoretically valid in the limit case $\varepsilon \to 0$, where the amplitudes of the perturbations go to zero and the size of the random system goes to infinity. In this section we aim at showing that the asymptotic behaviors of the soliton can be observed in numerical simulations in the case where ε is small, more precisely smaller than any other characteristic scale of the problem. We use a Fourier or pseudo-spectral method to simulate the perturbed KdV equation.[30] This numerical algorithm provides accurate and stable solutions to a large class of systems of ordinary and partial differential equations.[31] The physical interval $[0, L]$ is discretized by M equidistant points, with spacing $h = L/M$. The function u is numerically defined only on these points. With this scheme the space derivatives are determined by using Fast Fourier Transforms $\mathscr{F}$. u_x is evaluated as $\mathscr{F}^{-1}(iv\mathscr{F}u)$, u_{xxx} as $-\mathscr{F}^{-1}(iv^3\mathscr{F}u)$, and so on. Combined with a leap frog time step, the KdV equation is approximated by:

$$u(x, t+\Delta t) - u(x, t-\Delta t) + 2i(V_1(t) + 6(V_2(t)+1)\, u(t, x)))\, \Delta t \mathscr{F}^{-1}(v\mathscr{F}u)$$
$$- 2i\Delta t(1 - V_3(t))\, \mathscr{F}^{-1}(v^3 \mathscr{F}u) = 0.$$

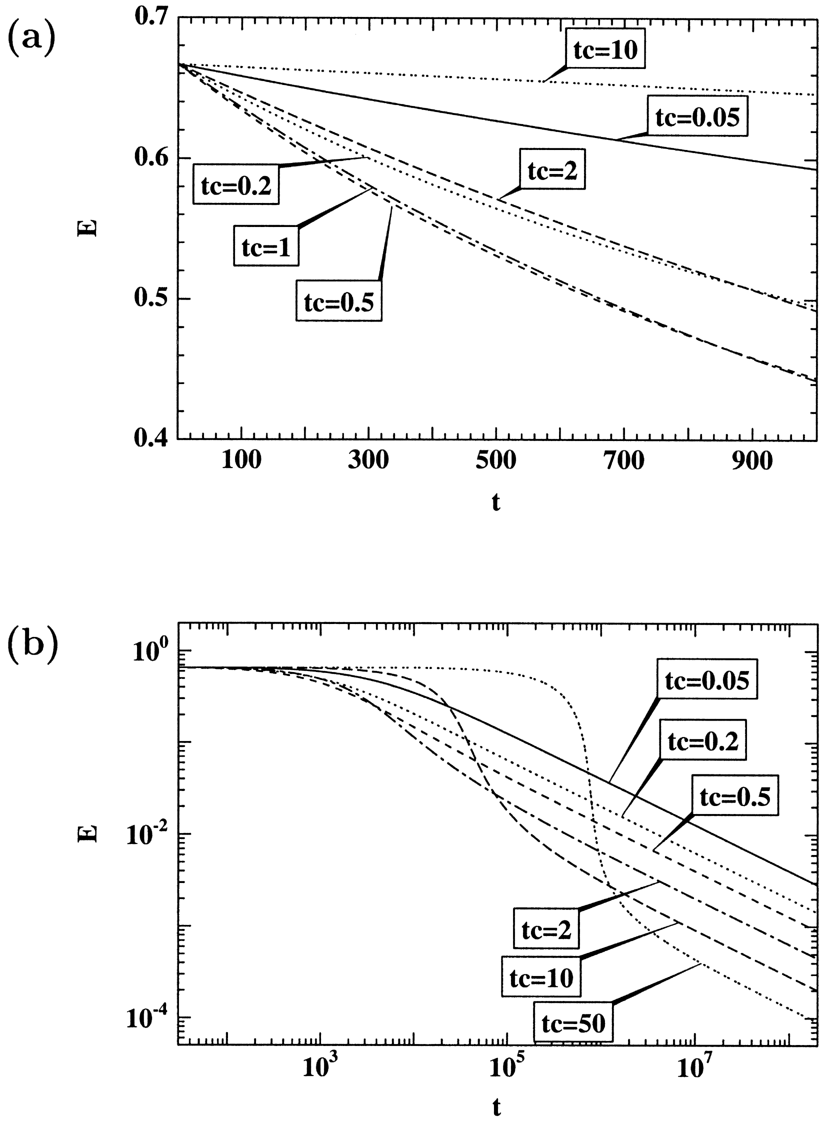


Fig. 3. Energy of the soliton during the propagation. The lines correspond to the theoretical values computed from the system (37). We assume that $\sigma^2 = 10^{-2}$. We take different values for the correlation time t_c. (a) (in lin-lin scale) the energy is shown to decay fast when $8\kappa^3 t_c$ is of order 1. (b) (in log-log scale), it is shown that the behavior of the energy for longer times obeys the algebraic decay $t^{-1/2}$ and is all the faster as t_c is larger.

Following ref. 32 we make a modification in the last term and take $\mathscr{F}^{-1}(\sin(\nu^3 \Delta t)\, \mathscr{F}u)$ instead of $\Delta t \mathscr{F}^{-1}(\nu^3 \mathscr{F}u)$, which gives more stability for the high wavenumbers ν.

Since the time domain is planned to be very long, of order ε^{-2}, the solution will propagate over distances of order ε^{-2}, so that we would have to take a computational domain of size $L \sim \varepsilon^{-2}$. In order to deal with a tractable problem, we use a shifting computational domain which is always centered at the center of the energy of the solution. Moreover we impose boundaries of this domain which absorb outgoing waves. This can be readily achieved by adding a negative potential which is smooth so as to reduce reflections:

$$V_{\text{abs}}(x) = \begin{cases} V_{\max} \tanh^2\left(\dfrac{L_0 - x}{L_0}\right), & \text{if } \; 0 \leqslant x < L_0, \\ 0, & \text{if } \; L_0 \leqslant x \leqslant L_1 - L_0, \\ V_{\max} \tanh^2\left(\dfrac{L_1 - L_0 - x}{L_0}\right), & \text{if } \; L_1 - L_0 < x \leqslant L_1, \end{cases} \tag{51}$$

where L_1 (resp. 0) is the left (resp. right) end of the computational domain, and $[0, L_0]$ (resp. $[L_1 - L_0, L_1]$) is the left (resp. right) absorbing slab.

We first assume in this section that the random perturbation V_2 is a random step-wise constant process, which takes values v_j over the elementary intervals $[nt_c, (n+1)\, t_c)$. Here $(v_j)_{j=0,\dots,J-1}$ is a sequence of independent and identically distributed variables, which obey uniform distributions over the interval $[-1, 1]$, so that $\hat{\gamma}_2(\omega) = (t_c/3)\, \text{sinc}^2(\omega t_c/2)$ where $\text{sinc}(s) = \sin(s)/s$. We take $\varepsilon = 0.1$. The time $T = Jt_c$ will be chosen so large (of order ε^{-2}) that we can observe the effect of the small perturbation $\varepsilon R_2(u)$. We measure the energy of the solution during the propagation, as well as the envelope of the transmitted solution, that we can compare with the envelope of the incident soliton.

We perform different simulations where the initial wave at time $t = 0$ is a pure soliton with parameter κ_0 centered at $x_s = L/2$. In the first one we simulate the homogeneous KdV equation (1), which admits as an exact solution (23). We can therefore check the accuracy of the numerical method, since we can see that the computed solution maintains a very close resemblance to the initial soliton (data not shown), while the mass, energy and Hamiltonian are almost constant (up to 10^{-3} after a propagation time of 10^5). The other simulations are carried out with various values of the coherence time t_c and different realizations of the random perturbation with $\varepsilon = 0.1$ or $\varepsilon = 0.2$. The simulated evolutions of the soliton parameters are presented in Fig. 4 and compared with the theoretical evolutions given by (37) in the scale t/ε^2. We can observe in particular in Fig. 4b the algebraic decay of the energy of the soliton.

We now assume that the random perturbation V_2 is a stationary random process with Gaussian statistics and Gaussian autocorrelation function:

$$\gamma_2(t) = \exp\left(-\frac{t^2}{2t_c^2}\right),$$

so that the power spectral density $\hat{\gamma}_2$ has also Gaussian shape. We take also $\varepsilon = 0.1$. Figure 5 plots the energy of the output soliton versus time.

It thus appears that the numerical simulations are in very good agreement with the theoretical results. The simulated plots follow very closely the theoretical ones. This is partly due to the fact that the perturbed equation preserves the total energy:

$$E_{\text{tot}} = \frac{16}{3}\sum_j \kappa_j^3 + \int_{-\infty}^{\infty} (2k)^2\, n(k)\, dk, \tag{52}$$

where $n(.)$ is the density of scattered mass. This implies stability for the parameter of the main soliton. All these results confirm that the effective ordinary differential equation (37) describes with accuracy the transmission of a soliton in a KdV system with small random time perturbations.

4. POSITION-DEPENDENT PERTURBATIONS OF THE KDV EQUATION THAT PRESERVE THE HAMILTONIAN

4.1. The Perturbed Model

We consider a perturbed Korteweg–de Vries equation with a non-zero right-hand side:

$$u_t + u_{xxx} - 6uu_x = \varepsilon R(u)(t, x). \tag{53}$$

The small parameter $\varepsilon \in (0, 1)$ characterizes the amplitude of the perturbation. Here we assume that

$$R(u) = (V_1(x)\, u)_x + (V_2(x)\, u^2)_x + (V_3(x)\, u_x)_{xx}, \tag{54}$$

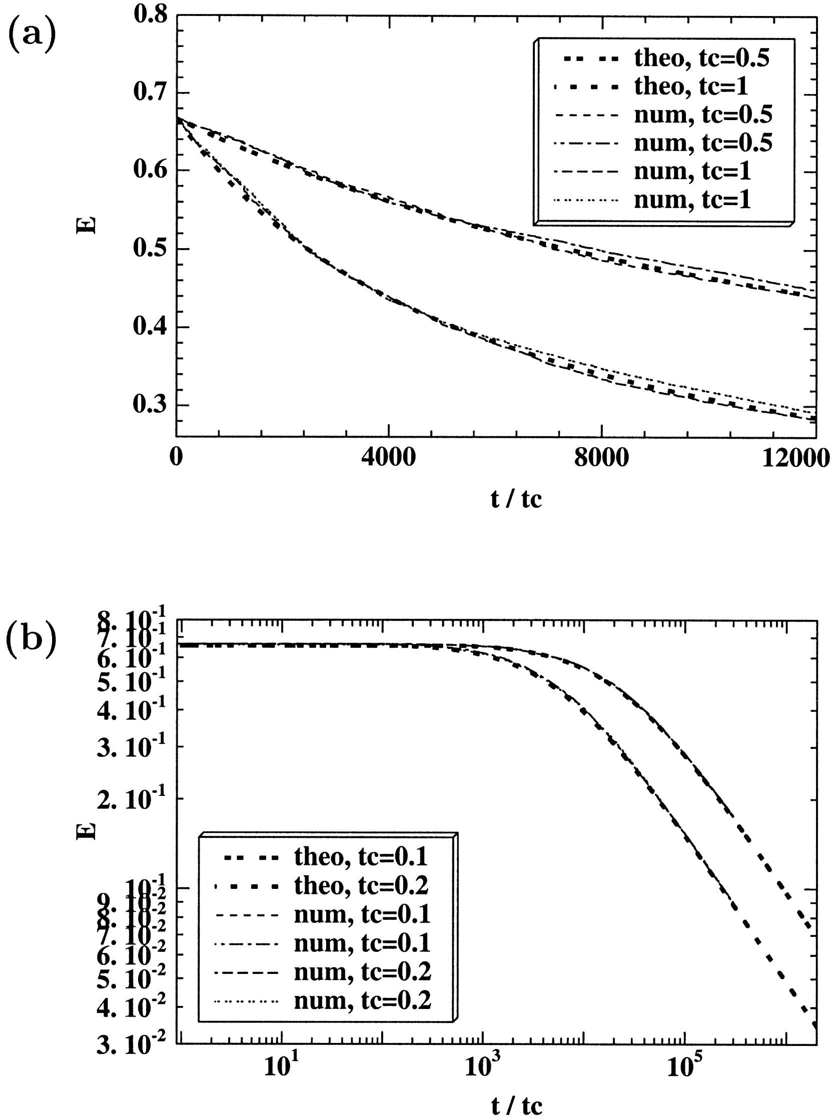


Fig. 4. Energy of the soliton whose initial parameter is $\kappa_0 = 0.5$ (energy $E_0 = 2/3$), driven by a random perturbation $6\varepsilon V_2 u u_x$ where V_2 is a random step-wise constant process with coherence time t_c and amplitude $\varepsilon = 0.1$. The scales are linear in (a) and logarithmic in (b). The thick dotted lines represent the theoretical coefficients of the transmitted soliton. In thin dashed and dotted lines are plotted the simulated energies of the solitons for different realizations of the random perturbation.

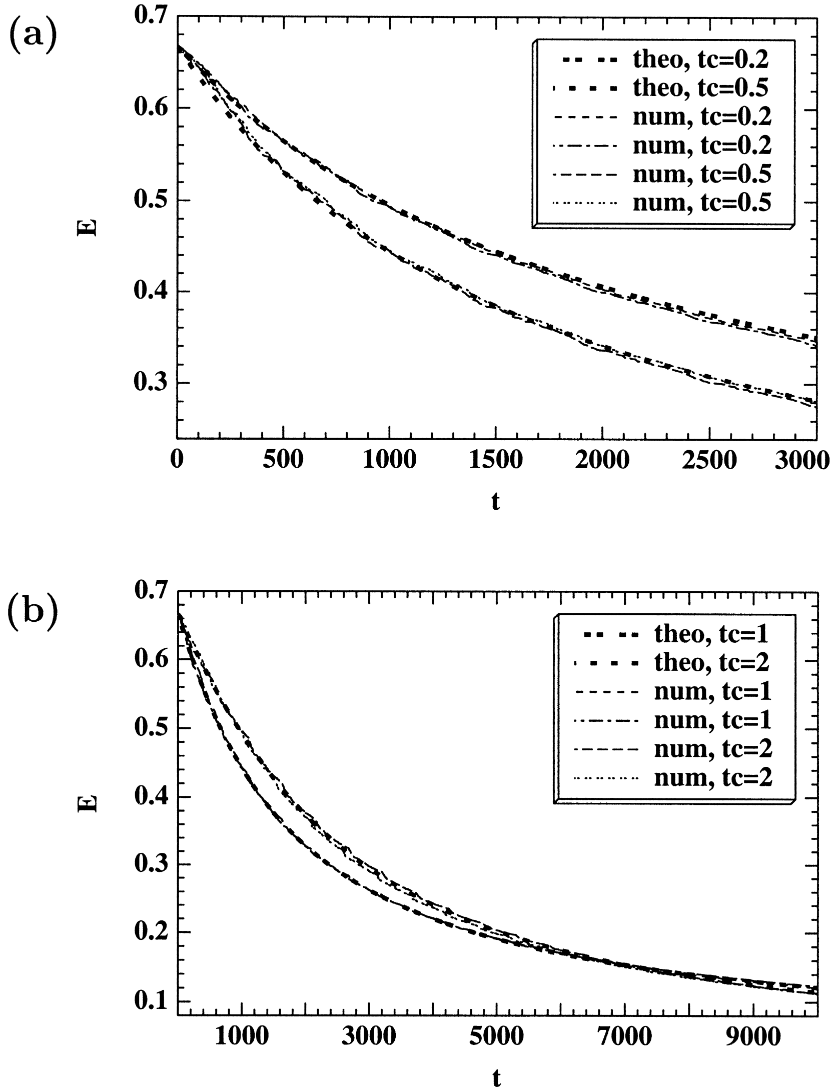


Fig. 5. Energy of the soliton whose initial parameter is $\kappa_0 = 0.5$ (energy $E_0 = 2/3$), driven by a random perturbation $6\varepsilon V_2 u u_x$ where V_2 is a random process with Gaussian statistics, amplitude $\varepsilon = 0.1$, and coherence time t_c. The thick dotted lines represent the theoretical coefficients of the transmitted soliton. In thin dashed and dotted lines are plotted the simulated energies of the solitons for different realizations of the random perturbation. (a) It can be seen that the decay is all the faster as t_c is larger, since this is the regime $8\kappa^3 t_c \leqslant 1$. (b) It can be seen that the decay is first fast for the smallest t_c, since this is the regime $8\kappa^3 t_c \geqslant 1$. However, as κ decays, the regime $8\kappa^3 t_c \leqslant 1$ is recovered and then the decay is all the faster as t_c is larger.

which means that Eq. (53) can be derived from the perturbed Hamiltonian:

$$H_{\text{per}} = \int \varepsilon V_1(x)\, u^2 + (2 + \tfrac{2}{3}\, \varepsilon V_2(x))\, u^3 + (1 - \varepsilon V_3(x))\, u_x^2, \tag{55}$$

where V_1 (resp. V_2, V_3) stands for position-dependent fluctuations of the velocity (resp. the nonlinear coefficient, the dispersion). The perturbations V_j are assumed to be zero-mean, stationary and ergodic processes. For the sake of simplicity we also assume that they are independent processes. The autocorrelation function of V_j is denoted by:

$$\gamma_j(x) := \mathbb{E}[V_j(0)\, V(x)] = \mathbb{E}[V_j(x')\, V_j(x' + x)], \tag{56}$$

where $\mathbb{E}$ stands for the statistical average with respect to the stationary distribution of V_j. We assume the technical ϕ-mixing condition with $\phi \in L^{1/2}$ (see Section 3.1). We introduce the Fourier transform of the autocorrelation function of the perturbation V_j:

$$\hat{\gamma}_j(k) := \int_{-\infty}^{\infty} \gamma_j(x) \cos(kx)\, dx, \tag{57}$$

which is nonnegative real-valued since it is proportional to the power spectral density of V_j. Here we have to discuss the continuity properties that are required for the processes V_j. In case of time-dependent perturbations no smoothness is required, but due to the particular form (54) derivatives of the processes V_j should exist. The processes V_1 and V_2 are required to possess continuous and differentiable realizations. For instance stationary processes with Gaussian statistics that satisfy $\int \hat{\gamma}_j(k)\, k^{2+\delta}\, dk < \infty$, $\delta > 0$, $j = 1, 2$, are suitable. The process V_3 is required to possess continuous and twice differentiable realizations. For instance stationary processes with Gaussian statistics that satisfy $\int \hat{\gamma}_3(k)\, k^{4+\delta}\, dk < \infty$, $\delta > 0$, are suitable.

In such conditions Proposition 3.1 holds true. The only difference is that the function F is here equal to:

$$F(\kappa) = \frac{1}{64\kappa^4} \sum_{j=1}^{3} \int_{\mathbb{R}} (2k)^4\, C_j(\kappa, k)\, dk, \tag{58}$$

where the function C_j is the spectral mass density scattered by the soliton with parameter κ per unit time due to the perturbation R_j. The exact expressions of the C_j's are the following:

$$C_1(\kappa, k) = \frac{4\pi k^6 \left(1+\frac{k^2}{\kappa^2}\right)^6}{9\kappa^6 \sinh^2\left(\pi \frac{k^3}{\kappa^3}\right)} \hat{\gamma}_1(K(\kappa, k)), \tag{59}$$

$$C_2(\kappa, k) = \frac{32\pi k^6 \left(1+\frac{k^6}{\kappa^6}\right)^2 \left(9+10\frac{k^2}{\kappa^2}+5\frac{k^4}{\kappa^4}+\frac{k^6}{\kappa^6}\right)^2}{45^2\kappa^2 \sinh^2\left(\pi \frac{k^3}{\kappa^3}\right)} \hat{\gamma}_2(K(\kappa, k)), \tag{60}$$

$$C_3(\kappa, k) = \frac{64\pi k^6 \left(1+\frac{k^2}{\kappa^2}\right)^2 \left(6-\frac{k^2}{\kappa^2}\left(1+\frac{k^2}{\kappa^2}\right)^4\right)^2}{15^2\kappa^2 \sinh^2\left(\pi \frac{k^3}{\kappa^3}\right)} \hat{\gamma}_3(K(\kappa, k)), \tag{61}$$

where K is the function:

$$K(\kappa, k) = \frac{2k(k^2+\kappa^2)}{\kappa^2}. \tag{62}$$

The function F is positive-real valued, which means that the parameter κ of the output soliton increases with time. This configuration is completely different from the time perturbations where the parameter of the soliton decays. Note also that spatial fluctuations of the velocity involve modifications of the soliton dynamics, unlike the time-dependent case. Furthermore spatial fluctuations of the nonlinear coefficient and dispersion are not equivalent, although they are both of the same form: $C_j(\kappa, k) = \kappa^4 C_j(1, k/\kappa)\, \hat{\gamma}_j(\kappa K(1, k/\kappa))$, $j = 2, 3$.

4.2. Effective Dynamics of the Soliton Parameter

Long-Range Correlation. We first assume that the typical wavenumber $2\kappa_0$ of the incoming soliton lies in the tail of the power spectral density of the perturbation. For the forthcoming analysis, we also assume that the power spectral density decays as $c_p l_c^{1-p}\, |k|^{-p}$, $|k|\, l_c \gg 1$. Note that $p > 3$ since we assume that the V_j's are differentiable.

In the case of a perturbation V_1 of the velocity, the parameter κ grows algebraically as $t^{1/p}$. In terms of the propagation distance L, the decay is algebraic with power $3/(p+2)$: $E(L) \sim L^{3/(p+2)}$.

In the case of a perturbation V_j, $j=2,3$ of the nonlinear coefficient or of the dispersion, the parameter κ grows algebraically as $t^{1/(p-4)}$ if $p>4$, and exponentially if $p=4$. If $p<4$, then the soliton will apparently blow up in finite time, as $(1-t/T_p)^{1/(p-4)}$. The blow-up time T_p is proportional to $l_c^{p-1}\kappa_0^{p-4}$. However, in terms of the propagation distance L, the growth of the energy is algebraic with power $3/(p-2)$: $E(L)\sim L^{3/(p-2)}$.

Short-Range Correlation. We now assume that the typical wavenumber $2\kappa_0$ of the incoming soliton is smaller than the bandwidth of the power spectral density of the perturbation. In the case of a perturbation V_1 of the velocity, the parameter κ first grows exponentially. However, as κ increases, the typical wavenumber 2κ also increases, and finally reaches the tail of the power spectral density of the perturbation. We then enter the long-range correlation regime.

In the case of a perturbation V_2 (resp. V_3) of the nonlinear coefficient (resp. the dispersion), the parameter κ first grows algebraically as $t^{1/4}$. Here also, as κ increases, the typical wavenumber 2κ reaches the tail of the power spectral density of the perturbation. We then enter the long-range correlation regime.

4.3. Numerical Simulations

The numerical scheme is the same as in Section 3.5. We consider perturbations V_j that are random processes with Gaussian statistics, Gaussian autocorrelation function, and correlation length l_c:

$$\gamma_j(x)=\exp\left(-\frac{x^2}{2l_c^2}\right)$$

The evolution of the energy of the soliton is plotted in Figs. 6 and 7 for different correlation lengths and different types of perturbations. The simulated values are compared with the theoretical predictions, which shows excellent agreement.

We would like to comment on the oscillations of the energy that can be observed in Figs. 6 and 7, while such fluctuations were absent in the time-dependent perturbations. Indeed, in the configuration at hand, the perturbed Hamiltonian H_{per} is preserved, which also reads as:

$$H_{\text{per}}=\int_{\mathbb{R}}(2k)^4\,n(k)\,dk-\frac{2}{5}\sum_{j=1}^{J}(2\kappa_j)^5+\varepsilon\int V_1(x)\,u^2+\tfrac{2}{3}V_2(x)\,u^3-V_3(x)\,u_x^2.$$

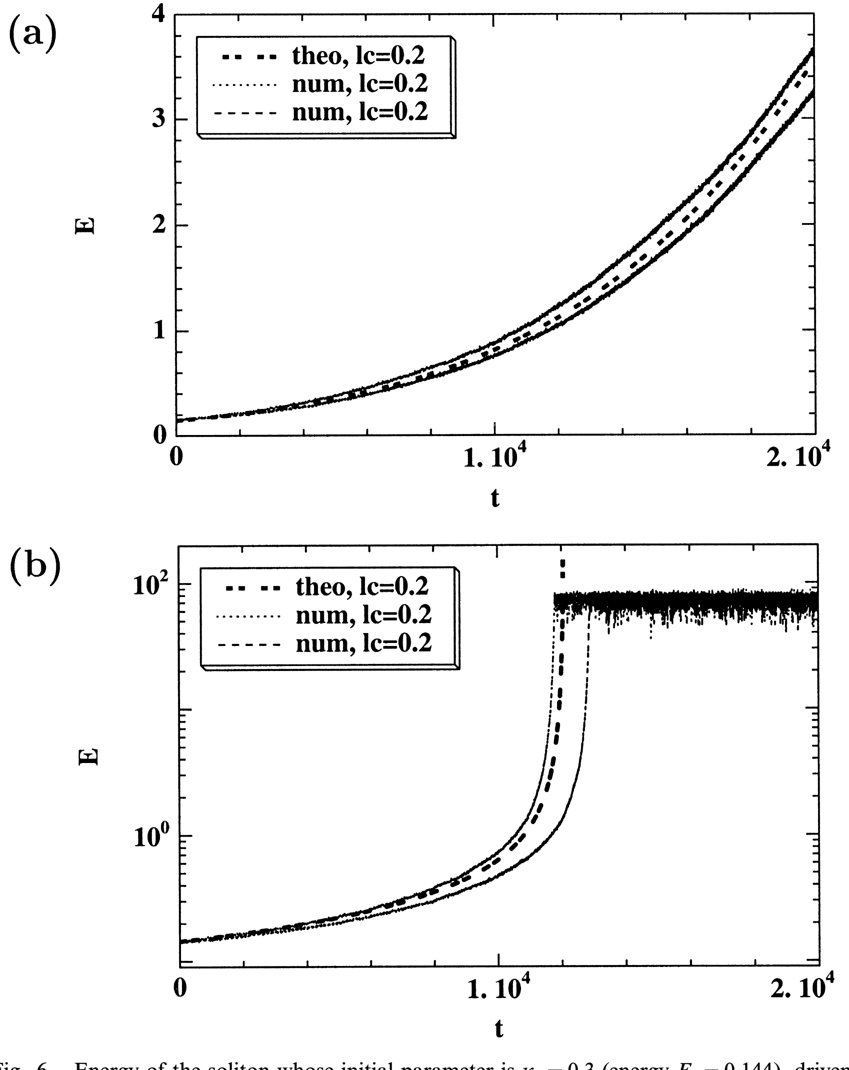


Fig. 6. Energy of the soliton whose initial parameter is $\kappa_0 = 0.3$ (energy $E_0 = 0.144$), driven by a random perturbation with Gaussian statistics and correlation length $l_c = 0.2$. The thick dotted lines represent the theoretical predictions. In thin dashed and dotted lines are plotted the simulated energies of the solitons for different realizations of the random perturbations. Figure 6(a) presents results corresponding to a perturbation V_1 of the velocity (see Eq. (54)) with $\varepsilon = 0.025$. We can observe the exponential growth of the soliton energy. Figure 6(b) presents results corresponding to a perturbation V_2 of the nonlinear coefficient with $\varepsilon = 0.1$. We can observe the blow up of the energy in finite time. The saturation of the energy at the value 80 is a numerical artifact, since the numerical scheme is not designed to deal with such an intense and narrow wave solution.

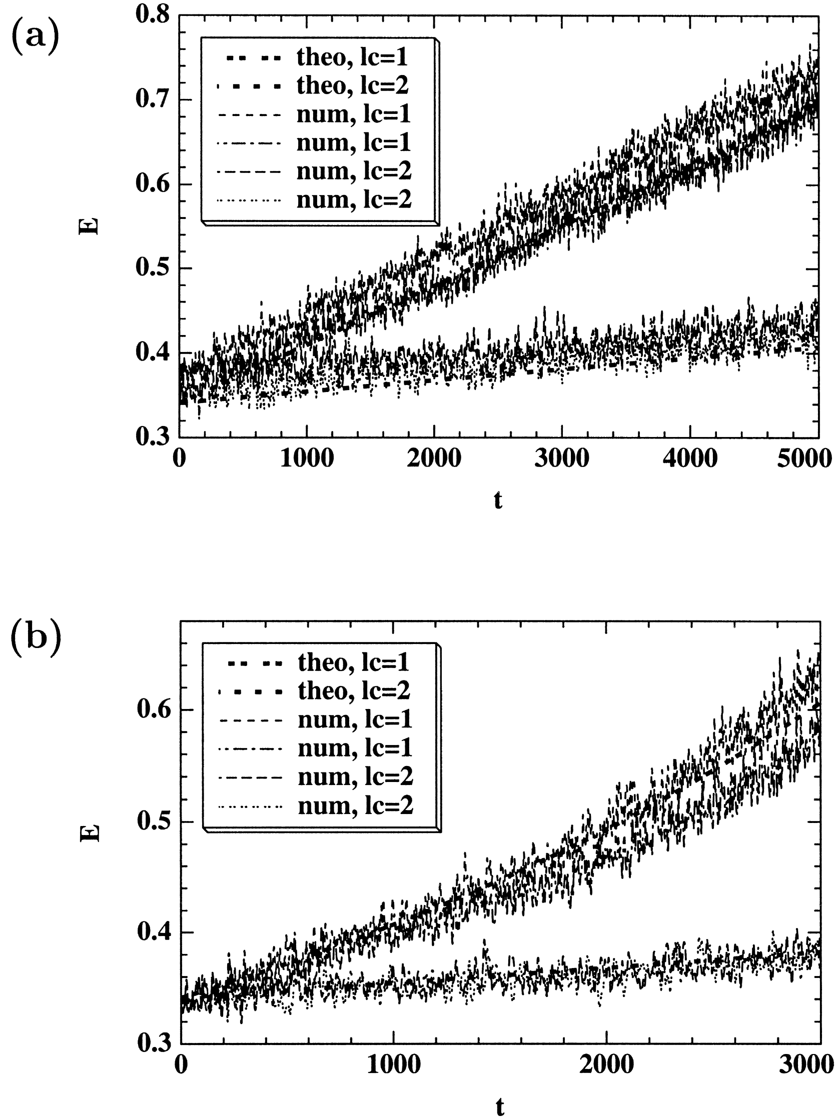


Fig. 7. The same as in Figure 6, but the random perturbation with Gaussian statistics has correlation length $l_c = 1$ or 2, and the incoming soliton has parameter is $\kappa_0 = 0.4$ (energy $E_0 = 0.341$).

The last term of the right-hand side is negligible in the asymptotic framework $\varepsilon \to 0$, but when $\varepsilon > 0$ it gives rise to local fluctuations of the unperturbed Hamiltonian as defined by (22), hence local fluctuations of the parameter κ and the corresponding energy $16\kappa^3/3$. The fluctuations are important when the correlation length of the medium is of the same order or even larger than the soliton width (see Fig. 7). They are smoothed when the correlation length is much smaller than the soliton width (see Fig. 6).

5. POSITION-DEPENDENT PERTURBATIONS OF THE KDV EQUATION THAT PRESERVE THE ENERGY

We consider a perturbed Korteweg–de Vries equation with a non-zero right-hand side:

$$u_t + u_{xxx} - 6uu_x = \varepsilon R(u)(t, x). \tag{63}$$

The small parameter $\varepsilon \in (0, 1)$ characterizes the amplitude of the perturbation. Here we assume that

$$\begin{aligned} R(u) = {} & (V_1(x)\, u)_x + V_1(x)\, u_x + (V_2(x)\, u^2)_x + V_2(x)(u^2)_x \\ & + (V_3(x)\, u_x)_{xx} + (V_3(x)\, u_{xx})_x, \end{aligned} \tag{64}$$

which means that Eq. (63) conserves the energy of the solution:

$$\int uR(u)\, dx = 0$$

Note here that the total mass is not preserved. Once again V_1 (resp. V_2, V_3) stands for position-dependent fluctuations of the velocity (resp. the nonlinear coefficient, the dispersion).

Proposition 3.1 holds true in such conditions, at the expense to take the following definition for the function F:

$$F(\kappa) = -\frac{1}{16\kappa^2} \sum_{j=1}^{3} \int_{\mathbb{R}} (2k)^2\, C_j(\kappa, k)\, dk, \tag{65}$$

where the functions C_j are the mass density scattered by the soliton with parameter κ per unit time due to the perturbation R_j. The exact expressions of the C_j's are the following:

$$C_1(\kappa, k) = \frac{4\pi k^6 \left(1+\frac{k^2}{\kappa^2}\right)^2}{\kappa^6 \sinh^2\left(\pi \frac{k^3}{\kappa^3}\right)} \hat{\gamma}_1(K(\kappa, k)), \tag{66}$$

$$C_2(\kappa, k) = \frac{64\pi k^6 \left(2+\frac{k^2}{\kappa^2}\right)^2 \left(1+\frac{k^6}{\kappa^6}\right)^2}{9\kappa^2 \sinh^2\left(\pi \frac{k^3}{\kappa^3}\right)} \hat{\gamma}_2(K(\kappa, k)), \tag{67}$$

$$C_3(\kappa, k) = \frac{64\pi k^6 \left(1+\frac{k^2}{\kappa^2}\right)^2 \left(3-\left(1+\frac{k^2}{\kappa^2}\right)^2\right)^2}{9\kappa^2 \sinh^2\left(\pi \frac{k^3}{\kappa^3}\right)} \hat{\gamma}_3(K(\kappa, k)), \tag{68}$$

where the function K is defined by Eq. (62).

Short-Range Correlation. In the case of a perturbation V_1 of the velocity, we get an exponential decay of the parameter κ. Note that, in terms of propagation distance, this means that the soliton will disintegrate in finite distance L_d,

$$L_d = \frac{2\kappa_0^2}{\sigma_1^2 l_c c_d}, \qquad c_d = \pi \int_{-\infty}^{\infty} \frac{s^8(1+s^2)^2}{\sinh^2(\pi s^3)} ds \simeq 0.24$$

or else that it cannot be transmitted through a slab whose length exceeds the critical value L_d. In the case of a perturbation V_2 (resp. V_3) of the non-linear coefficient (resp. the dispersion), we get an algebraic decay of the parameter κ as $t^{-1/4}$. In terms of the propagation distance L, the decay is algebraic with power $1/2$: $E(L) \sim L^{-1/2}$.

Long-Range Correlation. In the case of a perturbation V_1 of the velocity, we get an apparent disintegration of the soliton in finite time since the parameter κ behaves like $(1-t/T_p)^{1/p}$. However, as κ decays, the typical wavenumber 2κ becomes smaller than the bandwidth of the perturbation and leaves the tail of the power spectral density. Thus we get back the regime described in the previous section.

In the case of a perturbation V_2 (resp. V_3) of the nonlinear coefficient (resp. the dispersion), we can observe a slow decay of the soliton parameter as $t^{1/(p-4)}$ if $p<4$, and an exponential decay if $p=4$. If $p>4$, we put into evidence an apparent disintegration of the soliton if since the soliton parameter behaves like as $(1-t/T_p)^{1/(p-4)}$. However, by the same arguments as in the case of the perturbation V_1, we get back the short-range correlation regime when κ becomes small enough so that $2\kappa l_c < 1$.

Numerical Simulations. The numerical scheme is the same as in Section 3.5. We consider perturbations V_j that are random processes with Gaussian statistics, Gaussian autocorrelation function, and correlation length l_c. The evolution of the energy of the soliton is plotted in Figs. 8 and 9 for different correlation length and different types of perturbations. The simulated values are compared with the theoretical predictions, which shows excellent agreement.

Note that the total energy is preserved, so that the parameter κ can only decay and no local fluctuation can be observed in Figs. 8 and 9, unlike the configuration described in Section 4.3.

6. RANDOM PERTURBATIONS AND DAMPING

In this section we would like to show that the theoretical approach developed in the previous sections can still be applied in presence of damping, that is to say when there is no known integral of motion. Indeed a precise estimate of one quantity is actually sufficient to derive analogous results as those of Proposition 3.1. To illustrate this assertion we shall consider the case of a random potential:

$$u_t + u_{xxx} - 6uu_x = \varepsilon V(t)\, u, \tag{69}$$

where $\varepsilon \in (0, 1)$ characterizes the amplitude of the perturbation, and V is a time-dependent random potential. We shall denote by γ_V the autocorrelation function of V and by $\hat{\gamma}_V$ its power spectral density defined as the Fourier transform of γ_V. It is straightforward to establish that the energy of the wave is not preserved but varies randomly:

$$E_{\text{tot}}(t) = E_0 \exp\left(2\varepsilon \int_0^t V(s)\, ds\right), \tag{70}$$

where $E_0 = \frac{16}{3}\kappa_0^3$ is the energy of the incoming soliton. Once this identity is known, we can apply the very same approach as in Section 3. There then appears a dichotomy. Indeed the application of a standard diffusion approximation theorem shows that in the asymptotic $\varepsilon \to 0$, the total energy in the scale t/ε^2 evolves as $\bar{E}(t)$:

$$E_{\text{tot}}(t/\varepsilon^2) \xrightarrow{\varepsilon \to 0} \bar{E}(t) := E_0 \exp(2\sigma W_t),$$

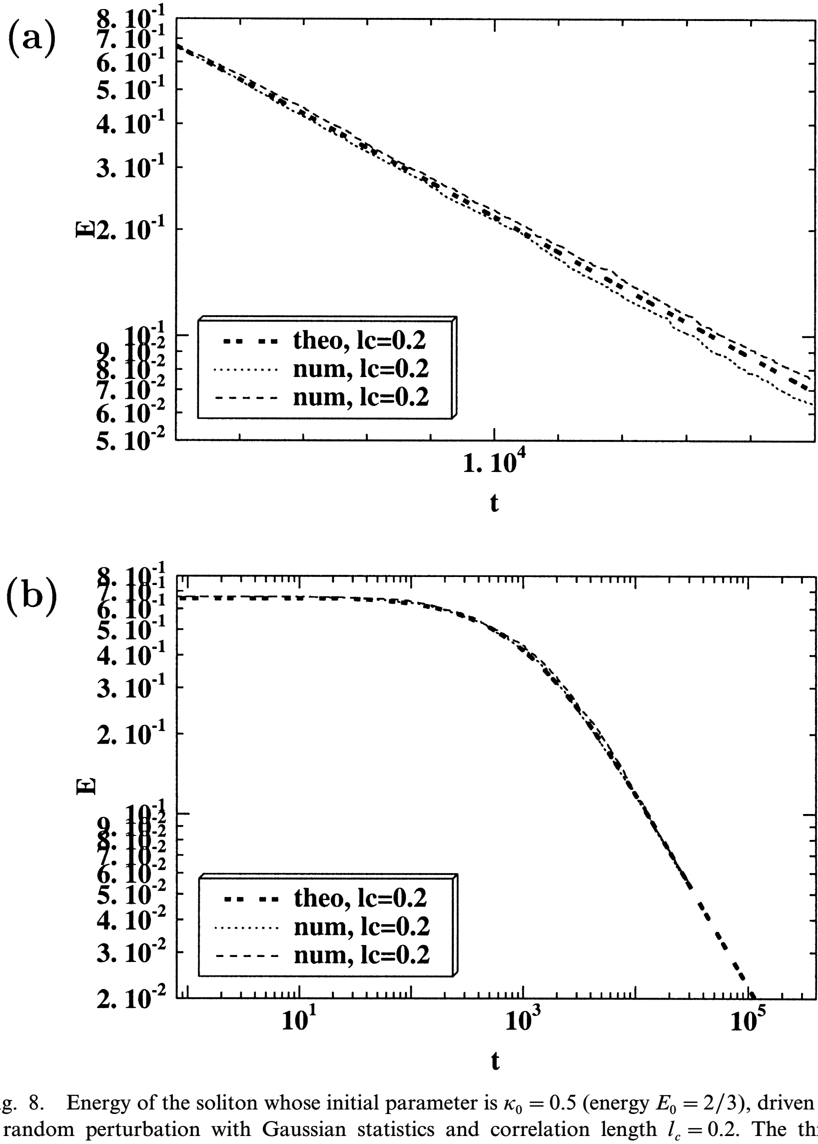


Fig. 8. Energy of the soliton whose initial parameter is $\kappa_0 = 0.5$ (energy $E_0 = 2/3$), driven by a random perturbation with Gaussian statistics and correlation length $l_c = 0.2$. The thick dotted lines represent the theoretical predictions. In thin dashed and dotted lines are plotted the simulated energies of the solitons for different realizations of the random perturbations. Figure 8(a) presents results corresponding to a perturbation V_1 of the velocity with $\varepsilon = 0.015$ (see Eq. (64)). We can observe the exponential decay of the energy. Figure 8(b) presents results corresponding to a perturbation V_2 of the nonlinear coefficient with $\varepsilon = 0.1$. We can observe the decay of the energy as $t^{-3/4}$.

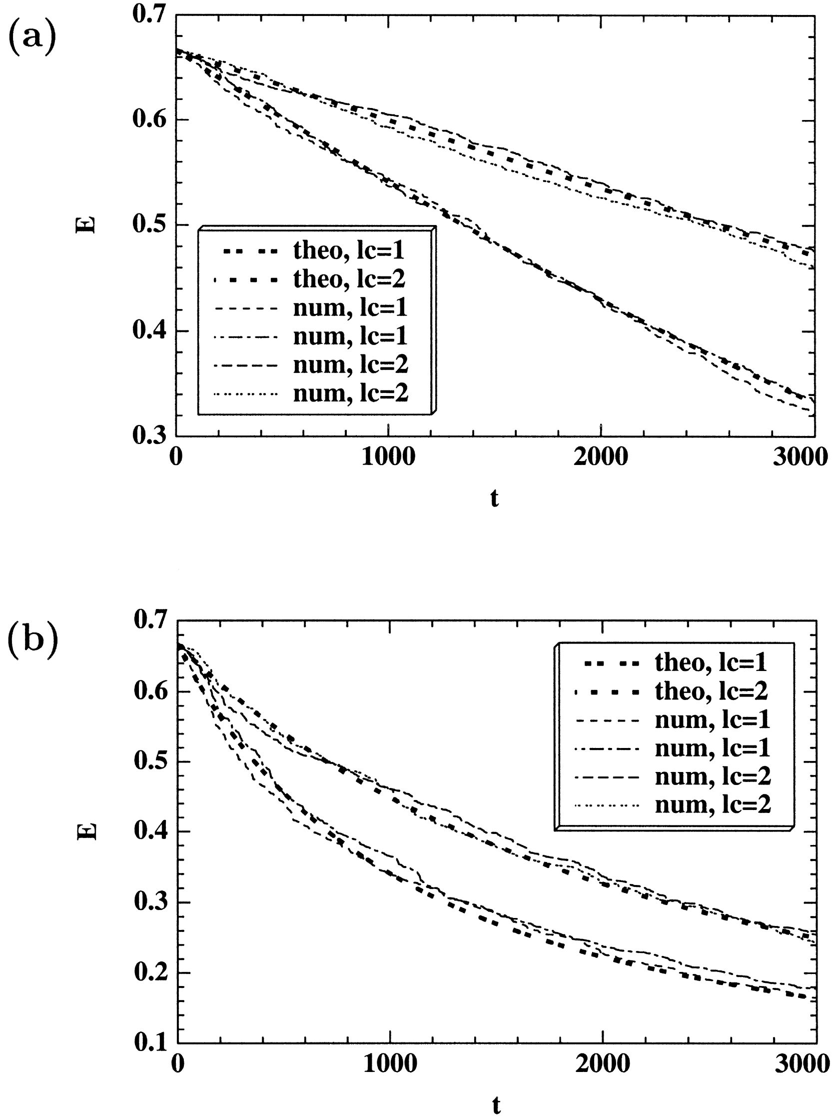


Fig. 9. The same as in Fig. 8, but the random perturbation has correlation length $l_c = 1$ or 2.

where W_t is a standard one-dimensional Brownian motion and $\sigma := \sqrt{\hat{\gamma}_V(0)}$. By the Wiener–Khintchine theorem it is known that $\hat{\gamma}_V(0)$ is non-negative. We should then distinguish between the case $\hat{\gamma}_V(0) = 0$ (no damping) and $\hat{\gamma}_V(0) > 0$ (presence of an effective damping).

6.1. Perturbation Without Damping

We assume in this section that $\hat{\gamma}_V(0) = 0$. We may think at the case where V is the time-derivative U' of some stationary process U with continuously differentiable realizations, autocorrelation function γ_U, and power spectral density $\hat{\gamma}_U$. Note that the autocorrelation functions of U and V are related through the identity:

$$\gamma_V(t) = \mathbb{E}[U'(t')\, U'(t'+t)] = -\mathbb{E}[U(t')\, U''(t'+t)] = -\gamma_U''(t)$$

so that their respective power spectral densities satisfy:

$$\hat{\gamma}_V(\omega) = \omega^2 \hat{\gamma}_U(\omega).$$

We then get that, with probability that goes to 1 as $\varepsilon \to 0$, the scattered wave at time t/ε^2 consists of one main soliton with parameter $\kappa^\varepsilon(t)$, a soliton gas, and radiation. The process $(\kappa^\varepsilon(t))_{t \in [0,T]}$ converges in probability to the deterministic function $(\kappa_l(t))_{t \in [0,T]}$ which satisfies the ordinary differential equation:

$$\frac{d\kappa_l}{dt} = F(\kappa_l), \qquad \kappa_l(0) = \kappa_0, \tag{71}$$

where

$$\begin{aligned} F(\kappa) &= -\frac{1}{16\kappa^2} \int_{\mathbb{R}} (2k)^2\, C_V(\kappa, k)\, dk, \\ C_V(\kappa, k) &= \frac{4\pi}{9 \sinh^2\left(\pi \dfrac{k}{\kappa}\right)}\, \hat{\gamma}_V(\Omega(k, \kappa)). \end{aligned} \tag{72}$$

If $V = U_t$ for some stationary process U with power spectral density $\hat{\gamma}_U$, then

$$F(\kappa) = -\frac{1}{16\kappa^2} \int_{\mathbb{R}} (2k)^2 C_U(\kappa, k)\, dk,$$

$$C_U(\kappa, k) = \frac{256\pi k^2 \kappa^4 \left(1 + \frac{k^2}{\kappa^2}\right)^2}{9 \sinh^2\left(\pi \frac{k}{\kappa}\right)} \hat{\gamma}_U(\Omega(k, \kappa)). \tag{73}$$

The result in this configuration is not surprising, and can be revisited in terms of the auxiliary field $\tilde{u}(t,x) := u(t,x) \exp(-\varepsilon \int_0^t V(s)\, ds)$. The field $\tilde{u}$ is solution of:

$$\tilde{u}_t + \tilde{u}_{xxx} - 6\tilde{u}\tilde{u}_x = 6\tilde{u}\tilde{u}_x \tilde{U}(t),$$

where $\tilde{U}(t) = \exp(\varepsilon \int_0^t V(s)\, ds) - 1$. From the identity $V = U_t$ we get that $\tilde{U}(t) = \exp \varepsilon(U(t) - U(0)) - 1 \simeq \varepsilon(U(t) - U(0))$. The auxiliary quantity $\tilde{u}$ thus obeys a KdV equation with a random time-dependent nonlinear coefficient. This configuration was studied in Section 3, and the results are consistent with each other (compare C_U and C_2 as defined by (40)). In Fig. 10 we compare numerical simulations with the theoretical predictions in the case $V = U'$ where U is a stationary random process with Gaussian statistics, Gaussian autocorrelation function $\gamma_U(t) = \exp(-t^2/(2t_c^2))$ with coherence time t_c. When we consider the energy of the soliton, we deals with a curve that is randomly modulated at time scale 1. Indeed the energy is preserved in the asymptotic framework $\varepsilon \to 0$, but for $\varepsilon > 0$:

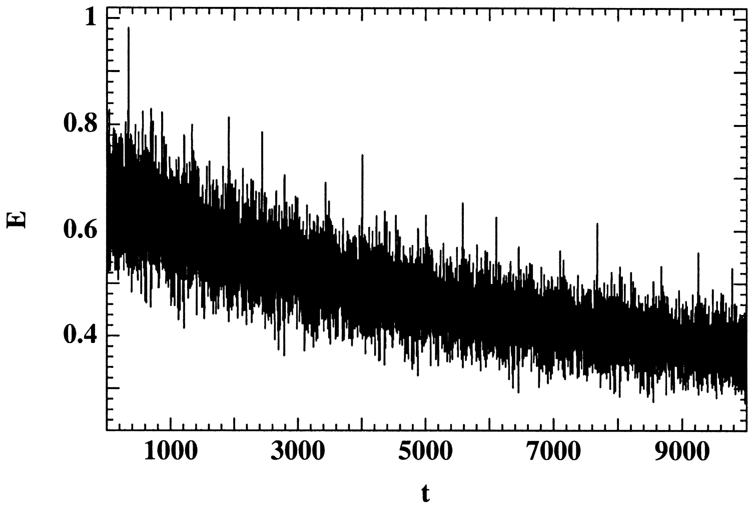
$$E_{\mathrm{tot}}(t) = E_0 \exp(2\varepsilon(U(t) - U(0))).$$

These local fluctuations (of amplitude $\sim \varepsilon$) involve the observed local fluctuations of the energy of the soliton (Fig. 10a). However these fluctuations have no importance when we consider the long-time behavior (at scale ε^{-2}) which is imposed by Eqs. (71) and (73) (Fig. 10b).

6.2. Perturbation with Damping

We assume in this section that $\hat{\gamma}_V(0) > 0$ and set $\sigma = \sqrt{\hat{\gamma}_V(0)}$. We then get that, with probability that goes to 1 as $\varepsilon \to 0$, the scattered wave at time t/ε^2 consists of one main soliton with parameter $\kappa^\varepsilon(t)$, a soliton gas, and

(a)



(b)

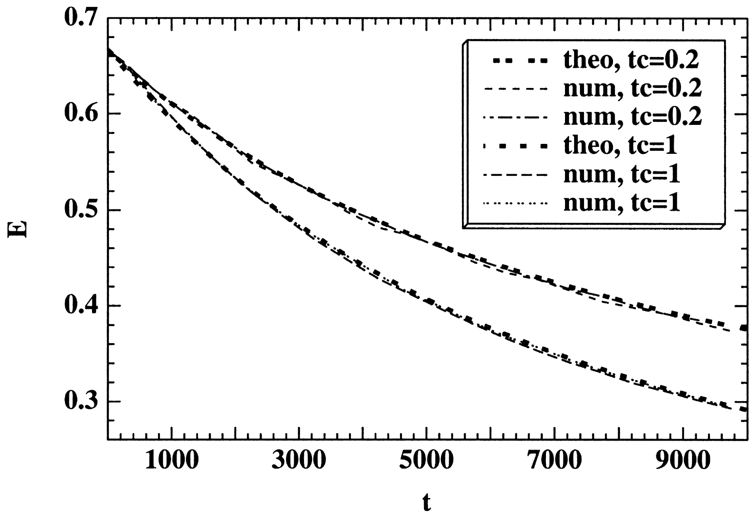


Fig. 10. Energy of the soliton whose initial parameter is $\kappa_0 = 0.5$ (energy $E_0 = 2/3$), driven by a random perturbation $\varepsilon V(t)\,u$ where V is the time derivative of a random process with Gaussian statistics, Gaussian autocorrelation function, coherence time t_c, and amplitude $\varepsilon = 0.1$. Figure 10(a): We can observe on this particular realization ($t_c = 0.2$) that the energy is fluctuating at time scale 1, but there is a noticeable trend at time scale ε^{-2}. Figure 10(b): The numerical results are smoothed by averaging the energies over a time interval of duration 100. They can then be compared with the theoretical curves.

radiation. The process $(\kappa^{\varepsilon}(t))_{t\in[0,T]}$ converges in probability to the random function $(\kappa_l(t))_{t\in[0,T]}$ which satisfies the *stochastic* differential equation:

$$d\kappa_l = F(\kappa_l)\,dt + \tfrac{2}{3}\,\sigma\kappa_l \circ dW_t, \qquad \kappa_l(0)=\kappa_0, \tag{74}$$

where

$$\begin{aligned} F(\kappa) &= -\frac{1}{16\kappa^2}\int_{\mathbb{R}} (2k)^2\, C_V(\kappa,k)\,dk, \\ C_V(\kappa,k) &= \frac{4\pi}{9\sinh^2\left(\pi\dfrac{k}{\kappa}\right)}\,\hat{\gamma}_V(\Omega(k,\kappa)), \end{aligned} \tag{75}$$

and $\circ$ stands for the Stratonovich integral. In case of a white noise $\gamma(t)=\sigma^2\delta(t)$, we have $F(\kappa)=-\frac{\sigma^2}{27}\,\kappa$, so that the solution of the stochastic differential equation has a closed-form expression:

$$\kappa_l(t)=\kappa_0\exp\left(-\frac{\sigma^2}{27}\,t+\frac{2}{3}\,\sigma W_t\right). \tag{76}$$

Note that the typical behavior of the parameter κ is to decay exponentially with t, since $W_t \sim \sqrt{t}$:

$$\frac{1}{t}\ln\kappa_l(t) \overset{\sigma^2 t\gg 1}{\simeq} -\frac{\sigma^2}{27}.$$

However the moments of κ_l grow exponentially:

$$\mathbb{E}[\kappa_l^n(t)]=\kappa_0^n\exp\left(\left(\frac{2n^2}{9}-\frac{n}{27}\right)\sigma^2 t\right).$$

In particular the mean energy ($n=3$) growths as $\exp(17\sigma^2 t/9)$, while its typical value decays as $\exp(-\sigma^2 t/9)$. The mean value is actually imposed by a very small set of realizations of the random potential V for which the energy takes exceptional and very large values. Note also that the ratio of the energy E_s of the soliton and of the total energy E_{tot} is non-random:

$$\frac{E_s}{E_{tot}}=\exp\left(-\frac{\sigma^2}{9}\,t\right). \tag{77}$$

In Fig. 11 we assume that V is a step-wise constant potential that takes the value v_n over the interval $[nt_c, (n+1)\,t_c)$, $n\in\mathbb{N}$, where $(v_n)_{n\in\mathbb{N}}$ is a sequence

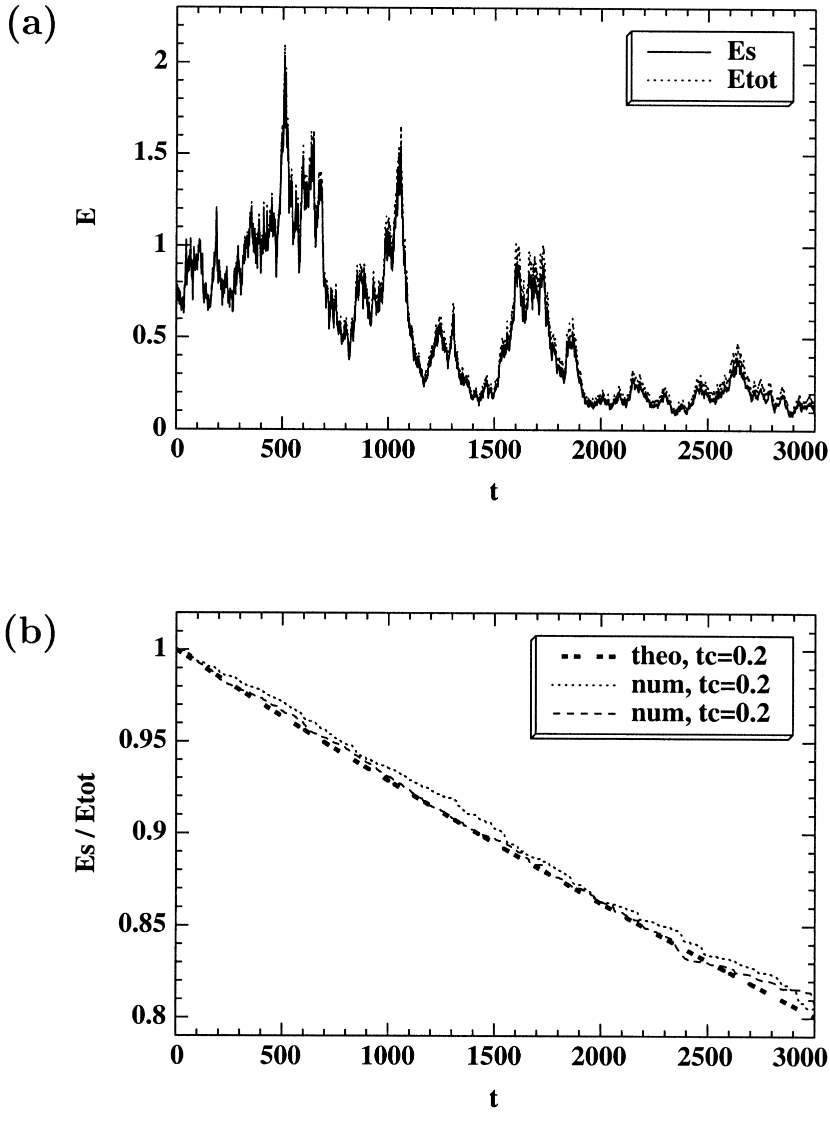


Fig. 11. Energy of the soliton whose initial parameter is $\kappa_0 = 0.5$ (energy $E_0 = 2/3$), driven by a random perturbation $\varepsilon V(t)\, u$ where V is stepwise constant with coherence time $t_c = 0.2$, and amplitude $\varepsilon = 0.1$. Figure 11(a): The energy of the soliton E_s is plotted together with the total energy E_{tot} as defined by (70). We can observe that these quantities are very close to each other, although a slight departure can be detected as time increases. Figure 11(b): The ratio between E_s and E_{tot} is plotted for two different realizations and compared to the theoretical predictions (77).

of independent and identically distributed random variables with uniform distributions over $[-1, 1]$. We consider configurations close to the white noise case by setting $t_c = 0.2$, so that the product of the typical frequency $8\kappa_0^3$ times t_c is smaller than 1. We can see that the energy of the soliton varies randomly (Fig. 11a), and follows rather closely the total energy. Actually the part of the total energy (70) that is contained in the soliton is clearly non-random and follows the theoretical curve (Fig. 11b).

7. CONCLUSIONS

We have applied different types of random perturbations to the integrable Korteweg–de Vries equation. We have studied the propagation of a soliton in these random systems. Our method can be applied to a large class of random perturbations that are in some sense stationary and ergodic. The only but important condition is that some integral of motion should be preserved by the perturbation, or else that it could be a priori computed. Indeed, if the amount of radiation can be estimated in very great generality, the feedback of this generation onto the evolution of the soliton parameter will be imposed by the underlying conservation relation and it will strongly depends on the nature of the conserved quantity. If the energy is preserved, then the soliton parameter will decay. This is the case for many realistic perturbation models with time-dependent or position-dependent coefficients. However we have also exhibited some models where our analysis shows that the soliton is speeded-up.

As a result our approach could be applied to any perturbation which produces a computable change in some integral of motion. However, as pointed out in Section 3.3, the final step of the derivation of the result consists in an a posteriori check of the so-called adiabatic hypothesis. So we could state a very general result only in a formal way, that is to say provided that an implicit condition is fulfilled. Our approach can thus be generalized, but a specific study will still be necessary for any new type of perturbations.

We have put into evidence that the scattering of the soliton generates not only radiation during its motion, but also a soliton gas, that is to say a collection of J^ε, of order ε^{-2}, solitons with small masses of order ε^2 (remember ε is the dimensionless parameter that governs the amplitude of the perturbation $\varepsilon R(u)$ and the duration of the propagation T/ε^2). In the asymptotic framework where ε goes to zero, the soliton gas has non-zero mass, but zero energy. The production of the soliton gas is interesting by itself as a new phenomenon that is not encountered when a random NLS equation is considered, but it should also be pointed out that this production is very important in that one cannot understand correctly the changes in the conservation equations without accounting for soliton production. Finally note that, in the framework of the inverse scattering transform, radiation corresponds to the absolutely continuous spectrum, the main soliton corresponds to the discrete pure point spectrum, and the soliton gas would correspond to the singular continuous spectrum. It should be interesting to study in more detail the structure of the soliton gas.

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